ON $\theta$-CENTRALIZERS OF SEMIPRIME RINGS (II)

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ABSTRACT. The following result is proved: Let $R$ be a 2-torsion free semiprime ring, and let $T : R \to R$ be an additive mapping, related to a surjective homomorphism $\theta : R \to R$, such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ for all $x \in R$. Then $T$ is both a left and a right $\theta$-centralizer.

§1. Introduction

This paper has been motivated by the work of Brešar [5], Vukman [10], and Zalar [11]. Throughout, $R$ will represent an associative ring with center $Z(R)$. Recall that $R$ is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and $R$ is semiprime if $aRa = (0)$ implies $a = 0$. A ring $R$ is 2-torsion free if $2x = 0$, $x \in R$ implies $x = 0$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for all pairs $x, y \in R$; $D$ is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. A derivation $D$ is inner if there exists an element $a \in R$ such that $D(x) = [a, x]$ for all $x \in R$. An additive mapping $D : R \to R$ related to a map $\theta : R \to R$ is called a $(\theta, \theta)$-derivation if $D(xy) = D(x)\theta(y) + \theta(x)D(y)$ for all pairs $x, y \in R$; $D$ is called a Jordan $(\theta, \theta)$-derivation if $D(x^2) = D(x)x + \theta(x)D(x)$ for all $x \in R$. A $(\theta, \theta)$-derivation $D$ is inner if there exists $a \in R$ such that $D(x) = [a, \theta(x)]$ for all $x \in R$. It is clear that if $\theta$ is the identity map on $R$, then every $(\theta, \theta)$-derivation is an ordinary derivation.

Every derivation is a Jordan derivation; the converse is in general not true. A classical result of Herstein [7] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein’s result can be found in [2]. Cusack [6] generalized Herstein’s result to 2-torsion free semiprime rings (see also [4] for an alternative proof). Zalar [11] gave the following definition: An additive mapping $T : R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) for all $x, y \in R$. If $R$ is a ring with involution $*$, then every additive mapping $E : R \to R$ that satisfies $E(x^2) = E(x)x^* + xE(x)$ for all $x \in R$ is called a Jordan $*$-derivation. These mappings are closely related to the question of the representability of quadratic forms by bilinear forms. Some algebraic properties of Jordan $*$-derivations were considered in [8], where further references can be found. For quadratic forms, see [9].

If the product in $R$ is given by $x \circ y = xy + yx$, then a Jordan derivation is an additive mapping $D$ satisfying $D(x \circ y) = D(x) \circ y + x \circ D(y)$ for all $x, y \in R$; a Jordan homomorphism is an additive mapping $A$ satisfying $A(x \circ y) = A(x) \circ A(y)$ for all $x, y \in R$. Zalar [11] defined the Jordan centralizer to be an additive mapping $T$ such that $T(x \circ y) = T(x) \circ y = x \circ T(y)$. Since the product $\circ$ is commutative, there is no difference between the left and the right Jordan centralizers.

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We follow Zalar [11] and call $T$ a centralizer if $T$ is both a left and a right centralizer. If $a \in R$, then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer if $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$). Following the ideas of [4], Zalar [11] proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a (right) centralizer. In [1], the definition of a $\theta$-centralizer was introduced. This is a generalization of the definition of a centralizer. Albash [1] followed the same direction as Zalar [11] and proved his result with this definition. In our paper, Vukman’s result [10] will be proved for the $\theta$-centralizer case under some additional conditions.

Definition 1.1 ([1]). An additive mapping $T : R \to R$ is called a left (right) $\theta$-centralizer associated with a function $\theta : R \to R$ if for all $x, y \in R$ we have

\[ T(xy) = T(x)\theta(y) \quad (T(xy) = \theta(x)T(y)). \]

$T$ is called a left (right) Jordan $\theta$-centralizer if for all $x, y \in R$ we have

\[ T(x^2) = T(x)\theta(x) \quad (T(x^2) = \theta(x)T(x)). \]

It is clear that if $T : R \to R$ is an additive mapping associated with a homomorphism $\theta : R \to R$, if $L_a(x) = a\theta(x)$ and $R_a(x) = \theta(x)a$ for all $x \in R$ and $a \in R$, then $L_a$ is a left $\theta$-centralizer and $R_a$ is a right $\theta$-centralizer.

Remark 1.1. Clearly, every centralizer is a special case of a $\theta$-centralizer with $\theta = I_R$, and every Jordan centralizer is a special case of a Jordan $\theta$-centralizer with $\theta = I_R$.

If $T : R \to R$ is a $\theta$-centralizer associated with a function $\theta : R \to R$, where $R$ is an arbitrary ring, then $T$ satisfies the relation

\[ 2T(x^2) = T(x)\theta(x) + \theta(x)T(x) \quad \text{for all } x \in R. \]

It seems natural to ask, as Vukman [10] did in the centralizer case, whether the converse is true. More precisely, we ask whether an additive mapping $T$ on a ring $R$ satisfying the above relation is a $\theta$-centralizer for all $x, y \in R$. It is our aim to prove that the answer is in the affirmative if $R$ is a 2-torsion free semiprime ring and $\theta$ is a surjective endomorphism with $\theta(Z) = Z$.

Theorem 1.2. Let $R$ be a 2-torsion free semiprime ring with identity element, and let $T : R \to R$ be an additive mapping such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ for all $x \in R$, where $\theta$ is a surjective endomorphism of $R$. Then $T$ is both a left and a right $\theta$-centralizer.

Theorem 1.3. Let $R$ be a 2-torsion free semiprime ring with center $Z$, and let $T : R \to R$ be an additive mapping such that $2T(x^2) = T(x)\theta(x) + \theta(x)T(x)$ for all $x \in R$, where $\theta$ is a surjective endomorphism of $R$ with $\theta(Z) = Z$. Then $T$ is both a left and a right $\theta$-centralizer.

§2. Proof of Theorem 1.2

We assume that $R$ is noncommutative (the theorem is trivial when $R$ is commutative). Replacing $x$ by $x + 1$ in the equation

\[ 2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \quad x \in R, \]

after some calculations we obtain

\[ 2T(x) = a\theta(x) + \theta(x)a, \quad x \in R, \]

where $a$ denotes $T(1)$. We intend to prove that $a \in Z(R)$. Combining (1) and (2), we obtain

\[ 2(a\theta(x^2) + \theta(x^2)a) = (a\theta(x) + \theta(x)a)\theta(x) + \theta(x)(a\theta(x) + \theta(x)a). \]
This reduces to
\begin{equation}
[D(x), x]_\theta = 0, \quad x \in R.
\end{equation}

Here $[m, n]_\theta$ always denotes $m\theta(n) - \theta(n)m$. It is well known that if $f(x) = [x, a]$ for every $x \in R$, then $f$ is a derivation on $R$, which is called an inner derivation. Similarly, $D(x)$ is a $(\theta, \theta)$-derivation; we call $D(x)$ an inner $(\theta, \theta)$-derivation on $R$. For an ordinary inner derivation $f$ defined on a 2-torsion free semiprime ring $R$, Herstein [8, Lemma 1.1.9] proved that if $a \in R$ commutes with the inner derivation $f$ defined by $a$ (i.e., if $f(x) = [a, x]$ and $[a, f(x)] = 0$ for all $x \in R$), then $a \in Z(R)$, where $Z(R)$ is the center of $R$. We want to prove the same result in the case of an inner $(\theta, \theta)$-derivation. For this, we put $x + y$ for $x$ in (3). We have
\begin{equation}
[D(x + y), x + y]_\theta = 0 = [D(x), x]_\theta + [D(x), y]_\theta + [D(y), x]_\theta + [D(y), y]_\theta = [D(x), y]_\theta + [D(y), x]_\theta, \quad x, y \in R.
\end{equation}

We rewrite the above equation in the form
\begin{equation}
[a, \theta(x)]\theta(y) + [a, \theta(y)]\theta(x) = \theta(y)[a, \theta(x)] + \theta(x)[a, \theta(y)].
\end{equation}

In particular, since $\theta$ is surjective, there exists $y \in R$ such that $\theta(y) = a$, so that we can put $\theta(y) = a$ in (4). We obtain
\begin{equation}
[a, \theta(x)]a = a[a, \theta(x)].
\end{equation}

Thus,
\begin{equation}
[D(x), a] = 0 = [[a, x]_\theta, a], \quad x \in R.
\end{equation}

We substitute $xy$ for $x$ in (5); since $D$ is a $(\theta, \theta)$-derivation, we get
\begin{align*}
0 &= [a, [a, \theta(xy)]] = [a, [a, \theta(x)\theta(y)]] = [a, \theta(x)[a, \theta(y)] + [a, \theta(x)]\theta(y)] \\
&= [a, \theta(x)[a, \theta(y)]] + [a, [a, \theta(x)]\theta(y)] \\
&= \theta(x)[a, [a, \theta(y)]] + [a, \theta(x)][a, \theta(y)] + [a, \theta(x)][a, \theta(y)] + [a, [a, \theta(x)]\theta(y)].
\end{align*}

By (5), the above relation reduces to
\begin{equation}
2[a, \theta(x)\theta(y)] = 0, \quad x, y \in R.
\end{equation}

In a 2-torsion free semiprime ring this gives
\begin{equation}
[a, \theta(x)][a, \theta(y)] = 0, \quad x, y \in R.
\end{equation}

Substituting $y = rx$, $x, y, r \in R$, in the above relation, we get
\begin{align*}
[a, \theta(x)][a, \theta(rx)] &= [a, \theta(x)][a, \theta(r)\theta(x)] \\
&= [a, \theta(x)]\theta(r)[a, \theta(x)] + [a, \theta(x)][a, \theta(r)]\theta(x) \\
&= [a, \theta(x)]\theta(r)[a, \theta(x)] = 0, \quad x, y \in R.
\end{align*}

Since $\theta$ is surjective and $R$ is a semiprime ring, it follows that
\begin{equation}
D(x) = [a, \theta(x)] = [a, x]_\theta = 0, \quad x \in R.
\end{equation}

This means that $a\theta(x) = \theta(x)a$ for all $x \in R$. Now from (2) it follows that $T(x) = a\theta(x) = \theta(x)a$. Thus, $T$ is both a left and a right $\theta$-centralizer. This completes the proof of the theorem in the case where $R$ has an identity element.
§3. Proof of Theorem 1.3

We intend to prove that

\[ [T(x), x]_\theta = 0, \quad x \in R. \]  

For this, first we prove a weaker result: \( T \) satisfies the relation

\[ [T(x), x^2]_\theta = 0, \quad x \in R. \]  

Since the above relation can be written in the form

\[ [T(x), x]_\theta \theta(x) + \theta(x)[T(x), x]_\theta = 0, \]  

it is obvious that \( T \) satisfies (7) if \( T \) satisfies (6). Substituting \( x + y \) for \( x \) in the relation

\[ 2T(x^2) = T(x)\theta(x) + \theta(x)T(x), \quad x \in R, \]  

we obtain

\[ 2T(xy + yx) = T(x)\theta(y) + \theta(x)T(y) + T(y)\theta(y) + \theta(y)T(x), \quad x, y \in R. \]  

Our next step is to prove the relation

\[ 8T(xy) = T(x)\theta(xy + 3yx) + \theta(yx + 3xy)T(x) \]
\[ + \theta(x)T(y)\theta(x) - \theta(x^2)T(y) - T(y)\theta(x^2), \quad x, y \in R. \]  

For this, we substitute \( 2(xy + yx) \) for \( y \) in (9). Using (9), we obtain

\[ 4T(x(xy + yx) + (yx + xy)x) \]
\[ = 2T(x)\theta(x + yx) + 2T(x)T(xy + yx) + 2T(xy + yx)\theta(y) + 2\theta(xy + yx)T(x) \]
\[ = 2T(x)\theta(y + 3yx) + \theta(3xy + 3yx)T(x) + \theta(x)T(y)\theta(y) + \theta(y)T(x)\theta(x) \]
\[ + \theta(x)T(y)\theta(x) + \theta(x)T(y)\theta(x) + T(y)\theta(x^2) \]
\[ + \theta(y)T(x)\theta(x) + 2\theta(xy + yx)T(x). \]

Then we have

\[ 4T(x(xy + yx) + (yx + xy)x) \]
\[ = T(x)\theta(2xy + 3yx) + \theta(3xy + 2yx)T(x) + \theta(x)T(y)\theta(y) + \theta(y)T(x)\theta(x) \]
\[ + 2\theta(x)T(y)\theta(x) + \theta(x^2)T(y) + T(y)\theta(x^2) \quad \forall x, y \in R. \]

On the other hand,

\[ 4T(x(xy + yx) + (yx + xy)x) = 4T(x^2y + yx^2) + 8T(xy) \]
\[ = 2T(x^2)\theta(y) + 2T(y)\theta(x^2) + 2\theta(x^2)T(y) + 2\theta(y)T(x^2) + 8T(xy) \]
\[ = T(x)\theta(xy) + \theta(xy)T(x)\theta(y) + 2\theta(x^2)T(y) \]
\[ + 2T(y)\theta(x^2) + \theta(y)T(x)\theta(x) + \theta(y)T(x) + 8T(xy). \]

Therefore,

\[ 4T(x(xy + yx) + (yx + xy)x) \]
\[ = T(x)\theta(xy) + \theta(xy)T(x) + \theta(x)T(x)\theta(y) + \theta(y)T(x)\theta(x) \]
\[ + 2\theta(x^2)T(y) + 2T(y)\theta(x^2) + 8T(xy), \quad x, y \in R. \]

Comparing (11) and (12), we arrive at (10). Now, we prove the relation

\[ T(x)\theta(xy + 2yx^2 - 2x^2y + \theta(xy - 2yx^2 - 2x^2y) + \theta(xy - 2yx^2 - 2x^2y)T(x) \]
\[ + \theta(x)T(x)\theta(x + yx + \theta(xy + yx)T(x)\theta(x) \]
\[ + \theta(x^2)T(x)\theta(y) + \theta(y)T(x)\theta(x^2) = 0, \quad x, y \in R. \]
Replacing $y$ by $8xyx$ in (9) and using (10), we obtain

$$16T(x^2yx + xyx^2) = 8T(x)\theta(xy) + 8\theta(x)T(xy) + 8T(xy)\theta(x) + 8\theta(xy)T(x)$$

$$= 8T(x)\theta(xy) + \theta(x)T(x)\theta(xy + 3yx) + \theta(xy + 3x^2y)T(x)$$

$$+ 2\theta(x^2)T(y)\theta(x) - \theta(x^3)T(y) - \theta(x)T(y)\theta(x^2)$$

$$+ T(x)\theta(xy + 3y^2) + \theta(xy + 3xy)T(x)\theta(x) + 2\theta(x)T(y)\theta(x^2)$$

$$- \theta(x^3)T(y)\theta(x) - T(y)\theta(x^3) + 8\theta(xy)T(x).$$

Therefore,

$$16T(x^2yx + xyx^2) = T(x)\theta(9xyx + 3y^2) + \theta(9xyx + 3x^2y)T(x)$$

$$+ \theta(x)T(x)\theta(xy + 3yx) + \theta(xy + 3xy)T(x)\theta(x) + \theta(x^2)T(y)\theta(x)$$

$$+ \theta(x)T(y)\theta(x^2) - T(y)\theta(x^3) - \theta(x^3)T(y), \quad x, y \in R.$$

On the other hand, first using (10), and after collecting some terms and using (9), we obtain

$$16T(x^2yx + xyx^2) = 16T(x(xy)x) + 16T(x(y)x)$$

$$= 2T(x)\theta(x^2y + 3xy) + 2\theta(x^2y + 3x^2y)T(x) + 4\theta(x)T(y)\theta(x)$$

$$2\theta(x^2)T(xy) - 2T(xy)\theta(x^2) + 2T(x)\theta(xy + 3xy)$$

$$+ 2\theta(x^2y + 3xyx)T(x) + 4\theta(x)T(y)\theta(x) - 2\theta(x^2)T(yx) - 2T(yx)\theta(x^2)$$

$$= T(x)\theta(2x^2y + 6yx^2 + 8xy) + \theta(8xyx + 2yx^2 + 6x^2y)T(x)$$

$$+ 4\theta(x)T(xy + yx)\theta(x) - 2\theta(x^2)T(xy + yx) - 2T(xy + yx)\theta(x^2)$$

$$= T(x)\theta(2x^2y + 6yx^2 + 8xy) + \theta(8xyx + 2yx^2 + 6x^2y)T(x)$$

$$+ 2\theta(x^2)T(y)\theta(x) + 2\theta(x^2y + 3xyx)T(y)\theta(x) + 2\theta(xy)T(y)\theta(x^2) + 2\theta(xy)T(x)\theta(x)$$

$$- \theta(x^3)T(x)\theta(y) - \theta(y^3)T(x)\theta(x) - \theta(x^2)T(y)\theta(x)$$

$$- \theta(x)T(y^2) - \theta(x)T(y)\theta(x^2) - T(y)\theta(x^3) - \theta(y)T(x)\theta(x^2).$$

Now we have

$$16T(x^2yx + xyx^2) = T(x)\theta(2x^2y + 5yx^2 + 8xy)$$

$$+ \theta(2x^2 + 5x^2y + 8yx)T(x) + 2\theta(xy)T(x)\theta(yx) + 2\theta(xy)T(x)\theta(xy)$$

$$+ \theta(x^2)T(y)\theta(x) + \theta(x)T(y)\theta(x^2) - \theta(x^2)T(y)\theta(x)$$

$$- \theta(x^3)T(y) - T(y)\theta(x^3), \quad x, y \in R.$$

Comparing (14) and (15), we arrive at (13).

Replacing $y$ by $xy$ in (13), we obtain

$$T(x)\theta(xy^2 - 2y^3 - 2x^2y) + \theta(xy^2 - 2x^2y - 2y^3)T(x)$$

$$+ \theta(x)T(xy + yx) + \theta(xy + yx)T(x)\theta(x) + \theta(x^2)T(xy) = 0$$

$$\forall x, y \in R.$$

Right multiplication of (13) by $\theta(x)$ yields

$$T(x)\theta(xy^2 - 2y^3 - 2x^2y) + \theta(xy^2 - 2x^2y - 2y^2)T(x)\theta(x)$$

$$+ \theta(xy)T(x)\theta(x^2) + \theta(xy + yx)T(x)\theta(x^2) + \theta(x^2)T(xy) = 0$$

$$\forall x, y \in R.$$
Subtracting (17) from (16), we get
\[
\theta(xy)[\theta(x), T(x)] + \theta(2x^2y)[T(x), \theta(x)] + \theta(2yx^2)[T(x), \theta(x)]
\]
\[
+ \theta(xy)[\theta(x), T(x)]\theta(x) + \theta(yx)[\theta(x), T(x)]\theta(x)
\]
\[
+ \theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R.
\]
After collecting the first and the fourth term together, this reduces to
\[
\theta(xy)[\theta(x^2), T(x)] + 2\theta(x^2y)[T(x), \theta(x)] + 2\theta(yx^2)[T(x), \theta(x)]
\]
\[
+ \theta(xy)[\theta(x), T(x)]\theta(x) + \theta(yx)[\theta(x), T(x)]\theta(x)
\]
\[
+ \theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R.
\]
We substitute \( T(x)\theta(y) \) for \( \theta(y) \) in (18) to obtain
\[
\theta(x)T(x)\theta(y)[\theta(x^2), T(x)] + 2\theta(x^2)T(x)\theta(y)[T(x), \theta(x)]
\]
\[
+ 2T(x)\theta(y)\theta(x^2)[T(x), \theta(x)] + T(x)\theta(y)\theta(x)[T(x), \theta(x)]
\]
\[
+ T(x)\theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R.
\]
Left multiplication of (18) by \( T(x) \) leads to the relation
\[
T(x)\theta(xy)[\theta(x), T(x)] + 2T(x)\theta(x^2y)[T(x), \theta(x)]
\]
\[
+ 2T(x)\theta(yx^2)[T(x), \theta(x)] + T(x)\theta(yx)[\theta(x), T(x)]\theta(x)
\]
\[
+ T(x)\theta(y)[\theta(x), T(x)]\theta(x^2) = 0, \quad x, y \in R.
\]
Subtracting (20) from (19), we get
\[
[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] - 2[T(x), \theta(x^2)]\theta(y)[T(x), \theta(x)] = 0, \quad x, y \in R.
\]
If we set
\[
a = [T(x), \theta(x)], \quad b = [T(x), \theta(x^2)], \quad c = -2[T(x), \theta(x^2)],
\]
the above relation becomes
\[
a\theta(y)b + c\theta(y)a = 0, \quad y \in R.
\]
Substituting \( yz \) for \( y \) in (21), we obtain
\[
a\theta(yz)b + c\theta(yz)a = a\theta(y)\theta(z)b + c\theta(y)\theta(z)a = 0, \quad z, y \in R.
\]
Next, substituting \( \theta(y)a \) for \( \theta(y) \) in the last equation, we see that
\[
a\theta(y)a\theta(z)b + c\theta(y)a\theta(z)a = 0, \quad z, y \in R.
\]
Left multiplication of (21) by \( a\theta(y) \) gives
\[
a\theta(y)a\theta(z)b + a\theta(y)c\theta(z)a = 0, \quad z, y \in R.
\]
After subtracting (23) from (22), we obtain
\[
(a\theta(y)c - c\theta(y)a)\theta(z)a = 0, \quad z, y \in R.
\]
After replacing \( \theta(z) \) by \( \theta(z)c\theta(y) \) in (24), we obtain
\[
(a\theta(y)c - c\theta(y)a)\theta(z)c\theta(y)a = 0, \quad z, y \in R.
\]
Right multiplication of (24) by \( \theta(y)c \) gives
\[
(a\theta(y)c - c\theta(y)a)\theta(z)a\theta(y)c = 0, \quad z, y \in R.
\]
After subtracting (25) from (26), we get
\[
(a\theta(y)c - c\theta(y)a)\theta(z)(a\theta(y)c - c\theta(y)a) = 0, \quad z, y \in R.
Since $R$ is semiprime and $\theta$ is surjective, it follows that

$$a\theta(y)c = c\theta(y)a, \quad y \in R.$$  

Combining (21) with (27), we arrive at the relation

$$a\theta(y)(b + c) = 0, \quad y \in R.$$

In other words,

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] = 0, \quad x, y \in R.$$  

Since

$$[T(x), \theta(x^2)]\theta(y)[T(x), \theta(x^2)]$$

$$= \theta(x)[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] + [T(x), \theta(x)]\theta(x)\theta(y)[T(x), \theta(x^2)]$$

$$= [T(x), \theta(x)]\theta(xy)[T(x), \theta(x^2)] = 0, \quad x, y \in R,$$

we get

$$[T(x), \theta(x^2)]\theta(y)[T(x), \theta(x^2)] = 0, \quad x, y \in R.$$  

Since $R$ is semiprime and $\theta$ is surjective, we obtain

$$[T(x), \theta(x^2)] = 0, \quad x \in R.$$

After substituting $x + y$ for $x$ in (31), we get

$$[T(x), \theta(y^2)] + [T(y), \theta(x^2)] + [T(x), \theta(xy + yx)] + [T(y), \theta(xy + yx)] = 0, \quad x, y \in R.$$  

Substituting $-x$ for $x$ and comparing the resulting relation obtained with the above, we obtain

$$[T(x), \theta(xy + yx)] + [T(y), \theta(x^2)] = 0, \quad x, y \in R,$$

because $R$ is torsion free. Substituting $2(xy + yx)$ for $y$ in (32) and using (9) and (31), we arrive at

$$0 = 2[T(x), \theta(x^2)2y + 2yx]\$$

$$+ [T(x), \theta(y)]\theta(y)T(y) + T(y)\theta(x) + \theta(y)T(x), \theta(x^2)]$$

$$= 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(xy)] + T(x)\theta(y, \theta(x^2)]$$

$$+ \theta(x)[T(x), \theta(x^2)] + [T(y), \theta(x^2)]\theta(x) + \theta(y), \theta(x^2)]T(x) = 0, \quad x, y \in R.$$  

Now we have

$$2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(xy)] + T(x)\theta(y), \theta(x^2)]$$

$$+ \theta(x)[T(x), \theta(x^2)] + [T(y), \theta(x^2)]\theta(x) + \theta(y), \theta(x^2)]T(x) = 0, \quad x, y \in R.$$  

For $y = x$ the above relation reduces to

$$\theta(x^2)[T(x), \theta(x)] + [T(x), \theta(x)]\theta(x^2) + 2[T(x), \theta(x^3)] = 0, \quad x \in R.$$  

Thus,

$$\theta(x^2)[T(x), \theta(x)] + 3[T(x), \theta(x)]\theta(x^2) = 0, \quad x \in R.$$  

In accordance with the relation $\theta(x)[T(x), \theta(x)] + [T(x), \theta(x)]\theta(x) = 0$ (see (31)), in the above equation we replace $\theta(x^2)[T(x), \theta(x)]$ by $-[T(x), \theta(x)]\theta(x^2)$. This gives

$$[T(x), \theta(x)]\theta(x^2) = 0, \quad x, y \in R,$$

and

$$\theta(x^2)[T(x), \theta(x)] = 0, \quad x, y \in R.$$
Because of (32), we can replace \([T(y), \theta(x^2)]\) by \([-T(x), \theta(xy + yx)]\) in (33), obtaining
\[
0 = 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(xy)] + T(x)[\theta(y), \theta(x^2)] \\
+ [\theta(y), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(xy + yx)] - [T(x), \theta(xy + yx)]\theta(x) \\
= 2\theta(x^2)[T(x), \theta(y)] + 2[T(x), \theta(y)]\theta(x^2) + 4[T(x), \theta(x)]\theta(yx) \\
+ 4\theta(x)[T(x), \theta(y)]\theta(x) + 4\theta(xy)[T(x), \theta(x)] + T(x)[\theta(y), \theta(x^2)] \\
+ [\theta(y), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(x)]\theta(y) - \theta(x^2)[T(x), \theta(y)] \\
- \theta(x)[T(x), \theta(y)]\theta(x) - \theta(xy)[T(x), \theta(x)] - [T(x), \theta(x)]\theta(yx) \\
- \theta(x)[T(x), \theta(y)]\theta(x) - [T(x), \theta(y)]\theta(x^2) - \theta(y)[T(x), \theta(x)]\theta(x).
\]
Now we have
\[
\theta(x^2)[T(x), \theta(y)] + [T(x), \theta(y)]\theta(x^2) + 3[T(x), \theta(x)]\theta(yx) \\
+ 3\theta(xy)[T(x), \theta(x)] + 2\theta(x)[T(x), \theta(y)]\theta(x) + T(x)[\theta(y), \theta(x^2)] \\
+ [\theta(y), \theta(x^2)]T(x) - \theta(x)[T(x), \theta(x)]\theta(yx) - \theta(yx)[T(x), \theta(x)]\theta(x) \\
= \theta(x^2)[T(x), \theta(y)]\theta(x) + \theta(x^2)[T(x), \theta(y)] + [T(x), \theta(y)]\theta(x^3) \\
+ \theta(y)[T(x), \theta(x)]\theta(x^2) + 3\theta(xy)[T(x), \theta(x)] + 3\theta(xy)[T(x), \theta(x)] \\
+ 2\theta(x)[T(x), \theta(y)]\theta(x^2) + 2\theta(xy)[T(x), \theta(x)]\theta(x) + T(x)[\theta(y), \theta(x^2)]\theta(x) \\
+ [\theta(y), \theta(x^2)]\theta(x)T(x) - \theta(x)[T(x), \theta(x)]\theta(xy) - \theta(xy)[T(x), \theta(x)]\theta(x), \\
x, y \in R.
\]
By (34) and (36), this reduces to
\[
\theta(x^2)[T(x), \theta(y)]\theta(x) + \theta(x^2)[T(x), \theta(x)] + [T(x), \theta(y)]\theta(x^3) \\
+ 3\theta(xy)[T(x), \theta(x)] + 3\theta(xy)[T(x), \theta(y)]\theta(x^2) \\
+ 2\theta(xy)[T(x), \theta(x)]\theta(x) + T(x)[\theta(y), \theta(x^2)]\theta(x) \\
+ [\theta(y), \theta(x^2)]\theta(x)T(x) - \theta(x)[T(x), \theta(x)]\theta(xy) = 0, \quad x, y \in R.
\]
Right multiplication of (37) by \(\theta(x)\) gives
\[
\theta(x^2)[T(x), \theta(y)]\theta(x) + [T(x), \theta(y)]\theta(x^3) + 3[T(x), \theta(x)]\theta(xy^2) \\
+ 3\theta(xy)[T(x), \theta(x)] + 2\theta(xy)[T(x), \theta(y)]\theta(x^2) \\
+ T(x)[\theta(y), \theta(x)]\theta(x) + [\theta(y), \theta(x^2)]T(x)\theta(x) \\
- \theta(x)[T(x), \theta(x)]\theta(xy) - \theta(y)[T(x), \theta(x)]\theta(x^2) = 0, \quad x, y \in R.
\]
By (34) we have $[T(x), \theta(x)]\theta(x^2) = 0$, and (39) becomes

$$\theta(x^2)[T(x), \theta(y)]\theta(x) + [T(x), \theta(y)]\theta(x^3) + 3[T(x), \theta(x)]\theta(yx^2)$$

(40) $$+ 3\theta(y)[T(x), \theta(x)]\theta(x) + 2\theta(x)[T(x), \theta(y)]\theta(x^2) + T(x)[\theta(y), \theta(x^2)]\theta(x) + [\theta(y), \theta(x^2)]T(x)\theta(x) - \theta(x)[T(x), \theta(x)]\theta(y) = 0, \quad x, y \in R.$$ 

After subtracting (40) from (38), we obtain

$$\theta(x^2)[T(x), \theta(x)] + 3\theta(xy)[\theta(x), [T(x), \theta(x)]] + 2\theta(xy)[T(x), \theta(x)]\theta(x) + [\theta(y), \theta(x^2)]\theta(x), T(x)] = 0.$$

By (35), this reduces to

$$2\theta(x^2)[T(x), \theta(x)] + 3\theta(xy)[T(x), \theta(x)] - \theta(xy)[T(x), \theta(x)]\theta(x) = 0, \quad x, y \in R.$$ 

Replacing $-[T(x), \theta(x)]\theta(x)$ by $\theta(x)[T(x), \theta(x)]$ in the above relation, we obtain

$$\theta(x^2)[T(x), \theta(x)] + 2\theta(xy)[T(x), \theta(x)] = 0, \quad x, y \in R.$$ 

Using (31), (34), (35), and (36), we see that (18) reduces to

$$\theta(x^2)[T(x), \theta(x)] = 0, \quad x, y \in R.$$ 

Combining this with the above relation, we get

$$\theta(x)\theta(y)\theta(x)[T(x), \theta(x)] = 0, \quad x, y \in R.$$ 

Since $\theta$ is surjective, we can replace $\theta(y)$ by $[T(x), \theta(x)]\theta(y)$; it follows that

$$\theta(x)[T(x), \theta(x)]\theta(y)\theta(x)[T(x), \theta(x)] = 0, \quad x, y \in R.$$ 

Since $\theta$ is surjective and $R$ is semiprime, we have

(41) $$\theta(x)[T(x), \theta(x)] = 0, \quad x \in R.$$ 

Also,

$$[T(x), \theta(x)]\theta(x) = -\theta(x)[T(x), \theta(x)] = 0, \quad x, y \in R.$$ 

Thus,

(42) $$[T(x), \theta(x)]\theta(x) = 0, \quad x \in R.$$ 

Substituting $x + y$ for $x$ in (41) and using (41), we get

$$\theta(x + y)[T(x + y), \theta(x + y)]$$

$$\quad = \theta(x)[T(x), \theta(y)] + \theta(x)[T(y), \theta(x)] + \theta(x)[T(y), \theta(y)]$$

$$\quad + \theta(y)[T(x), \theta(x)] + \theta(y)[T(x), \theta(y)] + \theta(y)[T(y), \theta(x)] = 0, \quad x, y \in R.$$ 

Substituting $-x$ for $x$, comparing the resulting relation with the above, and using the fact that $R$ is 2-torsion free, we obtain

$$\theta(y)[T(x), \theta(x)] + \theta(x)[T(x), \theta(y)] + \theta(x)[T(y), \theta(x)] = 0, \quad x, y \in R.$$ 

Multiplying the above relation by $[T(x), \theta(x)]$ and using (42), we arrive at the formula

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x)] = 0, \quad x, y \in R.$$ 

Since $R$ is semiprime and $\theta$ is surjective, we get

(43) $$[T(x), \theta(x)] = 0, \quad x \in R.$$ 

Combining (43) with (8), we see that

$$T(x^2) = \theta(x)T(x), \quad x \in R.$$ 

This means that $T$ is a left and also a right Jordan $\theta$-centralizer. It remains to use [1, Theorem 2]. The proof is complete.
Corollary 3.1 (\cite{10} Theorem 1). Let \( R \) be a 2-torsion free semiprime ring, and let \( T : R \to R \) be an additive mapping such that \( 2T(x^2) = T(x)x + xT(x) \) for all \( x \in R \). Then \( T \) is both a left and a right centralizer.

References


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