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ON THE NUMBER OF SOLUTIONS OF THE CONGRUENCE $xy \equiv l \pmod q$ UNDER THE GRAPH OF A TWICE CONTINUOUSLY DIFFERENTIABLE FUNCTION

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Abstract. A result by V. A. Bykovskiĭ (1981) on the number of solutions of the congruence $xy \equiv l \pmod q$ under the graph of a twice continuously differentiable function is refined. As an application, Porter's result (1975) on the mean number of steps in the Euclidean algorithm is sharpened and extended to the case of Gauss–Kuzmin statistics.

§1. Introduction

Notation.

1) For a natural number q, we denote by $\delta_q(n)$ the characteristic function of divisibility by q:
$$\delta_q(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod q, \\ 0 & \text{if } n \not\equiv 0 \pmod q. \end{cases}$$
2) The sum of powers of divisors of a natural number q is denoted by
$$\sigma_\alpha(q) = \sum_{d|q} d^\alpha.$$
3) If A is a statement, then $[A]$ means 1 if A is true and 0 otherwise.
4) For a rational number r, $r = [t_0; t_1, \dots, t_s]$ denotes the canonical continued fraction of length $s = s(r)$, where $t_0 = [r]$ (the integral part of r), $t_1, \dots, t_s$ are quotients (natural numbers), and $t_s \geq 2$ for $s \geq 1$.
5) For rational $r = [t_0; t_1, \dots, t_s]$ and real $x \in [0, 1]$, by $s^{(x)}(r)$ we denote the Gauss–Kuzmin statistics
$$s^{(x)}(r) = \#\{j : 1 \leq j \leq s, [0; t_j, \dots, t_s] \leq x\}.$$

In particular, the length of a continued fraction is $s = s(r) = s^{(1)}(r)$.

Let q be a natural number, l an integer, and f a nonnegative function. Denote by $T[f]$ the number of solutions of the congruence $xy \equiv l \pmod q$ that lie in the domain $P_1 < x \leq P_2$, $0 < y \leq f(x)$:
$$T[f] = \sum_{P_1 < x \leq P_2} \sum_{0 < y \leq f(x)} \delta_q(xy - l).$$

In a series of number-theoretic problems, the need for asymptotic formulas for $T[f]$ arises. They underlie results on convolutions of arithmetic functions [10, 12], on sums

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of arithmetic functions on the values of a quadratic polynomial [2, 12], on statistical properties of the Euclidean algorithm [1, 5, 13], etc.

We denote

$$S[f] = \frac{1}{q} \sum_{P_1 < x \le P_2} \mu_{q,l}(x) f(x),$$

where $\mu_{q,l}(x)$ is the number of solutions of the congruence $xy \equiv l \pmod q$ with respect to the variable y lying within the limits $1 \le y \le q$.

The following statement is the main result of the present paper.

Theorem 1. *Let P_1 and P_2 be real numbers, let $P = P_2 - P_1 \ge 2$, and let a nonnegative function $f(x)$ be twice continuously differentiable on the entire interval $[P_1, P_2]$. Suppose that*

$$\frac{1}{A} \le |f''(x)| \le \frac{w}{A}$$

for some $A > 0$ and $w \ge 1$. Then the following asymptotic formula is valid:

$$T[f] = S[f] - \frac{P}{2} \cdot \delta_q(l) + R[f], \tag{1}$$

where

$$\begin{aligned} R[f] \ll_w \sigma_0^{2/3}(q)\sigma_0^{5/3}(a)\sigma_{-1/2}^{4/3}(a)PA^{-1/3} + \sigma_0(q)\sigma_0(a)(A^{1/2}a^{1/2}\sigma_{-1}(q)\sigma_{-1/2}(a) \\ + q^{1/2}\sigma_0(a)\sigma_{-1/2}^2(a)\log^2 P + a\log P) \end{aligned} \tag{2}$$

and $a = (l, q)$.

This theorem refines a result of the paper [2], where formula (1) was proved with the remainder term

$$R[f] \ll a^{1/2} q^{\varepsilon}((PA^{-1/3} + A^{2/3})\log^{4/3} P + q^{1/2}\log^2 P).$$

As an application of Theorem 1, we prove a refinement of a result due to Porter [13] (see also [1]) extended to the case of the Gauss–Kuzmin statistic.

Theorem 2. *Suppose $b \ge 2$ is natural and $x \in (0, 1]$ is real. Then the sum*

$$N_x^*(b) = \sum_{\substack{1 \le a \le b \\ (a,b)=1}} s^{(x)}(a/b)$$

satisfies the following asymptotic formula:

$$N_x^*(b) = \frac{2\varphi(b)}{\zeta(2)}(\log(x+1)\log b + C(x)) + O_{\varepsilon,x}\left(b^{5/6}\log^{7/6+\varepsilon} b\right),$$

where $\varepsilon > 0$ is an arbitrarily small number,

$$\begin{aligned} C(x) = \log(1+x)\left(\log x - \frac{\log(x+1)}{2} + 2\gamma - 2\frac{\zeta'(2)}{\zeta(2)} - 1\right) \\ + h_1(x) + h_2(x) + \frac{\zeta(2)}{2}\left(x \cdot [x < 1] - \frac{x}{1+x}\right), \end{aligned} \tag{3}$$

$$h_1(x) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{m=1}^{n} \frac{x}{n+mx} - \log(1+x)\right), \tag{4}$$

$$h_2(x) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{\frac{n}{x} \le m < \frac{n}{x}+n} \frac{1}{m} - \log(1+x)\right). \tag{5}$$

Moreover, the estimate of the remainder term is uniform in x provided $x \in [x_0, 1]$ for some fixed $x_0 > 0$.

§2. Estimates of Kloosterman sums

Suppose q is a natural number and l, m, n are integers. We define the sums

$$K_q(l,m,n) = \sum_{x,y=1}^{q} \delta_q(xy-l) e^{2\pi i \frac{mx+ny}{q}}; \tag{6}$$

in a special case they coincide with the classical Kloosterman sums

$$K_q(m,n) = K_q(1,m,n) = \sum_{x,y=1}^{q} \delta_q(xy-1) e^{2\pi i \frac{mx+ny}{q}}.$$

We mention the simplest properties of the sums $K_q(l,m,n)$.

1° If $(l,q)=1$, then $K_q(l,m,n) = K_q(lm,n)$.

2° If $q = q_1 q_2$ and $(q_1,q_2)=1$, then

$$K_q(l,m,n) = K_{q_1}(\bar{q}_2^2 l, m, n) K_{q_2}(\bar{q}_1^2 l, m, n),$$

where $\bar{q}_1$ and $\bar{q}_2$ are solutions of the congruences $q_1\bar{q}_1 \equiv 1 \pmod{q_2}$ and $q_2\bar{q}_2 \equiv 1 \pmod{q_1}$.

3° For any permutation $\sigma \in S_3$, we have

$$K_q(n_1,n_2,n_3) = K_q(n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}).$$

4° $K_q(l,m,n) = K_q(l,-m,-n)$.

We obtain the first property if we put $x = lx_1$ in the definition (6). To prove the second property, it suffices to make the substitutions

$$x = x_1 q_2 + x_2 q_1, \quad y = y_1 q_2 + y_2 q_1 \quad (1 \le x_1, y_1 \le q_1, 1 \le x_2, y_2 \le q_2).$$

The third property follows from the relation

$$K_q(l,m,n) = \frac{1}{q} \sum_{x,y,z=1}^{q} e^{2\pi i \frac{mx+ny+lz-xyz}{q}}.$$

The fourth property is obtained with the help of the substitution $x \to -x$, $y \to -y$.

In [7], for the classical Kloosterman sums, Estermann proved the estimate

$$|K_q(m,n)| \le \sigma_0(q) \cdot (m,n,q)^{1/2} \cdot q^{1/2}. \tag{7}$$

A similar inequality remains valid for the sums $K_q(l,m,n)$.

Lemma 1. *Suppose q is natural, l, m, and n are integers, and $a = (l,q)$. Then*

$$|K_q(l,m,n)| \le f_q(l,m,n) \cdot q^{1/2}, \tag{8}$$

where

$$f_q(l,m,n) = \sigma_0(q)\sigma_0((l,m,n,q))(lm,ln,mn,q)^{1/2}.$$

Proof. Property 2° of the sums $K_q(l,m,n)$ and the multiplicativity of $f_q(l,m,n)$ as a function of q show that it suffices to prove (8) in the case where q is a power of a prime number.

Suppose p is prime, $\alpha \ge 1$, $q = p^\alpha$, and $(l,m,n,p^\alpha) = p^\lambda$. If $\lambda = \alpha$, then

$$K_q(l,m,n) = \sum_{x,y=1}^{p^\alpha} \delta_{p^\alpha}(xy) = (\alpha+1)p^\alpha - \alpha p^{\alpha-1} \le \sigma_0(q) q,$$

and (8) is fulfilled. Now we assume that $\lambda \le \alpha - 1$, $l = p^\lambda l_1$, $m = p^\lambda m_1$, and $n = p^\lambda n_1$. By the symmetry 3° of the Kloosterman sums, we may assume that $(l_1, p) = 1$. Then, after the changes $x = p^\beta x_1$, $y = p^{\lambda-\beta} y_1$ $(0 \le \beta \le \lambda)$, we obtain

$$\begin{aligned} K_{p^\alpha}(l,m,n) &= \sum_{\beta=0}^{\lambda} \sum_{x_1=1}^{p^{\alpha-\beta}} \sum_{y_1=1}^{p^{\alpha-\lambda+\beta}} \delta_{p^{\alpha-\lambda}}(x_1 y_1 - l_1) e^{2\pi i \frac{m_1 p^\beta x_1 + n_1 p^{\lambda-\beta} y_1}{p^{\alpha-\lambda}}} \\ &= p^\lambda \sum_{\beta=0}^{\lambda} \sum_{x_1=1}^{p^{\alpha-\lambda}} \sum_{y_1=1}^{p^{\alpha-\lambda}} \delta_{p^{\alpha-\lambda}}(x_1 y_1 - l_1) e^{2\pi i \frac{m_1 p^\beta x_1 + n_1 p^{\lambda-\beta} y_1}{p^{\alpha-\lambda}}} \\ &= p^\lambda \sum_{\beta=0}^{\lambda} K_{p^{\alpha-\lambda}}(m_1 p^\beta, n_1 p^{\lambda-\beta}, l_1). \end{aligned}$$

Using property 1° of the sums $K_q(l,m,n)$ and Estermann's estimate (7), we find that

$$\begin{aligned} &\left|K_{p^{\alpha-\lambda}}(m_1 p^\beta, n_1 p^{\lambda-\beta}, l_1)\right| = \left|K_{p^{\alpha-\lambda}}(l_1 m_1 p^\beta, n_1 p^{\lambda-\beta}, 1)\right| \\ &\quad = \left|K_{p^{\alpha-\lambda}}(l_1 m_1 p^\beta, n_1 p^{\lambda-\beta})\right| \le \sigma_0(p^\alpha)(m_1 p^\beta, n_1 p^{\lambda-\beta}, p^{\alpha-\lambda})^{1/2} p^{(\alpha-\lambda)/2}. \end{aligned}$$

Taking the relation $\lambda + 1 = \sigma_0((l,m,n,p^\alpha))$ into account, we arrive at the inequality

$$|K_{p^\alpha}(l,m,n)| \le \sigma_0(p^\alpha)\sigma_0((l,m,n,p^\alpha)) p^{\alpha/2} \max_{0 \le \beta \le \lambda} (mp^\beta, np^{\lambda-\beta}, p^\alpha)^{1/2}.$$

Observing that

$$(mp^\beta, np^{\lambda-\beta}, p^\alpha) \le (lm, ln, mn, p^\alpha),$$

we complete the proof of the lemma. □

Remark 1. The sharper estimate

$$(mp^\beta, np^{\lambda-\beta}, p^\alpha)^2 = (m^2 p^{2\beta}, n^2 p^{2\alpha-2\beta}, p^{2\alpha}) \le (l^2 m^2, l^2 n^2, m^2 n^2, lmn, p^{2\alpha})$$

shows that the inequality in Lemma 1 is valid with the coefficient

$$f_q(l,m,n) = \sigma_0(q)\sigma_0((l,m,n,q))(l^2 m^2, l^2 n^2, m^2 n^2, lmn, q^2)^{1/4}.$$

Corollary 1. *Suppose q is natural, l is an integer, and $a = (l,q)$. Then*

$$\sum_{m,n=1}^{q} \frac{|K_q(l,m,n)|}{m \cdot n} \ll \sigma_0(q)\sigma_0^2(a)\sigma_{-1/2}^2(a)\log^2(q+1)q^{1/2}.$$

Proof. By Lemma 1,

$$|K_q(l,m,n)| \le \sigma_0(q)\sigma_0(a)(lm, ln, mn, q)^{1/2} q^{1/2}.$$

Therefore, to prove the corollary it suffices to verify that the sum

$$S = \sum_{m,n=1}^{q} \frac{(lm, ln, mn, q)^{1/2}}{mn}$$

satisfies the estimate

$$S \ll \sigma_0(a)\sigma_{-1/2}^2(a)\log^2(q+1). \tag{9}$$

We transform the sum S:

$$\begin{aligned} S &= \sum_{\delta|q} \delta^{1/2} \sum_{m,n=1}^{q} \frac{[(lm, ln, mn) = \delta]}{mn} \\ &\le \sum_{\delta|q} \delta^{1/2} \sum_{m,n=1}^{q} \frac{1}{mn} \left[\frac{\delta}{(l,\delta)} \mid m, \frac{\delta}{(l,\delta)} \mid n, \delta \mid mn \right] \\ &\le \sum_{\delta|q} \delta^{1/2} \sum_{m_1,n_1=1}^{q} \frac{1}{m_1 n_1} \cdot \frac{(l,\delta)^2}{\delta^2} [(l,\delta)^2 \mid m_1 n_1 \delta]. \end{aligned}$$

Henceforth, $[A]$ denotes 1 if the assertion A is true and 0 otherwise.

Introducing the parameters $\delta_1 = (\delta, l)$ and $\delta_2 = (\delta, \delta_1^2)$, we find that

$$\begin{aligned} S &\le \sum_{\delta_1|(l,q)} \delta_1^2 \sum_{\substack{\delta|q \\ (\delta,l)=\delta_1}} \frac{1}{\delta^{3/2}} \sum_{m_1,n_1=1}^{q} \frac{[\delta_1^2 \mid m_1 n_1 \delta]}{m_1 n_1} \\ &\le \sum_{\delta_1|(l,q)} \delta_1^2 \sum_{\substack{\delta|q \\ (\delta,l)=\delta_1}} \frac{1}{\delta^{3/2}} \sum_{m_1,n_1=1}^{q} \frac{1}{m_1 n_1} \left[\frac{\delta_1^2}{\delta_2} \mid m_1 n_1 \right]. \end{aligned}$$

Note that $\delta_1 \mid \delta_2$ and $\delta_1^2/\delta_2 \mid \delta_1$. Consequently, the estimate

$$\sum_{m_1,n_1=1}^{q} \frac{1}{m_1 n_1} [d \mid m_1 n_1] \le \sum_{k=1}^{q^2} \frac{\sigma_0(dk)}{dk} \ll \frac{\sigma_0(d)}{d} \cdot \log^2(q+1)$$

implies that

$$S \ll \sigma_0(a) \log^2(q+1) \sum_{\delta_1|(l,q)} \sum_{\substack{\delta|q \\ (\delta,l)=\delta_1}} \frac{\delta_2}{\delta^{3/2}} = \sigma_0(a) \log^2(q+1) \sum_{\delta|q} \frac{\delta_2}{\delta^{3/2}}.$$

After the changes $\delta = \delta_1 \cdot \delta_0$ and $(\delta_1, \delta_0) = \delta'$, we have

$$\begin{aligned} \sum_{\delta|q} \frac{\delta_2}{\delta^{3/2}} &= \sum_{\delta_1|(l,q)} \sum_{\substack{\delta|q \\ (\delta,l)=\delta_1}} \frac{(\delta_1^2, \delta)}{\delta^{3/2}} \le \sum_{\delta_1|(l,q)} \delta_1^{-1/2} \sum_{\delta_0|q} \frac{(\delta_1, \delta_0)}{\delta_0^{3/2}} \\ &= \sum_{\delta_1|(l,q)} \delta_1^{-1/2} \sum_{\delta'|\delta_1} \delta' \sum_{\substack{\delta_0|q \\ (\delta_0,\delta_1)=\delta'}} \frac{1}{\delta_0^{3/2}} \ll \sum_{\delta_1|a} \delta_1^{-1/2} \sum_{\delta'|a} (\delta')^{-1/2}. \end{aligned}$$

Therefore,

$$\sum_{\delta|q} \frac{\delta_2}{\delta^{3/2}} \ll \sigma_{-1/2}^2(a). \tag{10}$$

Thus, estimate (9) is proved, together with the lemma. □

Separately, we estimate the sums $K_q(l, m, n)$ in the case where one of the arguments is equal to zero. Let

$$c_q(m, n) = K_q(0, m, n) = \sum_{x,y=1}^{q} \delta_q(xy) e^{2\pi i \frac{mx+ny}{q}}.$$

Such sums are generalizations of the Ramanujan sums

$$c_q(n) = \sum_{x=1}^{q}{}^{*} e^{2\pi i \frac{nx}{q}}$$

(from now on the asterisk means that the summation is over the reduced system of residues), because

$$c_q(n) = K_q(0,1,n) = c_q(1,n).$$

By property 2° of the Kloosterman sums $K_q(l,m,n)$ for $q = q_1q_2$, $(q_1,q_2) = 1$, we have

$$c_q(m,n) = c_{q_1}(m,n)c_{q_2}(m,n).$$

Therefore, to compute the sums $c_q(m,n)$, it suffices to consider the case where q is a power of a prime number.

Lemma 2. *Suppose p is prime, $\alpha \geq 1$, and $q = p^\alpha$. Then*

$$c_q(m,n) = g_{p^\alpha}(m,n) - g_{p^{\alpha-1}}(m,n),$$

where

$$g_q(m,n) = q \cdot \delta_q((m,q)(n,q))\sigma_0((m,q)(n,q)q^{-1}). \tag{11}$$

Proof. If $xy \equiv 0 \pmod{p^\alpha}$, then the relations

$$x = p^\beta x_1, \quad (x_1,p) = 1, \quad y = p^{\alpha-\beta}y_1$$

hold true for some β $(0 \leq \beta \leq \alpha)$. Therefore,

$$\begin{aligned} c_q(m,n) &= \sum_{\beta=0}^{\alpha} \sum_{x_1=1}^{p^{\alpha-\beta}}{}^{*} \sum_{y_1=1}^{p^\beta} e^{2\pi i \frac{mp^\beta x_1 + np^{\alpha-\beta}y_1}{p^\alpha}} \\ &= \sum_{\beta=0}^{\alpha} \sum_{x_1=1}^{p^{\alpha-\beta}} \sum_{y_1=1}^{p^\beta} e^{2\pi i \left(\frac{mx_1}{p^{\alpha-\beta}} + \frac{ny_1}{p^\beta}\right)} - \sum_{\beta=0}^{\alpha-1} \sum_{x_1=1}^{p^{\alpha-\beta-1}} \sum_{y_1=1}^{p^\beta} e^{2\pi i \left(\frac{mx_1}{p^{\alpha-\beta-1}} + \frac{ny_1}{p^\beta}\right)} \\ &= g_{p^\alpha}(m,n) - g_{p^{\alpha-1}}(m,n), \end{aligned}$$

where

$$g_{p^\alpha}(m,n) = p^\alpha \sum_{\beta=0}^{\alpha} \delta_{p^{\alpha-\beta}}(m)\delta_{p^\beta}(n).$$

To verify (11), we note that if $p^\alpha \nmid (m,p^\alpha)(n,p^\alpha)$, then $g_{p^\alpha}(m,n) = 0$. If $(m,p^\alpha) = p^{\nu_1}$, $(n,p^\alpha) = p^{\nu_2}$, and $\nu_1 + \nu_2 \geq \alpha$, then

$$\begin{aligned} g_{p^\alpha}(m,n) &= p^\alpha \sum_{\beta=0}^{\alpha} [\alpha - \nu_1 \leq \beta \leq \nu_2] \\ &= p^\alpha(\nu_1 + \nu_2 - \alpha + 1) = p^\alpha \cdot \sigma_0((m,p^\alpha)(n,p^\alpha)p^{-\alpha}). \end{aligned}$$

□

Corollary 2. *For any natural q and integers m and any n, we have*

$$|c_q(m,n)| \leq \sigma_0((m,q))(q,mn). \tag{12}$$

In particular,

$$K_q(m,0,0) = |c_q(m,0)| \leq \sigma_0((m,q))q. \tag{13}$$

Proof. Since the quantity on the right-hand side of (12) (as well as $c_q(m,n)$) is a multiplicative function of the parameter q, it suffices to prove (12) for powers of prime numbers. Suppose p is prime, $\alpha \geq 1$, and $q = p^\alpha$. Consider three cases:

(1) $p^\alpha \mid (m,p^\alpha)(n,p^\alpha)$, and thus $p^{\alpha-1} \mid (m,p^{\alpha-1})(n,p^{\alpha-1})$;

(2) $p^\alpha \nmid (m,p^\alpha)(n,p^\alpha)$, but $p^{\alpha-1} \mid (m,p^{\alpha-1})(n,p^{\alpha-1})$;

(3) $p^{\alpha-1} \nmid (m,p^{\alpha-1})(n,p^{\alpha-1})$.

In the first case, $g_{p^\alpha}(m,n) > 0$, $g_{p^{\alpha-1}}(m,n) > 0$, and for $(m,p^\alpha) = p^{\nu_1}$, $(n,p^\alpha) = p^{\nu_2}$ we have

$$\begin{aligned} g_{p^{\alpha-1}}(m,n) &= p^{\alpha-1}(\min\{\nu_1, \alpha-1\} + \min\{\nu_2, \alpha-1\} - \alpha + 2) \\ &\leq p^{\alpha-1}(\nu_1 + \nu_2 - \alpha + 2) \leq p^\alpha(\nu_1 + \nu_2 - \alpha + 1) = g_{p^\alpha}(m,n). \end{aligned}$$

Therefore, by Lemma 2,

$$\begin{aligned} 0 \leq c_{p^\alpha}(m,n) &= g_{p^\alpha}(m,n) - g_{p^{\alpha-1}}(m,n) \leq g_{p^\alpha}(m,n), \\ |c_{p^\alpha}(m,n)| \leq g_{p^\alpha}(m,n) &\leq p^\alpha \sigma_0((m,p^\alpha))\delta_{p^\alpha}((m,p^\alpha)(n,p^\alpha)) = \sigma_0((m,q))(q,mn). \end{aligned}$$

Similarly, in the second case we obtain

$$\begin{aligned} |c_q(m,n)| &= g_{p^{\alpha-1}}(m,n) \\ &\leq p^{\alpha-1}\sigma_0((m,p^\alpha))\delta_{p^{\alpha-1}}((m,p^{\alpha-1})(n,p^{\alpha-1})) = \sigma_0((m,q))(q,mn). \end{aligned}$$

Finally, in the third case we have $|c_q(m,n)| = 0$. □

Corollary 3. *Suppose q is a natural number, l is an integer, and $a = (l,q)$. Then*

$$\sum_{n=1}^{q} \frac{|c_q(l,n)|}{n} \ll \sigma_0(q)\sigma_0(a)\log(q+1) \cdot a.$$

Proof. By Corollary 2,

$$\begin{aligned} \sum_{n=1}^{q} \frac{|c_q(l,n)|}{n} &\leq \sigma_0(a) \sum_{n=1}^{q} \frac{1}{n}(q, ln) = a\sigma_0(a) \sum_{n=1}^{q} \frac{1}{n}\left(\frac{q}{a}, n\right) \\ &\leq a\sigma_0(a) \sum_{\delta \mid q/a} \delta \sum_{\substack{n=1 \\ \delta \mid n}}^{q} \frac{1}{n} \ll a\sigma_0(a)\sigma_0(q)\log(q+1). \end{aligned}$$ □

Lemma 3. *Suppose $q \geq 1$ is a natural number, l is an integer, Q_1, Q_2, P_1, P_2 are real, $0 \leq P_1, P_2 \leq q$, and $a = (l,q)$. Then for the sum*

$$\Phi_q(Q_1, Q_2; P_1, P_2) = \sum_{\substack{Q_1 < u \leq Q_1 + P_1 \\ Q_2 < v \leq Q_2 + P_2}} \delta_q(uv - l)$$

we have the asymptotic formula

$$\Phi_q(Q_1, Q_2; P_1, P_2) = \frac{K_q(l,0,0)}{q^2} \cdot P_1 P_2 + O\left(\psi_l(q)\right),$$

where

$$\psi_l(q) = \sigma_0(q)\sigma_0^2(a)\sigma_{-1/2}^2(a)\log^2(q+1)q^{1/2} + \sigma_0(q)\sigma_0(a)\log(q+1)a. \tag{14}$$

Proof. We define the integers

$$\begin{aligned} M_1 &= [Q_1], & M_2 &= [Q_2], \\ N_1 &= [Q_1 + P_1] - [Q_1], & N_2 &= [Q_2 + P_2] - [Q_2]. \end{aligned}$$

Then

$$\Phi_q(Q_1, Q_2; P_1, P_2) = \Phi_q(M_1, M_2; N_1, N_2) \tag{15}$$

and

$$0 \le N_1, N_2 \le q, \tag{16}$$

because

$$0 \le N_j = [Q_j + P_j] - [Q_j] = P_j + \{Q_j\} - \{Q_j + P_j\} \le q \quad (j = 1, 2).$$

First, we prove the lemma with M_1, M_2, N_1, N_2 in place of Q_1, Q_2, P_1, P_2. If one of the numbers N_1 and N_2 is equal to zero, then the lemma is trivial. For this reason, in the sequel we assume that N_1 and N_2 are natural numbers. We define two functions

$$F_1 : \{M_1 + 1, \dots, M_1 + q\} \to \{0, 1\}, \quad F_2 : \{M_2 + 1, \dots, M_2 + q\} \to \{0, 1\}$$

by setting

$$F_j(x) = \begin{cases} 1 & \text{if } \ M_j < x \le M_j + N_j, \\ 0 & \text{if } \ M_j + N_j < x \le M_j + q. \end{cases}$$

These functions have the following Fourier series expansions:

$$F_j(x) = \sum_{-q/2 < k \le q/2} \widehat{F}_j(k) e^{2\pi i \frac{kx}{q}},$$

where

$$\widehat{F}_j(k) = \frac{1}{q} \sum_{y=M_j+1}^{M_j+N_j} e^{-2\pi i \frac{ky}{q}}.$$

For $k = 0$ we have

$$\widehat{F}_j(0) = \frac{1}{q} N_j,$$

and for the other $k \in (-q/2, q/2]$ we can sum the geometric progression to find

$$\widehat{F}_j(k) = \frac{1}{q} \cdot \frac{1 - e^{-2\pi i k N_j/q}}{1 - e^{-2\pi i k/q}} \cdot e^{-2\pi i k (M_j+1)/q}.$$

Therefore,

$$\left|\widehat{F}_j(k)\right| = \frac{1}{q} \cdot \frac{|\sin(\pi k N_j/q)|}{|\sin(\pi k/q)|} \le \frac{1}{q|\sin(\pi k/q)|} \le \frac{1}{2|k|} \qquad (-q/2 < k \le q/2; \ \ k \ne 0). \tag{17}$$

In accordance with what has been said above,

$$\begin{aligned} \Phi_q(M_1, M_2; N_1, N_2) &= \sum_{u=M_1+1}^{M_1+q} \sum_{v=M_2+1}^{M_2+q} \delta_q(uv - l) F_1(u) F_2(v) \\ &= \sum_{u=M_1+1}^{M_1+q} \sum_{v=M_2+1}^{M_2+q} \delta_q(uv - l) \sum_{-q/2 < m,n \le q/2} \widehat{F}_1(m) \widehat{F}_2(n) e^{2\pi i \frac{mu+nv}{q}} \\ &= \sum_{-q/2 < m,n \le q/2} \widehat{F}_1(m) \widehat{F}_2(n) K_q(l, m, n). \end{aligned}$$

Separating out the term with $m = n = 0$, we obtain the relation

$$\Phi_q(M_1, M_2; N_1, N_2) = \frac{N_1 N_2}{q^2} \cdot K_q(l, 0, 0) + R_1 + R_2 + R_3,$$

where

$$R_1 = \sideset{}{'}\sum_{-q/2<n\le q/2} \widehat{F}_1(0)\widehat{F}_2(n)K_q(l,0,n),$$
$$R_2 = \sideset{}{'}\sum_{-q/2<m\le q/2} \widehat{F}_1(m)\widehat{F}_2(0)K_q(l,m,0),$$
$$R_3 = \sideset{}{'}\sum_{-q/2<m\le q/2}\ \sideset{}{'}\sum_{-q/2<n\le q/2} \widehat{F}_1(m)\widehat{F}_2(n)K_q(l,m,n).$$

Henceforth, the prime at the summation sign means that the summation index does not take the value zero. Using inequalities (17) for the Fourier coefficients and properties 3°–4° of the sums $K_q(l,m,n)$, we arrive at the estimates

$$R_{1,2} \ll \sum_{n=1}^{q} \frac{|K_q(l,0,n)|}{n},$$
$$R_3 \ll \sum_{m,n=1}^{q} \frac{|K_q(l,m,n)| + |K_q(-l,m,n)|}{mn}.$$

Applying Corollaries 1 and 3, we obtain the asymptotic formula

$$\Phi_q(M_1,M_2;N_1,N_2) = \frac{N_1N_2}{q^2}\cdot K_q(l,0,0) + O(\psi_l(q)),$$

where the function $\psi_l(q)$ is as in (14).

The definitions of M_1, M_2, N_1, and N_2 and condition (16) imply that

$$|P_1P_2 - N_1N_2| = |P_1(P_2-N_2) + N_2(P_1-N_1)| \le 2q.$$

By (13) and (15), this inequality implies that

$$\begin{aligned}\Big|\Phi_q(Q_1,Q_2;P_1,P_2) &- \frac{P_1P_2}{q^2}K_q(l,0,0)\Big| \\ &\ll \psi_l(q) + \Big|\frac{N_1N_2}{q^2}K_q(l,0,0) - \frac{P_1P_2}{q^2}K_q(l,0,0)\Big| \\ &\ll \psi_l(q) + \frac{K_q(l,0,0)}{q^2}q \ll \psi_l(q).\end{aligned}$$

The lemma is proved. □

Remark 2. For any P_1 and $P_2 = q$, the same arguments yield the formula

$$\Phi_q(Q_1,Q_2;P_1,P_2) = \frac{P_1}{q}K_q(l,0,0) + O(\sigma_0(q)\sigma_0(a)a).$$

§3. Application of van der Corput's method

Lemma 4. *Let P_1 and P_2 be real numbers, and let $P = P_2 - P_1 \ge 1$. Assume that a real function $f(x)$ is continuously differentiable on the entire interval $[P_1,P_2]$, and that $f'(x)$ is monotone, $\|f'(x)\| \ge \lambda > 0$. Then*

$$\sum_{P_1<x\le P_2} e^{2\pi i f(x)} \ll \lambda^{-1}.$$

Proof. See [8, Theorem 2.1]. □

Lemma 5. *Let P_1 and P_2 be real numbers, and let $P = P_2 - P_1 \geq 1$. Assume that a real function $f(x)$ is twice continuously differentiable on the entire interval $[P_1, P_2]$, and that, for some $A > 0$ and $w \geq 1$, we have*

$$\frac{1}{A} \leq |f''(x)| \leq \frac{w}{A}.$$

Then

$$\sum_{P_1 < x \leq P_2} e^{2\pi i f(x)} \ll_w \frac{P}{\sqrt{A}} + \sqrt{A}.$$

Proof. See [8, Theorem 2.2]. □

Lemma 6. *Let P_1 and P_2 be real numbers, and let $P = P_2 - P_1 \geq 2$. Assume that a real function $f(x)$ is twice continuously differentiable on the entire interval $[P_1, P_2]$, and that, for some $A \geq P$ and $w \geq 1$, we have*

$$\frac{1}{A} \leq |f''(x)| \leq \frac{w}{A}.$$

Then, for any natural number q,

$$\sum_{m=1}^{q} \left| \sum_{P_1 < x \leq P_2} e^{2\pi i (\frac{mx}{q} + f(x))} \right| \ll_w q \left(\frac{P}{\sqrt{A}} + \log A \right) + \sqrt{A}.$$

Proof. We note that it suffices to prove the lemma for the sum

$$S = \sum_{1 \leq m \leq q/8} \left| \sum_{P_1 < x \leq P_2} e^{2\pi i (\frac{mx}{q} + f(x))} \right|.$$

Without loss of generality, we may assume that $f'' > 0$ and, for $x \in [P_1, P_2]$, the values of the derivative of $f(x)$ lie inside an interval of length not exceeding $1/8$. Otherwise, the interval $[P_1, P_2]$ can be divided into $O(\frac{P}{A} + 1) = O(1)$ shorter intervals such that on each of them this condition is fulfilled. Thus, the values of the derivative of the function $\frac{mx}{q} + f(x)$ range inside an interval of length not exceeding $1/4$. If this interval contains a half-integer, then $\|\frac{m}{q} + f'(x)\| \geq \frac{1}{4}$ and, by Lemma 4,

$$S \ll q \ll q \left(\frac{P}{\sqrt{A}} + \log A \right) + \sqrt{A}.$$

Now we consider the case where, for some integer k, for all $1 \leq m \leq q/2$ and $x \in [P_1, P_2]$, the values of the derivative of $\frac{mx}{q} + f(x)$ lie inside the interval $[k - \frac{1}{2}, k + \frac{1}{2}]$, i.e.,

$$k - \frac{1}{2} \leq \frac{m}{q} + f'(x) \leq k + \frac{1}{2} \quad (1 \leq m \leq q/8,\ P_1 < x \leq P_2).$$

Let integers m_1 and m_2 be determined by the conditions

$$\frac{m_1}{q} + f'(P_2) \leq k - \frac{1}{\sqrt{A}}, \qquad \frac{m_1 + 1}{q} + f'(P_2) > k - \frac{1}{\sqrt{A}};$$
$$\frac{m_2}{q} + f'(P_1) \geq k + \frac{1}{\sqrt{A}}, \qquad \frac{m_2 - 1}{q} + f'(P_1) < k + \frac{1}{\sqrt{A}}.$$

We write the sum S in the form $S = S_1 + S_2 + S_3$, where

$$S_1 = \sum_{1\le m<m_1} \left| \sum_{P_1<x\le P_2} e^{2\pi i(\frac{mx}{q}+f(x))} \right|,$$

$$S_2 = \sum_{m_1<m<m_2} \left| \sum_{P_1<x\le P_2} e^{2\pi i(\frac{mx}{q}+f(x))} \right|,$$

$$S_3 = \sum_{m_2\le m\le q/2} \left| \sum_{P_1<x\le P_2} e^{2\pi i(\frac{mx}{q}+f(x))} \right|.$$

By Lemma 4,

$$S_1 \ll \sum_{1\le m<m_1} \frac{1}{k-\left(\frac{m}{q}+f'(P_2)\right)} < \int_0^{m_1} \frac{dm}{k-\left(\frac{m}{q}+f'(P_2)\right)} \ll q\cdot\log A.$$

Similarly,

$$S_3 \ll \sum_{m_2<m\le q/2} \frac{1}{\frac{m}{q}+f'(P_1)-k} < \int_{m_2}^{q/2} \frac{dm}{\frac{m}{q}+f'(P_1)-k} \ll q\cdot\log A.$$

We apply Lemma 5 to the sum S_2:

$$\begin{aligned} S_2 &\ll (m_2-m_1+1)\left(\frac{P}{\sqrt{A}}+\sqrt{A}\right) \ll (m_2-m_1+1)\sqrt{A} \\ &\ll \left(q\left(\frac{1}{\sqrt{A}}+\frac{P}{A}\right)+1\right)\sqrt{A} = q\left(1+\frac{P}{\sqrt{A}}\right)+\sqrt{A}. \end{aligned}$$

Summing the estimates for S_1, S_2, and S_3, we obtain the required estimate of the sum S. □

§4. Proof of the main result

Lemma 7 (Poisson summation formula)**.** *Let h be a real nonnegative function such that the integral*

$$\int_{-\infty}^{\infty} h(x)\,dx$$

exists as an improper Riemann integral. Assume also that h is monotone nondecreasing on the interval $(-\infty,0]$ and is monotone nonincreasing on $[0,\infty)$. Then

$$\sum_{m=-\infty}^{\infty} \frac{h(m+0)+h(m-0)}{2} = \sum_{n=-\infty}^{\infty} \widehat{h}(n),$$

where the two series converge absolutely, and

$$\widehat{h}(n) = \int_{-\infty}^{\infty} h(t)e^{-2\pi int}\,dt$$

is the Fourier transform of h.

Proof. See [6, 11.24]. □

Proof of Theorem 1. We assume that $A \gg 1$, $\max\{A,q\} \le P^2$, and $\log(Aq) \ll \log P$, because otherwise the estimate to be proved is worse than the trivial one.

Note that it suffices to prove the lemma under the assumption that the graph of the function $f(x)$ for $x \in [P_1,P_2]$ goes through no points of the integer lattice. Indeed, if this condition is not fulfilled, then ε can be chosen within the limits $0<\varepsilon\le A^{-1/3}$, so that for integers $x \in [P_1,P_2]$, the numbers $f(x)\pm\varepsilon$ are not integers. If we assume that for

the functions $f \pm \varepsilon$ relation (1) holds true with the remainder term (2), then, by using the relations

$$T[f-\varepsilon] < T[f] \le T[f+\varepsilon], \quad S[f \pm \varepsilon] = S[f] + O(PA^{-1/3}),$$

we obtain the required estimate of the remainder term for the function f as well.

We may also assume that $f \ge 2q$, because otherwise we can replace the function f by $f + 2q$.

For $\Delta < 1/4$ and for real α, β ($\beta - \alpha > \Delta$), we define the functions

$$\psi(\alpha, \beta; x) = [\alpha < x \le \beta],$$
$$\psi_{\mp}(\alpha, \beta; x) = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \psi\left(\alpha \pm \frac{\Delta}{2}, \beta \mp \frac{\Delta}{2}; x + t\right) dt;$$

obviously, we have

$$\psi_-(\alpha, \beta; x) \le \psi(\alpha, \beta; x) \le \psi_+(\alpha, \beta; x).$$

We denote by $N(x)$ the number of solutions of the congruence $xy \equiv l \pmod q$ with respect to the unknown y that lies within the limits $1 \le y \le f(x)$. Then

$$N_-(x) \le N(x) = \sum_{y=-\infty}^{\infty} \delta_q(xy - l)\psi\left(\frac{1}{2}, f(x); y\right) \le N_+(x),$$

where

$$\begin{aligned} N_{\mp}(x) &= \sum_{y=-\infty}^{\infty} \delta_q(xy - l)\psi_{\mp}\left(\frac{1}{2}, f(x); y\right) \\ &= \sum_{k=1}^{q} \delta_q(xk - l) \sum_{m=-\infty}^{\infty} \psi_{\mp}\left(\frac{1}{2}, f(x); mq + k\right). \end{aligned}$$

For any value of k, the functions

$$h_{\mp}(m) = \psi_{\mp}\left(\frac{1}{2}, f(x); mq + k\right)$$

are nonnegative, continuous, and Riemann integrable on the entire real line; also, they are monotone nondecreasing on the interval $(-\infty, 0]$ and monotone nonincreasing on $[0, \infty)$ (recall that $f \ge 2q$ by assumption). Therefore, the Poisson summation formula can be applied to them (see Lemma 7), yielding

$$\begin{aligned} N_{\mp}(x) &= \sum_{k=1}^{q} \delta_q(xk - l) \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi_{\mp}\left(\frac{1}{2}, f(x); qv + k\right) e^{-2\pi i n v}\, dv\right) \\ &= \frac{1}{q} \sum_{k=1}^{q} \delta_q(xk - l) \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \psi_{\mp}\left(\frac{1}{2}, f(x); u\right) e^{-2\pi i n \frac{u-k}{q}}\, du\right) \\ &= \frac{1}{q} \sum_{k=1}^{q} \delta_q(xk - l) \sum_{n=-\infty}^{\infty} e^{2\pi i \frac{nk}{q}} \widehat{\psi}_{\mp}\left(\frac{1}{2}, f(x); \frac{n}{q}\right). \end{aligned}$$

If $n = 0$, then

$$\widehat{\psi}_{\mp}\left(\frac{1}{2}, f(x); \frac{n}{q}\right) = f(x) - \frac{1}{2} \mp \Delta.$$

For $n \ne 0$, we have

$$\widehat{\psi}_{\mp}\left(\frac{1}{2}, f(x); \frac{n}{q}\right) = q \cdot t_\Delta(N) \frac{e^{-2\pi i \frac{n}{q}(\frac{1}{2} \pm \frac{\Delta}{2})} - e^{-2\pi i \frac{n}{q}(f(x) \mp \frac{\Delta}{2})}}{2\pi i n},$$

where

$$t_\Delta(n) = \frac{\sin \frac{\pi n \Delta}{q}}{\frac{\pi n \Delta}{q}}, \quad |t_\Delta(n)| \le T_\Delta(n) = \min\Big\{1, \frac{q}{|n|\Delta}\Big\}.$$

Consequently,

$$N_\mp(x) = \frac{\mu_{q,l}(x)}{q}\left(f(x) - \frac{1}{2} \mp \Delta\right) + N_\mp^{(1)}(x) + N_\mp^{(2)}(x), \tag{18}$$

where

$$N_\mp^{(1)}(x) = \sum_{k=1}^{q} \delta_q(xk-l) \sideset{}{'}\sum_{n=-\infty}^{\infty} t_\Delta(n) \frac{e^{2\pi i \frac{n}{q}(k - \frac{1}{2} \mp \frac{\Delta}{2})}}{2\pi i n},$$

$$N_\mp^{(2)}(x) = \sum_{k=1}^{q} \delta_q(xk-l) \sideset{}{'}\sum_{n=-\infty}^{\infty} t_\Delta(n) \frac{e^{-2\pi i \frac{n}{q}(f(x) \mp \frac{\Delta}{2} - k)}}{2\pi i n}.$$

Next,

$$T_-[f] \le T[f] = \sum_{P_1 < x \le P_2} N(x) \le T_+[f],$$

where

$$T_\mp[f] = \sum_{P_1 < x \le P_2} N_\mp(x).$$

By Remark 2,

$$\sum_{P_1 < x \le P_2} \mu_{q,l}(x) = \frac{P}{q} K_q(l,0,0) + O(\sigma_0(q)\sigma_0(a)a).$$

Therefore, after summation, formula (18) yields

$$\begin{aligned} T_\mp[f] = S[f] - \frac{P}{2q^2} K_q(l,0,0) + T_\mp^{(1)}[f] + T_\mp^{(2)}[f] \\ + O\Big(\frac{\Delta P}{q^2} K_q(l,0,0)\Big) + O\left(\sigma_0(q)\sigma_0(a)a\right), \end{aligned} \tag{19}$$

where

$$T_\mp^{(1,2)}[f] = \sum_{P_1 < x \le P_2} N_\mp^{(1,2)}(x).$$

It is well known that the function

$$r(x) = \begin{cases} \frac{1}{2} - \{x\} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

has the Fourier series expansion

$$r(x) = \frac{1}{2\pi i} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n x}}{n}.$$

This implies that the smoothed function

$$g(x) = \frac{q}{\Delta} \int_{-\Delta/(2q)}^{\Delta/(2q)} \left(\frac{1}{2} - \{x+t\}\right) dt$$

is representable in the form

$$g(x) = \frac{1}{2\pi i} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{t_\Delta(n)}{n} \cdot e^{2\pi i n x}.$$

For $x \in \left(\frac{\Delta}{2q}, 1-\frac{\Delta}{2q}\right)$ we have

$$g(x) = \frac{1}{2} - x + O\left(\frac{\Delta}{q}\right).$$

Consequently,

$$\begin{aligned} N_{\mp}^{(1)}(x) &= \sum_{k=1}^{q} \delta_q(xk-l) g\left(\frac{k}{q} - \frac{1}{2q} \mp \frac{\Delta}{2q}\right) \\ &= \sum_{k=1}^{q} \delta_q(xk-l)\left(\frac{1}{2} - \frac{k}{q} + \frac{1}{2q}\right) + O\left(\frac{\Delta \cdot \mu_{q,l}(x)}{q}\right). \end{aligned}$$

To find $T_{\mp}^{(1)}[f]$, we split the sum in the variable x into segments, the length of which does not exceed q. For the sums

$$S' = \sum_{x,k=1}^{q} \delta_q(xk-l)\frac{k}{q} \quad \text{and} \quad S'' = \sum_{x,k=1}^{q} \delta_q(xk-l)\frac{q-k}{q}$$

we have $S' + S'' = K_q(l,0,0)$ and $S'' = S' - q\delta_q(l)$. Therefore,

$$S' = \frac{1}{2}\left(K_q(l,0,0) + q\delta_q(l)\right),$$

$$\sum_{x=1}^{q} N_{\mp}^{(1)}(x) = \frac{1}{2}\left(\frac{K_q(l,0,0)}{q} - q\delta_q(l)\right) + O\left(\frac{\Delta \cdot K_q(l,0,0)}{q}\right). \tag{20}$$

Moreover, applying Lemma 3 to the double sums in the identity

$$\sum_{x=1}^{P'}\sum_{k=1}^{q} \delta_q(xk-l)\frac{k}{q} = \sum_{x=1}^{P'}\sum_{k=1}^{q} \delta_q(xk-l) - \frac{1}{q}\sum_{b=1}^{q}\sum_{x=1}^{P'}\sum_{k=1}^{b-1} \delta_q(xk-l),$$

for $1 \le P' \le q$ we obtain

$$\sum_{x=1}^{P'}\sum_{k=1}^{q} \delta_q(xk-l)\frac{k}{q} = \frac{P'}{2q}K_q(l,0,0) + O(\psi_l(q)).$$

Hence,

$$\sum_{x=1}^{P'} N_{\mp}^{(1)}(x) = \frac{1}{2}\Big(\frac{P'}{q^2}K_q(l,0,0) - P'\delta_q(l)\Big) + O(\psi_l(q)) + O\Big(\frac{\Delta P' \cdot K_q(l,0,0)}{q^2}\Big). \tag{21}$$

Thus, (20) and (21) show that

$$\sum_{P_1 < x \le P_2} N_{\mp}^{(1)}(x) = \frac{1}{2}\Big(\frac{P}{q^2}K_q(l,0,0) - P\delta_q(l)\Big) + O(\psi_l(q)) + O\Big(\frac{\Delta P \cdot K_q(l,0,0)}{q^2}\Big).$$

Substituting this in (19), we arrive at the relation

$$T_{\mp}[f] = S[f] - \frac{P}{2}\delta_q(l) + T_{\mp}^{(2)}[f] + O(\psi_l(q)) + O\left(\frac{\Delta P \cdot K_q(l,0,0)}{q^2}\right). \tag{22}$$

Now we estimate $T_{\mp}^{(2)}[f]$. Using the relation

$$\sum_{k=0}^{q-1} \delta_q(xk-l)e^{-2\pi i \frac{nk}{q}} = \frac{1}{q}\sum_{m=1}^{q} K_q(l,m,n)e^{2\pi i \frac{mx}{q}},$$

we write the quantity $N_{\mp}^{(2)}(x)$ in the form

$$N_{\mp}^{(2)}(x) = \frac{1}{q} \sum_{n=-\infty}^{\infty}{}' t_{\Delta}(n) \sum_{m=1}^{q} K_q(l,m,n) \frac{e^{2\pi i(\frac{n}{q}(f(x)\mp\frac{\Delta}{2})-\frac{mx}{q})}}{2\pi i n}.$$

Hence,

$$|T_{\mp}^{(2)}[f]| \ll \frac{1}{q} \sum_{n=1}^{\infty} \frac{T_{\Delta}(n)}{n} \sum_{m=1}^{q} |K_q(l,m,n)| \cdot \left| \sum_{P_1<x\le P_2} e^{2\pi i \frac{mx-nf(x)}{q}} \right|.$$

By Lemma 1,

$$|T_{\mp}^{(2)}[f]| \ll \frac{\sigma_0(q)\sigma_0(a)}{q^{1/2}} \cdot S,$$

where $a = (l,q)$,

$$S = \sum_{n=1}^{\infty} \frac{T_{\Delta}(n)}{n} \sum_{m=1}^{q} (lm, ln, mn, q)^{1/2} \cdot |S_q(m,n)|,$$

$$S_q(m,n) = \sum_{P_1<x\le P_2} e^{2\pi i \frac{mx-nf(x)}{q}}.$$

We set $\delta = (lm, ln, mn, q)$, $\delta_1 = (l,\delta)$ and transform the sum S:

$$\begin{aligned} S &= \sum_{\delta|q} \delta^{1/2} \sum_{n=1}^{\infty} \frac{T_{\Delta}(n)}{n} \sum_{m=1}^{q} [(lm, ln, mn) = \delta] \cdot |S_q(m,n)| \\ &\le \sum_{\delta|q} \delta^{1/2} \sum_{n=1}^{\infty} \frac{T_{\Delta}(n)}{n} \left[\frac{\delta}{\delta_1} \mid n\right] \sum_{m=1}^{q} \left[\frac{\delta}{\delta_1} \mid m, \delta \mid mn\right] \cdot |S_q(m,n)|. \end{aligned}$$

After the change of summation indices $m = \delta m_1/\delta_1$ and $n = \delta n_1/\delta_1$, we arrive at the following bound on S:

$$\begin{aligned} S &\le \sum_{\delta|q} \frac{\delta_1}{\delta^{1/2}} \sum_{n_1=1}^{\infty} \frac{T_{\Delta}(\delta n_1/\delta_1)}{n_1} \sum_{m_1=1}^{\delta_1 q/\delta} \left[\delta_1^2 \mid \delta m_1 n_1\right] \cdot |S_q(\delta m_1/\delta_1, \delta n_1/\delta_1)| \\ &= \sum_{\delta|q} \frac{\delta_1}{\delta^{1/2}} \sum_{n_1=1}^{\infty} \frac{T_{\Delta}(\delta n_1/\delta_1)}{n_1} \sum_{m_1=1}^{\delta_1 q/\delta} \left[\frac{\delta_1^2}{\delta_2} \mid m_1 n_1\right] \cdot \left|S_{\delta_1 q/\delta}(m_1,n_1)\right|, \end{aligned}$$

where $\delta_2 = (\delta_1^2, \delta)$. Next, we set $\delta_3 = (n_1, \delta_1^2/\delta_2)$. Then

$$\begin{aligned} S &\le \sum_{\delta|q} \frac{\delta_1}{\delta^{1/2}} \sum_{\delta_3|\delta_1^2/\delta_2} \sum_{n_1=1}^{\infty} \frac{T_{\Delta}(\delta n_1/\delta_1)}{n_1} [(n_1, \delta_1^2/\delta_2) = \delta_3] \\ &\quad \times \sum_{m_1=1}^{\delta_1 q/\delta} \left[\frac{\delta_1^2}{\delta_2\delta_3} \mid m_1\right] \cdot \left|S_{\delta_1 q/\delta}(m_1,n_1)\right| \\ &\le \sum_{\delta|q} \frac{\delta_1}{\delta^{1/2}} \sum_{\delta_3|\delta_1^2/\delta_2} \sum_{n_2=1}^{\infty} \frac{T_{\Delta}(\delta\delta_3 n_2/\delta_1)}{\delta_3 n_2} \sum_{m_2=1}^{\frac{\delta_2\delta_3 q}{\delta\delta_1}} \left|S_{\delta_1 q/\delta}\left(\frac{\delta_1^2}{\delta_2\delta_3} m_2, \delta_3 n_2\right)\right|, \end{aligned}$$

where $n_1 = \delta_3 n_2$, $m_1 = \frac{\delta_1^2}{\delta_2\delta_3} m_2$. We estimate the sum of $S_{\delta_1 q/\delta}\big(\frac{\delta_1^2}{\delta_2\delta_3} m_2, \delta_3 n_2\big)$. This sum depends on the function

$$F(x) = \left(\frac{\delta_1^2}{\delta_2\delta_3} m_2 x - \delta_3 n_2 f(x)\right) \frac{\delta}{\delta_1 q},$$

for which

$$F''(x) = \frac{\delta\delta_3 n_2}{\delta_1 q} f''(x) \asymp \frac{\delta\delta_3 n_2}{\delta_1 q A} = \frac{1}{A_1}, \quad A_1 = \frac{\delta_1 q A}{\delta\delta_3 n_2}.$$

We apply Lemma 6 if $A_1 \geq P$ and Lemma 5 if $A_1 < P$. This shows that $S \ll S_1 + S_2$, where

$$S_1 = \sum_{\delta|q} \frac{\delta_1}{\delta^{1/2}} \sum_{\delta_3|\delta_1^2/\delta_2} \sum_{n=1}^{\infty} \frac{T_\Delta(\delta\delta_3 n/\delta_1)}{\delta_3 n} \Big(\frac{\delta_2\delta_3 q}{\delta\delta_1} \Big(\frac{P}{A_1^{1/2}} + \log P \Big) + A_1^{1/2} \Big) [A_1 \geq P],$$

$$S_2 = \sum_{\delta|q} \frac{\delta_1}{\delta^{1/2}} \sum_{\delta_3|\delta_1^2/\delta_2} \sum_{n=1}^{\infty} \frac{T_\Delta(\delta\delta_3 n/\delta_1)}{\delta_3 n} \cdot \frac{\delta_2\delta_3 q}{\delta\delta_1} \Big(\frac{P}{A_1^{1/2}} + A_1^{1/2} \Big) [A_1 < P].$$

Collecting the terms of the same form in S_1 and S_2 and using the monotonicity of $T_\Delta(n)$, we get the estimate

$$S \ll S_3 + S_4 + S_5 + S_6, \tag{23}$$

where

$$S_3 = PA^{-1/2} q^{1/2} \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_2 \delta_3^{1/2}}{\delta \delta_1^{1/2}} \sum_{n=1}^{\infty} \frac{T_\Delta(\delta\delta_3 n/\delta_1)}{n^{1/2}},$$

$$S_4 = q \log P \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_2}{\delta^{3/2}} \sum_{n=1}^{\infty} \frac{T_\Delta(n)}{n},$$

$$S_5 = A^{1/2} q^{1/2} \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_1^{3/2}}{\delta \delta_3^{3/2}} \sum_{n=1}^{\infty} \frac{T_\Delta(n)}{n^{3/2}},$$

$$S_6 = A^{1/2} q^{3/2} \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_2 \delta_1^{1/2}}{\delta^2 \delta_3^{1/2}} \sum_{n=1}^{\infty} \frac{T_\Delta(\delta\delta_3 n/\delta_1)}{n^{3/2}} \left[n > \frac{\delta_1 q A}{\delta\delta_3 P} \right].$$

Making use of the inequality $T_\Delta(n) \leq \min\{1, q(\Delta|n|)^{-1}\}$ and considering the cases where $b\Delta > q$ and $b\Delta \leq q$, we obtain the estimate

$$\sum_{n=1}^{\infty} \frac{T_\Delta(bn)}{n^{1/2}} \ll \Big(\frac{q}{b\Delta} \Big)^{1/2}.$$

Hence,

$$S_3 \ll PA^{-1/2} \Delta^{-1/2} q \sum_{\delta|q} \frac{\delta_2}{\delta^{3/2}} \sum_{\delta_3|\delta_1^2/\delta_2} 1.$$

Since

$$\sum_{\delta_3|\delta_1^2/\delta_2} 1 \leq \sum_{\delta_3|\delta_1} 1 \leq \sigma_0(a),$$

inequality (10) implies that

$$\sum_{\delta|q} \frac{\delta_2}{\delta^{3/2}} \sum_{\delta_3|\delta_1^2/\delta_2} 1 \ll \sigma_0(a) \sigma_{-1/2}^2(a) \tag{24}$$

and

$$S_3 \ll \sigma_0(a) \sigma_{-1/2}^2(a) PA^{-1/2} \Delta^{-1/2} q. \tag{25}$$

Next, since

$$\sum_{n=1}^{\infty} \frac{T_\Delta(n)}{n} \ll \log P,$$

by (24) we have

$$S_4 \ll q \log^2 P \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_2}{\delta^{3/2}} \ll q\sigma_0(a)\sigma_{-1/2}^2(a) \log^2 P. \tag{26}$$

To estimate the sum S_5, we note that

$$\sum_{n=1}^{\infty} \frac{T_\Delta(n)}{n^{3/2}} \ll 1.$$

Therefore,

$$\begin{aligned} S_5 &\ll A^{1/2} q^{1/2} \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_1^{3/2}}{\delta \delta_3^{3/2}} \ll A^{1/2} q^{1/2} \sum_{\delta|q} \frac{\delta_1^{3/2}}{\delta} \\ &\ll A^{1/2} q^{1/2} \sum_{\delta_1|(q,l)} \delta_1^{3/2} \sum_{\substack{\delta|q \\ (\delta,l)=\delta_1}} \frac{1}{\delta} \ll A^{1/2} q^{1/2} \sigma_{-1}(q) \sigma_{1/2}(a). \end{aligned}$$

Consequently,

$$S_5 \ll A^{1/2} q^{1/2} a^{1/2} \sigma_{-1}(q) \sigma_{-1/2}(a). \tag{27}$$

For any N, $k > 0$, we have

$$\sum_{n>N} \frac{T_\Delta(kn)}{n^{3/2}} \le \sum_{n>N} \frac{T_\Delta(kN)}{n^{3/2}} \ll \frac{T_\Delta(kN)}{N^{1/2}}.$$

Thus,

$$S_6 \ll P^{1/2} q T_\Delta\left(\frac{Aq}{P}\right) \sum_{\delta|q} \sum_{\delta_3|\delta_1^2/\delta_2} \frac{\delta_2}{\delta^{3/2}},$$

and, by (24),

$$S_6 \ll \sigma_0(a)\sigma_{-1/2}^2(a) P^{1/2} q T_\Delta\left(\frac{Aq}{P}\right).$$

Now, we apply the inequality

$$T_\Delta\left(\frac{Aq}{P}\right) = \min\left\{1, \frac{P}{\Delta A}\right\} \le \left(\frac{P}{\Delta A}\right)^{1/2}$$

to get an estimate similar to (25) for the sum S_6:

$$S_6 \ll \sigma_0(a)\sigma_{-1/2}^2(a) P A^{-1/2} \Delta^{-1/2} q. \tag{28}$$

Substituting (25), (26), and (27) in (23), we arrive at an estimate of the sum S and of the remainder term $T_{\mp}^{(2)}[f]$:

$$\begin{aligned} S &\ll \sigma_0(a)\sigma_{-1/2}^2(a)(PA^{-1/2}\Delta^{-1/2}q + q\log^2 P) + \sigma_{-1}(q)\sigma_{-1/2}(a)A^{1/2}q^{1/2}a^{1/2}, \\ T_{\mp}^{(2)}[f] &\ll \sigma_0(q)\sigma_{-1/2}^2(a)\sigma_0^2(a)(PA^{-1/2}q^{1/2}\Delta^{-1/2} + q^{1/2}\log^2 P) \\ &\qquad + \sigma_0(q)\sigma_{-1}(q)\sigma_0(a)\sigma_{-1/2}(a)A^{1/2}a^{1/2}. \end{aligned}$$

Now substituting this in (22) and using (13), we obtain (1) with a remainder term

$$\begin{aligned} R[f] \ll{}& \Delta P \cdot q^{-1}\sigma_0(a) + a\sigma_0(q)\sigma_0(a)\log(q+1) \\ &+ \sigma_0(q)\sigma_{-1/2}^2(a)\sigma_0^2(a)(PA^{-1/2}q^{1/2}\Delta^{-1/2} + q^{1/2}\log^2 P) \\ &+ \sigma_0(q)\sigma_{-1}(q)\sigma_0(a)\sigma_{-1/2}(a)A^{1/2}a^{1/2}. \end{aligned}$$

The choice

$$\Delta = qA^{-1/3}\sigma_0^{2/3}(q)\sigma_0^{2/3}(a)\sigma_{-1/2}^{4/3}(a)$$

completes the proof of the theorem. □

Remark 3. In applications, as a rule, the greatest contribution is made by the first summand of the remainder term. For this reason, usually, a simpler estimate of the remainder term can be used:

$$R[f] \ll \sigma_0^{2/3}(q)\sigma_0^2(a)PA^{-1/3} + \left(A^{1/2}a^{1/2} + q^{1/2} + a\right)P^{\varepsilon}. \tag{29}$$

Remark 4. For $q = 1$, the theorem proved above converts to a known result on the number of points under the graph of a twice continuously differentiable function (see [3, Lemma 4], and also [14, Problem I.6.4]).

§5. A refinement of a result by Porter

Lemma 8. *For any natural number $b \geq 4$, the sums*

$$D_k = \sum_{a|b} \frac{\sigma_0^k(a)}{a} \quad (k \geq 0)$$

satisfy the estimate

$$D_k \ll (\log\log b)^{2^k}. \tag{30}$$

Proof. The relations $\sigma_1(n) = n\sigma_{-1}(n)$ and $\sigma_1(n) \ll n\log\log n$ (see, e.g., [9, Theorem 323]) imply that the lemma holds true for the sum $D_0 = \sigma_{-1}(b)$. If we assume that (30) is valid for some $k \geq 0$, then for $k+1$ we obtain

$$\begin{aligned} D_{k+1} &= \sum_{a|b} \frac{\sigma_0^k(a)}{a} \sum_{t|a} 1 = \sum_{t|b} \sum_{a_1|b/t} \frac{\sigma_0^k(ta_1)}{ta_1} \\ &\leq \sum_{t|b} \frac{\sigma_0^k(t)}{t} \sum_{a_1|b/t} \frac{\sigma_0^k(a_1)}{a_1} \leq D_k^2 \ll (\log\log b)^{2^{k+1}}. \end{aligned}$$

□

Proof of Theorem 2. We assume that $\varepsilon < 1/6$ and denote by $T_x(b)$ the number of solutions of the equation

$$m_1m_2 + n_1n_2 = b \tag{31}$$

with respect to the unknowns $1 \leq m_1 \leq n_1$ and $1 \leq m_2 \leq n_2x$. Let $T_x^*(b)$ denote the number of solutions of equation (31) in which $1 \leq m_1 \leq n_1$, $(m_1, n_1) = 1$, and $1 \leq m_2 \leq n_2x$. For the sum

$$N_x(b) = \sum_{a=1}^{b} s^{(x)}(a/b),$$

the following relation is valid (see the proof of Lemma 3 in [4]):

$$N_x(b) = 2T_x^*(b) + b\left(x \cdot [x < 1] - \frac{x}{x+1}\right) + O(1).$$

The quantities $N_x(b)$ and $T_x(b)$ are related to $N_x^*(b)$ and $T_x^*(b)$ by the Möbius inversion formula,

$$N_x^*(b) = \sum_{d|b} \mu(d)N_x(b/d), \quad T_x^*(b) = \sum_{d|b} \mu(d)T_x(b/d).$$

Therefore,

$$N_x^*(b) = 2 \sum_{d_1 d_2 | b} \mu(d_1)\mu(d_2) T_x\left(\frac{b}{d_1 d_2}\right) + \varphi(b)\left(x \cdot [x<1] - \frac{x}{x+1}\right) + O(b^\varepsilon). \tag{32}$$

To compute $T_x(b)$, we introduce the parameter $U = (b \log b)^{1/2}$ and divide all the solutions of equation (31) into two groups. We attribute the solutions with $n_1 < U$ to the first group and all other solutions to the second. Accordingly, $T_x(b)$ is represented in the form

$$T_x(b) = T_1(b, U) + T_2(b, U). \tag{33}$$

First, we find an asymptotic formula for $T_1(b, U)$. We note that, for fixed n_1, the variables m_1 and m_2 satisfy the congruence

$$m_1 m_2 \equiv b \pmod{n_1}. \tag{34}$$

If n_1, m_1, and m_2 are known, then n_2 is determined uniquely:

$$n_2 = \frac{b - m_1 m_2}{n_1}.$$

The restriction $m_2 \le n_2 x$ is equivalent to the inequality

$$m_2 \le \frac{bx}{n_1 + m_1 x} = f_{n_1}(m_1). \tag{35}$$

Thus, the problem reduces to the calculation of the number of solutions of the congruence (34) in which the variables satisfy the restrictions $0 < m_1 \le n_1$ and $m_2 \le f_{n_1}(m_1)$. We apply Theorem 1 with $P_1 = 0$, $P_2 = n_1$, $f = f_{n_1}$, and with a simpler estimate of the remainder term (see Remark 3). Since

$$f''_{n_1}(m_1) \asymp \frac{b}{n_1^3},$$

we obtain

$$T[f_{n_1}] = S[f_{n_1}] - \frac{n_1}{2} \cdot \delta_{n_1}(b) + R[f_{n_1}].$$

Hence,

$$T_1(b, U) = \sum_{n_1 < U} T[f_{n_1}] = S_1(b, U) + R_1(b, U) + O(b^{1/2+\varepsilon}), \tag{36}$$

where

$$S_1(b, U) = \sum_{n_1 < U} S[f_{n_1}] = \sum_{n_1 < U} \frac{1}{n_1} \sum_{m_1 \le n_1} \mu_{n_1, b}(m_1) f_{n_1}(m_1), \tag{37}$$

$$\begin{aligned} R_1(b, U) = \sum_{n_1 < U} R[f_{n_1}] &\ll b^{1/3} \sum_{n_1 < U} \sigma_0^{2/3}(n_1)\sigma_0^2(a_1) \\ &\quad + b^\varepsilon \sum_{n_1 < U} \left(n_1^{3/2} a_1^{1/2} b^{-1/2} + n_1^{1/2} + a_1\right), \end{aligned} \tag{38}$$

and $a_1 = (n_1, b)$. Applying the estimate $\sigma_0(xy) \le \sigma_0(x)\sigma_0(y)$ and the Hölder inequality, we see that

$$\begin{aligned} \sum_{n_1 < U} \sigma_0^{2/3}(n_1)\sigma_0^2(a_1) &\le \sum_{a_1 | b} \sigma_0^2(a_1) \sum_{n < U/a_1} \sigma_0^{2/3}(n a_1) \\ &\le \sum_{a_1 | b} \sigma_0^3(a_1) \left(\sum_{n < U/a_1} \sigma_0(n)\right)^{2/3} (U/a_1)^{1/3} \ll U \log^{2/3} b \sum_{a_1 | b} \frac{\sigma_0^3(a_1)}{a_1}. \end{aligned}$$

Next, applying Lemma 8, we arrive at the inequality

$$b^{1/3}\sum_{n_1<U}\sigma_0^{2/3}(n_1)\sigma_0^2(a_1)\ll b^{5/6}\log^{7/6+\varepsilon/2}b.$$

The contribution of the other terms occurring in the formula for $R_1(b,U)$ is smaller provided $\varepsilon<1/6$:

$$\begin{aligned}b^{-1/2+\varepsilon}\sum_{n_1<U}n_1^{3/2}a_1^{1/2}&\ll b^{-1/2+\varepsilon}\sum_{a_1|b}a_1^{1/2}\sum_{n<U/a_1}(a_1n)^{3/2}\\&\ll b^{-1/2+\varepsilon}U^{5/2}\sum_{a_1|b}a_1^{-1/2}\ll b^{3/4+2\varepsilon},\\ b^{\varepsilon}\sum_{n_1<U}n_1^{1/2}&\ll b^{\varepsilon}U^{3/2}\ll b^{3/4+2\varepsilon},\\ b^{\varepsilon}\sum_{n_1<U}a_1&\ll b^{\varepsilon}\sum_{a_1|b}a_1\sum_{n<U/a_1}1\le b^{\varepsilon}U\sigma_{-1}(b)\ll b^{1/2+2\varepsilon}.\end{aligned}$$

Thus,

$$T_1(b,U)=S_1(b,U)+O(b^{5/6}\log^{7/6+\varepsilon}b). \tag{39}$$

To find $S_1(b,U)$, first we consider the sum

$$\Phi(U)=\sum_{n<U}\frac{1}{n}\sum_{m\le n}\frac{x}{n+mx},$$

which can be written in the form

$$\begin{aligned}\Phi(U)&=\log(1+x)\sum_{n<U}\frac{1}{n}+\sum_{n<U}\frac{1}{n}\biggl(\sum_{m\le n}\frac{x}{n+mx}-\log(1+x)\biggr)\\&=\log(1+x)(\log U+\gamma)+h_1(x)+O(U^{-1}),\end{aligned}$$

where $h_1(x)$ is defined as in (4). Hence, applying the Möbius inversion formula to the sum

$$\Phi^*(U)=\sum_{n\le U}\frac{1}{n}\sum_{m\le n}{}^{*}\frac{x}{n+mx},$$

we get

$$\Phi^*(U)=\sum_{d\le U}\frac{\mu(d)}{d^2}\cdot\Phi\left(\frac{U}{d}\right),$$

which leads to the asymptotic formula

$$\Phi^*(U)=\frac{\log(1+x)}{\zeta(2)}\left(\log U+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+\frac{h_1(x)}{\zeta(2)}+O\left(\frac{\log(U+1)}{U}\right). \tag{40}$$

Substituting $\mu_{n_1,b}(m_1)=d_1\cdot\delta_{d_1}(b)$ with $d_1=(m_1,n_1)$ in (37), after the changes $m_1=d_1m$ and $n_1=d_1n$ we obtain

$$\begin{aligned}S_1(b,U)&=\sum_{n_1<U}\frac{1}{n_1}\sum_{m_1\le n_1}\frac{bx}{n_1+m_1x}d_1\cdot\delta_{d_1}(b)\\&=\sum_{d_1|b}\frac{b}{d_1}\sum_{n<U/d_1}\frac{1}{n}\sum_{m\le n}{}^{*}\frac{x}{n+mx}=\sum_{d|b}\frac{b}{d}\Phi^*\left(\frac{U}{d}\right).\end{aligned}$$

By (40), this can be written as

$$S_1(b,U)=\frac{1}{\zeta(2)}\sum_{d|b}\frac{b}{d}\left(\log(1+x)\left(\log\frac{U}{d}+\gamma-\frac{\zeta'(2)}{\zeta(2)}\right)+h_1(x)\right)+O\bigl(b^{1/2+\varepsilon}\bigr). \tag{41}$$

Substituting (41) in (39), we arrive at an asymptotic formula for $T_1(b,U)$:

$$T_1(b,U) = \frac{1}{\zeta(2)} \sum_{d|b} \frac{b}{d} \left(\log(1+x) \left(\log \frac{U}{d} + \gamma - \frac{\zeta'(2)}{\zeta(2)} \right) + h_1(x) \right) + O\big(b^{5/6} \log^{7/6+\varepsilon} b\big). \tag{42}$$

To find $T_2(b,U)$, we note that, for fixed n_2, the variables m_1 and m_2 satisfy the congruence

$$m_1 m_2 \equiv b \pmod{n_2}. \tag{43}$$

If n_2, m_1, and m_2 are known, then n_1 is determined uniquely:

$$n_1 = \frac{b - m_1 m_2}{n_2}.$$

The restriction $\max\{m_1, U\} \le n_1$ is equivalent to the inequality

$$m_1 \le \min\left\{ \frac{b}{m_2 + n_2}, \ \frac{b - U n_2}{m_2} \right\} = g_{n_2}(m_2).$$

We divide the interval $I = (0, n_2]$, inside which the variable m_2 changes, into shorter intervals by the points 1, 2, 2^2, …, 2^k ($k = [\log_2 n_2]$), and to this partition we add the point $m_0 = \frac{b}{U} - n_2$ at which the function g_{n_2} may be nondifferentiable:

$$I = \bigsqcup_{j=1}^{k'} I_j \quad (k' = k+2).$$

We assume that

$$g_{n_2}(m_2) = \begin{cases} \dfrac{b}{m_2 + n_2} & \text{if} \quad m_2 \in \bigsqcup\limits_{j=1}^{k''} I_j, \\ \dfrac{b - U n_2}{m_2} & \text{if} \quad m_2 \in \bigsqcup\limits_{j=k''+1}^{k'} I_j, \end{cases}$$

where $0 \le k'' \le k'$. We apply Theorem 1 to the function g_{n_2} on each of the intervals I_j. Then for the entire interval I we get

$$T[g_{n_2}] = S[g_{n_2}] + R''[g_{n_2}] + R'[g_{n_2}] + O\left(b^{1+\varepsilon} U^{-1}\right),$$

where

$$S[g_{n_2}] = \frac{1}{n_2} \sum_{1 \le m_2 \le n_2} \mu_{n_2,b}(m_2) g_{n_2}(m_2),$$

$$R''[g_{n_2}] = \sum_{j=1}^{k''} R^{(j)}[g_{n_2}], \quad R'[g_{n_2}] = \sum_{j=k''+1}^{k'} R^{(j)}[g_{n_2}],$$

and $R^{(j)}[g_{n_2}]$ is the remainder term obtained as in Theorem 1 on the interval I_j.

For $j = 1, \dots, k''$, we have

$$g_{n_2}(m_2) \asymp \frac{b}{n_2^3}$$

on the interval I_j. Therefore, the sum of the remainders $R''[g_{n_2}]$ is estimated as the sum in (38) (with the replacement of U by b/U):

$$\sum_{n_2 \le b/U} R''[g_{n_2}] \ll b^{5/6} \log^{7/6+\varepsilon} b. \tag{44}$$

If $j = k'' + 1, \dots, k'$, then

$$g''_{n_2}(m_2) \asymp \frac{b - Un_2}{m_2^3} \asymp \frac{b - Un_2}{2^{3j}}$$

on the interval I_j. Consequently, by (29),

$$R^{(j)}[g_{n_2}] \ll \sigma_0^{2/3}(n_2)\sigma_0^2(a_2)b^{1/3} + b^{\varepsilon/2}\left(2^{3j/2}a_2^{1/2}b^{-1/2} + n_2^{1/2} + a_2\right),$$

where $a_2 = (n_2, b)$. Thus,

$$R'[g_{n_2}] = \sum_{j=k''+1}^{k'} R^{(j)}[g_{n_2}] \ll \sigma_0^{2/3}(n_2)\sigma_0^2(a_2)\log b \cdot b^{1/3} + b^{\varepsilon}\left(n_2^{3/2}a_2^{1/2}(b - Un_2)^{-1/2} + n_2^{1/2} + a_2\right).$$

Hence, as in the case of $R[f_{n_1}]$, we arrive at the estimate

$$\sum_{n_2 \le b/U - 2} R'[g_{n_2}] \ll b^{5/6}\log^{7/6+\varepsilon} b. \tag{45}$$

If the value of the variable $n_2 > b/U - 2$ is fixed, then n_1 can take at most $b^{1/2+\varepsilon/2}$ values, and for fixed n_1 and n_2, at most $\sigma_0(b - n_1n_2) \ll b^{\varepsilon/2}$ values of m_1 and m_2 may exist. Therefore, by (44) and (45), we have

$$T_2(b, U) = \sum_{n_2 \le b/U} T[g_{n_2}] = \sum_{n_2 \le b/U - 2} T[g_{n_2}] + O(b^{1/2+\varepsilon}) = S_2(b, U) + O(b^{5/6}\log^{7/6+\varepsilon} b),$$

where

$$S_2(b, U) = \sum_{n_2 \le b/U} \frac{1}{n_2} \sum_{m_2 \le n_2} \mu_{n_2,b}(m_2) g_{n_2}(m_2).$$

As in the case of the sum $S_1(b, U)$, after the change

$$\mu_{n_2,b}(m_2) = d_2 \cdot \delta_{d_2}(b), \quad d_2 = (m_2, n_2),$$

the sum $S_2(b, U)$ takes the form

$$S_2(b, U) = \sum_{d|b} \frac{b}{d} \sum_{n \le b/(dU)} \frac{1}{n} \sideset{}{^*}\sum_{m \le nx} \min\left\{\frac{1}{m+n}, \frac{1}{m} - \frac{dUn}{bm}\right\} = \sum_{d|b} \frac{b}{d} \cdot F_x^*\Big(\frac{b}{dU}\Big),$$

where

$$F_x^*(\xi) = \sum_{n \le \xi} \frac{1}{n} \sideset{}{^*}\sum_{m \le nx} \min\left\{\frac{1}{m+n}, \frac{1}{m} - \frac{n}{m\xi}\right\}.$$

For the sum $F_x^*(\xi)$, the following asymptotic formula is valid (see [4, Lemma 10]):

$$F_x^*(\xi) = \frac{\log(x+1)}{\zeta(2)}\log\xi + \frac{H(x)}{\zeta(2)} + O\Big(\frac{\log^2(\xi+1)}{\xi}\Big),$$

where

$$H(x) = \log(1+x)\left(\log x - \frac{\zeta'(2)}{\zeta(2)} - \frac{\log(x+1)}{2} + \gamma - 1\right) + h_2(x)$$

and $h_2(x)$ is defined as in (5). Therefore,

$$S_2(b, U) = \frac{1}{\zeta(2)} \sum_{d|b} \frac{b}{d}\left(\log(x+1)\log\frac{b}{dU} + H(x)\right) + O(b^{1/2+\varepsilon}),$$

$$T_2(b, U) = \frac{1}{\zeta(2)} \sum_{d|b} \frac{b}{d}\left(\log(x+1)\log\frac{b}{dU} + H(x)\right) + O(b^{5/6}\log^{7/6+\varepsilon} b).$$

Substituting this expression for $T_2(b,U)$ and relation (42) in (33), we arrive at an asymptotic formula for $T_x(b)$:

$$T_x(b) = \frac{1}{\zeta(2)} \sum_{d|b} \frac{b}{d} \left(\log(x+1) \log \frac{b}{d^2} + C_1(x) \right) + O(b^{5/6} \log^{7/6+\varepsilon} b),$$

where

$$C_1(x) = H(x) + \log(1+x) \left(\gamma - \frac{\zeta'(2)}{\zeta(2)} \right) + h_1(x).$$

We substitute this result in (32). Then, since

$$\sum_{dd_1d_2|b} \frac{\mu(d_1)\mu(d_2)}{dd_1d_2} = \frac{\varphi(b)}{b}, \qquad \sum_{dd_1d_2|b} \frac{\mu(d_1)\mu(d_2)}{dd_1d_2} \log(d_1d_2d^2) = 0$$

(see [11]), we get

$$\sum_{d_1d_2|n} \mu(d_1)\mu(d_2) T_x\left(\frac{b}{d_1d_2}\right) = \frac{\varphi(b)}{\zeta(2)} (\log(x+1)\log b + C_1(x)) + O(b^{5/6} \log^{7/6+\varepsilon} b),$$

$$N_x^*(b) = \frac{2\varphi(b)}{\zeta(2)} (\log(x+1)\log b + C(x)) + O(b^{5/6} \log^{7/6+\varepsilon} b),$$

where $C(x)$ is as in (3). The theorem is proved. □

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