Алгебра и анализ
Том 21 (2009), № 4

St. Petersburg Math. J.
Vol. 21 (2010), No. 4, Pages 513–577
S 1061-0022(2010)01106-3
Article electronically published on May 20, 2010

GENERAL SOLUTION OF THE YANG–BAXTER EQUATION WITH SYMMETRY GROUP $\mathrm{SL}(n,\mathbb{C})$

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Abstract. The problem of constructing the R-matrix is considered in the case of an integrable spin chain with symmetry group $\mathrm{SL}(n,\mathbb{C})$. A fairly complete study of general R-matrices acting in the tensor product of two continuous series representations of $\mathrm{SL}(n,\mathbb{C})$ is presented. On this basis, R-matrices are constructed that act in the tensor product of Verma modules (which are infinite-dimensional representations of the Lie algebra $\mathrm{sl}(n)$), and also R-matrices acting in the tensor product of finite-dimensional representations of the Lie algebra $\mathrm{sl}(n)$.

Contents

§1. Introduction

A modern approach to the theory of quantum integrable systems is the quantum inverse problem method [1]–[5], which relates an integrable model to an arbitrary solution of the Yang–Baxter equation [6, 7, 9, 11] and gives a method of constructing eigenstates of the Hamiltonian of the model with the help of the algebraic Bethe ansatz [1, 2, 5]. The eigenvalues can be found by solving the system of Bethe equations. The advantage of the algebraic Bethe ansatz is its simplicity and universality, because it admits a generalization to groups of higher ranks [12]–[14].

Quite often in applications, we meet integrable models in which the symmetry representations have no lowest weight vectors. As examples of such models, we mention the quantum Toda chain [15, 16], the XXX-spin chain with symmetry group $\mathrm{SL}(2,\mathbb{C})$, which has applications in the theory of Yang–Mills fields [17, 19, 20, 21], and also the XXX-spin

2010 *Mathematics Subject Classification.* Primary 81R12.
Key words and phrases. R-matrix, quantum inverse problem method, algebraic Bethe ansatz, Baxter operator.
Supported by RFBR, grants 07-02-92166-CNRS_a and 09-01-93108-CNRS_a (the first and the second author), grants 08-01-00683_a and 09-01-12150-ofi_m (the first author), National project RNP 2.1.1/1575 and German Research Foundation (DFG) grant 9209282 (the second author).

chain with symmetry group $\mathrm{SL}(2,\mathbb{R})$ [22] or the modular magnet [23]. In such cases, an alternative approach is the Q-operator method [11, 16] or the method of separation of variables [4, 15]. These methods are sufficiently well developed in the case of a symmetry group of rank one, but, in contrast to the algebraic Bethe ansatz, admit no generalization to models with symmetry groups of higher ranks. Despite the fact that the Baxter operator is known for various models [26, 29, 30, 31, 33, 34], there is no generally accepted explicit method of constructing Q-operators.

For various versions of XXX or XXZ-spin chains with the symmetry algebra $\mathrm{sl}(2)$ or $U_q(\mathrm{sl}(2))$ [21]–[25], the Q-operator can be constructed by the Pasquier–Gaudin method; see [16]. However, it is hard to generalize this method to models with symmetry group of higher rank.

Another method of constructing Q-operators was given by Bazhanov, Lukyanov, and Zamolodchikov in [35] for a model with symmetry algebra $U_q(\mathrm{sl}(2))$ and was generalized to the symmetry algebra $U_q(\mathrm{sl}(3))$ in [36] and to the symmetry algebra $U_q(\mathrm{sl}(2|1))$ in [37]. In this method, the Q-operator is constructed as the trace of a special monodromy matrix over an auxiliary space of an infinite-dimensional representation of a q-oscillator algebra, with the use of an explicit form of the universal R-matrix [38]. An abstract representation of the universal R-matrix in terms of generators is quite complicated, which causes technical difficulties when we want to generalize this method of constructing the Q-operator to $U_q(\mathrm{sl}(n))$ [41]. Moreover, the case of a nondeformed symmetry algebra $\mathrm{sl}(n)$ must be obtained as a limit as $q \to 1$, which is a nontrivial technical problem in itself.

In the papers [42]–[45], Q-operators in models of spin chains with symmetry algebras $\mathrm{sl}(2), \mathrm{sl}(3)$, and $\mathrm{sl}(2|1)$ were constructed. In each case, the Q-operator is constructed similarly as the trace of a special monodromy matrix over an infinite-dimensional auxiliary space. The main difference from [35]–[37] is that the role of the auxiliary space is played by the space of a specific representation of the symmetry algebra that is not an analog of a representation of a q-oscillator algebra.

As in [35]–[37], the universal R-matrix plays a key role. In the paper [47], it was proved that the R-matrix acting in the tensor product of two representations in general position of the symmetry algebra can be represented as the product of two (for $\mathrm{sl}(2)$) or three (for $\mathrm{sl}(3)$ and $\mathrm{sl}(2|1)$) simpler operators. The transfer matrix is constructed as the trace of the product of universal R-matrices. In this product, all R-matrices act as operators in the same auxiliary space, over which the trace is calculated. The factorization property of the R-matrix is inherited by the transfer matrix [42]–[45], which also decomposes into a product of simpler building blocks, and these are Q-operators. If we choose some specific parameters of the representation in the auxiliary space, then in this product all Q-operators but one are identity operators. Thus, for the remaining Q-operator we obtain a representation in the form of the trace of a specific monodromy matrix over an infinite-dimensional auxiliary space.

We want to generalize this method of constructing Q-operators to the case of the symmetry algebra $\mathrm{sl}(n)$. It is natural to divide this problem into two parts. The first part is the construction of a universal R-matrix for the symmetry algebra $\mathrm{sl}(n)$, and the second is the study of the transfer matrices and Q-operators.

The continuous series representations of the group $\mathrm{SL}(n,\mathbb{C})$ are most simple and universal for constructing R-matrices. In the paper [46], an R-matrix acting in the tensor product of two continuous series representations of the group $\mathrm{SL}(n,\mathbb{C})$ was constructed. In the case of the group $\mathrm{SL}(2,\mathbb{C})$, this was done in [21].

In the present paper, we construct and study in detail the R-matrices acting in the tensor product of continuous series representations of $\mathrm{SL}(n,\mathbb{C})$. The R-matrix for continuous series is constructed as an integral operator acting on an appropriate function space.

Restricting this integral operator to the space of polynomials, we obtain an R-matrix acting in the tensor product of Verma modules, which are infinite-dimensional representations of the Lie algebra $\mathrm{sl}(n)$. Further restriction to the space of polynomials realizing a finite-dimensional representation gives an R-matrix for finite-dimensional representations of the group $\mathrm{SL}(n,\mathbb{C})$.

Heuristically, it is convenient to regard formulas for Verma modules or for finite-dimensional representations as "shadows" or "projections" of the corresponding formulas for continuous series. The general picture is extraordinarily attractive, and we tried to perform the reduction step-by-step: continuous series $\longmapsto$ Verma modules $\longmapsto$ finite-dimensional representations.

The paper is organized as follows. In the first part, we consider the case of the group $\mathrm{SL}(2,\mathbb{C})$ in detail and realize our plan of constructing R-matrices for different types of representations.

In the second part, we repeat practically the same steps in the general case, so that all changes and complications can be traced. In comparison with the first part, the exposition becomes less detailed and more technical, because the main steps are the same and it is only necessary to trace the required changes.

A forthcoming paper will be devoted to the construction of transfer matrices and Baxter operators.

§2. The group $\mathrm{SL}(2,\mathbb{C})$

In this section, we construct an $\mathrm{SL}(2,\mathbb{C})$-invariant solution of the Yang–Baxter equation. The simplest case is studied in detail for the following reasons: first, by this simple example we illustrate almost all our methods; second, in the general case of an $\mathrm{SL}(n,\mathbb{C})$-invariant Yang–Baxter equation, most key formulas can be reduced to the case of $\mathrm{SL}(2,\mathbb{C})$. Moreover, on the one hand, this example is sufficiently nontrivial and has all main features of the general case, and, on the other hand, this example is quite simple and makes it possible to perform a detailed study of the required objects. The formulas become more cumbersome for groups of higher rank.

The following analogy seems to be useful. The construction of the operator R looks like the assemblage of a model with the help of a constructor set: simple building blocks are used to assemble more and more complicated constructions by certain rules. The operators intertwining two equivalent principal series representations of the group $\mathrm{SL}(2,\mathbb{C})$ play the role of the simplest building blocks. The assembling rules are given by a permutation group. As the objects constructed become more and more complicated, the building blocks and the assembling rules change. Thus, at each level of construction, we need an instruction with a sufficiently full description of all building blocks and the corresponding assembling rules.

We shall proceed as follows. First, we recall basic facts about the representations of the group $\mathrm{SL}(2,\mathbb{C})$, namely, the construction of induced representations, the construction of intertwining operators for equivalent representations, and the classification of irreducible representations.

Next, we consider the Yang–Baxter equation. We construct a solution of this equation in two steps. First, we solve a simpler RLL-relation defining for the operator R, and then prove that the resulting operator satisfies the general Yang–Baxter equation. We give a

visual graphical proof, which employs the star-triangle transformation, and an algebraic proof, which is more convenient for generalization to the case of the group $\mathrm{SL}(n,\mathbb{C})$.

We use the continuous series representations of $\mathrm{SL}(2,\mathbb{C})$ and construct an integral operator R satisfying the Yang–Baxter equation. For specific values of the representation parameters, invariant subspaces of polynomials arise in the representation space of the continuous series, on which finite-dimensional irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ are realized. An intermediate level between the continuous series representations and finite-dimensional representations is occupied by the Verma modules, which are infinite-dimensional representations of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ in the space of polynomials. Constructing the restrictions of the integral operator R to spaces of polynomials, we obtain operators R acting on Verma modules and finite-dimensional representations of the group $\mathrm{SL}(2,\mathbb{C})$. Here, the building blocks become more complicated, because the intertwining operators for equivalent principal series representations of $\mathrm{SL}(2,\mathbb{C})$ are ill defined on the space of polynomials. We consider the properties of the operators playing the role of elementary building blocks in detail at each step of restriction of the representation space: continuous series $\longrightarrow$ Verma modules $\longrightarrow$ finite-dimensional representations.

All formulas are presented in detailed notation to avoid misunderstanding in the case of the group $\mathrm{SL}(n,\mathbb{C})$, where we use more concise notation. In this section, we tried to be informal and did not emphasize statements and proofs, intending to make this in the general case of the group $\mathrm{SL}(n,\mathbb{C})$. Moreover, almost all formulas for $\mathrm{SL}(2,\mathbb{C})$ can easily be proved by direct calculation.

2.1. Representations of complex group $\mathrm{SL}(2,\mathbb{C})$**.** We recall the construction of induced representations of the group $\mathrm{GL}(2,\mathbb{C})$ of nonsingular complex matrices of order 2 [49, 50]. We denote by Z the group of lower triangular complex matrices with 1 on the diagonal, and by H the group of upper triangular matrices,

$$\mathrm{z}=\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\in \mathrm{Z},\quad \mathrm{h}=\begin{pmatrix}h_{11} & h_{12}\\ 0 & h_{22}\end{pmatrix}\in \mathrm{H}. \tag{2.1}$$

A matrix $\mathrm{z}\in\mathrm{Z}$ is parametrized by a single complex number $z=x+iy$. We take the subgroup of upper triangular matrices H as an induced subgroup and the one-dimensional representation $\alpha(\mathrm{h})$ of H as the induced representation. Let h_{11} and h_{22} be the diagonal entries of h, and let $\overline{h}_{11}$ and $\overline{h}_{22}$ be the complex conjugates. The function $\alpha(\mathrm{h})$ is given by the formula

$$\alpha(\mathrm{h})=h_{11}^{-1-\sigma_1}\,\overline{h}_{11}^{-1-\bar\sigma_1}\cdot h_{22}^{-2-\sigma_2}\,\overline{h}_{22}^{-2-\bar\sigma_2}=[h_{11}]^{-1-\sigma_1}\cdot[h_{22}]^{-2-\sigma_2}. \tag{2.2}$$

In general position, the parameters (σ_1,σ_2) and $(\bar\sigma_1,\bar\sigma_2)$ are not complex conjugate, because the single-valuedness of the function $\alpha(\mathrm{h})$ implies only that the differences must be integers, $\sigma_k-\bar\sigma_k\in\mathbb{Z}$. It is convenient to use the following concise notation:

$$[z]^a\equiv z^a\cdot\bar z^{\bar a}=|z|^{2a}\cdot\bar z^{\bar a-a}=|z|^{2\bar a}\cdot z^{a-\bar a}, \tag{2.3}$$

where $z=x+iy$ and $\bar z=x-iy$ are mutually complex conjugate and $a-\bar a\in\mathbb{Z}$.

The following two forms of induced representations are commonly used. In the first form, the induced representation is realized as a subrepresentation of the left regular representation

$$\mathrm{T}(g)\Phi(\mathrm{x})=\Phi(g^{-1}\mathrm{x});\quad g,\ \mathrm{x}\in\mathrm{GL}(2,\mathbb{C}) \tag{2.4}$$

in the space of functions on the group $\mathrm{GL}(2,\mathbb{C})$ that satisfy

$$\Phi(\mathrm{x})=\Phi(\mathrm{z}\,\mathrm{h})=\alpha(\mathrm{h}^{-1})\cdot\Phi(\mathrm{z}). \tag{2.5}$$

Here we have used the Gauss decomposition: almost every matrix $\mathrm{x} \in \mathrm{GL}(2,\mathbb{C})$ admits a unique representation in the form $\mathrm{x} = \mathrm{z}\,\mathrm{h}$.

A function $\Phi(\mathrm{x})$ satisfying (2.5) is uniquely determined by its restriction $\Phi(\mathrm{z})$ to the subgroup Z of lower triangular matrices. Rewriting the action of the operator of the left regular representation (2.4) for the functions $\Phi(\mathrm{z})$, we obtain the second form of the induced representation. Let z be an arbitrary matrix in Z and g an arbitrary matrix in $\mathrm{GL}(2,\mathbb{C})$. Using the Gauss decomposition, we represent the product $g^{-1}\cdot \mathrm{z}$ in the form

$$g^{-1}\cdot \mathrm{z} = \mathrm{z}'\cdot \mathrm{h}. \tag{2.6}$$

The action of the operator $\mathrm{T}(g)$ on a function is defined as follows:

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = \alpha(\mathrm{h}^{-1})\cdot \Phi(\mathrm{z}'). \tag{2.7}$$

The space in which the representation operators act consists of functions $\Phi(\mathrm{z})$ on the subgroup Z, i.e., of functions of the variables z and $\bar{z}$: $\Phi(\mathrm{z}) = \Phi(z,\bar{z})$. We perform explicit calculations:

$$\begin{gathered} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-c+az}{d-bz} & 1 \end{pmatrix} \begin{pmatrix} \frac{d-bz}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & \frac{1}{d-bz} \end{pmatrix}, \end{gathered} \tag{2.8}$$

obtaining

$$z' = \frac{-c+az}{d-bz}; \quad \alpha(\mathrm{h}^{-1}) = [ad-bc]^{1-\sigma_1}\cdot[d-bz]^{\sigma_1-\sigma_2-1}.$$

Thus, the action of the operator $\mathrm{T}(g)$ on $\Phi(z,\bar{z})$ is given by the formula

$$\mathrm{T}(g)\,\Phi(z,\bar{z}) = [ad-bc]^{1-\sigma_1}\cdot[d-bz]^{\sigma_1-\sigma_2-1}\cdot\Phi\left(\frac{-c+az}{d-bz}, \frac{-\bar{c}+\bar{a}\bar{z}}{\bar{d}-\bar{b}\bar{z}}\right). \tag{2.9}$$

The representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ obtained above is parametrized by two sets of numbers (σ_1,σ_2) and $(\bar{\sigma}_1,\bar{\sigma}_2)$ satisfying $\sigma_k - \bar{\sigma}_k \in \mathbb{Z}$. To simplify the notation, we drop the dependence on $(\bar{\sigma}_1,\bar{\sigma}_2)$ in $\mathrm{T}^{(\sigma_1,\sigma_2)}$. If the representation is to be shown explicitly, we write the representation operators in the form $\mathrm{T}^{(\sigma_1,\sigma_2)}(g)$ instead of $\mathrm{T}(g)$.

In the case of the group $\mathrm{SL}(2,\mathbb{C})$, the determinant of g is equal to 1 ($ad-bc=1$), and to characterize the representation we only need the difference $\sigma_{12} = \sigma_1 - \sigma_2$ ($\bar{\sigma}_{12} = \bar{\sigma}_1 - \bar{\sigma}_2$),

$$\mathrm{T}(g)\,\Phi(z,\bar{z}) = [d-bz]^{\sigma_{12}-1}\cdot\Phi\left(\frac{-c+az}{d-bz}, \frac{-\bar{c}+\bar{a}\bar{z}}{\bar{d}-\bar{b}\bar{z}}\right). \tag{2.10}$$

Thus, two sets of parameters (σ_1,σ_2) and $(\sigma_1+\sigma,\sigma_2+\sigma)$ differing by a common shift σ characterize the same representation of the group $\mathrm{SL}(2,\mathbb{C})$. We use the symmetric parametrization (σ_1,σ_2) by imposing an extra condition on the sum, $\sigma_1+\sigma_2=1$, which fixes the common translation.

2.2. Irreducible representations of the group $\mathrm{SL}(2,\mathbb{C})$**.** To this point, we considered a purely algebraic method of constructing representation operators and did not impose any restrictions on the functions $\Phi(z,\bar{z})$ in (2.10). Now, we specify the function space and assume that the functions $\Phi(z,\bar{z})$ are infinitely differentiable with respect to z and $\bar{z}$ in $\mathbb{C}$. We consider the behavior of infinitely differentiable functions under group transformations. Every transformation belonging to $\mathrm{SL}(2,\mathbb{C})$ can be obtained as

a composition of translations, dilations, and inversion,

$$
\begin{aligned}
&\mathrm{T}(g)\,\Phi(z,\bar z) = \Phi\left(z-c,\bar z-\bar c\right), && g=\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix};\\
&\mathrm{T}(g)\,\Phi(z,\bar z) = [a]^{1-\sigma_{12}}\cdot\Phi\left(az,\bar a\bar z\right), && g=\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix};\\
&\mathrm{T}(g)\,\Phi(z,\bar z) = z^{\sigma_{12}-1}\bar z^{\bar\sigma_{12}-1}\cdot\Phi\left(-\frac{1}{z},-\frac{1}{\bar z}\right), && g=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.
\end{aligned}
\tag{2.11}
$$

If a function $\Phi(z,\bar z)$ is infinitely differentiable with respect to z and $\bar z$, then its image under translations and dilations has the same property. Inversion takes a neighborhood of the point $z=0$ to a neighborhood of the point $z=\infty$, which imposes an additional restriction on the behavior of the function as $|z|\to\infty$. It can easily be proved [50] that the function $\Phi\left(z,\bar z\right)$ and its image under inversion are differentiable with respect to z and $\bar z$ if and only if the function $\Phi\left(z,\bar z\right)$ is infinitely differentiable with respect to z and $\bar z$ in the complex plane $\mathbb{C}$ and is expandable into an asymptotic series of the form

$$
\Phi\left(z,\bar z\right) \sim z^{\sigma_{12}-1}\bar z^{\bar\sigma_{12}-1}\cdot\sum_{n,m=0}^{+\infty}\Phi_{nm}\,z^{-n}\bar z^{-m}
\tag{2.12}
$$

as $|z|\to\infty$. Thus, we obtain a representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ of the group $\mathrm{SL}(2,\mathbb{C})$ in the space $\mathrm{C}^{\infty}_{\sigma_1,\sigma_2}(\mathbb{C})$ of functions infinitely differentiable with respect to z and $\bar z$ in the complex plane and satisfying the asymptotic relation (2.12). A point (σ_1,σ_2) in the parameter space is singular if both σ_{12} and $\bar\sigma_{12}$ are integers. We say that (σ_1,σ_2) is a point in general position if none of the numbers σ_{12} and $\bar\sigma_{12}$ is an integer.

For the parameters (σ_1,σ_2) in general position, the representation constructed is irreducible and belongs to the most general principal series of nonunitary representations. For a specific choice of (σ_1,σ_2), the representation operators are unitary, and we obtain representations of the principal unitary series. At singular points (σ_1,σ_2) such that σ_{12} and $\bar\sigma_{12}$ are positive integers, there is a finite-dimensional invariant subspace in the representation space on which a finite-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ [49, 50, 55] is realized. We restrict ourselves to irreducible representations of the three series listed above, leaving aside the representations arising at singular points such that σ_{12} and $\bar\sigma_{12}$ are either negative integers or integers of opposite signs.

- The principal series of nonunitary representations.

The action of the operators of the representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ on functions is defined by formula (2.10). The representation parameters satisfy the same restriction $\sigma_{12}-\bar\sigma_{12}\in\mathbb{Z}$. The representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ of the group $\mathrm{SL}(2,\mathbb{C})$ in the space $\mathrm{C}^{\infty}_{\sigma_1,\sigma_2}(\mathbb{C})$ is irreducible if (σ_1,σ_2) is in general position. The representations $\mathrm{T}^{(\sigma_1,\sigma_2)}$ and $\mathrm{T}^{(\sigma_2,\sigma_1)}$ of the principal nonunitary series are equivalent [50].

- The principal series of unitary representations [49, 50, 55].

For the specific values of the representation parameters (n is integral and λ is real)

$$
\sigma_{12}=-\frac{n}{2}+i\lambda;\quad \bar\sigma_{12}=\frac{n}{2}+i\lambda;\quad n\in\mathbb{Z},\quad \lambda\in\mathbb{R},
\tag{2.13}
$$

we can define the following invariant scalar product in the space $\mathrm{C}^{\infty}_{\sigma_1,\sigma_2}(\mathbb{C})$:

$$
\langle\Phi_1|\Phi_2\rangle=\int \mathrm{d}^2z\,\overline{\Phi_1(z,\bar z)}\,\Phi_2(z,\bar z);\quad \langle\mathrm{T}(g)\Phi_1|\mathrm{T}(g)\Phi_2\rangle=\langle\Phi_1|\Phi_2\rangle,
\tag{2.14}
$$

where $z=x+iy, \bar z=x-iy, \mathrm{d}^2z=\mathrm{d}x\mathrm{d}y$, and integration is taken over the entire complex plane. The invariance requirement leads to the condition $|\alpha(\mathrm{h}^{-1})|^2=|d-bz|^{-4}$, which

relates the function $\alpha(\mathrm{h}^{-1})$ to the Jacobian of the transformation

$$z \to \frac{-c+az}{d-bz}; \quad \bar{z} \to \frac{-\bar{c}+\bar{a}\bar{z}}{\bar{d}-\bar{b}\bar{z}}.$$

The equation $|\alpha(\mathrm{h}^{-1})|^2 = |d-bz|^{-4}$ is equivalent to $\sigma_{12} + \bar{\sigma}_{12}^* = 0$, where $*$ stands for complex conjugation, so that we obtain the parametrization (2.13) for the pair $\sigma_{12}, \bar{\sigma}_{12}$. Considering the completion $L^2(\mathbb{C})$ of the space $\mathrm{C}^\infty_{\sigma_1,\sigma_2}(\mathbb{C})$ with respect to the norm $\|\Phi\|^2 = \langle\Phi|\Phi\rangle$ and extending the operators $\mathrm{T}(g)$ by continuity to unitary operators in $L^2(\mathbb{C})$, we obtain representations of the principal unitary series in the space $L^2(\mathbb{C})$. The unitary representations of the principal series are irreducible, and the representations $\mathrm{T}^{(\sigma_1,\sigma_2)}$ and $\mathrm{T}^{(\sigma_2,\sigma_1)}$ are unitarily equivalent [49, 50, 55].

- Finite-dimensional irreducible representations.

If the parameters σ_{12} and $\bar{\sigma}_{12}$ are positive integers, then the representation space of the principal nonunitary series has a finite-dimensional invariant subspace of polynomials in z and $\bar{z}$. Indeed, the formula

$$\mathrm{T}(g)\,\Phi(z,\bar{z}) = (d-bz)^{\sigma_{12}-1}\left(\bar{d}-\bar{b}\bar{z}\right)^{\bar{\sigma}_{12}-1} \cdot \Phi\left(\frac{-c+az}{d-bz}, \frac{-\bar{c}+\bar{a}\bar{z}}{\bar{d}-\bar{b}\bar{z}}\right)$$

shows that the space of polynomials in z and $\bar{z}$ of degree at most $\sigma_{12}-1$ in z and at most $\bar{\sigma}_{12}-1$ in $\bar{z}$ is invariant under the action of the operators $T^{(\sigma_1,\sigma_2)}(g)$. It can be proved that the representation in question is irreducible and that every finite-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ can be obtained in this way. The dimension of a finite-dimensional representation is equal to $\sigma_{12}\cdot\bar{\sigma}_{12}$, so that there exist two two-dimensional fundamental representations. Namely, for $\sigma_{12} = 2$ and $\bar{\sigma}_{12} = 1$, the polynomials 1 and z form a basis of the representation space and, for $\sigma_{12} = 1$ and $\bar{\sigma}_{12} = 2$, the basis polynomials are 1 and $\bar{z}$.

2.3. Intertwining operators. The intertwining operators are the elementary building blocks in the construction of the general $\mathrm{SL}(2,\mathbb{C})$-invariant solution of the Yang–Baxter equation. In this section, we study the properties of these operators. First, we present various representations of intertwining operators and prove that they are equivalent. Then, we consider in detail the main relation for the intertwining operators, namely, the star–triangle relation.

It can be proved [49, 50] that, for two operators $\mathrm{T}^{(\sigma_1,\sigma_2)}$ and $\mathrm{T}^{(\rho_1,\rho_2)}$, an intertwining operator S exists if and only if $(\rho_1,\rho_2) = (\sigma_2,\sigma_1)$, i.e., if and only if the sets of parameters are related by a permutation. For the operator S, the defining relation has the form

$$\mathrm{S}\cdot\mathrm{T}^{(\sigma_1,\sigma_2)}(g) = \mathrm{T}^{(\sigma_2,\sigma_1)}(g)\cdot\mathrm{S}. \tag{2.15}$$

For the first form of an induced representation, the action of the intertwining operator on a function is given by the following formula [55] (the normalization factor $A(\sigma_{12})$ will be fixed later):

$$\begin{gathered}\mathrm{S}\,\Phi(\mathrm{x}) = A(\sigma_{12})\int d^2y\,\Phi(\mathrm{x}\,\mathrm{w}\,\mathrm{y});\\ \mathrm{w} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad \mathrm{y} = \begin{pmatrix}1 & 0\\ y & 1\end{pmatrix} \in \mathrm{Z}, \quad \mathrm{x}\in\mathrm{SL}(2,\mathbb{C}),\end{gathered} \tag{2.16}$$

where w is an element of the Weyl group. To prove (2.15), we verify that the operator S transforms the functions satisfying the condition

$$\Phi(\mathrm{z}\,\mathrm{h}) = [h_{11}]^{1+\sigma_1}\cdot[h_{22}]^{2+\sigma_2}\cdot\Phi(\mathrm{z}) \tag{2.17}$$

to functions satisfying the condition

$$\mathrm{S}\,\Phi(\mathrm{z}\,\mathrm{h}) = [h_{11}]^{1+\sigma_2}\cdot[h_{22}]^{2+\sigma_1}\cdot\mathrm{S}\;\Phi(\mathrm{z}), \tag{2.18}$$

i.e., that the parameters σ_1 and σ_2 are permuted. The proof is immediate: we substitute $\mathrm{x} = \mathrm{z}\,\mathrm{h}$ in $\Phi(\mathrm{x}\,\mathrm{w}\,\mathrm{y})$, transform appropriately the argument

$$\mathrm{z}\,\mathrm{h}\,\mathrm{w}\,\mathrm{y} = \mathrm{z}\,\mathrm{w}\,\mathrm{h}_{\mathrm{w}}\,\mathrm{y} = \mathrm{z}\,\mathrm{w}\,\mathrm{y}'\,\mathrm{h}'_{\mathrm{w}},$$

$$\mathrm{h} = \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix} \to \mathrm{h}_{\mathrm{w}} = \mathrm{w}^{-1}\mathrm{h}\,\mathrm{w} = \begin{pmatrix} h_{22} & 0 \\ -h_{12} & h_{11} \end{pmatrix} \to \mathrm{h}'_{\mathrm{w}} = \begin{pmatrix} h_{22} & 0 \\ 0 & h_{11} \end{pmatrix};$$

$$\mathrm{y}' = \begin{pmatrix} 1 & 0 \\ \frac{h_{11}}{h_{22}} \cdot y - \frac{h_{12}}{h_{22}} & 1 \end{pmatrix},$$

and use a simple rule to transform the measure d^2y,

$$\begin{aligned} \mathrm{S}\,\Phi(\mathrm{zh}) \to \int d^2y\,\Phi(\mathrm{zh}\,\mathrm{w}\,\mathrm{y}) &= \int d^2y\,\Phi(\mathrm{z}\,\mathrm{w}\,\mathrm{y}'\,\mathrm{h}'_{\mathrm{w}}) \\ &= \int d^2y\,[h_{22}][h_{11}]^{-1} \cdot \Phi(\mathrm{z}\,\mathrm{w}\,\mathrm{y}) \cdot [h_{22}]^{1+\sigma_1}[h_{11}]^{2+\sigma_2}. \end{aligned}$$

Rewriting formula (2.16) for the second form of an induced representation, we obtain (see [50])

$$\mathrm{S}\,\Phi(z,\bar z) = A(\sigma_{12}) \int d^2y\,(z-y)^{-1-\sigma_{12}}(\bar z-\bar y)^{-1-\bar\sigma_{12}}\,\Phi(y,\bar y) \tag{2.19}$$

or, in a more concise form,

$$\mathrm{S} = (i\partial_z)^{\sigma_{12}}\,(i\partial_{\bar z})^{\bar\sigma_{12}} \equiv [i\partial_z]^{\sigma_{12}}\,. \tag{2.20}$$

The simple normalization in this formula corresponds to a specific choice of the normalization factor $A(\sigma_{12})$ in (2.16). We return to the proof of formula (2.20) later. Now, we note that the concise representation (2.20) is very convenient when studying the properties of intertwining operators.

We consider two examples. First, this formula can be used to prove the unitary property of the intertwining operator in the case of representations of the principal unitary series. Integrating by parts, we can easily prove that $(i\partial_z)^\dagger = i\partial_{\bar z}$ and $(i\partial_{\bar z})^\dagger = i\partial_z$, and therefore, the operator adjoint to S coincides with its inverse,

$$\mathrm{S} = (i\partial_z)^{-\frac{n}{2}+i\lambda}\,(i\partial_{\bar z})^{\frac{n}{2}+i\lambda} \to \mathrm{S}^\dagger = (i\partial_{\bar z})^{-\frac{n}{2}-i\lambda}\,(i\partial_z)^{\frac{n}{2}-i\lambda} = \mathrm{S}^{-1}.$$

Second, with the help of (2.20) it is easy to trace the finite-dimensional invariant subspaces arising at the integral points (σ_1,σ_2) and to describe them constructively. The intertwining formula $\mathrm{S}\cdot\mathrm{T}^{(\sigma_1,\sigma_2)}(g) = \mathrm{T}^{(\sigma_2,\sigma_1)}(g)\cdot\mathrm{S}$ shows that the nontrivial kernel of the operator S is a subspace invariant under the action of the representation operators $\mathrm{T}^{(\sigma_1,\sigma_2)}(g)$. If the parameters σ_{12} and $\bar\sigma_{12}$ are positive integers, then $\mathrm{S} = [i\partial_z]^{\sigma_{12}}$ is a differential operator that has a nontrivial kernel consisting of functions satisfying the equation $\partial_z^{\sigma_{12}}\partial_{\bar z}^{\bar\sigma_{12}}\Phi(z,\bar z) = 0$. Thus, the representation space of the principal nonunitary series has an invariant subspace of polynomials in z and $\bar z$ of degree at most $\sigma_{12}-1$ in z and at most $\bar\sigma_{12}-1$ in $\bar z$. This example will be generalized to the case of the group $\mathrm{SL}(n,\mathbb{C})$ of higher rank, where the intertwining operators also allow us to trace the finite-dimensional invariant subspaces arising at integral points and to describe them constructively [51]–[53].

The fact that the representations (2.19) and (2.20) for the intertwining operator are equivalent can be proved via the Fourier transformation. As usual, the space of functions $\Phi(z,\bar z)$ is called the coordinate space and the space of functions $\widehat\Phi(p,\bar p)$ obtained by the Fourier transformation,

$$\Phi(z,\bar z) \stackrel{\mathrm{def}}{=} \int \frac{d^2p}{\pi^2}\,\mathrm{e}^{-ipz-i\bar p\bar z}\,\widehat\Phi(p,\bar p); \quad \widehat\Phi(p,\bar p) \stackrel{\mathrm{def}}{=} \int d^2z\,\mathrm{e}^{ipz+i\bar p\bar z}\,\Phi(z,\bar z), \tag{2.21}$$

is called the momentum space. For general values of the parameters α and $\bar\alpha$ ($\alpha-\bar\alpha\in\mathbb{Z}$), we define the operator $[i\partial_z]^\alpha \equiv (i\partial_z)^\alpha\,(i\partial_{\bar z})^{\bar\alpha}$ as the operator of multiplication by the function $[p]^\alpha = p^\alpha\bar p^{\bar\alpha}$ in the momentum space. The Fourier transform of a power function is calculated by the standard formula

$$\int \frac{d^2p}{\pi^2}\,\mathrm{e}^{-ipz-i\bar p\bar z}\,p^{\,\alpha}\bar p^{\,\bar\alpha} = \frac{A(\alpha)}{z^{1+\alpha}\,\bar z^{1+\bar\alpha}};\quad A(\alpha)\overset{\mathrm{def}}{=}\frac{i^{\alpha-\bar\alpha}}{\pi}\,\frac{\Gamma(1+\alpha)}{\Gamma(-\bar\alpha)}, \tag{2.22}$$

$$[p]^\alpha\mapsto\frac{A(\alpha)}{[z]^{1+\alpha}},\quad 1\mapsto\delta^{(2)}(z);\quad [z]^\alpha\mapsto\frac{A^{-1}(-1-\alpha)}{[p]^{1+\alpha}}. \tag{2.23}$$

Since the Fourier transform of a product is a convolution, we obtain an integral operator of the required form in the coordinate space,

$$[i\partial_z]^\alpha\,\Phi(z,\bar z) = A(\alpha)\int d^2y\,(z-y)^{-1-\alpha}(\bar z-\bar y)^{-1-\bar\alpha}\,\Phi(y,\bar y). \tag{2.24}$$

The operators $[i\partial_z]^\alpha$ have all the expected properties:

$$[i\partial_z]^\alpha\cdot[i\partial_z]^\beta = [i\partial_z]^{\alpha+\beta};\quad [i\partial_z]^\alpha\cdot[i\partial_z]^{-\alpha} = \mathbb{1},$$

which follows from similar formulas in the momentum space,

$$p^\alpha\bar p^{\bar\alpha}\cdot p^\beta\bar p^{\bar\beta} = p^{\alpha+\beta}\bar p^{\bar\alpha+\bar\beta};\quad p^\alpha\bar p^{\bar\alpha}\cdot p^{-\alpha}\bar p^{-\bar\alpha} = 1.$$

For the kernels of the operators, the first formula is equivalent to an integral identity, which can be written in two equivalent forms:

$$\int d^2w\frac{1}{[w-x]^{1+\alpha}\,[z-w]^{1+\beta}} = \frac{A(\alpha+\beta)}{A(\alpha)A(\beta)}\,\frac{1}{[z-x]^{1+\alpha+\beta}}, \tag{2.25}$$

$$[i\partial_z]^\alpha:\ \frac{1}{[z-x]^{1+\beta}}\mapsto\frac{A(\alpha+\beta)}{A(\beta)}\,\frac{1}{[z-x]^{1+\alpha+\beta}}. \tag{2.26}$$

In terms of kernels of operators, the second formula is equivalent to the identity

$$\int d^2w\frac{1}{[w-x]^{1+\alpha}\,[z-w]^{1-\alpha}} = \frac{1}{A(\alpha)A(-\alpha)}\,\delta^{(2)}(x-z), \tag{2.27}$$

obtained from (2.25) for $\beta=-\alpha+\varepsilon$ as $\varepsilon\to0$ by the following representation of the delta function:

$$\delta^{(2)}(z) = \lim_{\varepsilon\to0}\int\frac{d^2p}{\pi^2}\,\mathrm{e}^{-ipz-i\bar p\bar z}\,[p]^\varepsilon = \lim_{\varepsilon\to0}\frac{A(\varepsilon)}{[w]^{1+\varepsilon}}. \tag{2.28}$$

In the sequel, an important role will be played by an identity often used in the calculation of the Feynman diagrams in field theory and called the integration rule for a unique vertex, or briefly, the star-triangle relation [57]. This relation can be represented in three equivalent forms: the integral identity

$$\begin{aligned}&\int d^2w\frac{1}{[z-w]^\alpha\,[w-x]^\beta\,[w-y]^\gamma}\\ &\quad=\frac{A(-\beta)}{A(\alpha-1)A(\gamma-1)}\,\frac{1}{[z-x]^{1-\gamma}\,[z-y]^{1-\beta}\,[y-x]^{1-\alpha}},\end{aligned} \tag{2.29}$$

the action of the operator on a function

$$[i\partial_z]^{\alpha-1}:\ \frac{1}{[z-x]^\beta[z-y]^\gamma}\mapsto\frac{A(-\beta)}{A(\gamma-1)}\cdot\frac{1}{[z-x]^{1-\gamma}[z-y]^{1-\beta}[y-x]^{1-\alpha}}, \tag{2.30}$$

and the operator identity [59]

$$[i\partial_z]^\alpha\cdot[z]^{\alpha+\beta}\cdot[i\partial_z]^\beta = [z]^\beta\cdot[i\partial_z]^{\alpha+\beta}\cdot[z]^\alpha. \tag{2.31}$$

In the deformed-symmetry case, similar operator identities and their links with the solution of the Yang–Baxter equation were considered in detail in [58]. The first and the second formulas are valid under the following restriction on the sum of exponents, called the uniqueness condition:

$$\alpha+\beta+\gamma=\bar\alpha+\bar\beta+\bar\gamma=2.$$

To verify that (2.31) and (2.29) are equivalent, it suffices to calculate the kernels of the operators on both sides of (2.31) and obtain the star-triangle relation (2.29). Formula (2.30) is simply a restatement of (2.29) and can be derived from the (already proved) general intertwining formula for the representations of the group $\mathrm{GL}(2,\mathbb{C})$,

$$[i\partial_z]^{\sigma_1-\sigma_2}\cdot \mathrm{T}^{(\sigma_1,\sigma_2)}(g)\,\Phi(z,\bar z)=\mathrm{T}^{(\sigma_2,\sigma_1)}(g)\cdot[i\partial_z]^{\sigma_1-\sigma_2}\,\Phi(z,\bar z),$$

if we take

$$\Phi(z,\bar z)=[z]^{-\beta};\quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto\begin{pmatrix}1&1\\x&y\end{pmatrix},$$

use formulas (2.9) and (2.26), and introduce the notation $\alpha=1+\sigma_1-\sigma_2$, $\gamma=1+\sigma_2-\sigma_1-\beta$.

We note that all formulas of types (2.25) and (2.29) are understood in the sense of the regularized value of an integral [48] or, which is the same, in the sense of analytic continuation from the domain of the parameter values for which the integrals converge absolutely.

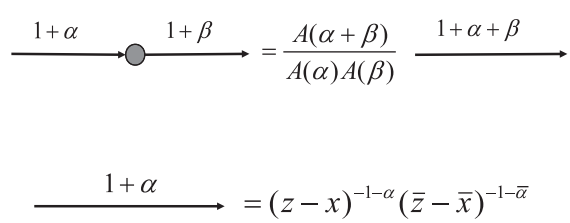

FIGURE 1. Graphical representation of identity (2.25).

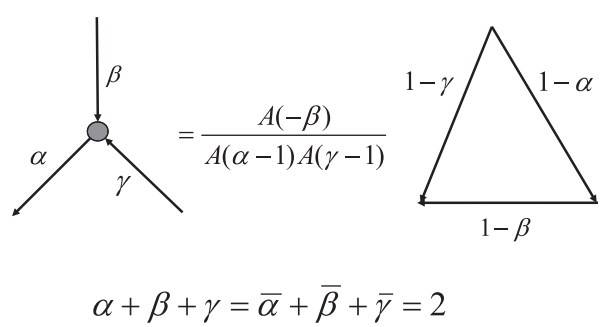

FIGURE 2. Star-triangle relation.

It is convenient to represent identity (2.25) graphically (Figure 1), using the following rules. We display the function $[z-x]^{-\alpha}$ by an arrow with index α that goes from x to z. A vertex at which several vertices meet is marked by the bold dot if integration with respect to the corresponding variable is performed.

A graphical representation of the star-triangle relation is displayed in Figure 2. The left-hand side of the identity is depicted by a star formed by three arrows going from

the same point. The vertex is marked to show that integration with respect to w is performed. The vertex at which three arrows meet is said to be *unique* if the sum of the indices of the three lines is equal to 2. The right-hand side of the identity is depicted by a triangle formed by three arrows. The triangle is *unique* if the sum of the indices of the three lines is equal to 1. Thus, the star-triangle relation allows us to transform a unique vertex to a unique triangle and *vice versa*.

2.4. Generators of the Lie algebras $\mathrm{gl}(2,\mathbb{C})$ **and** $\mathrm{sl}(2,\mathbb{C})$**.** The following matrix $\mathrm{L}(u)$, which coincides up to a trivial summand with the matrix E constructed from generators of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$, is a main object in the study of solutions of the Yang–Baxter equation:

$$\mathrm{L}(u) = u \cdot \mathbb{1} + \mathrm{E} = \begin{pmatrix} u + \mathrm{E}_{11} & \mathrm{E}_{21} \\ \mathrm{E}_{12} & u + \mathrm{E}_{22} \end{pmatrix}.$$

In this section, we obtain explicit expressions for the generators and a convenient representation of the matrix E.

The Lie algebra $\mathrm{gl}(2,\mathbb{C})$ consists of complex matrices of order 2. As a basis of the space $\mathrm{Mat}(2\times 2)$, we take the standard matrix units $\mathbf{e}_{ik}$, $(\mathbf{e}_{ik})_{nm} = \delta_{in}\delta_{km}$. The generators E_{ik} and $\bar{\mathrm{E}}_{ik}$ of the Lie algebra $\mathrm{gl}(2,\mathbb{C})$ in the representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ are defined by the relation

$$\mathrm{T}(1+\epsilon\cdot\mathbf{e}_{ik})\,\Phi(z,\bar z) = \Phi(z,\bar z) + \left(\epsilon\cdot\mathrm{E}_{ik} + \bar\epsilon\cdot\bar{\mathrm{E}}_{ik}\right)\,\Phi(z,\bar z) + O(\epsilon^2). \tag{2.32}$$

From the definition (2.9), we obtain explicit formulas for generators, which, in the representation in question, are realized by the first order differential operators

$$\begin{aligned} &\mathrm{E}_{11} = z\partial_z + 1 - \sigma_1; \quad \mathrm{E}_{12} = z^2\partial_z + (1-\sigma_1+\sigma_2)z; \\ &\mathrm{E}_{21} = -\partial_z; \quad \mathrm{E}_{22} = -z\partial_z - \sigma_2; \\ &\bar{\mathrm{E}}_{11} = \bar z\partial_{\bar z} + 1 - \bar\sigma_1; \quad \bar{\mathrm{E}}_{12} = \bar z^2\partial_{\bar z} + (1-\bar\sigma_1+\bar\sigma_2)\bar z; \\ &\bar{\mathrm{E}}_{21} = -\partial_{\bar z}; \quad \bar{\mathrm{E}}_{22} = -\bar z\partial_{\bar z} - \bar\sigma_2. \end{aligned} \tag{2.33}$$

We consider the matrix E with the entries E_{ik} (similarly, the matrix $\bar{\mathrm{E}}$ with the entries $\bar{\mathrm{E}}_{ik}$):

$$\mathrm{E} = \mathrm{E}^{(\sigma_1,\sigma_2)} = \begin{pmatrix} \mathrm{E}_{11} & \mathrm{E}_{21} \\ \mathrm{E}_{12} & \mathrm{E}_{22} \end{pmatrix} = \begin{pmatrix} z\partial_z + 1 - \sigma_1 & -\partial_z \\ z^2\partial_z + (1-\sigma_1+\sigma_2)\cdot z & -z\partial_z - \sigma_2 \end{pmatrix}. \tag{2.34}$$

The generators E_{ik} and $\bar{\mathrm{E}}_{ik}$ of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ are calculated in the same way by the formula

$$\mathrm{T}(1+\epsilon\cdot\mathcal{E}_{ik})\,\Phi(z,\bar z) = \Phi(z,\bar z) + \left(\epsilon\cdot\mathrm{E}_{ik} + \bar\epsilon\cdot\bar{\mathrm{E}}_{ik}\right)\,\Phi(z,\bar z) + O(\epsilon^2);$$
$$\mathcal{E}_{ik} = \mathbf{e}_{ik} - \frac{\delta_{ik}}{2}\cdot\mathbb{1},$$

where the $\mathcal{E}_{ik}$ are the generators of $\mathrm{sl}(2,\mathbb{C})$ in the fundamental representation, i.e., matrices with trace 0. We consider the matrix whose entries are the generators obtained,

$$\begin{pmatrix} \mathrm{E}_{11} & \mathrm{E}_{21} \\ \mathrm{E}_{12} & \mathrm{E}_{22} \end{pmatrix} = \begin{pmatrix} z\partial_z + \frac{1-\sigma_{12}}{2} & -\partial_z \\ z^2\partial_z + (1-\sigma_{12})\cdot z & -z\partial_z - \frac{1-\sigma_{12}}{2} \end{pmatrix}. \tag{2.35}$$

In contrast to (2.34), the generators of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ satisfy the relation $\mathrm{E}_{11} + \mathrm{E}_{22} = 0$. We use the general formula (2.34) because it is convenient to consider both cases simultaneously: if we impose the additional condition $\sigma_1+\sigma_2 = 1$ on the parameters σ_1, σ_2 in (2.34), we obtain (2.35).

There are convenient representations for E and $\bar{\mathrm{E}}$ as products of triangular matrices,

$$\begin{aligned} \mathrm{E}^{(\sigma_1,\sigma_2)} &= \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} -\sigma_1 & -\partial_z \\ 0 & -\sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}; \\ \bar{\mathrm{E}}^{(\bar\sigma_1,\bar\sigma_2)} &= \begin{pmatrix} 1 & 0 \\ \bar z & 1 \end{pmatrix} \begin{pmatrix} -\bar\sigma_1 & -\partial_{\bar z} \\ 0 & -\bar\sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar z & 1 \end{pmatrix}. \end{aligned} \tag{2.36}$$

In this decomposition of the matrix E, information about the representation in question of the Lie algebra $\mathrm{gl}(2,\mathbb{C})$ is encoded in a concise and visual way.

- The parameters σ_1 and σ_2 that characterize the representation are the diagonal entries of the middle matrices in (2.36). In the case of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$, we have $\sigma_1+\sigma_2=1$. If this relation is not satisfied, then we have a representation of the Lie algebra $\mathrm{gl}(2,\mathbb{C})$.
- The intertwining operator $\mathrm{S}=(i\partial_z)^{\sigma_1-\sigma_2}\,(i\partial_{\bar z})^{\bar\sigma_1-\bar\sigma_2}$ is constructed by a plain rule from the entries of the middle matrices in (2.36).
- The intertwining relation (2.15) is equivalent to the relations

$$\mathrm{S}\cdot\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} -\sigma_1 & -\partial_z \\ 0 & -\sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} -\sigma_2 & -\partial_z \\ 0 & -\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}\cdot\mathrm{S}, \tag{2.37a}$$

$$\mathrm{S}\cdot\begin{pmatrix} 1 & 0 \\ \bar z & 1 \end{pmatrix} \begin{pmatrix} -\bar\sigma_1 & -\partial_{\bar z} \\ 0 & -\bar\sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar z & 1 \end{pmatrix} \begin{pmatrix} -\bar\sigma_2 & -\partial_{\bar z} \\ 0 & -\bar\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar z & 1 \end{pmatrix}\cdot\mathrm{S}, \tag{2.37b}$$

  i.e., the diagonal entries of the middle matrices are interchanged under the action of S.

We note that, by integrating by parts, we can represent relations (2.37) as a system of differential equations for the kernel of the operator S. A unique solution is the expression (2.19), which was obtained earlier from other considerations.

§3. $\mathrm{SL}(2,\mathbb{C})$-invariant R-matrix

3.1. The Yang–Baxter equation. Our goal is to construct $\mathrm{SL}(2,\mathbb{C})$-invariant solutions of the Yang–Baxter equation

$$\mathrm{R}_{12}(u-v,\bar u-\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{23}(v,\bar v)=\mathrm{R}_{23}(v,\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{12}(u-v,\bar u-\bar v). \tag{3.1}$$

The linear operator $\mathrm{R}(u,\bar u)$ depends on two spectral parameters u and $\bar u$, which *a priori* are not related to each other. It will be proved that the difference $u-\bar u$ must be an integer.[1] The fact that the operator $\mathrm{R}(u,\bar u)$ is $\mathrm{SL}(2,\mathbb{C})$-invariant means that it acts in the space of the tensor product of two representations of the group $\mathrm{SL}(2,\mathbb{C})$ and commutes with the operators of the representation

$$\mathrm{T}(g)\otimes\mathrm{T}(g)\cdot\mathrm{R}(u,\bar u)=\mathrm{R}(u,\bar u)\cdot\mathrm{T}(g)\otimes\mathrm{T}(g).$$

In the Yang–Baxter equation, all operators are defined in the general tensor product of three spaces $\mathbb{V}_1\otimes\mathbb{V}_2\otimes\mathbb{V}_3$. The indices ik show that the operator R_{ik} acts nontrivially in the tensor product $\mathbb{V}_i\otimes\mathbb{V}_k$ and is extended to the remaining part of the tensor product $\mathbb{V}_1\otimes\mathbb{V}_2\otimes\mathbb{V}_3$ as the identity operator. In the sequel, we drop the index ik if the space in which the operator R acts is clear from the context.

We construct a solution of the Yang–Baxter equation in two steps. First, we find a solution of a simpler defining RLL-relation [56]. Then, we prove that the resulting operator $\mathrm{R}(u,\bar u)$ satisfies the general equation (3.1).

To make the origin of the RLL-relation clear, we outline how to derive it from the general Yang–Baxter equation. For the Yang–Baxter equation (3.1), as the space $\mathbb{V}_3$ we take the space of the fundamental two-dimensional representation of the group $\mathrm{SL}(2,\mathbb{C})$.

[1] For the first time, this condition was stated in the paper [21].

In this case, the operators $\mathrm{R}_{13}(u,\bar{u})$ and $\mathrm{R}_{23}(v,\bar{v})$ transform into the standard quantum L-operators, which are the matrices $\mathrm{L}_1(u)$ and $\mathrm{L}_2(v)$ [56], [4]–[6] that have the following general form:

$$\mathrm{L}(u) = u\cdot\mathbb{1} + \mathrm{E} = \begin{pmatrix} u+\mathrm{E}_{11} & \mathrm{E}_{21} \\ \mathrm{E}_{12} & u+\mathrm{E}_{22} \end{pmatrix} \tag{3.2}$$

and depend on the generators E_{ik} of the representations in the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively. Thus, we obtain the following simple defining RLL-relation for the operator R_{12} (see [56]):

$$\mathrm{R}_{12}(u-v,\bar{u}-\bar{v})\mathrm{L}_1(u)\mathrm{L}_2(v) = \mathrm{L}_2(v)\mathrm{L}_1(u)\mathrm{R}_{12}(u-v,\bar{u}-\bar{v}). \tag{3.3}$$

However, this equation is insufficient because the matrix $\mathrm{L}(u)$ depends only on the holomorphic variable and on the spectral parameter u. The missing equation can be obtained similarly, but for the role of the space $\mathbb{V}_3$ we must take the space of the second two-dimensional representation of the group $\mathrm{SL}(2,\mathbb{C})$. In this case, the operator $\mathrm{R}_{13}(u,\bar{u})$ is the matrix $\bar{\mathrm{L}}_1(\bar{u})$, the operator $\mathrm{R}_{23}(v,\bar{v})$ is the matrix $\bar{\mathrm{L}}_2(\bar{v})$, and we obtain

$$\mathrm{R}_{12}(u-v,\bar{u}-\bar{v})\bar{\mathrm{L}}_1(\bar{u})\bar{\mathrm{L}}_2(\bar{v}) = \bar{\mathrm{L}}_2(\bar{v})\bar{\mathrm{L}}_1(\bar{u})\mathrm{R}_{12}(u-v,\bar{u}-\bar{v}). \tag{3.4}$$

Up to a trivial summand, the matrix $\bar{\mathrm{L}}(\bar{u})$ coincides with the matrix $\bar{\mathrm{E}}$,

$$\bar{\mathrm{L}}(\bar{u}) = \bar{u}\cdot\mathbb{1} + \bar{\mathrm{E}} = \begin{pmatrix} \bar{u}+\bar{\mathrm{E}}_{11} & \bar{\mathrm{E}}_{21} \\ \bar{\mathrm{E}}_{12} & \bar{u}+\bar{\mathrm{E}}_{22} \end{pmatrix}. \tag{3.5}$$

We solve the resulting system of two RLL-equations, taking as $\mathbb{V}_1$ and $\mathbb{V}_2$ the representation spaces of continuous series. Let $\mathbb{V}_1$ be the space of functions of z_1 and $\bar{z}_1$ in which the operators of the representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ act, and let $\mathbb{V}_2$ be the space of functions of z_2 and $\bar{z}_2$ in which the operators of the representation $\mathrm{T}^{(\rho_1,\rho_2)}$ act. Using formula (2.36) for the matrices E from $\mathrm{L}_1(u)$ and $\mathrm{L}_2(v)$, we see that

$$\begin{aligned} \mathrm{L}_1(u) &= \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} u_1 & -\partial_{z_1} \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z_1 & 1 \end{pmatrix}; \\ \mathrm{L}_2(v) &= \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} v_1 & -\partial_{z_2} \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z_2 & 1 \end{pmatrix}. \end{aligned} \tag{3.6}$$

The matrix $\mathrm{L}_1(u)$ depends on the spectral parameter u and the parameters σ_1 and σ_2 of the representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$. Formula (3.6) shows that all the parameters can naturally be combined into the set of numbers $(u_1,u_2) = (u-\sigma_1, u-\sigma_2)$, which are the diagonal entries of the middle matrix, so that $\mathrm{L}_1(u) = \mathrm{L}_1(u_1,u_2)$. In the same way, we have $\mathrm{L}_2(v) = \mathrm{L}_2(v_1,v_2)$, where $(v_1,v_2) = (v-\rho_1, v-\rho_2)$.

It is convenient to extract the permutation P_{ik} from the operator R_{ik}, $\mathrm{R}_{ik} = \mathrm{P}_{ik}\,\check{\mathrm{R}}_{ik}$. Acting on a function, the operator P_{ik} permutes the arguments z_i and z_k. For example, the operator R_{12} acts in the space $\mathbb{V}_1\otimes\mathbb{V}_2$, the elements of which are functions $\Phi(z_1,\bar{z}_1,z_2,\bar{z}_2)$. In this case, the permutation operator P_{12} interchanges the variables z_1 and z_2, $\mathrm{P}_{12}\,\Phi(z_1,\bar{z}_1,z_2,\bar{z}_2) = \Phi(z_2,\bar{z}_2,z_1,\bar{z}_1)$. For the operator $\check{\mathrm{R}}$, the general Yang–Baxter equation is modified as follows:

$$\check{\mathrm{R}}_{23}(u-v,\bar{u}-\bar{v})\check{\mathrm{R}}_{12}(u,\bar{u})\check{\mathrm{R}}_{23}(v,\bar{v}) = \check{\mathrm{R}}_{12}(v,\bar{v})\check{\mathrm{R}}_{23}(u,\bar{u})\check{\mathrm{R}}_{12}(u-v,\bar{u}-\bar{v}), \tag{3.7}$$

and the defining relations take the form

$$\check{\mathrm{R}}(u-v,\bar{u}-\bar{v})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(v_1,v_2)\,\mathrm{L}_2(u_1,u_2)\,\check{\mathrm{R}}(u-v,\bar{u}-\bar{v}), \tag{3.8a}$$

$$\check{\mathrm{R}}(u-v,\bar{u}-\bar{v})\,\bar{\mathrm{L}}_1(\bar{u}_1,\bar{u}_2)\,\bar{\mathrm{L}}_2(\bar{v}_1,\bar{v}_2) = \bar{\mathrm{L}}_1(\bar{v}_1,\bar{v}_2)\,\bar{\mathrm{L}}_2(\bar{u}_1,\bar{u}_2)\,\check{\mathrm{R}}(u-v,\bar{u}-\bar{v}). \tag{3.8b}$$

In the sequel, we use a more concise notation, suppressing the antiholomorphic part of the equations and the explicit indication to the dependence on the variables $\bar{z}$ and $\bar{u}$ if this does not cause misunderstanding.

Equation (3.8) has a natural interpretation; namely, the operator $\check{\mathrm{R}}$ interchanges the parameters (u_1, u_2) of the first Lax matrix and the parameters (v_1, v_2) of the second matrix. It is convenient to combine these four parameters into one set in the following order: $\mathbf{u} \equiv (v_1, v_2, u_1, u_2)$. Thus, the operator $\check{\mathrm{R}}$ corresponds to a specific permutation s belonging to the permutation group of the four parameters,

$$s \to \check{\mathrm{R}}(u-v); \quad s(v_1, v_2, u_1, u_2) = (u_1, u_2, v_1, v_2).$$

Let us investigate how far we can proceed in parallel with the permutation group. An arbitrary permutation in the symmetry group $\mathfrak{S}_4$ can be constructed from the elementary transpositions s_1, s_2, and s_3,

$$s_1\mathbf{u} = (v_2, v_1, u_1, u_2), \quad s_2\mathbf{u} = (v_1, u_1, v_2, u_2), \quad s_3\mathbf{u} = (v_1, v_2, u_2, u_1),$$

permuting only two nearest neighbors. For example, for the specific permutation in question, the decomposition into elementary transpositions has the form $s = s_2 s_1 s_3 s_2$. It is natural to try to find the operators $\mathrm{S}_i(\mathbf{u})$ that realize these transpositions:

$$(\overbrace{v_1,\ v_2}^{\mathrm{S}_1}, u_1, u_2)\ :\ \mathrm{S}_1(\mathbf{u})\,\mathrm{L}_2(v_1, v_2) = \mathrm{L}_2(v_2, v_1)\mathrm{S}_1(\mathbf{u}),$$

$$(v_1\,\overbrace{v_2, u_1}^{\mathrm{S}_2},\ u_2)\ :\ \mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) = \mathrm{L}_1(v_2, u_2)\,\mathrm{L}_2(v_1, u_1)\mathrm{S}_2(\mathbf{u}),$$

$$(v_1, v_2, \overbrace{u_1,\ u_2}^{\mathrm{S}_3})\ :\ \mathrm{S}_3(\mathbf{u})\,\mathrm{L}_1(u_1, u_2) = \mathrm{L}_1(u_2, u_1)\mathrm{S}_3(\mathbf{u}).$$

The operators S_1 and S_3 act much in the same way since each of them permutes only parameters of a single L-operator. We note that the intertwining operator S realizes the required permutation of the parameters, because formula (2.37) is equivalent to the required relation for the matrix $\mathrm{L}(u_1, u_2)$,

$$\mathrm{S}\,\mathrm{L}(u_1, u_2) = \mathrm{L}(u_2, u_1)\,\mathrm{S}.$$

Thus, two of the three operators that realize elementary permutations are intertwining operators acting separately on their L-matrices,

$$\begin{aligned} \mathrm{S}_1(\mathbf{u}) &= (i\partial_{z_2})^{v_2-v_1}(i\partial_{\bar{z}_2})^{\bar{v}_2-\bar{v}_1} \equiv [i\partial_2]^{v_2-v_1}, \\ \mathrm{S}_3(\mathbf{u}) &= (i\partial_{z_1})^{u_2-u_1}(i\partial_{\bar{z}_1})^{\bar{u}_2-\bar{u}_1} \equiv [i\partial_1]^{u_2-u_1}. \end{aligned} \tag{3.9}$$

Using the representation (3.6) for the matrices L_1 and L_2 ($z_{ik} \equiv z_i - z_k$), we write the equation

$$\mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_1, u_2)\,\mathrm{L}_2(v_1, v_2) = \mathrm{L}_1(v_2, u_2)\,\mathrm{L}_2(v_1, u_1)\mathrm{S}_2(\mathbf{u})$$

for the last missing operator $\mathrm{S}_2(\mathbf{u})$ in an expanded form as follows:

$$\begin{aligned} &\mathrm{S}_2\underline{\begin{pmatrix}1 & 0\\ z_1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & u_2\end{pmatrix}}\begin{pmatrix}u_1 & -\partial_{z_1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ z_{21} & 1\end{pmatrix}\begin{pmatrix}1 & -\partial_{z_2}\\ 0 & v_2\end{pmatrix}\underline{\begin{pmatrix}v_1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -z_2 & 1\end{pmatrix}} \\ &\quad = \underline{\begin{pmatrix}1 & 0\\ z_1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & u_2\end{pmatrix}}\begin{pmatrix}v_2 & -\partial_{z_1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ z_{21} & 1\end{pmatrix}\begin{pmatrix}1 & -\partial_{z_2}\\ 0 & u_1\end{pmatrix}\underline{\begin{pmatrix}v_1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -z_2 & 1\end{pmatrix}}\mathrm{S}_2. \end{aligned}$$

For clearness, we explicitly singled out the diagonal matrices $\mathrm{diag}(1, u_2)$ and $\mathrm{diag}(v_1, 1)$ in the above formula. Despite a large number of factors, each factor is simple and the structure of the equation hints at a natural method for its solution. The parameters u_2 and v_1 enter the equation in such a way that, under the condition $[\mathrm{S}_2, z_1] = [\mathrm{S}_2, z_2] = 0$,

we can cancel out the underlined matrices, so that everything simplifies considerably, and we obtain

$$\mathrm{S}_2 \begin{pmatrix} u_1 & -\partial_{z_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\partial_{z_2} \\ 0 & v_2 \end{pmatrix} = \begin{pmatrix} v_2 & -\partial_{z_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\partial_{z_2} \\ 0 & u_1 \end{pmatrix} \mathrm{S}_2. \tag{3.10}$$

The conditions $[\mathrm{S}_2, z_1] = [\mathrm{S}_2, z_2] = 0$ mean that S_2 is the operator of multiplication by a function of the variables z_1 and z_2 that can easily be found from equation (3.10) and a similar antiholomorphic equation

$$\mathrm{S}_2(\mathbf{u}) = z_{12}^{u_1-v_2}\, \bar{z}_{12}^{\bar{u}_1-\bar{v}_2} = [z_{12}]^{u_1-v_2}. \tag{3.11}$$

Now, we have all operators realizing the elementary permutations $s_i \to \mathrm{S}_i(\mathbf{u})$, and it remains to construct a composite operator realizing the permutation $s = s_2 s_1 s_3 s_2$. It should be taken into account that, in the product $\mathrm{S}_i \mathrm{S}_j$, the operator S_i acts on the product of L-matrices in which the parameters are already permuted properly after the action of the operator S_j. As a result, the realization of the product is defined by the formula $s_i s_j \mapsto \mathrm{S}_i(s_j\mathbf{u})\mathrm{S}_j(\mathbf{u})$, and, finally, we get

$$\begin{aligned} s = s_2 s_1 s_3 s_2 \mapsto \check{\mathrm{R}}(u-v) &= \mathrm{S}_2(s_1 s_3 s_2 \mathbf{u})\mathrm{S}_1(s_3 s_2 \mathbf{u})\mathrm{S}_3(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u}) \\ &= [z_{12}]^{u_2-v_1}\, [i\partial_2]^{u_1-v_1}\, [i\partial_1]^{u_2-v_2}\, [z_{12}]^{u_1-v_2}. \end{aligned} \tag{3.12}$$

The explicit formula for the integral operator $\check{\mathrm{R}}(u-v)$,

$$\check{\mathrm{R}}(u-v)\Phi(z_1,\bar{z}_1,z_2,\bar{z}_2) \equiv \int d^2x_1 \int d^2x_2\, \check{\mathrm{R}}_{u-v}(z_1,z_2|x_1,x_2)\, \Phi(x_1,\bar{x}_1,x_2,\bar{x}_1)\,, \tag{3.13}$$

can easily be obtained from (3.12) by formula (2.24) for the middle operators:

$$\begin{aligned} \check{\mathrm{R}}_{u-v}(z_1,z_2|x_1,x_2) = A(u_1-v_1)A(u_2-v_2)[z_{12}]^{u_2-v_1} \\ \times [z_2-x_2]^{v_1-u_1-1}\, [z_1-x_1]^{v_2-u_2-1}\, [x_{12}]^{u_1-v_2} \end{aligned}$$

(the dependence of the kernel on the conjugate variables is suppressed for simplicity). The operator $\mathrm{R}(u-v)$ is obtained from the operator $\check{\mathrm{R}}(u-v)$ by left multiplication by a permutation operator, and the kernel of the first operator is obtained from the kernel of the second by the permutation $z_1 \rightleftarrows z_2$,

$$\begin{aligned} \mathrm{R}_u(z_1,z_2|x_1,x_2) = A(u-\sigma_1+\rho_1)\cdot A(u-\sigma_2+\rho_2)[z_{21}]^{u+\rho_1-\sigma_2} \\ \times [z_1-x_2]^{\sigma_1-\rho_1-u-1}\, [z_2-x_1]^{\sigma_2-\rho_2-u-1}\, [x_{12}]^{u-\sigma_1+\rho_2}. \end{aligned} \tag{3.14}$$

The spectral parameters u and $\bar{u}$, the parameters σ_1 and σ_2 of the representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$ in the space of functions of the variables z_1 and $\bar{z}_1$, and the parameters ρ_1 and ρ_2 of the representation $\mathrm{T}^{(\rho_1,\rho_2)}$ in the space of functions of the variables z_2 and $\bar{z}_2$ are involved in the formula for the kernel of the operator $\mathrm{R}(u)$ explicitly. This detailed writing shows that the kernel is a one-valued function of the complex variables z_1, z_2, x_1, and x_2 only if $u - \bar{u} \in \mathbb{Z}$. The kernel of the integral operator $\mathrm{R}(u-v)$ is represented graphically by a quadrangle, as shown in Figure 3.

We have found an operator R that solves the RLL-relations. It remains to prove that this R is $\mathrm{SL}(2,\mathbb{C})$-invariant and satisfies the general Yang–Baxter relation.

- $\mathrm{SL}(2,\mathbb{C})$-invariance.

The $\mathrm{SL}(2,\mathbb{C})$-invariance of R can easily be proved by direct calculation, but we give a general proof, which can easily be generalized to the case of the group $\mathrm{SL}(n,\mathbb{C})$. The proof is based on the following two facts. First, the operator $\mathrm{R}(u-v)$ as in (3.14) is a solution of the RLL-relations, and, second, this operator depends only on the difference $u-v$ of the spectral parameters, i.e., is invariant under the translations $u \to u+\lambda$ and $v \to v+\lambda$.

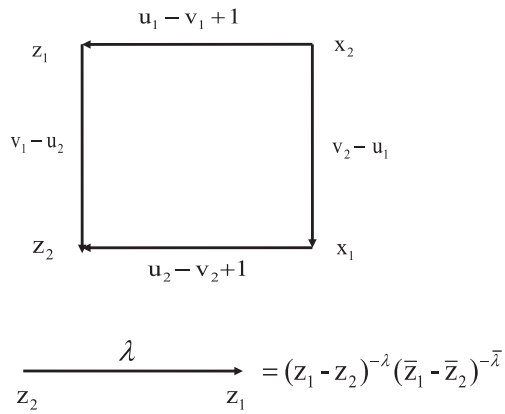

FIGURE 3. Graphical representation of the kernel of the operator $\mathrm{R}(u-v)$ (see 3.14).

Performing the translation $u \to u+\lambda$, $v \to v+\lambda$ in the relation

$$\mathrm{R}(u-v)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_2(v_1,v_2)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{R}(u-v)$$

and selecting the contributions linear in λ, we obtain

$$\mathrm{R}(u-v)\ (\mathrm{L}_1(u_1,u_2)+\mathrm{L}_2(v_1,v_2)) = (\mathrm{L}_2(v_1,v_2)+\mathrm{L}_1(u_1,u_2))\ \mathrm{R}(u-v). \tag{3.15}$$

We recall that

$$\mathrm{L}_1(u_1,u_2) = u\cdot\mathbb{1}+\mathrm{E}_1, \quad \mathrm{L}_2(v_1,v_2) = v\cdot\mathbb{1}+\mathrm{E}_2,$$

and therefore, equation (3.15) is obviously equivalent to

$$\mathrm{R}(u-v)\ (\mathrm{E}_1+\mathrm{E}_2) = (\mathrm{E}_1+\mathrm{E}_2)\ \mathrm{R}(u-v). \tag{3.16}$$

The matrix E_1 contains the generators of the representation $\mathrm{T}^{(\sigma_1,\sigma_2)}$, and the matrix E_2 contains the generators of the representation $\mathrm{T}^{(\rho_1,\rho_2)}$. Therefore, the operator $\mathrm{R}(u-v)$ commutes with the generators of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ that act in the tensor product $\mathrm{T}^{(\sigma_1,\sigma_2)}\otimes\mathrm{T}^{(\rho_1,\rho_2)}$. Finally, we obtain

$$\mathrm{R}(u-v)\,\mathrm{T}^{(\sigma_1,\sigma_2)}\otimes\mathrm{T}^{(\rho_1,\rho_2)} = \mathrm{T}^{(\sigma_1,\sigma_2)}\otimes\mathrm{T}^{(\rho_1,\rho_2)}\,\mathrm{R}(u-v). \tag{3.17}$$

- The Yang–Baxter relation.

We present a visual graphical proof [21] of the relation

$$\mathrm{R}_{12}(u-v,\bar u-\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{23}(v,\bar v) = \mathrm{R}_{23}(v,\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{12}(u-v,\bar u-\bar v)$$

for the integral operator R with the kernel (3.14). The kernel of the operator on the left-hand side of the Yang–Baxter equation, which is the product of three R-operators, is constructed from the kernels of the factors,

$$\begin{aligned}&\Big[\mathrm{R}_{12}(u-v,\bar u-\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{23}(v,\bar v)\Big](z_1,z_2,z_3|x_1,x_2,x_3)\\ &\qquad = \int d^2y_1\,d^2y_2\,d^2y_3\,R_{u-v}(z_1,z_2|y_1,y_2)R_u(y_1,z_3|x_1,y_3)R_v(y_2,y_3|x_2,x_3).\end{aligned} \tag{3.18}$$

Illustrating the kernel of each R-operator with the corresponding picture, where $z_1\to 1'$, $x_1\to 1$, etc., we obtain a graphical representation of the kernel of the product in the left

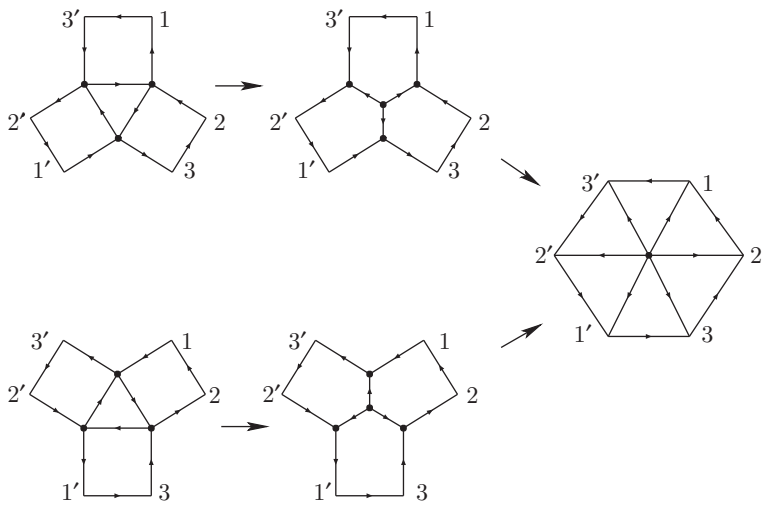

FIGURE 4. Graphical proof of the Yang–Baxter relation with the help of the star-triangle relation.

upper corner in Figure 4. The kernel of the operator involved on the right-hand side of the Yang–Baxter equation is constructed similarly from the kernels of the factors,

$$\begin{aligned}\Big[\mathrm{R}_{23}(v,\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{12}(u-v,\bar u-\bar v)\Big](z_1,z_2,z_3|x_1,x_2,x_3)\\ =\int d^2y_1\,d^2y_2\,d^2y_3\,R_v(z_2,z_3|y_2,y_3)R_u(z_1,y_3|y_1,x_3)R_{u-v}(y_1,y_2|x_1,x_2),\end{aligned}\tag{3.19}$$

and the pattern obtained is displayed in the left lower corner in Figure 4. These pictures are convenient because for each transformation of integrals we have a transformation of the corresponding pictures, and the cumbersome formulas are replaced by visual pictures. In the case in question, the transformations are based on the star-triangle identity and are displayed in Figure 4. Each central triangle is unique and transforms into a star. After that, three vertices become unique and are transformed into triangles. As a result, we obtain one and the same hexagon for the upper and for the lower picture. Unfortunately, it is hard to generalize such visual graphical language to the case of the group $\mathrm{SL}(n,\mathbb{C})$. Therefore, we restate the proof given above in more convenient algebraic terms.

3.2. The permutation group and relations for the operators $\mathrm{S}_k(\mathbf{u})$. The elementary transpositions s_1, s_2, and s_3 are the generators of the permutation group $\mathfrak{S}_4$; i.e., every permutation in $\mathfrak{S}_4$ can be represented as a product of the generators s_k. This representation is not unique, because distinct products can yield the same permutation. We consider a specific example needed in the sequel. The permutation s of the pair v_1, v_2 with the pair u_1, u_2 in (v_1, v_2, u_1, u_2) can be decomposed into a product of the generators in the following ways:

- the decomposition with a minimal number of factors: $s = s_2 s_1 s_3 s_2$;
- the decomposition into two commuting permutations of special type,

$$s = r_1 \cdot r_2 = r_2 \cdot r_1; \quad r_1 = s_2 s_1 s_2, \quad r_2 = s_2 s_3 s_2.$$

The permutations r_1 and r_2 have a simple geometric meaning,

$$r_1\mathbf{u} = (u_1, v_2, v_1, u_2); \quad r_2\mathbf{u} = (v_1, u_2, u_1, v_2),$$

i.e., r_1 interchanges v_1 and u_1, and r_2 interchanges v_2 and u_2. In this case, the permutation s of the pair v_1, v_2 with the pair u_1, u_2 is performed in two steps. First, v_1 are u_1

interchanged, and then v_2 and u_2 are interchanged, or *vice versa*. We have the following relations among the generators [54]:

$$s_1^2 = s_2^2 = s_3^2 = \mathbb{1}, \quad s_1 s_3 = s_3 s_1; \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3,$$

which makes it possible to transform different decompositions of the same permutation into each other. In our example, we use the relations $s_2^2 = \mathbb{1}$ and $s_1 s_3 = s_3 s_1$:

$$\begin{aligned} s = s_2 s_1 s_3 s_2 &= s_2 s_1 \underline{s_2 \cdot s_2} s_3 s_2 = r_1 \cdot r_2 \\ &= s_2 \underline{s_3 \cdot s_1} s_2 = s_2 \underline{s_1 \cdot s_3} s_2 = s_2 s_3 \underline{s_2 \cdot s_2} s_1 s_2 = r_2 \cdot r_1. \end{aligned} \tag{3.20}$$

Under the action of the product $\mathrm{L}_1(u_1, u_2)\ \mathrm{L}_2(v_1, v_2)$, the operators S_1, S_2, and S_3 perform the same permutations of parameters in the set (v_1, v_2, u_1, u_2) as the generators s_1, s_2, and s_3. We have the correspondence

$$s_i \to \mathrm{S}_i(\mathbf{u}); \quad s_i s_j \mapsto \mathrm{S}_i(s_j \mathbf{u}) \mathrm{S}_j(\mathbf{u}). \tag{3.21}$$

Now, if we prove that the relations

$$s_i s_i = \mathbb{1} \to \mathrm{S}_i(s_i \mathbf{u}) \mathrm{S}_i(\mathbf{u}) = \mathbb{1}; \quad s_1 s_3 = s_3 s_1 \to \mathrm{S}_1(s_3 \mathbf{u}) \mathrm{S}_3(\mathbf{u}) = \mathrm{S}_3(s_1 \mathbf{u}) \mathrm{S}_1(\mathbf{u}) \tag{3.22}$$

are valid for the operators $\mathrm{S}_i(\mathbf{u})$, then the identity for the generators s_k implies a similar identity for the operators $\mathrm{S}_k(\mathbf{u})$. For example, the operators corresponding to the permutations r_1 and r_2,

$$\begin{aligned} r_1 &\to \mathrm{R}_1(\mathbf{u}) = \mathrm{S}_2(s_1 s_2 \mathbf{u}) \mathrm{S}_1(s_2 \mathbf{u}) \mathrm{S}_2(\mathbf{u}), \\ r_2 &\to \mathrm{R}_2(\mathbf{u}) = \mathrm{S}_2(s_3 s_2 \mathbf{u}) \mathrm{S}_3(s_2 \mathbf{u}) \mathrm{S}_2(\mathbf{u}) \end{aligned} \tag{3.23}$$

commute,

$$r_1 r_2 = r_2 r_1 \to \mathrm{R}_1(r_2 \mathbf{u}) \mathrm{R}_2(\mathbf{u}) = \mathrm{R}_2(r_1 \mathbf{u}) \mathrm{R}_1(\mathbf{u}), \tag{3.24}$$

and the operator $\check{\mathrm{R}}(u - v)$ can be decomposed into a product in the same way as the permutation s,

$$\begin{aligned} \check{\mathrm{R}}(u - v) &= \mathrm{S}_2(s_1 s_3 s_2 \mathbf{u}) \mathrm{S}_1(s_3 s_2 \mathbf{u}) \mathrm{S}_3(s_2 \mathbf{u}) \mathrm{S}_2(\mathbf{u}) \\ &= \mathrm{R}_1(r_2 \mathbf{u}) \mathrm{R}_2(\mathbf{u}) = \mathrm{R}_2(r_1 \mathbf{u}) \mathrm{R}_1(\mathbf{u}). \end{aligned} \tag{3.25}$$

For the proof, we must perform the sequence of transformations (3.20) for the operators $\mathrm{S}_k(\mathbf{u})$. Relations (3.22) become obvious if we use the explicit formulas (3.9) and (3.11):

$$\begin{aligned} s_1 s_1 &= \mathbb{1} \to \mathrm{S}_1(s_1 \mathbf{u}) \mathrm{S}_1(\mathbf{u}) = [i\partial_2]^{v_2 - v_1} \cdot [i\partial_2]^{v_1 - v_2} = \mathbb{1}, \\ s_2 s_2 &= \mathbb{1} \to \mathrm{S}_2(s_2 \mathbf{u}) \mathrm{S}_2(\mathbf{u}) = [z_{12}]^{v_2 - u_1} \cdot [z_{12}]^{u_1 - v_2} = \mathbb{1}, \\ s_3 s_3 &= \mathbb{1} \to \mathrm{S}_3(s_3 \mathbf{u}) \mathrm{S}_3(\mathbf{u}) = [i\partial_1]^{u_2 - u_1} \cdot [i\partial_1]^{u_1 - u_2} = \mathbb{1}, \\ s_1 s_3 &= s_3 s_1 \to [i\partial_2]^{v_1 - v_2} \cdot [i\partial_1]^{u_1 - u_2} = [i\partial_1]^{u_1 - u_2} \cdot [i\partial_2]^{v_1 - v_2}. \end{aligned}$$

In the simple example of different decompositions of the permutation s and the operator $\check{\mathrm{R}}$ considered above, we used only the double relations (3.22). However, these double relations alone do not suffice, and a complete set of relations for the generators of the group $\mathfrak{S}_4$ includes triple relations. Thus, to prove that we have obtained a representation of the generators of the group $\mathfrak{S}_4$ by the operators $\mathrm{S}_k(\mathbf{u})$, it remains to verify that the following triple relations are valid:

$$s_1 s_2 s_1 = s_2 s_1 s_2 \to \mathrm{S}_1(s_2 s_1 \mathbf{u}) \mathrm{S}_2(s_1 \mathbf{u}) \mathrm{S}_1(\mathbf{u}) = \mathrm{S}_2(s_1 s_2 \mathbf{u}) \mathrm{S}_1(s_2 \mathbf{u}), \mathrm{S}_2(\mathbf{u}), \tag{3.26}$$

$$s_2 s_3 s_2 = s_3 s_2 s_3 \to \mathrm{S}_2(s_3 s_2 \mathbf{u}) \mathrm{S}_3(s_2 \mathbf{u}) \mathrm{S}_2(\mathbf{u}) = \mathrm{S}_3(s_2 s_3 \mathbf{u}) \mathrm{S}_2(s_3 \mathbf{u}) \mathrm{S}_3(\mathbf{u}). \tag{3.27}$$

In fact, the triple relations for the operators S_k are valid indeed, and are equivalent to the star-triangle identity. For example, if we substitute explicit expressions for the operators

$\mathrm{S}_k(\mathbf{u})$ in (3.26), we obtain a counterpart of the operator version of the star-triangle relation,

$$[i\partial_2]^{u_1-v_2}\cdot[z_{12}]^{u_1-v_1}\cdot[i\partial_2]^{v_2-v_1}=[z_{12}]^{v_2-v_1}\cdot[i\partial_2]^{u_1-v_1}\cdot[z_{12}]^{u_1-v_2}.$$

Calculating the kernels of the operators on both sides of this relation, we arrive at the integral star-triangle identity

$$\int d^2y_2\,\frac{A(u_1-v_2)A(v_2-v_1)}{[z_2-y_2]^{1+u_1-v_2}\,[z_1-y_2]^{v_1-u_1}\,[y_2-x_2]^{1+v_2-v_1}}$$
$$=\frac{A(u_1-v_1)}{[z_1-z_2]^{v_1-v_2}\,[z_2-x_2]^{1+u_1-v_1}\,[z_1-x_2]^{v_2-u_1}}.$$

As we saw above, the graphical proof of the Yang–Baxter relation for the operator R was based on the star-triangle relation. The triple relations (3.26) and (3.27) represent an algebraic counterpart of the star-triangle relation. Therefore, it is natural to expect that the Yang–Baxter relation is a consequence of the triple relations (3.26) and (3.27).

To demonstrate links between these relations, we consider the product of three L-operators. In the product $\mathrm{L}_1(u_1,u_2)\mathrm{L}_2(v_1,v_2)\ \mathrm{L}_3(w_1,w_2)$, we group the parameters in one set in the following order: $\mathbf{u}\equiv(w_1,w_2,v_1,v_2,u_1,u_2)$ and consider the group of permutations of all six parameters. The group $\mathfrak{S}_6$ already has five generators s_k, and the corresponding operators $\mathrm{S}_k(\mathbf{u})$ are

$$(\overbrace{w_1,\ w_2}^{\mathrm{S}_1},\overbrace{v_1,\ v_2}^{\mathrm{S}_3},\overbrace{u_1,\ u_2}^{\mathrm{S}_5});$$
$$\mathrm{S}_1\,\mathrm{L}_3(w_1,w_2)=\mathrm{L}_3(w_2,w_1)\mathrm{S}_1;$$
$$\mathrm{S}_3\,\mathrm{L}_2(v_1,v_2)=\mathrm{L}_2(v_2,v_1)\mathrm{S}_3;\quad \mathrm{S}_5\,\mathrm{L}_1(u_1,u_2)=\mathrm{L}_2(u_2,u_1)\mathrm{S}_5;$$
$$(w_1,\overbrace{w_2,\ v_1}^{\mathrm{S}_2},v_2,u_1,u_2);$$
$$\mathrm{S}_2(\mathbf{u})\,\mathrm{L}_2(v_1,v_2)\,\mathrm{L}_3(w_1,w_2)=\mathrm{L}_2(w_2,v_2)\,\mathrm{L}_3(w_1,v_1)\mathrm{S}_2(\mathbf{u});$$
$$(w_1,w_2,v_1,\overbrace{v_2,\ u_1}^{\mathrm{S}_4},u_2);$$
$$\mathrm{S}_4(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2)=\mathrm{L}_1(v_2,u_2)\,\mathrm{L}_2(v_1,u_1)\mathrm{S}_4(\mathbf{u}).$$

The construction of the operators S_k involves nothing new in comparison with the previous case, and everything reduces to the corresponding formulas for the product of two Lax matrices. The operators $\mathrm{S}_1,\mathrm{S}_2$, and S_3 are constructed for the product $\mathrm{L}_2(v_1,v_2)\mathrm{L}_3(w_1,w_2)$, and the operators $\mathrm{S}_3,\mathrm{S}_4$, and S_5 are constructed for the product $\mathrm{L}_1(u_1,u_2)\mathrm{L}_2(v_1,v_2)$. The operators $\mathrm{S}_1,\mathrm{S}_3$, and S_5 are intertwining operators, each for its own representation of the group $\mathrm{SL}(2,\mathbb{C})$. As in the previous case, the defining relations

$$\begin{aligned}
s_is_i&=1\to\mathrm{S}_i(s_i\mathbf{u})\mathrm{S}_i(\mathbf{u})=1;\\
s_is_j&=s_js_i\to\mathrm{S}_i(s_j\mathbf{u})\mathrm{S}_j(\mathbf{u})=\mathrm{S}_j(s_i\mathbf{u})\mathrm{S}_i(\mathbf{u}),\qquad |i-j|>1,\\
s_is_{i+1}s_i&=s_{i+1}s_is_{i+1}\to\mathrm{S}_i(s_{i+1}s_i\mathbf{u})\mathrm{S}_{i+1}(s_i\mathbf{u})\mathrm{S}_i(\mathbf{u})\\
&=\mathrm{S}_{i+1}(s_is_{i+1}\mathbf{u})\mathrm{S}_i(s_{i+1}\mathbf{u})\mathrm{S}_{i+1}(\mathbf{u}),
\end{aligned}$$

either are fulfilled trivially, or reduce to the star-triangle relation.

The Yang–Baxter relation is a triple relation for specific permutations in the group $\mathfrak{S}_6$ [7, 5] (Figure 5). The operator $\check{\mathrm{R}}_{12}(u-v)$ is a solution of the equation

$$\check{\mathrm{R}}_{12}(u-v)\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2)=\mathrm{L}_1(v_1,v_2)\,\mathrm{L}_2(u_1,u_2)\,\check{\mathrm{R}}_{12}(u-v),$$

$$\begin{tikzcd}
L_1(v_1\,v_2)L_2(u_1\,u_2)L_3(w_1\,w_2) \arrow[r, "R_{23}(u-w)"'] & L_1(v_1\,v_2)L_2(w_1\,w_2)L_3(u_1\,u_2) \arrow[d, "R_{12}(v-w)"] \\
L_1(u_1\,u_2)L_2(v_1\,v_2)L_3(w_1\,w_2) \arrow[u, "R_{12}(u-v)"] \arrow[d, "R_{23}(v-w)"] & L_1(w_1\,w_2)L_2(v_1\,v_2)L_3(u_1\,u_2) \\
L_1(u_1\,u_2)L_2(w_1\,w_2)L_3(v_1\,v_2) \arrow[r, "R_{12}(u-w)"'] & L_1(w_1\,w_2)L_2(u_1\,u_2)L_3(v_1\,v_2) \arrow[u, "R_{23}(u-v)"']
\end{tikzcd}$$

FIGURE 5. Triple relation for the operators $\check{\mathrm{R}}$.

and it corresponds to the permutation $s_4s_3s_5s_4$,

$$\check{\mathrm{R}}_{12}(u-v) = \mathrm{S}_4(s_3s_5s_4\mathbf{u})\mathrm{S}_3(s_5s_4\mathbf{u})\mathrm{S}_5(s_4\mathbf{u})\mathrm{S}_4(\mathbf{u}).$$

The operator $\check{\mathrm{R}}_{23}(v-w) = \mathrm{S}_2(s_1s_3s_2\mathbf{u})\mathrm{S}_1(s_3s_2\mathbf{u})\mathrm{S}_3(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u})$ corresponds to the permutation $s_2s_1s_3s_2$ and solves the equation

$$\check{\mathrm{R}}_{23}(v-w)\,\mathrm{L}_2(v_1,v_2)\,\mathrm{L}_3(w_1,w_2) = \mathrm{L}_2(w_1,w_2)\,\mathrm{L}_3(v_1,v_2)\,\check{\mathrm{R}}_{23}(v-w).$$

There are two equivalent decompositions of a specific s (Figure 5):

$$s(w_1,w_2,v_1,v_2,u_1,u_2) = (u_1,u_2,v_1,v_2,w_1,w_2),$$
$$s = s_4s_3s_5s_4\cdot s_2s_1s_3s_2\cdot s_4s_3s_5s_4 \to \check{\mathrm{R}}_{12}(v-w)\check{\mathrm{R}}_{23}(u-w)\check{\mathrm{R}}_{12}(u-v),$$
$$s = s_2s_1s_3s_2\cdot s_4s_3s_5s_4\cdot s_2s_1s_3s_2 \to \check{\mathrm{R}}_{23}(u-v)\check{\mathrm{R}}_{12}(u-w)\check{\mathrm{R}}_{23}(v-w).$$

These two different products of generators s_k result in the same permutation. Therefore, there exists a sequence of transformations that involves the entire set of defining relations and takes one product to the other; for example,

$$\begin{aligned}
s_2s_3\underline{s_1s_2}\cdot\underline{s_4}s_3s_5\underline{s_4}\cdot\underline{s_2s_1}s_3s_2 &= s_2s_3s_4\cdot s_1\underline{s_2s_3s_2}s_1\cdot s_5\cdot s_4s_3s_2\\
&= s_2\underline{s_3s_4s_3}\cdot\underline{s_1s_2s_1}\cdot s_5\cdot\underline{s_3s_4s_3}s_2 = \underline{s_2s_4}s_3\underline{s_4}\cdot\underline{s_2}s_1\underline{s_2}\cdot s_5\cdot\underline{s_4}s_3\underline{s_4s_2}\\
&= s_4\underline{s_2s_3s_2}\cdot s_1\cdot\underline{s_4s_5s_4}\cdot\underline{s_2s_3s_2}s_4 = s_4s_3\underline{s_2s_3}\cdot s_1\cdot\underline{s_5}s_4\underline{s_5}\cdot\underline{s_3s_2}s_3s_4\\
&= s_4s_3s_5\cdot s_2s_1\underline{s_3s_4s_3}s_2\cdot s_5s_3s_4 = s_4s_3s_5\cdot s_2s_1\underline{s_4}s_3\underline{s_4}s_2\cdot s_5s_3s_4\\
&= s_4s_3s_5s_4\cdot s_2s_1s_3s_2\cdot s_4s_5s_3s_4.
\end{aligned}$$

In contrast to (3.20), here we cannot do without triple relations. Repeating similar transformations for the operators S_k, we obtain an algebraic proof of the Yang–Baxter relation

$$\check{\mathrm{R}}_{23}(u-v)\check{\mathrm{R}}_{12}(u-w)\check{\mathrm{R}}_{23}(v-w) = \check{\mathrm{R}}_{12}(v-w)\check{\mathrm{R}}_{23}(u-w)\check{\mathrm{R}}_{12}(u-v).$$

If we translate all the transformations into the graphical language of kernels of operators (Figure 4), we get a sequence of transformations of one picture to another with a hexagon as an intermediate step (we must reverse half of the arrows in Figure 4). In the initial sequence of transformations, we use both double and triple relations for the generators

s_k. In terms of the kernels of S_k, the double relations turn into trivial transformations of pictures, and therefore, we do not "see" them. The triple relations correspond to the star-triangle transformations.

Thus, the operators $\mathrm{S}_k(\mathbf{u})$ corresponding to the generators s_k of the group $\mathfrak{S}_4$ of permutations of the parameters in the set $\mathbf{u} = (v_1, v_2, u_1, u_2)$ play the role of elementary building blocks in the construction of a general $\mathrm{SL}(2,\mathbb{C})$-invariant solution of the Yang–Baxter equation. The operators $\mathrm{S}_k(\mathbf{u})$ satisfy the required double and triple relations, which guarantees the Yang–Baxter relation for an R-matrix.

3.3. The operators R_1 and R_2. The role of the elementary building blocks in the construction of transfer matrices, Q-operators, or solutions of the Yang–Baxter equations for other types of representations will be played by the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ (see (3.23) and (3.25)) corresponding to the permutations $u_1 \leftrightarrow v_1$ and $u_2 \leftrightarrow v_2$. When passing to R_1 and R_2, symmetry breaks down, $\mathfrak{S}_4 \to \mathfrak{S}_2 \times \mathfrak{S}_2$, because only permutations from distinct L-operators are needed, and we "forget" the permutations $u_1 \leftrightarrow u_2$ or $v_1 \leftrightarrow v_2$ inside one L-operator.

In the present section, the symmetry group $\mathfrak{S}_2 \times \mathfrak{S}_2$ and the properties of the corresponding operators are considered in detail. Now, instead of the operators $\mathrm{S}_k(\mathbf{u})$, the elementary building blocks are the operators R_1 and R_2 corresponding to the permutations r_1 and r_2, where r_1 interchanges only u_1 and v_1, and r_2 interchanges only u_2 and v_2,

$$
\begin{gathered}
r_1(v_1, v_2, u_1, u_2) = (u_1, v_2, v_1, u_2), \quad r_2(v_1, v_2, u_1, u_2) = (v_1, u_2, u_1, v_2), \\
r_1 = s_2 s_1 s_2 \to \mathrm{R}_1(\mathbf{u}) = \mathrm{S}_2(s_1 s_2 \mathbf{u})\mathrm{S}_1(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u}) = [z_{12}]^{v_2 - v_1}\,[i\partial_2]^{u_1 - v_1}\,[z_{12}]^{u_1 - v_2}, \\
r_2 = s_2 s_3 s_2 \to \mathrm{R}_2(\mathbf{u}) = \mathrm{S}_2(s_3 s_2 \mathbf{u})\mathrm{S}_3(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u}) = [z_{12}]^{u_2 - u_1}\,[i\partial_1]^{u_2 - v_2}\,[z_{12}]^{u_1 - v_2}.
\end{gathered}
$$

The action of these operators on functions is defined by the following explicit formulas (the dependence of the function Φ on the conjugate variables is suppressed for simplicity):

$$
\begin{aligned}
\mathrm{R}_1(\mathbf{u})\Phi(z_1, z_2) &= A(u_1 - v_1)[z_{12}]^{v_2 - v_1} \\
&\quad \times \int d^2 x_2\, [z_2 - x_2]^{v_1 - u_1 - 1}\,[z_1 - x_2]^{u_1 - v_2}\,\Phi(z_1, x_2),
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
\mathrm{R}_2(\mathbf{u})\Phi(z_1, z_2) &= A(u_2 - v_2)[z_{12}]^{u_2 - u_1} \\
&\quad \times \int d^2 x_1\, [z_1 - x_1]^{v_2 - u_2 - 1}\,[x_1 - z_2]^{u_1 - v_2}\,\Phi(x_1, z_2).
\end{aligned} \tag{3.29}
$$

The permutations r_1 and r_2 commute, and the permutation s decomposes into the product $s = r_1 r_2$. Thus, for the operator $\check{\mathrm{R}}(u - v)$ we obtain

$$
\begin{gathered}
r_1 \to \mathrm{R}_1(\mathbf{u}), \quad r_2 \to \mathrm{R}_2(\mathbf{u}), \\
s = r_1 r_2 = r_2 r_1 \to \check{\mathrm{R}}(u - v) = \mathrm{R}_1(r_2\mathbf{u})\mathrm{R}_2(\mathbf{u}) = \mathrm{R}_2(r_1\mathbf{u})\mathrm{R}_1(\mathbf{u}).
\end{gathered}
$$

The kernels of the integral operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ are displayed graphically by triangles as in Figure 6. In the same figure we show how for the product $\mathrm{R}_1(r_2\mathbf{u})\mathrm{R}_2(\mathbf{u})$ we obtain a quadrangle displaying the kernel of the operator $\check{\mathrm{R}}(u - v)$ (see (3.13)).

- $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ as intertwining operators.

The operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ play the role of intertwining operators for the tensor product of two representations of the group $\mathrm{SL}(2,\mathbb{C})$. This can be verified with the help of an explicit construction in terms of the operators $\mathrm{S}_k(\mathbf{u})$, but we obtain the intertwining relations by the method used above in the proof that the operator $\mathrm{R}(u) = \mathrm{P}\check{\mathrm{R}}(u)$ is

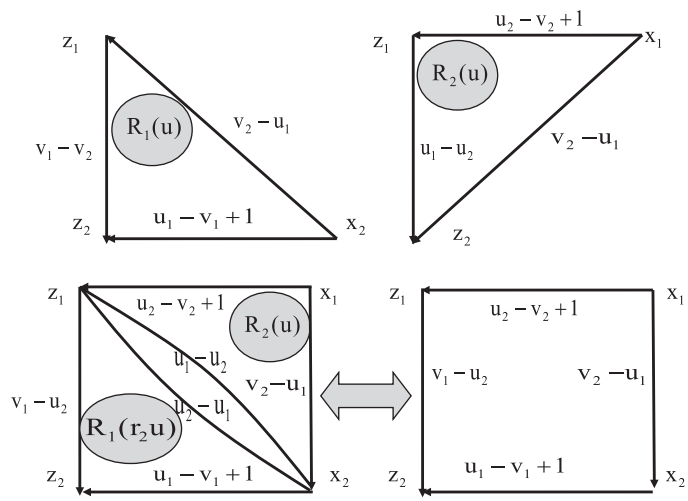

FIGURE 6. Graphical representation of the kernels of the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$.

$\mathrm{SL}(2,\mathbb{C})$-invariant. We need only the defining relations for the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$,

$$\mathrm{R}_1(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(v_1,u_2)\,\mathrm{L}_2(u_1,v_2)\,\mathrm{R}_1(\mathbf{u}),$$
$$\mathrm{R}_2(\mathbf{u})\,\mathrm{L}_1(u_1,u_2)\,\mathrm{L}_2(v_1,v_2) = \mathrm{L}_1(u_1,v_2)\,\mathrm{L}_2(v_1,u_2)\,\mathrm{R}_2(\mathbf{u}),$$

and the fact that both operators depend on the difference $u-v$ of spectral parameters, i.e., are invariant under the translations $u\mapsto u+\lambda$, $v\mapsto v+\lambda$. Performing the translation $u\mapsto u+\lambda$, $v\mapsto v+\lambda$ in the defining relations and collecting the contributions linear in λ, we obtain the required relations

$$\mathrm{R}_1(\mathbf{u})\ (\mathrm{L}_1(u_1,u_2)+\mathrm{L}_2(v_1,v_2)) = (\mathrm{L}_1(v_1,u_2)+\mathrm{L}_2(u_1,v_2))\ \mathrm{R}_1(\mathbf{u}),$$
$$\mathrm{R}_2(\mathbf{u})\ (\mathrm{L}_1(u_1,u_2)+\mathrm{L}_2(v_1,v_2)) = (\mathrm{L}_1(u_1,v_2)+\mathrm{L}_2(v_1,u_2))\ \mathrm{R}_2(\mathbf{u}).$$

Up to trivial terms, the matrices L_1 and L_2 coincide with the matrices constructed from generators. Thus, it remains to identify the parameters of the representations by requiring that the relation $\alpha_1+\alpha_2=1$ be valid for each matrix $\mathrm{E}^{(\alpha_1,\alpha_2)}$:

$$\mathrm{L}_1(u_1,u_2)+\mathrm{L}_2(v_1,v_2) = (u+v)\cdot\mathbb{1} + \mathrm{E}_1^{(\sigma_1,\sigma_2)} + \mathrm{E}_2^{(\rho_1,\rho_2)};$$
$$\mathrm{L}_1(v_1,u_2)+\mathrm{L}_2(u_1,v_2) = (u+v)\cdot\mathbb{1} + \mathrm{E}_1^{(\sigma_1+\lambda_1,\sigma_2-\lambda_1)} + \mathrm{E}_2^{(\rho_1-\lambda_1,\rho_2+\lambda_1)},$$
$$\lambda_1 = \frac{u_1-v_1}{2};$$
$$\mathrm{L}_1(u_1,v_2)+\mathrm{L}_2(v_1,u_2) = (u+v)\cdot\mathbb{1} + \mathrm{E}_1^{(\sigma_1-\lambda_2,\sigma_2+\lambda_2)} + \mathrm{E}_2^{(\rho_1+\lambda_2,\rho_2-\lambda_2)},$$
$$\lambda_2 = \frac{u_2-v_2}{2}.$$

As a result, we obtain the following intertwining relations:

$$\mathrm{R}_1(\mathbf{u})\ \mathrm{T}^{(\sigma_1,\sigma_2)}\otimes\mathrm{T}^{(\rho_1,\rho_2)} = \mathrm{T}^{(\sigma_1+\lambda_1,\sigma_2-\lambda_1)}\otimes\mathrm{T}^{(\rho_1-\lambda_1,\rho_2+\lambda_1)}\,\mathrm{R}_1(\mathbf{u}), \tag{3.30}$$

$$\mathrm{R}_2(\mathbf{u})\,\mathrm{T}^{(\sigma_1,\sigma_2)}\otimes\mathrm{T}^{(\rho_1,\rho_2)} = \mathrm{T}^{(\sigma_1-\lambda_2,\sigma_2+\lambda_2)}\otimes\mathrm{T}^{(\rho_1+\lambda_2,\rho_2-\lambda_2)}\,\mathrm{R}_2(\mathbf{u}). \tag{3.31}$$

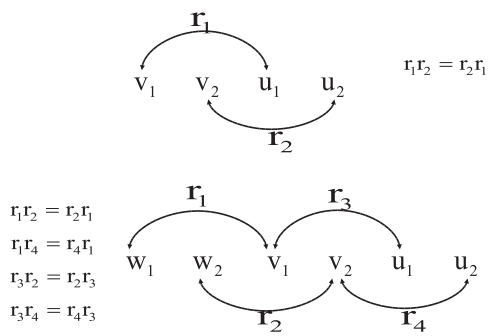

FIGURE 7. Parameter permutations.

It is of interest to trace explicitly how the three steps give the intertwining relations reflecting the $\mathrm{SL}(2,\mathbb{C})$-invariance of the operator $\mathrm{R}(u-v) = \mathrm{P}\mathrm{R}_1(r_2\mathbf{u})\mathrm{R}_2(\mathbf{u})$:

$$\mathrm{T}^{(\sigma_1,\sigma_2)} \otimes \mathrm{T}^{(\rho_1,\rho_2)} \xrightarrow{\mathrm{R}_2(\mathbf{u})} \mathrm{T}^{\left(\sigma_1-\frac{u_2-v_2}{2},\sigma_2+\frac{u_2-v_2}{2}\right)} \otimes \mathrm{T}^{\left(\rho_1+\frac{u_2-v_2}{2},\rho_2-\frac{u_2-v_2}{2}\right)}$$
$$\xrightarrow{\mathrm{R}_1(r_2\mathbf{u})} \mathrm{T}^{(\rho_1,\rho_2)} \otimes \mathrm{T}^{(\sigma_1,\sigma_2)} \xrightarrow{\mathrm{P}} \mathrm{T}^{(\sigma_1,\sigma_2)} \otimes \mathrm{T}^{(\rho_1,\rho_2)}.$$

- Defining relations.

As in the case of the operators S_k, the operators R_1 and R_2 must satisfy the same relations as the generators of the corresponding permutation group. In the case of the product of two L-operators, the permutation group is $\mathfrak{S}_2 \times \mathfrak{S}_2$ and the complete set of generators is the set of relations for two commuting groups $\mathfrak{S}_2$:

$$r_1 r_2 = r_2 r_1 \to \mathrm{R}_1(r_2\mathbf{u})\mathrm{R}_2(\mathbf{u}) = \mathrm{R}_2(r_1\mathbf{u})\mathrm{R}_1(\mathbf{u}), \tag{3.32}$$

$$r_1^2 = \mathbb{1} \to \mathrm{R}_1(r_1\mathbf{u})\mathrm{R}_1(\mathbf{u}) = \mathbb{1}; \quad r_2^2 = \mathbb{1} \to \mathrm{R}_2(r_2\mathbf{u})\mathrm{R}_2(\mathbf{u}) = \mathbb{1}. \tag{3.33}$$

We consider the product $\mathrm{L}_1(u_1,u_2)\mathrm{L}_2(v_1,v_2)\mathrm{L}_3(w_1,w_2)$ of three L-operators. In this case, triple relations arise for the operators R_1 and R_2 (an analog of the Yang–Baxter relation). Instead of the group $\mathfrak{S}_6$ with five generators s_k, now we have two commuting groups $\mathfrak{S}_3$ with four generators r_k (Figure 7):

$$r_1 = s_2 s_1 s_2, \quad r_2 = s_2 s_3 s_2, \quad r_3 = s_4 s_3 s_4, \quad r_4 = s_4 s_5 s_4.$$

The permutations r_1 and r_3 interchange the parameters $w_1 \leftrightarrow v_1$ and $v_1 \leftrightarrow u_1$ and are the generators of the first group $\mathfrak{S}_3$ of permutations of the parameters (w_1, v_1, u_1),

$$r_1\mathbf{u} = s_2 s_1 s_2 \mathbf{u} = (v_1, w_2, w_1, v_2, u_1, u_2),$$
$$r_3\mathbf{u} = s_4 s_3 s_4 \mathbf{u} = (w_1, w_2, u_1, v_2, v_1, u_2).$$

For each of these permutations, we have an operator R_1 that acts in different spaces. We denote the corresponding operator by R^1_{ij}, preserving the standard lower indices ij to indicate the space in which the operator acts nontrivially and moving the index 1 upward,

$$r_1 \to \mathrm{R}^1_{23}(\mathbf{u}) = \mathrm{S}_2(s_1 s_2\mathbf{u})\mathrm{S}_1(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u});$$
$$r_3 \to \mathrm{R}^1_{12}(\mathbf{u}) = \mathrm{S}_4(s_3 s_4\mathbf{u})\mathrm{S}_3(s_4\mathbf{u})\mathrm{S}_4(\mathbf{u}).$$

The permutations r_2 and r_4 interchange the parameters $w_2 \leftrightarrow v_2$ and $v_2 \leftrightarrow u_2$ and are the generators of the second group $\mathfrak{S}_3$ of permutations of the parameters (w_2, v_2, u_2),

$$r_2\mathbf{u} = s_2 s_3 s_2 \mathbf{u} = (w_1, v_2, v_1, w_2, u_1, u_2),$$
$$r_4\mathbf{u} = s_4 s_5 s_4 \mathbf{u} = (w_1, w_2, v_1, u_2, u_1, v_2).$$

For each of these permutations, we have an operator R_2 that acts in different spaces,

$$r_2 \to \mathrm{R}^2_{23}(\mathbf{u}) = \mathrm{S}_2(s_3 s_2 \mathbf{u})\mathrm{S}_3(s_2\mathbf{u})\mathrm{S}_2(\mathbf{u}),$$
$$r_4 \to \mathrm{R}^2_{12}(\mathbf{u}) = \mathrm{S}_4(s_5 s_4 \mathbf{u})\mathrm{S}_5(s_4\mathbf{u})\mathrm{S}_4(\mathbf{u}).$$

We list all relations for the operators R_1 and R_2. The four double relations reflect the fact that the generators of different groups $\mathfrak{S}_3$ commute,

$$r_2 r_1 = r_1 r_2, \quad r_4 r_3 = r_3 r_4, \quad r_4 r_1 = r_1 r_4, \quad r_2 r_3 = r_3 r_2.$$

- The first two relations give nothing new because they are equivalent to (3.32):

$$r_2 r_1 = r_1 r_2 \to \mathrm{R}^2_{23}(r_1\mathbf{u})\mathrm{R}^1_{23}(\mathbf{u}) = \mathrm{R}^1_{23}(r_2\mathbf{u})\mathrm{R}^2_{23}(\mathbf{u}),$$
$$r_4 r_3 = r_3 r_4 \to \mathrm{R}^2_{12}(r_3\mathbf{u})\mathrm{R}^1_{12}(\mathbf{u}) = \mathrm{R}^1_{12}(r_4\mathbf{u})\mathrm{R}^2_{12}(\mathbf{u}).$$

- The last two relations are new:

$$\begin{aligned} r_4 r_1 = r_1 r_4 &\to \mathrm{R}^2_{12}(r_1\mathbf{u})\mathrm{R}^1_{23}(\mathbf{u}) = \mathrm{R}^1_{23}(r_4\mathbf{u})\mathrm{R}^2_{12}(\mathbf{u}), \\ r_2 r_3 = r_3 r_2 &\to \mathrm{R}^2_{23}(r_3\mathbf{u})\mathrm{R}^1_{12}(\mathbf{u}) = \mathrm{R}^1_{12}(r_2\mathbf{u})\mathrm{R}^2_{23}(\mathbf{u}). \end{aligned} \tag{3.34}$$

The triple relations follow from the triple relations for each of the groups $\mathfrak{S}_3$,

$$r_3 r_1 r_3 = r_1 r_3 r_1, \quad r_2 r_4 r_2 = r_4 r_2 r_4,$$

$$r_3 r_1 r_3 = r_1 r_3 r_1 \to \mathrm{R}^1_{12}(r_1 r_3\mathbf{u})\mathrm{R}^1_{23}(r_3\mathbf{u})\mathrm{R}^1_{12}(\mathbf{u}) = \mathrm{R}^1_{23}(r_3 r_1\mathbf{u})\mathrm{R}^1_{12}(r_1\mathbf{u})\mathrm{R}^1_{23}(\mathbf{u}), \tag{3.35}$$

$$r_2 r_4 r_2 = r_4 r_2 r_4 \to \mathrm{R}^2_{23}(r_4 r_2\mathbf{u})\mathrm{R}^2_{12}(r_4\mathbf{u})\mathrm{R}^2_{23}(\mathbf{u}) = \mathrm{R}^2_{12}(r_2 r_4\mathbf{u})\mathrm{R}^2_{23}(r_4\mathbf{u})\mathrm{R}^2_{12}(\mathbf{u}). \tag{3.36}$$

A complete set of relations for the generators of the group $\mathfrak{S}_n$ includes only double and triple relations. Therefore, the relations listed above exhaust all relations for the operators in question. On this stage, we do not need an independent proof of the above-mentioned relations for the operators R_1 and R_2. As before, we deal with representations of continuous series, where everything is constructed from the operators $\mathrm{S}_k(\mathbf{u})$ satisfying the required relations of the permutation group. Therefore, two operators constructed from $\mathrm{S}_k(\mathbf{u})$ and giving rise to the same permutation of parameters in the product of L-operators coincide automatically.

Despite this, explicit proofs are useful for various reasons. Therefore, as an example of application of formula (2.31), we present a detailed proof of the triple relation (3.36) for the operator R_2. It is convenient to use the representation (2.20) for the intertwining operator, the operator reformulation of the star-triangle relation [59]

$$[i\partial_z]^a \cdot [z]^{a+b} \cdot [i\partial_z]^b = [z]^b \cdot [i\partial_z]^{a+b} \cdot [z]^a, \tag{3.37}$$

and various consequences from this formula obtained by translations, e.g.,

$$[i\partial_z]^a \cdot [z-x]^{a+b} \cdot [i\partial_z]^b = [z-x]^b \cdot [i\partial_z]^{a+b} \cdot [z-x]^a.$$

All formulas of this type are proved in the same way. Namely, calculating the kernel of the operators on the left-hand and on the right-hand side, we arrive at the integral star-triangle equation.

In the triple relation (3.36), we substitute the following concise representations for the operators R_2 acting in different spaces:

$$\mathrm{R}_{12}^2(\mathbf{u}) = [z_{12}]^{u_2-u_1}\,[i\partial_1]^{u_2-v_2}\,[z_{12}]^{u_1-v_2} \overset{(2.31)}{=} [i\partial_1]^{u_1-v_2}\cdot[z_{12}]^{u_2-v_2}\cdot[i\partial_1]^{u_2-u_1},$$
$$\mathrm{R}_{23}^2(\mathbf{u}) = [z_{23}]^{v_2-v_1}\,[i\partial_2]^{v_2-w_2}\,[z_{23}]^{v_1-w_2},$$

where $z_{ik} = z_i - z_k$, and consider the resulting relation

$$\begin{aligned}&[i\partial_1]^{u_1-w_2}[z_{12}]^{v_2-w_2}\underline{[i\partial_1]^{v_2-u_1}}\cdot[z_{23}]^{u_2-v_1}[i\partial_2]^{u_2-w_2}[z_{23}]^{v_1-w_2}\\ &\quad\times\underline{[i\partial_1]^{u_1-v_2}}[z_{12}]^{u_2-v_2}[i\partial_1]^{u_2-u_1} = [z_{23}]^{u_2-v_1}[i\partial_2]^{u_2-v_2}\underline{[z_{23}]^{v_1-v_2}}\\ &\quad\times[i\partial_1]^{u_1-w_2}[z_{12}]^{u_2-w_2}[i\partial_1]^{u_2-u_1}\cdot\underline{[z_{23}]^{v_2-v_1}}[i\partial_2]^{v_2-w_2}[z_{23}]^{v_1-w_2}.\end{aligned}$$

Here, we have underlined the operators that cancel out if they are moved towards each other. After the first stage of cancellations, we obtain a simpler equation:

$$\begin{aligned}&\underline{[i\partial_1]^{u_1-w_2}}[z_{12}]^{v_2-w_2}\cdot\underline{[z_{23}]^{u_2-v_1}}[i\partial_2]^{u_2-w_2}\underline{[z_{23}]^{v_1-w_2}}\cdot[z_{12}]^{u_2-v_2}\underline{[i\partial_1]^{u_2-u_1}}\\ &\quad=\underline{[z_{23}]^{u_2-v_1}}[i\partial_2]^{u_2-v_2}\cdot\underline{[i\partial_1]^{u_1-w_2}}[z_{12}]^{u_2-w_2}\underline{[i\partial_1]^{u_2-u_1}}\cdot[i\partial_2]^{v_2-w_2}\underline{[z_{23}]^{v_1-w_2}}.\end{aligned}$$

Now, we underline the operators that cancel separately from the left and from the right if they are moved outwards. This is the last stage of cancellations because, as a result, we obtain the operator relation (2.31),

$$[z_{12}]^{v_2-w_2}\cdot[i\partial_2]^{u_2-w_2}\cdot[z_{12}]^{u_2-v_2} = [i\partial_2]^{u_2-v_2}\cdot[z_{12}]^{u_2-w_2}\cdot[i\partial_2]^{v_2-w_2}.$$

§4. Verma modules and finite-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$

Up to this point, we have only considered the principal continuous series of representations of the group $\mathrm{SL}(2,\mathbb{C})$. In this case, the operators $\mathrm{S}_k(\mathbf{u})$, which are elementary building blocks, are well defined on the function space in which the representation of the group $\mathrm{SL}(2,\mathbb{C})$ is realized. For irreducible representations of other types, the situation is more complicated.

In the case where the parameters σ_{12} and $\bar\sigma_{12}$ are positive integers, the representation space of the principal nonunitary series representation of the group $\mathrm{SL}(2,\mathbb{C})$ has an invariant subspace of polynomials in z and $\bar z$ of degree at most $\sigma_{12}-1$ in z and at most $\bar\sigma_{12}-1$ in $\bar z$. We can restrict ourselves to the representations of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ and consider irreducible representations in the space $\mathbb{C}[z,\bar z]$ of polynomials in z and $\bar z$. If the parameters σ_{12} and $\bar\sigma_{12}$ are numbers in general position, then the representation is irreducible. If σ_{12} and $\bar\sigma_{12}$ are positive integers, then there is an invariant subspace in the space of polynomials in which a finite-dimensional Lie algebra representation extendible to the group representation is realized.

Thus, the Verma modules, which are infinite-dimensional representations of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ in the space of polynomials, occupy an intermediate level between the representations of the principal series and the finite-dimensional representations of the group $\mathrm{SL}(2,\mathbb{C})$.

In this case, the operators $\mathrm{S}_k(\mathbf{u})$ need an additional regularization. Therefore, it is convenient to combine them into larger building blocks $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$, which are well defined on the space of polynomials.

The finite-dimensional representations of the group $\mathrm{SL}(2,\mathbb{C})$ are realized in the space of polynomials of a fixed degree. This space is not invariant under the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$. Therefore, on this stage, the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ are combined naturally in the largest building block, an R-matrix.

There is a specific hierarchy of symmetry, depending on the type of the representations of the group $\mathrm{SL}(2,\mathbb{C})$ and of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ for which a solution of the Yang–Baxter equation is constructed.

- For the principal series representations (both unitary and nonunitary) of the group $\mathrm{SL}(2,\mathbb{C})$, the elementary building blocks are the operators $\mathrm{S}_k(\mathbf{u})$ corresponding to the generators s_k of the group $\mathfrak{S}_4$ of permutations of the parameters in the set $\mathbf{u} = (v_1, v_2, u_1, u_2)$.
- For general representations of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ in the space of polynomials (Verma modules), the operators $\mathrm{S}_k(\mathbf{u})$ need an additional regularization. Therefore, the well-defined operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ (see (3.23) and (3.25)) corresponding to the permutations $u_1 \leftrightarrow v_1$ and $u_2 \leftrightarrow v_2$ are used as elementary building blocks. In this case, symmetry breaks down, $\mathfrak{S}_4 \to \mathfrak{S}_2 \times \mathfrak{S}_2$, because we only need the parameter permutations from distinct L-operators, and we "forget" the parameter permutations $u_1 \leftrightarrow u_2$ and $v_1 \leftrightarrow v_2$ inside one and the same L-operator.
- The finite-dimensional representations of the group $\mathrm{SL}(2,\mathbb{C})$ are realized on the space of polynomials of a fixed degree. The representation space is not invariant under the action of the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$. Thus, only the R-matrix itself is an operator for which the representation space is invariant. The further symmetry breakdown occurs, $\mathfrak{S}_2 \times \mathfrak{S}_2 \to \mathfrak{S}_2$, and one permutation $s(v_1, v_2, u_1, u_2) = (u_1, u_2, v_1, v_2)$ of parameters from distinct L-operators is used. In this case, we "forget" the parameter permutations $u_1 \leftrightarrow v_1$ or $u_2 \leftrightarrow v_2$ separately.

As a result, we obtain a sequence of symmetry breakdowns with increasing restrictions on the representation space, $\mathfrak{S}_4 \to \mathfrak{S}_2 \times \mathfrak{S}_2 \to \mathfrak{S}_2$. From this point of view, the infinite-dimensional principal series representations are the simplest, and the finite-dimensional representations are the most complicated. The corresponding parts of the constructor set become more and more complicated, and the assembling rules are determined by a permutation group in a sequence of symmetry breakdowns. In the present section, we take a closer look at the transition from the principal series representations to Verma modules and then to finite-dimensional representations.

4.1. Verma modules. We recall the definition of a Verma module of the Lie algebra $\mathrm{sl}(2)$. In the standard notation, the commutation relations for the generators $\mathbf{e}, \mathbf{f}, \mathbf{h}$ have the following form in the Chevalley basis:

$$[\mathbf{e}, \mathbf{f}] = \mathbf{h}, \quad [\mathbf{h}, \mathbf{e}] = 2\mathbf{e}, \quad [\mathbf{h}, \mathbf{f}] = -2\mathbf{f}.$$

The Verma module V_λ is a free $\mathrm{sl}(2)$-module generated by a lowest weight vector $|0\rangle$,

$$\mathbf{h}|0\rangle = \lambda \cdot |0\rangle, \quad \lambda \in \mathbb{C}; \quad \mathbf{f}|0\rangle = 0.$$

As a linear space, the module V_λ is the linear span of the basis vectors $\{|k\rangle\}_{k=0}^{\infty}$,

$$|k\rangle = \mathbf{e}^k|0\rangle, \quad \mathbf{h}|k\rangle = (\lambda + k) \cdot |k\rangle, \quad \mathbf{f}|k\rangle = -k(\lambda + k - 1) \cdot |k-1\rangle. \tag{4.1}$$

The Verma module V_λ is irreducible for a general $\lambda \in \mathbb{C}$. The exceptional points are $\lambda = 1 - n$, where $n \in \mathbb{N}$ is a positive integer. For these and only these values of λ, an additional lowest weight vector $|n\rangle$ arises in the Verma module,

$$\mathbf{h}|n\rangle = (\lambda + n) \cdot |n\rangle, \quad \mathbf{f}|n\rangle = -n(1 - n + n - 1) \cdot |n-1\rangle = 0,$$

and, as a consequence, we have an invariant subspace (the Verma submodule $\mathrm{V}_{\lambda+n}$) that is the linear span of the basis vectors $\{|n+k\rangle\}_{k=0}^{\infty}$. The quotient space $\mathrm{V}_\lambda/\mathrm{V}_{\lambda+n}$ is a finite-dimensional space of dimension $n+1$ in which a finite-dimensional representation of the Lie algebra $\mathrm{sl}(2)$ is realized. The vectors $\{|k\rangle\}_{k=0}^{n}$ form a basis of this subspace.

In the realization in question, the generators of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ in the Cartan–Weyl basis are first order differential operators in z and $\bar{z}$:

$$\mathrm{E}_{11} = z\partial_z + 1 - \sigma_1;\ \ \mathrm{E}_{12} = z^2\partial_z + (1-\sigma_1+\sigma_2)z;\ \ \mathrm{E}_{21} = -\partial_z;\ \ \mathrm{E}_{22} = -z\partial_z - \sigma_2,$$
$$\bar{\mathrm{E}}_{11} = \bar{z}\partial_{\bar{z}} + 1 - \bar{\sigma}_1;\ \ \bar{\mathrm{E}}_{12} = \bar{z}^2\partial_{\bar{z}} + (1-\bar{\sigma}_1+\bar{\sigma}_2)\bar{z};\ \ \bar{\mathrm{E}}_{21} = -\partial_{\bar{z}};\ \ \bar{\mathrm{E}}_{22} = -\bar{z}\partial_{\bar{z}} - \bar{\sigma}_2.$$

Thus, for the generators in the Chevalley basis, we have

$$\mathbf{h} = \mathrm{E}_{11} - \mathrm{E}_{22} = 2\,z\partial_z + 1 - \sigma_{12};\quad \mathbf{e} = \mathrm{E}_{12} = z^2\partial_z + (1-\sigma_{12})z;\quad \mathbf{f} = \mathrm{E}_{21} = -\partial_z,$$
$$\bar{\mathbf{h}} = \bar{\mathrm{E}}_{11} - \bar{\mathrm{E}}_{22} = 2\,\bar{z}\partial_{\bar{z}} + 1 - \bar{\sigma}_{12};\quad \bar{\mathbf{e}} = \bar{\mathrm{E}}_{12} = \bar{z}^2\partial_{\bar{z}} + (1-\bar{\sigma}_{12})\bar{z};\quad \bar{\mathbf{f}} = \bar{\mathrm{E}}_{21} = -\partial_{\bar{z}}.$$

The lowest weight vector must be killed by the generators $\mathbf{f}$ and $\bar{\mathbf{f}}$, $\mathbf{f}|0\rangle = \bar{\mathbf{f}}|0\rangle = 0$. The corresponding function $|0\rangle = \Phi(z,\bar{z})$ satisfies the equations $\partial_z\Phi(z,\bar{z}) = \partial_{\bar{z}}\Phi(z,\bar{z}) = 0$, the solution of which is $\Phi(z,\bar{z}) = 1$. For the Verma module $\mathrm{V}_{\lambda,\bar{\lambda}}$ with $\lambda = 1-\sigma_{12}$, $\bar{\lambda} = 1-\bar{\sigma}_{12}$, so that

$$\mathbf{h}|0\rangle = (1-\sigma_{12})\cdot|0\rangle,\quad \bar{\mathbf{h}}|0\rangle = (1-\bar{\sigma}_{12})\cdot|0\rangle,$$

we have a basis formed by the vectors $|n,\bar{n}\rangle = \mathbf{e}^n\bar{\mathbf{e}}^{\bar{n}}|0\rangle$, where n and $\bar{n}$ are nonnegative integers, and the explicit form of the corresponding functions can easily be found:

$$\begin{aligned}|n,\bar{n}\rangle &= \left(z^2\partial_z + (1-\sigma_{12})z\right)^n\left(\bar{z}^2\partial_{\bar{z}} + (1-\bar{\sigma}_{12})\bar{z}\right)^{\bar{n}}\cdot 1\\ &= (1-\sigma_{12})_n\,(1-\bar{\sigma}_{12})_{\bar{n}}\cdot z^n\bar{z}^{\bar{n}},\end{aligned}$$

where $(a)_k$ is the Pochhammer symbol, $(a)_k \equiv a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$.

Since the holomorphic operators $\mathbf{h},\mathbf{e}$, and $\mathbf{f}$ commute with the antiholomorphic operators $\bar{\mathbf{h}},\bar{\mathbf{e}}$, and $\bar{\mathbf{f}}$, complete separation of the holomorphic and antiholomorphic sectors is possible: $\mathrm{V}_{\lambda,\bar{\lambda}} = \mathrm{V}_\lambda \otimes \mathrm{V}_{\bar{\lambda}}$. We note that it is natural to assume that the parameters σ_{12} and $\bar{\sigma}_{12}$ are independent, because the polynomials are single-valued functions, and there is no need to require that the difference $\sigma_{12}-\bar{\sigma}_{12}$ be an integer. In the general case, i.e., if none of the numbers σ_{12} and $\bar{\sigma}_{12}$ is integral, the Verma module in question is isomorphic to the space of polynomials in z and $\bar{z}$, and the isomorphism is given by the natural mapping of bases: $\mathbf{e}^n\bar{\mathbf{e}}^{\bar{n}}|0\rangle \leftrightarrow [z]^n \equiv z^n\bar{z}^{\bar{n}}$. In this case, $\mathrm{V}_{\lambda,\bar{\lambda}} = \mathbb{C}[z,\bar{z}]$, $\mathrm{V}_\lambda = \mathbb{C}[z]$, and $\mathrm{V}_{\bar{\lambda}} = \mathbb{C}[\bar{z}]$. Thus, we obtain the following separation of the holomorphic and antiholomorphic sectors: $\mathbb{C}[z,\bar{z}] = \mathbb{C}[z]\otimes\mathbb{C}[\bar{z}]$. If the parameters σ_{12} and $\bar{\sigma}_{12}$ are positive integers, then the space of polynomials $\mathbb{C}[z,\bar{z}]$ has an invariant subspace of polynomials of degree at most $\sigma_{12}-1$ in z and at most $\bar{\sigma}_{12}-1$ in $\bar{z}$ in which a finite-dimensional representation of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is realized. The holomorphic and antiholomorphic sectors are separated similarly: the invariant subspace is the tensor product of the space of polynomials in z of degree not exceeding $\sigma_{12}-1$ and the space of polynomials in $\bar{z}$ of degree not exceeding $\bar{\sigma}_{12}-1$. It turns out that the required operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$, acting on polynomials, split into products of operators acting on z and $\bar{z}$ independently. Therefore, as a result, the holomorphic and antiholomorphic sectors separate from each other.

The operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ are defined as the integral operators (3.28) and (3.29) that act in the space of functions $\Phi(z_1,z_2)$ depending on the variables $z_1,\bar{z}_1$ and $z_2,\bar{z}_2$. Now, we consider the action of these operators in the space of polynomials $\mathbb{C}[z_1,\bar{z}_1]\otimes\mathbb{C}[z_2,\bar{z}_2]$. In this new situation, it is convenient to choose another normalization. We drop all normalizing factors for a while and make the changes of variables $x_2 = \alpha z_2 + (1-\alpha)z_1$

and $x_1 = \alpha z_1 + (1-\alpha) z_2$ in the integrals (3.28) and (3.29), respectively:

$$\mathrm{R}_1(\mathbf{u}) \,:\, \Phi(z_1, z_2) \to \int \mathrm{d}^2\alpha\, [\alpha]^{u_1 - v_2}\, [1-\alpha]^{v_1 - u_1 - 1}\, \Phi\left(z_1, \alpha z_2 + (1-\alpha) z_1\right), \tag{4.2}$$

$$\mathrm{R}_2(\mathbf{u}) \,:\, \Phi(z_1, z_2) \to \int \mathrm{d}^2\alpha\, [\alpha]^{u_1 - v_2}\, [1-\alpha]^{v_2 - u_2 - 1}\, \Phi\left(\alpha z_1 + (1-\alpha) z_2, z_2\right). \tag{4.3}$$

We can combine these formulas by introducing an operator $\mathbf{R}_{ik}^{a,b}$ that acts nontrivially only on the arguments z_i and z_k of the function $\Phi(z_1, \dots, z_n)$, namely,

$$\begin{aligned} \mathbf{R}_{ik}^{a,b} \,&:\, \Phi(z_1, \dots, z_i, \dots, z_k, \dots, z_n) \\ &\mapsto \mathrm{N}_{a,b}^{-1} \int \mathrm{d}^2\alpha\, [\alpha]^{a-1}\, [1-\alpha]^{b-a-1}\, \Phi\left(z_1, \dots, \alpha z_i + (1-\alpha) z_k, \dots, z_k, \dots, z_n\right). \end{aligned}$$

It is convenient to choose the normalizing coefficient $\mathrm{N}_{a,b}$ so that

$$\mathbf{R}_{ik}^{a,b} \,:\, 1 \mapsto 1 \iff \mathrm{N}_{a,b} = \int \mathrm{d}^2\alpha\, [\alpha]^{a-1}\, [1-\alpha]^{b-a-1}. \tag{4.4}$$

The lower indices in $\mathbf{R}_{ik}^{a,b}$ indicate that the operator acts nontrivially only on the arguments z_i and z_k, and a and b are the parameters on which the operator depends. For specific values of the parameters a and b, we obtain the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$:

$$\mathrm{R}_1(\mathbf{u}) = \mathbf{R}_{21}^{u_1 - v_2 + 1, v_1 - v_2 + 1}; \quad \mathrm{R}_2(\mathbf{u}) = \mathbf{R}_{12}^{u_1 - v_2 + 1, u_1 - u_2 + 1}. \tag{4.5}$$

Obviously, the operator $\mathbf{R}_{ik}^{a,b}$ takes polynomials to polynomials and acts trivially on the variable z_k. We present an explicit formula for the action of $\mathbf{R}_{ik}^{a,b}$ on the monomials of the form $[z_i]^n$.

Proposition 1. *The action of the operator $\mathbf{R}_{ik}^{a,b}$ on the monomials $[z_i]^n = z_i^n \bar{z}_i^{\bar{n}}$, where n and $\bar{n}$ are nonnegative integers, is given by the following formula*:

$$\mathbf{R}_{ik}^{a,b} \,:\, [z_i]^n \mapsto \sum_{j=0}^{n} \sum_{\bar{j}=0}^{\bar{n}} \mathrm{R}_j^n \cdot \bar{\mathrm{R}}_{\bar{j}}^{\bar{n}} \cdot [z_i]^j \cdot [z_k]^{n-j}.$$

Thus, we have the factorization $\mathbf{R}_{ik}^{a,b} = \mathrm{R}_{ik}^{a,b} \cdot \bar{\mathrm{R}}_{ik}^{\bar{a},\bar{b}}$, where $\mathrm{R}_{ik}^{a,b}$ acts only on the holomorphic variables and $\bar{\mathrm{R}}_{ik}^{\bar{a},\bar{b}}$ acts only on the antiholomorphic variables,

$$\mathrm{R}_{ik}^{a,b} \,:\, z_i^n \mapsto \sum_{j=0}^{n} \mathrm{R}_j^n \cdot z_i^j z_k^{n-j}, \quad \bar{\mathrm{R}}_{ik}^{\bar{a},\bar{b}} \,:\, \bar{z}_i^{\bar{n}} \mapsto \sum_{\bar{j}=0}^{\bar{n}} \bar{\mathrm{R}}_{\bar{j}}^{\bar{n}} \cdot \bar{z}_i^{\bar{j}} \bar{z}_k^{\bar{n}-\bar{j}}. \tag{4.6}$$

The explicit expression for the coefficients R_j^n and $\bar{\mathrm{R}}_{\bar{j}}^{\bar{n}}$ has the following form:

$$\mathrm{R}_j^n = \binom{j}{n} \cdot \frac{\mathrm{B}(a+j; b-a+n-j)}{\mathrm{B}(a; b-a)}, \quad \bar{\mathrm{R}}_{\bar{j}}^{\bar{n}} = \binom{\bar{j}}{\bar{n}} \cdot \frac{\mathrm{B}(\bar{a}+\bar{j}; \bar{b}-\bar{a}+\bar{n}-\bar{j})}{\mathrm{B}(\bar{a}; \bar{b}-\bar{a})}.$$

For the proof, we use the binomial formula to expand the only nontrivial factor in the integral that defines the action of the operator $\mathbf{R}_{ik}^{a,b}$ on a function,

$$[\alpha z_i + (1-\alpha) z_k]^n = \sum_{j=0}^{n} \sum_{\bar{j}=0}^{\bar{n}} \binom{j}{n} \binom{\bar{j}}{\bar{n}} \cdot [\alpha z_i]^j \cdot [(1-\alpha) z_k]^{n-j}; \quad \binom{j}{n} = \frac{n!}{j!(n-j)!},$$

which gives the following expression for the coefficient of $[z_i]^j \cdot [z_k]^{n-j}$:

$$\frac{\binom{j}{n}\binom{\bar{j}}{\bar{n}}}{\mathrm{N}_{a,b}} \int \mathrm{d}^2\alpha\, [\alpha]^{a+j-1}\, [1-\alpha]^{b-a+n-j-1}.$$

It only remains to calculate the integrals by formula (2.25),

$$\int \mathrm{d}^2\alpha\, [\alpha]^{a+j-1}\, [1-\alpha]^{b-a+n-j-1}$$
$$= S(a,b)\cdot \mathrm{B}(a+j;b-a+n-j)\mathrm{B}(\bar{a}+\bar{j};\bar{b}-\bar{a}+\bar{n}-\bar{j}),$$
$$S(a,b) = \frac{\sin(\pi a)\sin\left(\pi(b-a)\right)}{\sin(\pi b)}, \quad \mathrm{B}(a;b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

For the principal series representations of the group $\mathrm{SL}(2,\mathbb{C})$, all important relations for the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ can be proved by a purely technical application of the properties of the operator $[i\partial]^\alpha$, as listed at the beginning of the section devoted to the intertwining operator. In the case of Verma modules, the operator $\mathrm{R}_{ik}^{a,b}$ plays the role of the basic building block. We have no way to extract simpler building blocks from this operator. Therefore, we consider the properties of the operator $\mathrm{R}_{ik}^{a,b}$ in detail and give independent proofs of the required results to show as clearly as possible how all this works. Moreover, to this point, we used the language of integral operators, which apparently is most appropriate for function spaces of continuous series. In the case of operators on spaces of polynomials, we can use different ways to describe the operators.

Since we have a complete separation between the holomorphic and the antiholomorphic sector, there is no loss of generality in considering only representations in the space of holomorphic polynomials. To make combinatorial formulas shorter, it is convenient to use the generating function for the basis vectors $|n\rangle \sim z^n$ of the Verma module V_λ, which can easily be expressed explicitly with the help of the binomial expansion

$$(1-x)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, x^k,$$
$$\sum_{k=0}^{\infty} \frac{x^k}{k!}\, |k\rangle = \sum_{k=0}^{\infty} \frac{x^k}{k!}\, (1-\sigma_{12})_k\, z^k = (1-xz)^{\sigma_{12}-1}.$$

We note that, for the parameter σ_{12} in general position, the above formula gives a generating function for all basis monomials z^n without restrictions on the degree. If σ_{12} is a positive integer, then we have a generating function for all basis monomials z^n in the space of monomials in z of degree at most $\sigma_{12}-1$, i.e., in the space of a finite-dimensional representation of the Lie algebra $\mathrm{sl}(2)$.

Proposition 2. *The following formulas for the action of the operator* $\mathrm{R}_{ik}^{a,b}$ *on polynomials are equivalent*:

$$\mathrm{R}_{ik}^{a,b} \,:\, z_i^n \mapsto \sum_{j=0}^{n} \binom{j}{n} \frac{(a)_j (b-a)_{n-j}}{(b)_n} \cdot z_i^j z_k^{n-j}, \tag{4.7}$$

$$\mathrm{R}_{ik}^{a,b} \,:\, (1-x_i z_i)^{-b} \mapsto (1-x_i z_i)^{-a}(1-x_i z_k)^{a-b}, \tag{4.8}$$

$$\mathrm{R}_{ik}^{a,b} = \frac{\Gamma(b)}{\Gamma(a)} \cdot \frac{\Gamma(z_{ik}\partial_i + a)}{\Gamma(z_{ik}\partial_i + b)}, \tag{4.9}$$

$$\begin{aligned} &\mathrm{R}_{ik}^{a,b}\Phi(z_i,\dots,z_k) \\ &\qquad = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \cdot \int_0^1 \mathrm{d}\alpha\, \alpha^{a-1}(1-\alpha)^{b-a-1}\Phi(\alpha z_i + (1-\alpha)z_k,\dots,z_k). \end{aligned} \tag{4.10}$$

Formula (4.7) is a restatement of (4.6) in terms of the Pochhammer symbols. Formula (4.8) is a concise form of (4.7) in terms of the generating function for the monomials

z_i^n, because

$$(1-x_iz_i)^{-b}=\sum_{n=0}^{\infty}\frac{(b)_n}{n!}x_i^nz_i^n\xrightarrow{\mathrm{R}_{ik}^{a,b}}\sum_{n=0}^{\infty}\frac{(b)_n}{n!}x_i^n\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}\frac{(a)_j(b-a)_{n-j}}{(b)_n}\cdot z_i^jz_k^{n-j}$$
$$=\sum_{j=0}^{\infty}\frac{(a)_j}{j!}(x_iz_i)^j\cdot\sum_{m=0}^{\infty}\frac{(b-a)_m}{m!}(x_iz_i)^m=(1-x_iz_i)^{-a}(1-x_iz_k)^{a-b}.$$

Formula (4.10) can be obtained from (4.6) by the integral representation

$$\mathrm{B}(a-j;b-a+n-j)=\int_0^1\mathrm{d}\alpha\,\alpha^{a-j-1}(1-\alpha)^{b-a+n-j-1}$$

for the beta function if we calculate the sum of the binomial expansion under the integral sign.

Formula (4.9) is a restatement of (4.10) in terms of operators. First, we use the integral representation for the beta function of the operator argument,

$$\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\cdot\mathrm{e}^{-z_k\partial_i}\cdot\mathrm{B}(z_i\partial_i+a,b-a)\cdot\mathrm{e}^{z_k\partial_i}$$
$$=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\cdot\mathrm{e}^{-z_k\partial_i}\cdot\int_0^1\mathrm{d}\alpha\,\alpha^{a-1+z_i\partial_i}(1-\alpha)^{b-a-1}\cdot\mathrm{e}^{z_k\partial_i},$$

and then verify that the operator $\mathrm{e}^{-z_k\partial_i}\cdot\alpha^{z_i\partial_i}\cdot\mathrm{e}^{z_k\partial_i}$ acts on functions in the required way,

$$\mathrm{e}^{-z_k\partial_i}\cdot\alpha^{z_i\partial_i}\cdot\mathrm{e}^{z_k\partial_i}\Phi(z_i)=\mathrm{e}^{-z_k\partial_i}\cdot\alpha^{z_i\partial_i}\Phi(z_i+z_k)$$
$$=\mathrm{e}^{-z_k\partial_i}\cdot\Phi(\alpha z_i+z_k)=\Phi\left(\alpha(z_i-z_k)+z_k\right).$$

The next proposition gives intertwining relations for the operator $\mathrm{R}_{ik}^{a,b}$. It is convenient to indicate the representation parameters a and b explicitly, together with the spaces in which the raising generators of the Lie algebra sl(2) act. We introduce the following concise notation for the raising generators:

$$\mathbf{e}_i^a=z_i^2\partial_i+a\,z_i,\quad\mathbf{e}_{ik}^{a,b}=\mathbf{e}_i^a+\mathbf{e}_k^b.$$

Proposition 3. *The intertwining relations* (3.30) *and* (3.31) *for the operators* $\mathrm{R}_1(\mathbf{u})$ *and* $\mathrm{R}_2(\mathbf{u})$ *are equivalent to the following commutation relations for the operator* $\mathrm{R}_{ik}^{a,b}$:

$$\left[\mathrm{R}_{ik}^{a,b},\partial_i+\partial_k\right]=0,\quad\left[\mathrm{R}_{ik}^{a,b},z_i\partial_i+z_k\partial_k\right]=0,\tag{4.11}$$

$$\left[\mathrm{R}_{ik}^{a,b},z_k\right]=0,\quad\mathrm{R}_{ik}^{a,b}\,\mathbf{e}_{ik}^{b,c}=\mathbf{e}_{ik}^{a,b+c-a}\,\mathrm{R}_{ik}^{a,b}.\tag{4.12}$$

Regardless of (3.30) and (3.31), all relations can easily be verified by direct calculation and with the help of formula (4.9).

Remark. We have combined the two last relations together, because they are defining relations for the operator $\mathrm{R}_{ik}^{a,b}$; i.e., formula (4.8) for the action on the generating function is a consequence of these two relations. It is convenient to represent the expression for the generating function in the following exponential form:

$$\exp\left(x\mathbf{e}^a\right)\cdot1=\sum_{n=0}\frac{x^n}{n!}\left(z^2\partial+az\right)^n\cdot1=\sum_{n=0}\frac{x^n}{n!}\left(a\right)_n z^n=(1-xz)^{-a}.$$

Using the obvious identity

$$\exp\left(x\mathbf{e}_{ik}^{b,c}\right)\cdot1=(1-xz_i)^{-b}(1-xz_k)^{-c}$$

and the condition $\mathrm{R}_{ik}^{a,b} : 1 \mapsto 1$, we obtain

$$\begin{aligned}\mathrm{R}_{ik}^{a,b}(1-xz_i)^{-b}(1-xz_k)^{-c} &= \mathrm{R}_{ik}^{a,b}\,\exp\left(x\mathbf{e}_{ik}^{b,c}\right)\cdot 1 = \exp\left(x\mathbf{e}_{ik}^{a,b+c-a}\right)\,\mathrm{R}_{ik}^{a,b}\cdot 1\\ &= (1-xz_i)^{-a}(1-xz_k)^{a-b-c}.\end{aligned}$$

Since $\mathrm{R}_{ik}^{a,b}$ commutes with z_k, we can cancel out $(1-xz_k)^{-c}$, obtaining (4.8).

The operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ can be expressed in terms of $\mathrm{R}_{ik}^{a,b}$. Therefore, relations for $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ can be restated as those for the operator $\mathrm{R}_{ik}^{a,b}$.

Proposition 4. *The operator* $\mathrm{R}_{ik}^{a,b}$ *satisfies the following relations*: *the double relations equivalent to* (3.33) *and* (3.32):

$$\mathrm{R}_{ik}^{a,b}\,\mathrm{R}_{ik}^{b,a} = \mathbb{1};\quad \mathrm{R}_{21}^{a,b}\,\mathrm{R}_{12}^{c,a} = \mathrm{R}_{12}^{b-a+c,b}\,\mathrm{R}_{21}^{c,b-a+c},$$

the double relations equivalent to (3.34):

$$\mathrm{R}_{13}^{a,b}\,\mathrm{R}_{23}^{c,d} = \mathrm{R}_{23}^{c,d}\,\mathrm{R}_{13}^{a,b};\quad \mathrm{R}_{32}^{a,b}\,\mathrm{R}_{31}^{b,c} = \mathrm{R}_{31}^{a,a-b+c}\,\mathrm{R}_{32}^{a-b+c,c},$$

and the triple relation

$$\mathrm{R}_{23}^{b,c}\,\mathrm{R}_{12}^{a+d,c+d}\,\mathrm{R}_{23}^{a,b} = \mathrm{R}_{12}^{a+d,b+d}\,\mathrm{R}_{23}^{a,c}\,\mathrm{R}_{12}^{b+d,c+d}.$$

Besides the relations listed above, we mention the convenient identities

$$\mathrm{R}_{ik}^{a,a} = \mathbb{1},\quad \mathrm{R}_{ik}^{a,b}\,\mathrm{R}_{ik}^{b,c} = \mathrm{R}_{ik}^{a,c},$$

which follow directly from (4.9).

In the case of principal series representations of the group $\mathrm{SL}(2,\mathbb{C})$, the relations for the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ can be proved by an application of the relations for the operators $\mathrm{S}_k(\mathbf{u})$. Now the operator $\mathrm{R}_{ik}^{a,b}$ plays the same role, and we present an independent proof of these relations to show how all this works in the new situation. To prove that two operators acting in the space of polynomials are equal, it suffices to check that these operators coincide on the basis monomials.

It is convenient to use generating functions instead of individual monomials. For example, the action of the operator $\mathrm{R}_{32}^{a,b}\,\mathrm{R}_{31}^{b,c}$ on the generating function gives the same result as the action of the operator $\mathrm{R}_{31}^{a,a-b+c}\,\mathrm{R}_{32}^{a-b+c,c}$:

$$\begin{aligned}(1-xz_3)^{-c} &\xrightarrow{\mathrm{R}_{31}^{b,c}} (1-xz_3)^{-b}(1-xz_1)^{b-c}\\ &\xrightarrow{\mathrm{R}_{32}^{a,b}} (1-xz_3)^{-a}(1-xz_2)^{a-b}(1-xz_1)^{b-c},\\ (1-xz_3)^{-c} &\xrightarrow{\mathrm{R}_{32}^{a-b+c,c}} (1-xz_3)^{b-a-c}(1-xz_2)^{a-b}\\ &\xrightarrow{\mathrm{R}_{31}^{a,a-b+c}} (1-xz_3)^{-a}(1-xz_1)^{b-c}(1-xz_2)^{a-b}.\end{aligned}$$

In this universal way, we can prove all relations except the following two:

$$\mathrm{R}_{21}^{a,b}\,\mathrm{R}_{12}^{c,a} = \mathrm{R}_{12}^{b-a+c,b}\,\mathrm{R}_{21}^{c,b-a+c},\quad \mathrm{R}_{23}^{b,c}\,\mathrm{R}_{12}^{a+d,c+d}\,\mathrm{R}_{23}^{a,b} = \mathrm{R}_{12}^{a+d,b+d}\,\mathrm{R}_{23}^{a,c}\,\mathrm{R}_{12}^{b+d,c+d}.$$

To prove these, besides the intertwining relations or, which is the same, formulas (4.8) and (4.9), we must use the Pfaff–Saalschütz summation formula [61]

$$\sum_{k=0}^{n}\binom{k}{n}\frac{(a)_k\,(b)_k\,(c-a-b)_{n-k}}{(c)_k} = \frac{(c-a)_n\,(c-b)_n}{(c)_n}. \tag{4.13}$$

As an example, we consider the proof of the relation

$$\mathrm{R}_{21}^{a,b}\,\mathrm{R}_{12}^{c,a}\,\frac{1}{(1-xz_1)^a}\cdot\frac{1}{(1-yz_2)^{b-a+c}} = \mathrm{R}_{12}^{b-a+c,b}\,\mathrm{R}_{21}^{c,b-a+c}\frac{1}{(1-xz_1)^a}\cdot\frac{1}{(1-yz_2)^{b-a+c}},$$

from which a similar relation for an arbitrary monomial $z_1^n z_2^m$ follows. It is convenient to split the proof into two parts. The intertwining relations (4.12) allow us to simplify the problem and to reduce everything to the verification of an operator identity for monomials that only depend on one variable, i.e., to make the reduction $z_1^n z_2^m \to z_1^n$ or $z_1^n z_2^m \to z_2^m$. For the monomials in one variable, the resulting identity follows from the Pfaff–Saalschütz summation formula [61].

By (4.12), it is easy to verify that the operators on the left-hand and the right-hand side commute in one and the same way with the raising generator $\mathbf{e}_{12}^{a,b-a+c}$: $\mathrm{A}\,\mathbf{e}_{12}^{a,b-a+c} = \mathbf{e}_{12}^{b-a+c,a}\,\mathrm{A}$, where A is one of the operators $\mathrm{R}_{21}^{a,b}\,\mathrm{R}_{12}^{c,a}$ or $\mathrm{R}_{12}^{b-a+c,b}\,\mathrm{R}_{21}^{c,b-a+c}$. We transform the generating function ($\lambda = x - y$) as follows:

$$\exp\left(x\mathbf{e}_1^a + y\mathbf{e}_2^{b-a+c}\right)\cdot 1 = \exp\bigl(y\mathbf{e}_{12}^{a,b-a+c}\bigr)\exp\bigl(\lambda\mathbf{e}_1^a\bigr)\cdot 1 = \exp\bigl(y\mathbf{e}_{12}^{a,b-a+c}\bigr)(1-\lambda z_1)^{-a}.$$

Now, we can move the exponential with the generator $\mathbf{e}_{12}^{a,b-a+c}$ to the left, obtaining the same exponent but with the generator $\mathbf{e}_{12}^{b-a+c,a}$. Canceling this common factor out, we obtain

$$\mathrm{R}_{21}^{a,b}\,\mathrm{R}_{12}^{c,a}\;(1-\lambda z_1)^{-a} = \mathrm{R}_{12}^{b-a+c,b}\,\mathrm{R}_{21}^{c,b-a+c}(1-\lambda z_1)^{-a}.$$

It suffices to check the corresponding identity for an arbitrary monomial z_1^n. The right-hand side of the identity can be simplified further:

$$\mathrm{R}_{21}^{a,b}\,\mathrm{R}_{12}^{c,a}\,z_1^n = \mathrm{R}_{12}^{b-a+c,b}\,\mathrm{R}_{21}^{c,b-a+c}z_1^n = \mathrm{R}_{12}^{b-a+c,b}\,z_1^n.$$

Applying formula (4.7), we can easily verify that the relation obtained is a consequence of the Pfaff–Saalschütz summation formula [61]. The triple relation is proved in the same way, namely, the intertwining formulas (4.12) allow us to reduce the problem to the verification of an operator identity for monomials that depend on one variable only, and the resulting identity turns out to be equivalent to the Pfaff–Saalschütz summation formula [61].

The above calculation shows that the representation (4.8), which is a direct consequence of the intertwining relations (4.12), is a convenient representation for the operator $\mathrm{R}_{ik}^{a,b}$. In the sequel, we show that this representation can be generalized to the case of the group $\mathrm{SL}(n,\mathbb{C})$.

4.2. Finite-dimensional representations. The operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ used in the construction of solutions of the Yang–Baxter equation

$$\mathrm{R}(u-v) = \mathrm{P}_{12}\mathrm{R}_1(r_2\mathbf{u})\mathrm{R}_2(\mathbf{u}) = \mathrm{P}_{12}\mathrm{R}_2(r_1\mathbf{u})\mathrm{R}_1(\mathbf{u}) \tag{4.14}$$

are defined as operators acting on the space of polynomials $\mathbb{C}[z_1,z_2] = \mathbb{C}[z_1]\otimes\mathbb{C}[z_2]$:

$$\begin{aligned}&\mathrm{R}_1(\mathbf{u}) = \mathrm{R}_{21}^{u_1-v_2+1,v_1-v_2+1}\ :\\ &\qquad z_1^i z_2^k \mapsto \sum_{j=0}^{k}\binom{j}{k}\cdot\frac{(u_1-v_2+1)_j\,(v_1-u_1)_{k-j}}{(v_1-v_2+1)_k}\cdot z_1^{i+k-j}z_2^j,\end{aligned} \tag{4.15}$$

$$\begin{aligned}&\mathrm{R}_2(\mathbf{u}) = \mathrm{R}_{12}^{u_1-v_2+1,u_1-u_2+1}\ :\\ &\qquad z_1^i z_2^k \mapsto \sum_{j=0}^{i}\binom{j}{i}\cdot\frac{(u_1-v_2+1)_j\,(v_2-u_2)_{i-j}}{(u_1-u_2+1)_i}\cdot z_1^j z_2^{i+k-j}.\end{aligned} \tag{4.16}$$

In the case of irreducible Verma modules, the representation parameters (σ_1,σ_2) and (ρ_1,ρ_2), where

$$(u_1,u_2) = (u-\sigma_1, u-\sigma_2)\,,\quad (v_1,v_2) = (v-\rho_1, v-\rho_2)\,,$$

are complex numbers in general position. If $\sigma_{12} = 1 + n$, $n = 0, 1, 2, \dots$, then the space $\mathbb{C}[z_1]$ of polynomials has an invariant subspace with the basis $1, z_1, z_1^2, \dots, z_1^n$ on which an $(n+1)$-dimensional representation of the Lie algebra $\mathrm{sl}(2, \mathbb{C})$ extendible to a representation of the group $\mathrm{SL}(2, \mathbb{C})$ is realized. In the same way, if $\rho_{12} = 1 + m$, $m = 0, 1, 2, \dots$, the space $\mathbb{C}[z_2]$ has an invariant $(m+1)$-dimensional subspace of polynomials with the basis $1, z_2, \dots, z_2^m$. Thus, for the integral points $\sigma_{12} = 1 + n$ and $\rho_{12} = 1 + m$, the space $\mathbb{C}[z_1, z_2]$ of polynomials has an invariant subspace of polynomials of degree at most n in z_1 and at most m in z_2. It turns out that, at integral points, the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ are well defined on the invariant subspace; however, they map this subspace to a finite-dimensional space of larger dimension.

Proposition 5. *For integers $\sigma_{12} = 1 + n$ and $\rho_{12} = 1 + m$, the operators $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ are well defined on the space of polynomials of degree at most n in z_1 and at most m in z_2. The operator $\mathrm{R}_1(\mathbf{u})$ maps this subspace to the space of polynomials of degree at most $(n+m)$ in z_1 and at most m in z_2,*

$$\mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,m} \xrightarrow{\mathrm{R}_1} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n+m,m},$$

and the operator $\mathrm{R}_2(\mathbf{u})$ maps the same subspace to the space of polynomials of degree at most n in z_1 and at most $(n+m)$ in z_2,

$$\mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,m} \xrightarrow{\mathrm{R}_2} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,n+m}.$$

Since the spectral parameters are arbitrary complex numbers, nothing happens to the differences $u_i - v_k$ at integral points. They remain arbitrary complex numbers. In the differences $u_i - u_k$ and $v_i - v_k$, the spectral parameters cancel out. These expressions only depend on the representation parameters, and these parameters determine what happens to the operator at an integral point. Only the denominators in (4.15) and (4.16) do not depend on the difference of the spectral parameters,

$$(u_1 - u_2 + 1)_i = (1 - \sigma_{12})_i = \frac{(-)^i n!}{(n-i)!}, \quad (v_1 - v_2 + 1)_k = (1 - \rho_{12})_k = \frac{(-)^k m!}{(m-k)!},$$

which vanish if $i > n$ or $k > m$. Therefore, the operators themselves are well defined on the space of polynomials of degree at most n in z_1 and at most m in z_2. The restrictions on the degrees of z_1 and z_2 in the polynomial obtained after the action of the operator $\mathrm{R}_1(\mathbf{u})$ or $\mathrm{R}_2(\mathbf{u})$ can easily be seen from the explicit formulas (4.15) and (4.16) or (4.8).

Consider the remaining operators $\mathrm{R}_1(r_2\mathbf{u})$ and $\mathrm{R}_2(r_1\mathbf{u})$ in formula (4.14) for $\mathrm{R}(u-v)$:

$$\begin{aligned}
&\mathrm{R}_1(r_2\mathbf{u}) = \mathrm{R}_{21}^{u_1-u_2+1,v_1-u_2+1} : \\
&\qquad z_1^i z_2^k \mapsto \sum_{j=0}^{k} \binom{j}{k} \cdot \frac{(u_1 - u_2 + 1)_j \, (v_1 - u_1)_{k-j}}{(v_1 - u_2 + 1)_k} \cdot z_1^{i+k-j} z_2^j, \\
&\mathrm{R}_2(r_1\mathbf{u}) = \mathrm{R}_{12}^{v_1-v_2+1,v_1-u_2+1} : \\
&\qquad z_1^i z_2^k \mapsto \sum_{j=0}^{i} \binom{j}{i} \cdot \frac{(v_1 - v_2 + 1)_j \, (v_2 - u_2)_{i-j}}{(v_1 - u_2 + 1)_i} \cdot z_1^j z_2^{i+k-j}.
\end{aligned}$$

Now, the factors $(u_1 - u_2 + 1)_j = \frac{(-)^j n!}{(n-j)!}$ and $(v_1 - v_2 + 1)_j = \frac{(-)^j m!}{(m-j)!}$, independent of the spectral parameters, are in the numerator and vanish if $j > n$ and $j > m$, respectively. This gives rise to restrictions in the sums over j, namely, $j \le n$ and $j \le m$, and, as a

consequence, to restrictions on the degrees of polynomials. As a result, we obtain

$$\mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,m} \xrightarrow{\mathrm{R}_1(\mathbf{u})} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n+m,m} \xrightarrow{\mathrm{R}_2(r_1\mathbf{u})} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{m,n},$$
$$\mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,m} \xrightarrow{\mathrm{R}_2(\mathbf{u})} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,n+m} \xrightarrow{\mathrm{R}_1(r_2\mathbf{u})} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{m,n}.$$

The remaining permutation operator returns everything to the initial position,

$$\mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{m,n} \xrightarrow{\mathrm{P}_{12}} \mathrm{Span}\left\{z_1^i z_2^k\right\}_{i=0,k=0}^{n,m},$$

and the initial finite-dimensional space is invariant under $\mathrm{R}(u-v)$.

Thus, in the case of finite-dimensional representations, the decomposition of the operator $\mathrm{R}(u)$ into a product of simpler blocks is similar to the decomposition of a square matrix into a product of rectangular matrices. Formulas for the action of the compound operator $\mathrm{R}(u)$ on polynomials are obtained from formulas (4.7)–(4.10) for $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$.

To show that the operator $\mathrm{R}(u)$ obtained for finite-dimensional representations coincides with that known from the literature, first we present some standard formulas [4, 5, 60].

The $(2j+1)$-dimensional representation D_j of half-integer spin j of the Lie algebra $s\ell_2$ is realized on the space of polynomials in z with the basis $\{z^k\}_{k=0}^{2j}$. In the representation D_j, the action of the generators is defined by the formulas

$$\mathrm{S} = z\partial_z - j, \quad \mathrm{S}^- = -\partial_z, \quad \mathrm{S}^+ = z^2\partial_z - 2jz$$

or

$$\mathrm{S}z^k = (k-j)z^k, \quad \mathrm{S}^- z^k = -kz^{k-1}, \quad \mathrm{S}^+ z^k = (k-2j)z^{k+1}, \tag{4.17}$$

and, for $j = 1/2$ and the basis $\{-z, 1\}$, we obtain the usual two-dimensional representation

$$2\mathrm{S} \simeq \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{S}^- \simeq \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{S}^+ \simeq \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The operator $\mathrm{R}(u)$ (see [4, 5, 60]) acts in the tensor product of two representations $D_{j_1} \otimes D_{j_2}$, is $s\ell_2$-invariant,

$$[\mathrm{S}_{j_1} + \mathrm{S}_{j_2}, \mathrm{R}(u)] = \left[\mathrm{S}_{j_1}^{\pm} + \mathrm{S}_{j_2}^{\pm}, \mathrm{R}(u)\right] = 0,$$

and is defined by the spectral expansion

$$\mathrm{R}(u) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathrm{R}_j(u) \cdot \mathrm{P}_j, \quad \mathrm{R}_j = \frac{\Gamma(u+j_1+j_2)}{\Gamma(u-j_1-j_2)} \frac{\Gamma(u-j)}{\Gamma(u+j)}, \tag{4.18}$$

where P_j is the projection onto the irreducible representation D_j in the following decomposition of the tensor product:

$$D_{j_1} \otimes D_{j_2} = D_{j_1+j_2} \oplus D_{j_1+j_2-1} \oplus \cdots \oplus D_{|j_1-j_2|} = \sum_{j=|j_1-j_2|}^{j_1+j_2} D_j.$$

We consider some well-known specific cases. For $j_1 = \frac{1}{2}$ and $j_2 = j$, the operator $\mathrm{R}(u)$ acts in $\mathbb{C}^2 \otimes D_j$ and, up to normalization and a translation of the spectral parameter, coincides with the Lax operator,

$$\mathrm{R}(u) \to (u + j + \tfrac{1}{2})^{-1} \cdot \mathrm{L}(u + \tfrac{1}{2}). \tag{4.19}$$

For $j_1 = j_2 = \frac{1}{2}$, the operator $\mathrm{R}(u)$ acts in $\mathbb{C}^2 \otimes \mathbb{C}^2$, and, up to normalization, coincides with the well-known R-matrix,

$$\mathrm{R}(u) \to (u+1)^{-1} \cdot (u + \mathrm{P}), \tag{4.20}$$

where P is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$. We recall that the L-operator is defined on the space $\mathbb{C}^2 \otimes D_j$ as follows:

$$\mathrm{L}(u) = u + \left(\sigma_3 \otimes S_j + \sigma_- \otimes S_j^+ + \sigma_+ \otimes S_j^-\right) = \begin{pmatrix} u + z\partial_z - j & -\partial_z \\ z^2\partial_z - 2jz & u - z\partial_z + j \end{pmatrix}.$$

It satisfies the following commutation relations with the R-matrix, $\mathrm{R}(u) = u + \mathrm{P}$ (see [4, 5, 56]):

$$\mathrm{R}(u-v)\mathrm{L}^{(1)}(u)\mathrm{L}^{(2)}(v) = \mathrm{L}^{(2)}(v)\mathrm{L}^{(1)}(u)\mathrm{R}(u-v),$$

where the operators $\mathrm{L}^{(1)}(u)$ and $\mathrm{L}^{(2)}(u)$ are defined as the extensions of the operator $\mathrm{L}(u)$ to the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes D_j$ that act by σ_3 and $\sigma_\pm$ on the first and the second space $\mathbb{C}^2$, respectively.

Comparing the formula for the L-operator with (2.34), we obtain the relation $(\sigma_1, \sigma_2) = (1+j, -j)$ for the parameters (σ_1, σ_2) of the representation with spin j. Now, using the standard notation

$$(\sigma_1, \sigma_2) = (1+j_1, -j_1), \quad (\rho_1, \rho_2) = (1+j_2, -j_2), \quad u - v \to u,$$

we present an integral representation of the action of the operator $\mathrm{R}(u)$ (see (4.14)) on polynomials.

Proposition 6. *The action of the operator* $\mathrm{R}(u)$ *on polynomials is given by the following formula*:

$$\begin{aligned} \mathrm{R}(u)\Phi(z_1, z_2) = \rho \int_0^1 \mathrm{d}\alpha \int_0^{1-\alpha} \mathrm{d}\beta\, \alpha^{j_2-j_1-u-1}\beta^{j_1-j_2-u-1} \\ \times (1-\alpha-\beta)^{u-j_1-j_2-1} \cdot \Phi\left(\alpha z_1 + (1-\alpha)z_2, \beta z_2 + (1-\beta)z_1\right), \end{aligned} \tag{4.21}$$

where the normalization factor is chosen from the condition $\mathrm{R}(u) : 1 \mapsto 1$:

$$\rho = \frac{\Gamma\left(-j_1 - j_2 - u\right)}{\Gamma\left(j_2 - j_1 - u\right)\Gamma\left(j_1 - j_2 - u\right)\Gamma\left(u - j_1 - j_2\right)}.$$

It can easily be verified by direct calculation that the integral operator presented above coincides with (4.14). It remains to prove that this operator coincides with the standard $\mathrm{R}(u)$ acting on $D_{j_1} \otimes D_{j_2}$. First, we consider specific cases. For $j_1 = \frac{1}{2}$, $j_2 = j$, and monomials of the form z_2^m and $z_1 z_2^m$, formula (4.21) gives

$$\begin{aligned} &\mathrm{R}(u) \,:\, (u + j + \tfrac{1}{2})z_2^m \mapsto (u + j - m + \tfrac{1}{2})z_2^m + m z_1 z_2^{m-1}, \\ &\mathrm{R}(u) \,:\, (u + j + \tfrac{1}{2})z_1 z_2^m \mapsto (2j - m)z_2^{m+1} + (u - j + m + \tfrac{1}{2})z_1 z_2^m. \end{aligned}$$

Representing the action of $\mathrm{R}(u)$ in the matrix form in the basis $\{-z_1, 1\}$ and using (4.17), first for an arbitrary j, and then for $j = \frac{1}{2}$, we can easily check that, for $j_1 = \frac{1}{2}$ and $j_2 = j$, the operator $\mathrm{R}(u)$ coincides with the Lax operator (4.19) up to normalization and a translation of the spectral parameter, and that, for $j_1 = j_2 = \frac{1}{2}$, it coincides with the standard R-matrix (4.20). The fact that the operator $\mathrm{R}(u)$ is $s\ell_2$-invariant was proved earlier for the more general representations (3.30)–(3.31). The fact that a finite-dimensional irreducible space of polynomials is invariant under the action of $\mathrm{R}(u)$ was proved at the beginning of the present section. To obtain the spectral expansion (4.18), we need to find the eigenvalues of the operator $\mathrm{R}(u)$ on the representation space D_j in the decomposition of the tensor product $D_{j_1} \otimes D_{j_2}$. It is easy to show that $(z_1 - z_2)^k$ is a lowest weight vector for the representation $D_{j_1+j_2-k}$ and that the eigenvalue is calculated by (4.21) or (4.9):

$$R(u) \,:\, (z_1 - z_2)^k \mapsto (-1)^k \frac{(u - j_1 - j_2)_k}{(-j_1 - j_2 - u)_k} \cdot (z_1 - z_2)^k,$$

which yields the spectral expansion (4.18). Thus, the operator $\mathrm{R}(u)$ obtained by restriction of the general solution of the Yang–Baxter equation to finite-dimensional invariant subspaces for the principal series representations of the group $\mathrm{SL}(2,\mathbb{C})$ coincides with the standard operator $\mathrm{R}(u)$ on the space of the tensor product $D_{j_1}\otimes D_{j_2}$.

§5. The group $\mathrm{SL}(n,\mathbb{C})$

In this section, we consider the construction of an $\mathrm{SL}(n,\mathbb{C})$-invariant solution of the Yang–Baxter equation, which is a direct generalization of the construction of an $\mathrm{SL}(2,\mathbb{C})$-invariant solution of the Yang–Baxter equation studied in the first part of the present paper.

We follow the same lines as in the case of the group $\mathrm{SL}(2,\mathbb{C})$. Therefore, at each step, it is easy to trace the changes required in the general case of the group $\mathrm{SL}(n,\mathbb{C})$ in comparison with $\mathrm{SL}(2,\mathbb{C})$. Moreover, we decode the general formulas for $\mathrm{SL}(n,\mathbb{C})$ in the simplest cases where $n=2$ and $n=3$ to show that, in the case of $n=2$, we return to formulas from the first part and to demonstrate how everything changes with the increase of the group rank. To avoid cumbersome formulas, we use more concise notation in comparison with the case of the group $\mathrm{SL}(2,\mathbb{C})$.

5.1. Representations of the group $\mathrm{SL}(n,\mathbb{C})$. In this section, we follow [49, 53] and describe a construction of induced representations of the group of nonsingular complex matrices of order n. We denote by Z the group of lower triangular complex matrices $\mathrm{z}=\|z_{ik}\|$ of order n for which $z_{kk}=1$ and $z_{ik}=0$ for $i<k$, and by H the group of lower triangular complex matrices of order n,

$$\mathrm{z}=\begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ z_{21} & 1 & 0 & \dots & 0\\ z_{31} & z_{32} & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ z_{n1} & z_{n2} & z_{n3} & \dots & 1\end{pmatrix}\in \mathrm{Z},\quad \mathrm{h}=\begin{pmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1n}\\ 0 & h_{22} & h_{23} & \dots & h_{2n}\\ 0 & 0 & h_{33} & \dots & h_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & h_{n,n}\end{pmatrix}\in \mathrm{H}. \tag{5.1}$$

As an induced subgroup, we take the subgroup of upper triangular matrices H, and as the induced representation, we take the one-dimensional representation $\alpha(\mathrm{h})$ of H,

$$\alpha(\mathrm{h})=[h_{11}]^{-1-\sigma_1}[h_{22}]^{-2-\sigma_2}\cdots[h_{nn}]^{-n-\sigma_n}=\prod_{k=1}^{n}[h_{kk}]^{-k-\sigma_k}, \tag{5.2}$$

where $h_{11},h_{22},\dots,h_{nn}$ are the diagonal entries of the matrix h. Again, we use the concise notation $[h_{kk}]^{-k-\sigma_k}=h_{kk}^{-k-\sigma_k}\bar{h}_{kk}^{-k-\bar{\sigma}_k}$, where $\bar{h}_{kk}$ is the complex conjugate to h_{kk} and $\bar{\sigma}_k-\sigma_k\in\mathbb{Z}$. In the first picture, the induced representation is realized as a subrepresentation of the left regular representation

$$\mathrm{T}(g)\Phi(\mathrm{x})=\Phi(g^{-1}\mathrm{x}) \tag{5.3}$$

in the space of functions on the group $\mathrm{GL}(n,\mathbb{C})$ that satisfy the condition

$$\Phi(\mathrm{x})=\Phi(\mathrm{z}\,\mathrm{h})=\alpha\left(\mathrm{h}^{-1}\right)\cdot\Phi(\mathrm{z}). \tag{5.4}$$

In this equation, we have used the Gauss decomposition (almost every matrix $\mathrm{x}\in\mathrm{GL}(\mathrm{n},\mathbb{C})$ can be represented uniquely in the form $\mathrm{x}=\mathrm{z}\,\mathrm{h}$).

A function $\Phi(\mathrm{x})$ satisfying condition (5.4) is uniquely determined by its restriction $\Phi(\mathrm{z})$ to the subgroup Z of lower triangular matrices. Writing the action of the left regular representation operator (5.3) on the functions $\Phi(\mathrm{z})$, we obtain the second picture of the induced representation.

Let z be an arbitrary matrix in Z, and let g be an arbitrary matrix in $\mathrm{GL}(n,\mathbb{C})$. Using the Gauss decomposition, we represent the product $g^{-1}\cdot \mathrm{z}$ in the form

$$g^{-1}\cdot \mathrm{z} = \mathrm{z}'\cdot \mathrm{h}. \tag{5.5}$$

When calculating the generator, it is convenient to explicitly separate the initial matrix z out of the resulting matrix $\mathrm{z}'\in \mathrm{Z}$ by decomposing z' into the product $\mathrm{z}\,\bar g$, $\mathrm{z}' = \mathrm{z}\,\bar g$. For example, in the case of the group $\mathrm{GL}(2,\mathbb{C})$, we obtain the following formula for the matrix $\bar g$ (for simplicity, we write z for z_{21}, and the notation is the same as in (2.8)):

$$\begin{pmatrix} 1 & 0\\ z' & 1\end{pmatrix} = \begin{pmatrix} 1 & 0\\ \frac{-c+az}{d-bz} & 1\end{pmatrix} = \begin{pmatrix} 1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \frac{-c+(a-d)z+bz^2}{d-bz} & 1\end{pmatrix}.$$

The space in which the representation operators $\mathrm{T}(g)$ act, consists of the functions $\Phi(\mathrm{z})$, where $\mathrm{z}\in \mathrm{Z}$, i.e., $\Phi(\mathrm{z})$ is a function of $\frac{n(n-1)}{2}$ variables, $\Phi(\mathrm{z}) = \Phi(z_{21}, z_{31},\dots, z_{n,n-1})$. We do not assume that the function is holomorphic; it also depends on the variables $\bar z_{21}, \bar z_{31},\dots,\bar z_{n,n-1}$. In the case of the group $\mathrm{GL}(2,\mathbb{C})$, we indicated the conjugate variables explicitly, but, in the case of the group $\mathrm{GL}(n,\mathbb{C})$, this becomes too cumbersome. Therefore, for simplicity, we indicate only the holomorphic part of the variables in all formulas. In the second picture of the induced representation, the action of the representation operator $\mathrm{T}(g)$ on a function is defined as follows:

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = \alpha\left(\mathrm{h}^{-1}\right)\cdot \Phi(\mathrm{z}') = [h_{11}]^{\sigma_1+1}[h_{22}]^{\sigma_2+2}\cdots[h_{nn}]^{\sigma_n+n}\cdot \Phi(\mathrm{z}'), \tag{5.6}$$

where the h_{kk} are the diagonal entries of the matrix h in (5.5). In the case of the group $\mathrm{GL}(2,\mathbb{C})$, the matrices z' and h can easily be calculated, and we obtain the explicit formula (2.9). In the general case, the entries of the required matrices can be expressed in terms of the minors of the matrix $g^{-1}\mathrm{z}$ (see [49, 53]):

$$z'_{ik} = \frac{\Delta_{ik}}{\Delta_k},\quad h_{kk} = \frac{\Delta_k}{\Delta_{k-1}}, \tag{5.7}$$

where Δ_{ik} is the minor formed by the entries of $g^{-1}\mathrm{z}$ that are in the intersection of the rows with indices $1,2,\dots,k-1,i$ $(i\ge k)$ and the columns with indices $1,2,\dots,k-1,k$, and $\Delta_k\equiv\Delta_{kk}$ is the minor formed by the entries of $g^{-1}\cdot\mathrm{z}$ that are in the intersection of the first k rows and the first k columns. It is convenient to use the following equivalent representation of the function $\alpha\left(\mathrm{h}^{-1}\right)$ in terms of the minors Δ_k $(\sigma_{k,k+1}\equiv\sigma_k-\sigma_{k+1})$:

$$\begin{aligned}\alpha\left(\mathrm{h}^{-1}\right) &= [\Delta_n]^{\sigma_n+n}\cdot[\Delta_1]^{\sigma_{12}-1}[\Delta_2]^{\sigma_{23}-1}\cdots[\Delta_{n-1}]^{\sigma_{n-1,n}-1}\\ &= [\Delta_n]^{\sigma_n+n}\cdot\prod_{k=1}^{n-1}[\Delta_k]^{\sigma_{k,k+1}-1}.\end{aligned} \tag{5.8}$$

The resulting representation is given by two sets of numbers, $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ and $\bar{\boldsymbol{\sigma}} = (\bar\sigma_1,\dots,\bar\sigma_n)$, satisfying the additional condition $\bar\sigma_k-\sigma_k\in\mathbb{Z}$. To simplify the notation, we do not indicate explicitly the dependence of the representation $\mathrm{T}^{\boldsymbol{\sigma}}$ on $\bar{\boldsymbol{\sigma}}$.

In the case of the group $\mathrm{SL}(n,\mathbb{C})$, the determinant of the matrix h is equal to 1, $\Delta_n = \det\mathrm{h} = 1$, and we need only the differences $\sigma_{k,k+1} = \sigma_k-\sigma_{k+1}$ $(\bar\sigma_{k,k+1} = \bar\sigma_k-\bar\sigma_{k+1})$ to characterize the representation,

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = [\Delta_1]^{\sigma_{12}-1}[\Delta_2]^{\sigma_{23}-1}\cdots[\Delta_{n-1}]^{\sigma_{n-1,n}-1}\cdot\Phi(\mathrm{z}'). \tag{5.9}$$

It is convenient to use a symmetric parametrization $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ of the representation $\mathrm{T}^{\boldsymbol{\sigma}}$ of the group $\mathrm{SL}(n,\mathbb{C})$ by imposing the additional condition

$$\sigma_1+\sigma_2+\cdots+\sigma_n = \frac{n(n-1)}{2}. \tag{5.10}$$

5.2. Irreducible representations of $\mathrm{SL}(n,\mathbb{C})$**.** In perfect analogy with the case of the group $\mathrm{SL}(2,\mathbb{C})$, we consider three types of irreducible representations of the group $\mathrm{SL}(n,\mathbb{C})$. In all cases, the action of the representation operators on functions is defined by the general formula (5.9); however, the operators $\mathrm{T}(g)$ act in different functional spaces, depending on the representation parameters. A point $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ in the space of parameters is singular if the differences $\sigma_{k,k+1}$ are integers. We say that a point $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ is in general position if none of the numbers $\sigma_{k,j}$ is an integer for $j > k$.

- The principal series of nonunitary representations of the group $\mathrm{SL}(n,\mathbb{C})$.

For such representations, which are of the most general type, the point $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ is in general position. The action of the operators on a function is defined by the general formula (5.9). The function space is a space of infinitely differentiable functions with a fixed asymptotic behavior at infinity. The asymptotic conditions are obtained in the same way as for the group $\mathrm{SL}(2,\mathbb{C})$. There are $n-1$ conditions in all, because, as in the case of the group $\mathrm{SL}(n,\mathbb{C})$, there exist $n-1$ basis inversions. We do not present these conditions since they are not used anywhere explicitly. It can be proved that the principal series representation $\mathrm{T}^{\boldsymbol{\sigma}}$ is irreducible and that two representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\boldsymbol{\sigma}'}$ are equivalent if and only if there exists a permutation that takes σ_k to σ'_k [49, 53].

- The principal series of unitary representations of the group $\mathrm{SL}(n,\mathbb{C})$.

To determine unitary representations of the group $\mathrm{SL}(n,\mathbb{C})$, we must define an inner product in the space of functions $\Phi(\mathrm{z})$ and to find the values of the parameters σ_k and $\bar\sigma_k$ for which the representation operators are unitary. The principal series of unitary representations of the group $\mathrm{SL}(n,\mathbb{C})$ is defined as follows [49, 55]. The representation space is the function space $L^2(\mathrm{Z})$ with the inner product

$$\langle\Phi_1|\Phi_2\rangle = \int \mathrm{dz}\,\overline{\Phi_1(\mathrm{z})}\,\Phi_2(\mathrm{z}); \quad \mathrm{dz} \equiv \prod_{1\le i<k\le n} d^2 z_{ki}. \tag{5.11}$$

For every $g \in \mathrm{SL}(n,\mathbb{C})$, we have an operator $\mathrm{T}(g)$ given by formula (5.6). It can be proved [49] that the representation operators are unitary,

$$\langle \mathrm{T}(g)\Phi_1|\mathrm{T}(g)\Phi_2\rangle = \langle\Phi_1|\Phi_2\rangle,$$

if and only if

$$|\alpha(\mathrm{h}^{-1})|^2 = |h_{11}|^2|h_{22}|^4\cdots|h_{nn}|^{2n},$$

where the right-hand side is the Jacobian of the transition from z to the new variables $\mathrm{z}' = \mathrm{z}\,\bar g$, $\mathrm{d}z' = \mathrm{d}z\cdot|h_{11}|^2\cdots|h_{nn}|^{2n}$. This condition leads to the system of equations $\sigma_{k,k+1} + \bar\sigma^*_{k,k+1} = 0$, $k = 1,2,\dots,n-1$. Thus, we arrive at the following representation for the parameters that define a unitary representation:

$$\sigma_{k,k+1} = -\frac{n_k}{2} + i\lambda_k, \quad \bar\sigma_{k,k+1} = \frac{n_k}{2} + i\lambda_k, \quad k = 1,2,\dots,n-1. \tag{5.12}$$

The numbers n_k are integers and the λ_k are real. The principal series unitary representation $\mathrm{T}^{\boldsymbol{\sigma}}$ is irreducible, and two representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\boldsymbol{\sigma}'}$ are unitarily equivalent if and only if there exists a permutation that takes σ_k to σ'_k [49].

- Finite-dimensional irreducible representations of the group $\mathrm{SL}(n,\mathbb{C})$ [51]–[53].

If all parameters $\sigma_{k,k+1}$ and $\bar\sigma_{k,k+1}$ are positive integers, then the representation space of the principal nonunitary series has a finite-dimensional invariant subspace of polynomials in z_{ik} and $\bar z_{ik}$.

In the case of the group $\mathrm{SL}(2,\mathbb{C})$, the invariant subspace is the kernel of the intertwining operator $\mathrm{S} = [i\partial_z]^{\sigma_{12}}$. At a singular point, i.e., when the parameters σ_{12} and $\bar\sigma_{12}$ are positive integers, the operator S is a differential operator, and its kernel consists of the

functions satisfying $\partial_z^{\sigma_{12}}\partial_{\bar z}^{\bar\sigma_{12}}\Phi(z,\bar z)=0$; i.e., it is formed by polynomials in z and $\bar z$ of degree at most $\sigma_{12}-1$ in z and at most $\bar\sigma_{12}-1$ in $\bar z$.

Intertwining operators remain convenient for a constructive description of finite-dimensional representations at integral points also in the case of the group $\mathrm{SL}(n,\mathbb{C})$. Here, all intertwining operators can be constructed from $n-1$ basis operators S_k that intertwine the representations $\mathrm{T}^{\boldsymbol\sigma}$ and $\mathrm{T}^{\boldsymbol\sigma_k}$: $\mathrm{S}_k\,\mathrm{T}^{\boldsymbol\sigma}=\mathrm{T}^{\boldsymbol\sigma_k}\mathrm{S}_k$, where $\boldsymbol\sigma_k=\mathrm{s}_k\boldsymbol\sigma$ is obtained from $\boldsymbol\sigma$ by interchanging two nearest neighbors,

$$\mathrm{s}_k\,(\dots,\sigma_k,\sigma_{k+1},\dots)=(\dots,\sigma_{k+1},\sigma_k,\dots),$$
$$\mathrm{s}_k\,(\dots,\bar\sigma_k,\bar\sigma_{k+1},\dots)=(\dots,\bar\sigma_{k+1},\bar\sigma_k,\dots).$$

As will be proved below, the explicit formula for the operator S_k has the following form (see (5.34b)):

$$\begin{gathered}\mathrm{S}_k=[i\mathrm{D}_k]^{\sigma_{k,k+1}}\equiv(i\mathrm{D}_k)^{\sigma_{k,k+1}}\,\left(i\bar{\mathrm{D}}_k\right)^{\bar\sigma_{k,k+1}},\quad k=1,\dots,n-1,\\ \mathrm{D}_k=\frac{\partial}{\partial z_{k+1,k}}+\sum_{m=k+2}^{n}z_{m,k+1}\frac{\partial}{\partial z_{mk}},\quad \bar{\mathrm{D}}_k=\frac{\partial}{\partial\bar z_{k+1,k}}+\sum_{m=k+2}^{n}\bar z_{m,k+1}\frac{\partial}{\partial\bar z_{mk}}.\end{gathered}\tag{5.13}$$

The intertwining relation $\mathrm{S}_k\,\mathrm{T}^{\boldsymbol\sigma}=\mathrm{T}^{\boldsymbol\sigma_k}\mathrm{S}_k$ implies that the nontrivial kernels of the operators S_k are invariant subspaces under the action of the representation operators $\mathrm{T}^{\boldsymbol\sigma}(g)$. The operator S_k has a nontrivial kernel if the parameters $\sigma_{k,k+1}$ and $\bar\sigma_{k,k+1}$ are positive integers, because S_k is a differential operator in this case. The kernel of an individual operator S_k is infinite-dimensional. However, it can be proved [51]–[53] that, at a singular point where all the parameters $\sigma_{k,k+1}$ and $\bar\sigma_{k,k+1}$ are positive integers, the intersection of the kernels of all operators S_k is a finite-dimensional space of polynomials. Thus, the space of a finite-dimensional irreducible representation is the space of polynomials $\Phi(\mathrm{z},\bar{\mathrm{z}})$ satisfying the system of equations (see [51]–[53])

$$\mathrm{D}_k^{\sigma_{k,k+1}}\,\bar{\mathrm{D}}_k^{\bar\sigma_{k,k+1}}\Phi(\mathrm{z},\bar{\mathrm{z}})=0,\quad k=1,\dots,n-1.\tag{5.14}$$

The holomorphic and antiholomorphic sectors admit separation, and an irreducible finite-dimensional representation is the tensor product of a holomorphic and an antiholomorphic representation, $\pi^{\boldsymbol\sigma,\bar{\boldsymbol\sigma}}=\pi^{\boldsymbol\sigma}\otimes\pi^{\bar{\boldsymbol\sigma}}$. The dimension of a finite-dimensional holomorphic representation can be calculated by the Weil characters formula [53],

$$\dim\pi^{\boldsymbol\sigma}=\frac{\prod_{i<k}\sigma_{ik}}{\prod_{i<k}(k-i)}.$$

For example, for the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{C})$ we obtain

$$\dim\pi^{\sigma_{12}}=\sigma_{12},\quad \dim\pi^{\sigma_{12},\sigma_{23}}=\frac{\sigma_{12}\sigma_{23}(\sigma_{12}+\sigma_{23})}{2}.$$

In the general case of $\mathrm{SL}(n,\mathbb{C})$, the fundamental holomorphic representation of dimension n is determined by the following parameters: $\sigma_{12}=2$, $\sigma_{23}=\dots=\sigma_{n-1,n}=1$, and for the fundamental antiholomorphic representation, by the parameters $\bar\sigma_{12}=2$, $\bar\sigma_{23}=\dots=\bar\sigma_{n-1,n}=1$.

5.3. The generators of the Lie algebra $\mathrm{gl}(n,\mathbb{C})$ **and the generators of the right translations.** The Lie algebra $\mathrm{gl}(n,\mathbb{C})$ of the group $\mathrm{GL}(n,\mathbb{C})$ consists of matrices of order n and has n^2 independent generators; for the role of these it is convenient to take the matrices $\mathbf{e}_{ik}$,

$$(\mathbf{e}_{ik})_{nm}=\delta_{in}\delta_{km}.\tag{5.15}$$

The matrices $\mathbf{e}_{ik}$ satisfy the standard commutation relations

$$[\mathbf{e}_{ik},\mathbf{e}_{nm}]=\delta_{kn}\mathbf{e}_{im}-\delta_{im}\mathbf{e}_{nk}.\tag{5.16}$$

In the representation $\mathrm{T}^{\boldsymbol{\sigma}}$, the generators E_{ik} of the Lie algebra $\mathrm{gl}(n,\mathbb{C})$ are the first order differential operators with respect to z_{ik}, $1 \le k < i \le n$, defined as follows. We take an element $g \in \mathrm{GL}(n,\mathbb{C})$ of the specific form $g = \mathbb{1} + \epsilon \cdot \mathbf{e}_{ik}$. The action of the generator E_{ik} on a function in the representation $\mathrm{T}^{\boldsymbol{\sigma}}$ is defined by the first term in the power series expansion in ϵ:

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = \Phi(\mathrm{z}) + \big(\epsilon \cdot \mathrm{E}_{ik} + \bar{\epsilon} \cdot \bar{\mathrm{E}}_{ik}\big)\,\Phi(\mathrm{z}) + O(\epsilon^2), \quad g = \mathbb{1} + \epsilon \cdot \mathbf{e}_{ik}, \tag{5.17}$$

where the action of the operator $\mathrm{T}(g)$ is defined by formula (5.6). From the definition of the generators E_{ik}, we immediately see that they satisfy the same standard commutation relations as the matrices $\mathbf{e}_{ik}$,

$$[\mathrm{E}_{ik}, \mathrm{E}_{nm}] = \delta_{kn}\mathrm{E}_{im} - \delta_{im}\mathrm{E}_{nk}. \tag{5.18}$$

For the generators E_{ik} with $i > k$, the corresponding matrix g is lower triangular. Thus, in the expansion (5.5), h is the identity matrix. The operators E_{ik} with $i > k$ are generators of left translations defined by (5.6), in view of the fact that $\mathrm{h} = \mathbb{1}$:

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = \Phi\left(g^{-1}\mathrm{z}\right) \quad \text{for} \quad g \in \mathrm{Z}.$$

Similarly, we can define the operators of right translation:

$$\mathrm{D}(g)\,\Phi(\mathrm{z}) = \Phi\left(\mathrm{z}\,g\right) \quad \text{for} \quad g \in \mathrm{Z}.$$

We note that the matrix $\bar{g}$ in (5.5) is lower triangular ($\bar{g} \in \mathrm{Z}$). Therefore, the transformation $\mathrm{z} \to \mathrm{z}' = \mathrm{z}\,\bar{g}$ is a right translation. Thus, we obtain the following relation between the operator $\mathrm{T}(g)$ and the right translation operator $\mathrm{D}(\bar{g})$:

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = \alpha(\mathrm{h}) \cdot \Phi(\mathrm{z}\bar{g}) = \alpha(\mathrm{h}) \cdot \mathrm{D}(\bar{g})\,\Phi(\mathrm{z}). \tag{5.19}$$

The right translation operators D_{ik} are defined by analogy with (5.17),

$$\Phi\big(z(\mathbb{1} + \epsilon \cdot \mathbf{e}_{ik})\big) = \Phi(z) + \epsilon \cdot \mathrm{D}_{ik}\Phi(z) + O(\epsilon^2); \quad i > k. \tag{5.20}$$

From this definition, we obtain the following useful formulas:

$$\mathrm{D}_{ik} \,:\, \mathrm{z} \mapsto \mathrm{z}\,\mathbf{e}_{ik}, \tag{5.21a}$$

$$\mathrm{D}_{ik} \,:\, \mathrm{z}^{-1} \mapsto -\mathbf{e}_{ik}\,\mathrm{z}^{-1}. \tag{5.21b}$$

The operators of right and left translation commute, and, as a consequence, the corresponding generators also commute, $[\mathrm{E}_{ik}, \mathrm{D}_{nm}] = 0$ ($i > k$, $n > m$). Moreover, the generators D_{ik}, as well as the generators E_{ik}, satisfy the same commutation relations as the matrices $\mathbf{e}_{ik}$:

$$[\mathrm{D}_{ik}, \mathrm{D}_{nm}] = \delta_{kn}\,\mathrm{D}_{im} - \delta_{im}\,D_{nk}. \tag{5.22}$$

The following explicit expressions for the generators of right and left translations (we recall that $i > k$ and $z_{ii} = 1$ in the formulas below) can easily be deduced from the definitions:

$$\mathrm{E}_{ik} = -\sum_{m=1}^{k} z_{km}\,\frac{\partial}{\partial z_{im}}; \quad \mathrm{D}_{ik} = \sum_{m=i}^{n} z_{mi}\,\frac{\partial}{\partial z_{mk}}. \tag{5.23}$$

Observe that the operator D_{ik} depends on the entries of the ith and kth columns of the matrix z. For the generators E_{ik} with $i \le k$, the expressions similar to (5.23) are more cumbersome. However, there is a simple formula that makes it possible to express all generators E_{ik} in terms of the generators of right translations.

Proposition 7. *The generators* E_{ik} *can be expressed in terms of the right translation operators* D_{ik} *by the following formula*:

$$\mathrm{E}_{ik} = -\sum_{mn} z_{km} \left(\mathrm{D}_{nm} + \delta_{nm}\,\sigma_m\right) \left(\mathrm{z}^{-1}\right)_{ni}, \tag{5.24}$$

where the D_{nm} *are nonzero only if* $n > m$. *Using the matrices* $\mathrm{E}^{\boldsymbol{\sigma}} = \sum_{mn} \mathrm{E}_{nm}\mathbf{e}_{mn}$, $\mathrm{D} = \sum_{n>m} \mathrm{D}_{nm}\mathbf{e}_{mn}$, *and* $\sigma = \sum_n \sigma_n \mathbf{e}_{nn}$, *we can represent formula* (5.24) *in a convenient matrix form*

$$\mathrm{E}^{\boldsymbol{\sigma}} = -\mathrm{z}\,(\mathrm{D} + \sigma)\,\mathrm{z}^{-1}. \tag{5.25}$$

The matrices z *and* z^{-1} *are lower triangular and the matrix* D *is strictly upper triangular. The dependence on the numbers* σ_k *that determine the representation* $\mathrm{T}^{\boldsymbol{\sigma}}$ *occurs only for the diagonal matrix* σ,

$$\mathrm{D} + \sigma = \begin{pmatrix} \sigma_1 & \mathrm{D}_{21} & \mathrm{D}_{31} & \dots & \mathrm{D}_{n1} \\ 0 & \sigma_2 & \mathrm{D}_{32} & \dots & \mathrm{D}_{n2} \\ 0 & 0 & \sigma_3 & \dots & \mathrm{D}_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n \end{pmatrix}. \tag{5.26}$$

Formula (5.25) *can be applied in the case of the Lie algebra* $\mathrm{gl}(n,\mathbb{C})$ *as well as in the case of the Lie algebra* $\mathrm{sl}(n,\mathbb{C})$. *The case of the Lie algebra* $\mathrm{sl}(n,\mathbb{C})$ *differs only in that the parameters satisfy* $\sigma_1 + \sigma_2 + \dots + \sigma_n = \frac{n(n-1)}{2}$.

As an example, we present factorization formulas for $n = 2, 3$:

$$\begin{pmatrix} \mathrm{E}_{11} & \mathrm{E}_{21} \\ \mathrm{E}_{12} & \mathrm{E}_{22} \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & \partial_{z_{21}} \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_{21} & 1 \end{pmatrix}^{-1}, \tag{5.27}$$

$$\begin{aligned} \begin{pmatrix} \mathrm{E}_{11} & \mathrm{E}_{21} & \mathrm{E}_{31} \\ \mathrm{E}_{12} & \mathrm{E}_{22} & \mathrm{E}_{32} \\ \mathrm{E}_{13} & \mathrm{E}_{23} & \mathrm{E}_{33} \end{pmatrix} &= -\begin{pmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{pmatrix} \\ &\quad \times \begin{pmatrix} \sigma_1 & \partial_{z_{21}} + z_{32}\partial_{z_{31}} & \partial_{z_{31}} \\ 0 & \sigma_2 & \partial_{z_{32}} \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{pmatrix}^{-1}. \end{aligned} \tag{5.28}$$

To obtain a formula for the generator E_{ik}, we consider the matrix $g = \mathbb{1} + \epsilon\,\mathbf{e}_{ik}$ in the Gauss decomposition $g^{-1}\,\mathrm{z} = \mathrm{z}\,\bar{g}\,\mathrm{h}$ and find the contributions linear in ϵ to the expansions of the matrices $\bar{g}$ and h in ϵ-series. Performing the substitutions $g^{-1} \to \mathbb{1} - \epsilon\,\mathbf{e}_{ik}$, $\bar{g} \to \mathbb{1} + \epsilon\,\bar{g}_0$, and $\mathrm{h} \to \mathbb{1} + \epsilon\,\mathrm{h}_0$ in the equation $\mathrm{z}^{-1}\,g^{-1}\,\mathrm{z} = \bar{g}\,\mathrm{h}$, in the first order in ϵ we get $\bar{g}_0 + \mathrm{h}_0 = -\mathrm{z}^{-1}\,\mathbf{e}_{ik}\,\mathrm{z}$. Thus, $\bar{g}_0$ coincides with the strictly lower triangular part of the matrix $-\mathrm{z}^{-1} \cdot \mathbf{e}_{ik} \cdot \mathrm{z}$, and h_0 coincides with the upper triangular part of this matrix,

$$(\bar{g}_0)_{pm} = -z_{km} \cdot \left(\mathrm{z}^{-1}\right)_{pi}, \quad p > m; \quad (\mathrm{h}_0)_{pp} = -z_{kp} \cdot \left(\mathrm{z}^{-1}\right)_{pi}. \tag{5.29}$$

Using the relationship (5.19) between the operator $\mathrm{T}(g)$ and the right translation $\mathrm{D}(\bar{g})$, we obtain the general formula

$$\mathrm{E}_{ik} = \sum_{mp} (\bar{g}_0)_{pm}\,\mathrm{D}_{pm} + \sum_p (\mathrm{h}_0)_{pp}\,(\sigma_p + p - n).$$

Substitution of the expressions calculated above for $\bar{g}_0$ and h_0 yields

$$\mathrm{E}_{ik} = -\sum_{mp} z_{km} \left(\mathrm{z}^{-1}\right)_{pi} \mathrm{D}_{pm} - \sum_p z_{kp} \left(\mathrm{z}^{-1}\right)_{pi} (\sigma_p + p - n).$$

The second formula in (5.21) implies the relation

$$\sum_{p=m+1}^{n} \mathrm{D}_{pm} \left(\mathrm{z}^{-1}\right)_{pi} = -(n-m) \left(\mathrm{z}^{-1}\right)_{mi}$$

with the help of which we can move the matrix $\left(\mathrm{z}^{-1}\right)_{pi}$ to the right of the operator D_{pm}, obtaining

$$\mathrm{E}_{ik} = -\sum_{mp} z_{km} \mathrm{D}_{pm} \left(\mathrm{z}^{-1}\right)_{pi} - \sum_{p} z_{kp} \left(\mathrm{z}^{-1}\right)_{pi} \left(\sigma_p + p - n + (n-p)\right),$$

or, in matrix notation, $\mathrm{E}^{\boldsymbol{\sigma}} = -\mathrm{z}\,(\mathrm{D}+\sigma)\,\mathrm{z}^{-1}$.

The Lie algebra $\mathrm{sl}(n,\mathbb{C})$ of the group $\mathrm{SL}(n,\mathbb{C})$ consists of all matrices of order n with zero trace and contains n^2-1 independent generators. As its generators, it is convenient to take the matrices

$$\mathcal{E}_{ik} = \mathbf{e}_{ik} - \frac{\delta_{ik}}{n}\mathbb{1}, \quad (\mathcal{E}_{ik})_{nm} = \delta_{in}\delta_{km} - \frac{1}{n}\delta_{ik}\delta_{nm}. \tag{5.30}$$

These matrices satisfy the same commutation relations as the generators of the Lie algebra $\mathrm{gl}(n,\mathbb{C})$,

$$[\mathcal{E}_{ik}, \mathcal{E}_{nm}] = \delta_{kn}\mathcal{E}_{im} - \delta_{im}\mathcal{E}_{nk}. \tag{5.31}$$

In the representation $\mathrm{T}^{\boldsymbol{\sigma}}$, the generators E_{ik} of the Lie algebra $\mathrm{sl}(n,\mathbb{C})$ are given by the formula

$$\mathrm{T}(g)\,\Phi(\mathrm{z}) = \Phi(\mathrm{z}) + \left(\epsilon\cdot\,\mathrm{E}_{ik} + \bar{\epsilon}\cdot\,\bar{\mathrm{E}}_{ik}\right)\Phi(\mathrm{z}) + O(\epsilon^2), \quad g = \mathbb{1} + \epsilon\mathcal{E}_{ik}.$$

The generators E_{ik} satisfy the same commutation relations as the matrices $\mathcal{E}_{ik}$ and an additional condition that follows from the fact that $\mathcal{E}_{ik}$ has zero trace,

$$[\mathrm{E}_{ik}, \mathrm{E}_{nm}] = \delta_{kn}\mathrm{E}_{im} - \delta_{im}\mathrm{E}_{nk}; \quad \mathrm{E}_{11} + \mathrm{E}_{22} + \cdots + \mathrm{E}_{nn} = 0. \tag{5.32}$$

This additional condition is equivalent to the condition $\sigma_1 + \sigma_2 + \cdots + \sigma_n = \frac{n(n-1)}{2}$, and the entire deduction of the factorization formula remains valid.

5.4. Intertwining operators. The principal series representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\boldsymbol{\sigma}'}$ are equivalent if and only if there exists a permutation that takes σ_k to σ'_k (see [49, 55]): $\boldsymbol{\sigma}' = \mathrm{s}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\boldsymbol{\sigma}' = (\sigma'_1, \ldots, \sigma'_n)$. The same permutation takes $\bar{\sigma}_k$ to $\bar{\sigma}'_k$, $\bar{\boldsymbol{\sigma}}' = \mathrm{s}\,\bar{\boldsymbol{\sigma}}$. An arbitrary permutation from the group $\mathfrak{S}_n$ can be constructed by elementary transpositions s_k $(k = 1, \ldots, n-1)$ that interchange only two neighboring components of the vector $\boldsymbol{\sigma}$,

$$\begin{aligned} &\mathrm{s}_k\left(\ldots, \sigma_k, \sigma_{k+1}, \ldots\right) = \left(\ldots, \sigma_{k+1}, \sigma_k, \ldots\right); \\ &\mathrm{s}_k\left(\ldots, \bar{\sigma}_k, \bar{\sigma}_{k+1}, \ldots\right) = \left(\ldots, \bar{\sigma}_{k+1}, \bar{\sigma}_k, \ldots\right). \end{aligned} \tag{5.33}$$

In the same way, the operator intertwining the representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\boldsymbol{\sigma}'}$ is obtained from the operators intertwining the representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\boldsymbol{\sigma}_k}$, where $\boldsymbol{\sigma}_k = \mathrm{s}_k\boldsymbol{\sigma}$. Therefore, we must start with constructing these elementary intertwining operators. They are obtained from the generators of right translations $\mathrm{D}_{k+1,k}$ that are above the main diagonal in (5.26). The operators $\mathrm{D}_k \equiv \mathrm{D}_{k+1,k}$ were already used in the construction of the finite-dimensional representations (5.13).

Proposition 8. *The operator* S_k *intertwining the representations* $\mathrm{T}^{\boldsymbol{\sigma}}$ *and* $\mathrm{T}^{\boldsymbol{\sigma}_k}$, $\mathrm{S}_k\,\mathrm{T}^{\boldsymbol{\sigma}} = \mathrm{T}^{\boldsymbol{\sigma}_k}\mathrm{S}_k$, *where* $\boldsymbol{\sigma}_k = \mathrm{s}_k\boldsymbol{\sigma}$, *is defined by the following equivalent formulas*:

$$\mathrm{S}_k(\boldsymbol{\sigma})\Phi(\mathrm{z}) = A\left(\sigma_{k,k+1}\right)\int d^2w\,[w]^{-1-\sigma_{k,k+1}}\,\Phi\left(\mathrm{z}\left(\mathbb{1} - w\,\mathbf{e}_{k+1,k}\right)\right), \tag{5.34a}$$

$$\mathrm{S}_k(\boldsymbol{\sigma}) = [i\mathrm{D}_k]^{\sigma_{k,k+1}} \equiv (i\mathrm{D}_k)^{\sigma_{k,k+1}}\,\left(i\bar{\mathrm{D}}_k\right)^{\bar{\sigma}_{k,k+1}}, \quad k = 1, \ldots, n-1, \tag{5.34b}$$

where $\mathrm{D}_k \equiv \mathrm{D}_{k+1,k}$ *is the right translation operator. The operator* S_k *is unitary if* $\sigma^*_{k,k+1} + \bar{\sigma}_{k,k+1} = 0$*, which is fulfilled automatically for the principal series representations.*

To prove this statement, first we verify that the operator S_k as in (5.34b) satisfies the required commutation relations with the generators of the representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\boldsymbol{\sigma}_k}$. It is convenient to use the matrix representation (5.25), so that it only remains to check the relation $\mathrm{S}_k\mathrm{E}^{\boldsymbol{\sigma}} = \mathrm{E}^{\boldsymbol{\sigma}_k}\mathrm{S}_k$. For this, we need to know the permutation relations between the operator S_k and the matrices z, D, and z^{-1} into which the matrix $\mathrm{E}^{\boldsymbol{\sigma}}$ factorizes. The permutation relations between S_k and the matrices z and z^{-1} follow from formula (5.21) ($\sigma_{k,k+1} \to \alpha$),

$$\mathrm{S}_k\,\mathrm{z} = \mathrm{z}\,\left(1 + \alpha\,\mathrm{D}_k^{-1}\,\mathbf{e}_{k+1,k}\right)\,\mathrm{S}_k; \quad \mathrm{S}_k\,\mathrm{z}^{-1} = \left(1 - \alpha\,\mathrm{D}_k^{-1}\mathbf{e}_{k+1,k}\right)\,\mathrm{z}^{-1}\,\mathrm{S}_k, \tag{5.35}$$

and the permutation relation with the matrix D follows from the commutation relations for the generators of right translations (5.22),

$$\mathrm{S}_k\,\mathrm{D} = \Big(\mathrm{D} + \alpha\,\mathrm{D}_k^{-1}\,\Big(\sum_{n<k}\mathrm{D}_{k+1,n}\,\mathbf{e}_{nk} - \sum_{n>k+1}\mathbf{e}_{k+1,n}\,\mathrm{D}_{n,k}\Big)\Big)\,\mathrm{S}_k. \tag{5.36}$$

We note that the operator D_k^{-1} commutes with the matrix $\mathbf{e}_{k+1,k}\mathrm{z}^{-1}$ and with the operators in the sum in (5.36). Using relations (5.35) and (5.36), we move the operator S_k in the initial product $\mathrm{S}_k\mathrm{E}^{\boldsymbol{\sigma}}$ to the right and then, multiplying the matrices located between z and z^{-1}, we obtain

$$\begin{aligned}\mathrm{S}_k\,\mathrm{E}^{\boldsymbol{\sigma}} &= -\mathrm{z}\Big(\mathrm{D} + \sigma + \alpha(\mathbf{e}_{k+1,k+1} - \mathbf{e}_{k,k}) - \alpha(\alpha - \sigma_k + \sigma_{k+1})\mathbf{e}_{k+1,k}\mathrm{D}_k^{-1}\Big)\,\mathrm{z}^{-1}\mathrm{S}_k \\ &= -\mathrm{z}\,(\mathrm{D} + \sigma')\,\mathrm{z}^{-1}\,\mathrm{S}_k = \mathrm{E}^{\boldsymbol{\sigma}_k}\,\mathrm{S}_k.\end{aligned} \tag{5.37}$$

The matrix $\sigma' = \sigma + \sigma_{k,k+1}\,(\mathbf{e}_{k+1,k+1} - \mathbf{e}_{k,k})$ differs from σ only in the permutation $\sigma_k \leftrightarrow \sigma_{k+1}$, and the nonhomogeneous contribution vanishes if $\alpha = \sigma_k - \sigma_{k+1}$.

Formula (5.34b) is a convenient concise representation for the intertwining operator $\mathrm{S}_k(\boldsymbol{\sigma})$. In the case of the group $\mathrm{SL}(2,\mathbb{C})$, the operator $[i\partial_z]^\lambda$ is defined as the integral operator

$$[i\partial_z]^\lambda\Phi(z,\bar{z}) = A(\lambda)\int d^2w\,[z-w]^{-1-\lambda}\,\Phi(w,\bar{w}).$$

A similar formula for the intertwining operator S_k is a direct consequence of this representation. From the definition of the right translation operator D_k, we see that the following formula for the corresponding global transformation is true:

$$\exp\left(x\,\mathrm{D}_k + \bar{x}\,\bar{\mathrm{D}}_k\right)\,\Phi(\mathrm{z}) = \Phi\left(\mathrm{z}\left(\mathbb{1} + x\,\mathbf{e}_{k+1,k}\right)\right),$$

which allows us to obtain the required integral representation (for $\lambda \to \sigma_{k,k+1}$):

$$\begin{aligned}[i\mathrm{D}_k]^\lambda\,\Phi(\mathrm{z}) &= [i\partial_x]^\lambda\exp\big(x\,\mathrm{D}_k + \bar{x}\,\bar{\mathrm{D}}_k\big)\,\Phi(\mathrm{z})\Big|_{x=0} = [i\partial_x]^\lambda\Phi\left(\mathrm{z}\left(\mathbb{1} + x\,\mathbf{e}_{k+1,k}\right)\right)\Big|_{x=0} \\ &= A(\lambda)\int d^2w\,[x-w]^{-1-\lambda}\,\Phi\left(\mathrm{z}\left(\mathbb{1} + w\,\mathbf{e}_{k+1,k}\right)\right)\Big|_{x=0} \\ &= A(\lambda)\int d^2w\,[w]^{-1-\lambda}\,\Phi\left(\mathrm{z}\left(\mathbb{1} - w\,\mathbf{e}_{k+1,k}\right)\right).\end{aligned}$$

We presented an independent proof of the formula for the intertwining operator in the second picture of the induced representation. We prove that the resulting formula coincides with the canonical representation [55] for the intertwining operator. In the

first picture of the induced representation, the formula for the operator intertwining the representations $\mathrm{T}^{\boldsymbol{\sigma}}$ and $\mathrm{T}^{\mathrm{s}\boldsymbol{\sigma}}$: $\mathrm{S}\,\mathrm{T}^{\boldsymbol{\sigma}} = \mathrm{T}^{\mathrm{s}\boldsymbol{\sigma}}\mathrm{S}$ has the form (see [55])

$$\mathrm{S}(\boldsymbol{\sigma})\,\Phi(\mathrm{x}) = A\,(\boldsymbol{\sigma}) \int_{\mathrm{Z_w}} \mathrm{dy}\,\Phi(\mathrm{x\,w\,y}); \quad \mathrm{y} \in \mathrm{Z_w} = \mathrm{Z} \cap \mathrm{w}^{-1}\mathrm{Hw}, \quad \mathrm{x} \in \mathrm{SL}(n,\mathbb{C}).$$

In this formula, $A(\boldsymbol{\sigma})$ is a normalizing factor, w is the Weyl group element that corresponds to a permutation $\mathrm{s} \in \mathfrak{S}_n$, and integration is performed over the lower triangular matrices belonging to the intersection $\mathrm{Z} \cap \mathrm{w}^{-1}\mathrm{Hw}$. For the elementary intertwining operators $\mathrm{S}_k(\boldsymbol{\sigma})$, the general formula yields

$$\mathrm{S}_k(\boldsymbol{\sigma})\,\Phi(\mathrm{x}) = A\,(\sigma_{k,k+1}) \int_{\mathrm{Z_{w_k}}} \mathrm{dy}\,\Phi(\mathrm{x\,w}_k\,\mathrm{y}); \quad \mathrm{y} \in \mathrm{Z}, \quad \mathrm{x} \in \mathrm{SL}(n,\mathbb{C}),$$

where the Weyl group element corresponding to the permutation s_k is an obvious generalization of the corresponding matrix

$$\mathrm{w}_k = \mathbb{1} - \mathbf{e}_{k,k} - \mathbf{e}_{k+1,k+1} + \mathbf{e}_{k,k+1} - \mathbf{e}_{k+1,k}$$

of the corresponding matrix for the group $\mathrm{SL}(2,\mathbb{C})$, and is obtained from the identity matrix by the change $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \to \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ of the kth diagonal block. To verify that this formula is equivalent to (5.34a), we write it in a more explicit form. The set $\mathrm{Z} \cap \mathrm{w}_k^{-1}\mathrm{Hw}_k$ consists of matrices of the form $\mathbb{1} + y_{k+1,k}\mathbf{e}_{k+1,k}$. Thus, integration is performed with respect to $d^2y_{k+1,k}$ over the entire complex plane. The following transformation of the kth diagonal block demonstrates clearly how an upper triangular factor is separated from the function argument,

$$\mathrm{w}_k\,\mathrm{y} \to \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\y_{k+1,k}&1\end{pmatrix} = \begin{pmatrix}1&0\\-y_{k+1,k}^{-1}&1\end{pmatrix}\begin{pmatrix}y_{k+1,k}&1\\0&y_{k+1,k}^{-1}\end{pmatrix},$$
$$\mathrm{z\,w}_k\,\mathrm{y} \to \mathrm{z}\left(\mathbb{1} - y_{k+1,k}^{-1}\mathbf{e}_{k+1,k}\right),$$

which gives the factor $[h_{kk}]^{\sigma_k+k}[h_{k+1,k+1}]^{\sigma_{k+1}+k+1} = [y_{k+1,k}]^{-1-\sigma_{k,k+1}}$, and we obtain

$$\int d^2y_{k+1,k}\,\Phi\,(\mathrm{z\,w}_k\,\mathrm{y}) = \int d^2y_{k+1,k}\,\Phi\left(\mathrm{z}\left(\mathbb{1} - y_{k+1,k}^{-1}\mathbf{e}_{k+1,k}\right)\right)[y_{k+1,k}]^{-1+\sigma_{k,k+1}}.$$

After the obvious change of variables $y_{k+1,k}^{-1} = w$, we arrive at formula (5.34a).

We have constructed elementary intertwining operators, and now it remains to prove that they satisfy the required relations. The complete set of relations for the generators of the permutation group $\mathfrak{S}_n$ has the form (see [54])

$$\mathrm{s}_k\mathrm{s}_k = \mathbb{1}, \quad \mathrm{s}_i\,\mathrm{s}_k = \mathrm{s}_k\,\mathrm{s}_i \qquad \text{for} \quad |i-k| > 1, \tag{5.38a}$$

$$\mathrm{s}_k\,\mathrm{s}_{k+1}\,\mathrm{s}_k = \mathrm{s}_{k+1}\,\mathrm{s}_k\,\mathrm{s}_{k+1}. \tag{5.38b}$$

As in the case of the group $\mathrm{SL}(2,\mathbb{C})$, the complete set of relations for intertwining operators copies the relations for the group $\mathfrak{S}_n$.

Proposition 9. *The intertwining operators* $\mathrm{S}_k(\boldsymbol{\sigma})$ *satisfy the following relations*:

$$\mathrm{S}_k(s_k\boldsymbol{\sigma})\mathrm{S}_k(\boldsymbol{\sigma}) = \mathbb{1}, \tag{5.39a}$$

$$\mathrm{S}_i(s_k\boldsymbol{\sigma})\,\mathrm{S}_k(\boldsymbol{\sigma}) = \mathrm{S}_k(s_i\boldsymbol{\sigma})\,\mathrm{S}_i(\boldsymbol{\sigma}) \ \textit{for}\ |i-k| > 1, \tag{5.39b}$$

$$\mathrm{S}_k(s_{k+1}s_k\boldsymbol{\sigma})\,\mathrm{S}_{k+1}(s_k\boldsymbol{\sigma})\,\mathrm{S}_k(\boldsymbol{\sigma}) = \mathrm{S}_{k+1}(s_ks_{k+1}\boldsymbol{\sigma})\,\mathrm{S}_k(s_{k+1}\boldsymbol{\sigma})\,\mathrm{S}_{k+1}(\boldsymbol{\sigma}). \tag{5.39c}$$

The detailed form of the first two lines in (5.39) is as follows:

$$[i\mathrm{D}_k]^{\sigma_{k+1}-\sigma_k}\,[i\mathrm{D}_k]^{\sigma_k-\sigma_{k+1}} = \mathbb{1}, \tag{5.40a}$$

$$[i\mathrm{D}_i]^{\sigma_i-\sigma_{i+1}}\,[i\mathrm{D}_k]^{\sigma_k-\sigma_{k+1}} = [i\mathrm{D}_k]^{\sigma_k-\sigma_{k+1}}\,[i\mathrm{D}_i]^{\sigma_i-\sigma_{i+1}}, \quad |i-k| > 1. \tag{5.40b}$$

The first relation is obvious. The second relation follows from the fact that the operators D_i and D_k commute if $|i-k|>1$. We consider the last relation in (5.39) in more detail,

$$[i\mathrm{D}_k]^a\,[i\mathrm{D}_{k+1}]^{a+b}\,[i\mathrm{D}_k]^b = [i\mathrm{D}_{k+1}]^b\,[i\mathrm{D}_k]^{a+b}\,[i\mathrm{D}_{k+1}]^a\,,$$
$$a=\sigma_{k+1,k+2},\quad b=\sigma_{k,k+1},$$

and prove that this is equivalent to the star-triangle relation. The operator $\mathrm{D}_k=\mathrm{D}_{k+1,k}$ depends only on the variables of the kth and $(k+1)$th columns of the matrix z. Thus, after the change of variables

$$z_{k+1,k}=x,\;\; z_{m,k+1}=x_{m,k+1},\;\; z_{m,k}=x_{m,k}+x\cdot x_{m,k+1},\;\; k<m\le n, \tag{5.41}$$

the operator D_k transforms into the derivative with respect to x, $\mathrm{D}_k=\partial_x$, and the operator D_{k+1} becomes a linear function of x,

$$\mathrm{D}_{k+2,k+1}=A\cdot x+B=A\cdot(x-x_0),\quad x_0=-A^{-1}B,$$

where the operator coefficients A and B are independent of x and ∂_x. In the new variables, the identity in question takes a familiar form of the star-triangle relation:

$$[i\partial]^a\,[x-x_0]^{a+b}\,[i\partial]^b\,=[x-x_0]^b\cdot\,[i\partial]^{a+b}\,[x-x_0]^a.$$

As in the case of the group $\mathrm{SL}(2,\mathbb{C})$, the intertwining operators S_k are elementary building blocks in the construction of the R-matrix.

§6. $\mathrm{SL}(n,\mathbb{C})$-invariant R-matrix

We construct $\mathrm{SL}(n,\mathbb{C})$-invariant solutions of the Yang–Baxter equation:

$$\mathrm{R}_{12}(u-v,\bar u-\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{23}(v,\bar v)=\mathrm{R}_{23}(v,\bar v)\mathrm{R}_{13}(u,\bar u)\mathrm{R}_{12}(u-v,\bar u-\bar v).$$

The fact that the operator $\mathrm{R}(u,\bar u)$ is $\mathrm{SL}(n,\mathbb{C})$-invariant means that it acts in the space of the tensor product of two representations of the group $\mathrm{SL}(n,\mathbb{C})$ and commutes with the representation operators,

$$\mathrm{T}(g)\otimes\mathrm{T}(g)\cdot\mathrm{R}(u,\bar u)=\mathrm{R}(u,\bar u)\cdot\mathrm{T}(g)\otimes\mathrm{T}(g).$$

As in the case of the group $\mathrm{SL}(2,\mathbb{C})$, we solve the problem in two steps. First, we obtain the RLL-equations and their solution. Second, we prove that the R-operator obtained satisfies the Yang–Baxter equation. If we take the n-dimensional space of the fundamental representation as one of the spaces in the tensor product on which the operator R acts, then we can realize the operator $\mathrm{R}(u)$ as an nth order matrix whose entries are operators acting in the second space. This matrix coincides with the L-operator [56]

$$\mathrm{L}(u)=u+\sum_{ik}\mathrm{E}_{ik}\mathbf{e}_{ki}=\begin{pmatrix} u+\mathrm{E}_{11} & \mathrm{E}_{21} & \cdots & \mathrm{E}_{n1}\\ \mathrm{E}_{12} & u+\mathrm{E}_{22} & \cdots & \mathrm{E}_{n2}\\ \cdots & \cdots & \cdots & \cdots\\ \mathrm{E}_{1n} & \mathrm{E}_{2n} & \cdots & u+\mathrm{E}_{nn}\end{pmatrix}, \tag{6.1}$$

where the E_{ik} are the generators of $\mathrm{gl}(n,\mathbb{C})$ that realize the representation in the second space.

In the Yang–Baxter equation, as the space $\mathbb{V}_3$ we choose the space of the n-dimensional fundamental representations. The operator $\mathrm{R}_{13}(u)$ becomes the matrix $\mathrm{L}_1(u)$, the entries of which are the generators of the representation in the space $\mathbb{V}_1$. In the same way, the operator $\mathrm{R}_{23}(v)$ becomes the matrix $\mathrm{L}_2(v)$, the entries of which are the generators of the representation in the space $\mathbb{V}_2$. As a result, we obtain the following defining equation for the operator R (see [56]):

$$\mathrm{R}_{12}(u-v)\mathrm{L}_1(u)\mathrm{L}_2(v)=\mathrm{L}_2(v)\mathrm{L}_1(u)\mathrm{R}_{12}(u-v).$$

For the role of $\mathbb{V}_1$, we take the space of functions of the variables of the matrix z, where the representation $\mathrm{T}^{\boldsymbol{\sigma}}$ acts, and as $\mathbb{V}_2$ we take the space of functions of the variables of the matrix w, where the representation $\mathrm{T}^{\boldsymbol{\rho}}$ acts. In this case, we use formula (5.25) to represent the matrices $\mathrm{L}_1(u)$ and $\mathrm{L}_2(v)$ in the following convenient form:

$$\mathrm{L}_1(u) = u + \mathrm{E}^{\boldsymbol{\sigma}} = \mathrm{z}\left(u - \sigma - \mathrm{D}_z\right)\mathrm{z}^{-1}, \quad \mathrm{L}_2(v) = v + \mathrm{E}^{\boldsymbol{\rho}} = \mathrm{w}\left(v - \rho - \mathrm{D}_w\right)\mathrm{w}^{-1}. \tag{6.2}$$

The matrix $\mathrm{L}_1(u)$ depends on the spectral parameter u and the n parameters σ_k characterizing the representation $\mathrm{T}^{\boldsymbol{\sigma}}$. Formula (6.2) shows that all the parameters can naturally be combined into the set of numbers $u_k = u - \sigma_k$, $k = 1, \dots, n$, which are the diagonal entries of the middle matrix, so that $\mathrm{L}_1(u) = \mathrm{L}_1(u_1, \dots, u_n)$. Similarly, $\mathrm{L}_2(v) = \mathrm{L}_2(v_1, \dots, v_n)$, where $v_k = v - \rho_k$.

From the R-matrix, we split off the permutation operator $\mathrm{R} = \mathrm{P}\,\check{\mathrm{R}}$, where the operator P interchanges the arguments of the function, $\mathrm{P}\,\Phi(\mathrm{z}, \mathrm{w}) = \Phi(\mathrm{w}, \mathrm{z})$. Now we present a complete set of defining relations for the operator $\check{\mathrm{R}}(u - v, \bar{u} - \bar{v})$, explicitly showing the antiholomorphic parts of the equations,

$$\begin{aligned} &\check{\mathrm{R}}(u - v, \bar{u} - \bar{v})\,\mathrm{L}_1(u_1, \dots, u_n)\,\mathrm{L}_2(v_1, \dots, v_n) \\ &\qquad = \mathrm{L}_1(v_1, \dots, v_n)\,\mathrm{L}_2(u_1, \dots, u_n)\,\check{\mathrm{R}}(u - v, \bar{u} - \bar{v}), \end{aligned} \tag{6.3a}$$

$$\begin{aligned} &\check{\mathrm{R}}(u - v, \bar{u} - \bar{v})\,\bar{\mathrm{L}}_1(\bar{u}_1, \dots, \bar{u}_n)\,\bar{\mathrm{L}}_2(\bar{v}_1, \dots, \bar{v}_n) \\ &\qquad = \bar{\mathrm{L}}_1(\bar{v}_1, \dots, \bar{v}_n)\,\bar{\mathrm{L}}_2(\bar{u}_1, \dots, \bar{u}_n)\,\check{\mathrm{R}}(u - v, \bar{u} - \bar{v}), \end{aligned} \tag{6.3b}$$

where the matrix $\bar{\mathrm{L}}(\bar{u})$ is defined in the following obvious way:

$$\bar{\mathrm{L}}(\bar{u}) = \bar{u} + \sum_{ik} \bar{\mathrm{E}}_{ik}\mathbf{e}_{ki}.$$

6.1. The operators $\mathbb{S}_k$. Equation (6.3) has the following natural interpretation: the operator $\check{\mathrm{R}}$ interchanges the set of parameters $(u_1, \dots, u_n)$ in the first L-operator with the set of parameters $(v_1, \dots, v_n)$ in the second L-operator. It is convenient to combine all parameters to one big set in the following order: $\mathbf{u} = (v_1, \dots, v_n, u_1, \dots, u_n)$. Thus, the operator $\check{\mathrm{R}}$ realizes the following specific permutation s in the permutation group of $2n$ parameters $(v_1, \dots, v_n, u_1, \dots, u_n)$:

$$(v_1, \dots, v_n, u_1, \dots, u_n) \xrightarrow{s} (u_1, \dots, u_n, v_1, \dots, v_n).$$

An arbitrary permutation in the group $\mathfrak{S}_{2n}$ can be constructed by elementary transpositions s_k $(k = 1, \dots, 2n - 1)$ that interchange only two nearest neighbors in the set $(v_1, \dots, v_n, u_1, \dots, u_n)$. Therefore, we shall seek the operators that realize these elementary permutations. The intertwining operators S_k are the operators of elementary permutations of parameters inside one L-operator, because the matrix relation $\mathrm{S}_k\,\mathrm{E}^{\boldsymbol{\sigma}} = \mathrm{E}^{\boldsymbol{\sigma}_k}\,\mathrm{S}_k$ yields exactly the required permutation in the L-operator,

$$\mathrm{S}_k\,\mathrm{L}(u_1, \dots, u_k, u_{k+1}, \dots, u_n) = \mathrm{L}(u_1, \dots, u_{k+1}, u_k, \dots, u_n)\,\mathrm{S}_k.$$

Thus, among all $2n - 1$ elementary permutations acting on $(v_1, \dots, v_n, u_1, \dots, u_n)$, we already have $2n - 2$ permutations acting separately on $v_1, \dots, v_n$ and on $u_1, \dots, u_n$:

$$(\overbrace{v_1, \dots, v_n}^{\mathbb{S}_1, \dots, \mathbb{S}_{n-1}}, \overbrace{u_1, \dots, u_n}^{\mathbb{S}_{n+1}, \dots, \mathbb{S}_{2n-1}}), \quad \mathbb{S}_k = \begin{cases} \mathbb{1} \otimes \mathrm{S}_k, & k = 1, \dots, n-1, \\ \mathrm{S}_{k-n} \otimes \mathbb{1}, & k = n+1, \dots, 2n-1, \end{cases} \tag{6.4}$$

$$\mathbb{S}_k \mathrm{L}_2(v_1, \dots, v_k, v_{k+1}, \dots, v_n) = \mathrm{L}_2(v_1, \dots, v_k, v_{k+1}, \dots, v_n)\,\mathbb{S}_k, \quad k = 1, \dots, n-1,$$

$$\mathbb{S}_k \mathrm{L}_1(u_1, \dots, u_k, u_{k+1}, \dots, u_n) = \mathrm{L}_1(u_1, \dots, u_{k+1}, u_k, \dots, u_n)\,\mathbb{S}_k, \quad k = n+1, \dots, 2n-1.$$

It only remains to find one operator $\mathbb{S}_n$ interchanging v_n and u_1,

$$(v_1, \dots, \overbrace{v_n, u_1}^{\mathbb{S}_n}, \dots, u_n).$$

The defining equation for the last missing operator $\mathbb{S}_n$ has the following form:

$$\begin{aligned} &\mathbb{S}_n\, \mathrm{L}_1(u_1, u_2, \dots, u_n)\, \mathrm{L}_2(v_1, \dots, v_{n-1}, v_n) \\ &\qquad = \mathrm{L}_1(v_n, u_2, \dots, u_n)\, \mathrm{L}_2(v_1, \dots, v_{n-1}, u_1)\, \mathbb{S}_n. \end{aligned} \tag{6.5}$$

This equation has an unexpectedly simple solution.

Proposition 10. *$\mathbb{S}_n$ is the operator of multiplication by a function,*

$$\mathbb{S}_n \Phi(\mathrm{z}, \mathrm{w}) = \mathrm{S}(\mathrm{z}, \mathrm{w}) \cdot \Phi(\mathrm{z}, \mathrm{w}). \tag{6.6}$$

The function $\mathrm{S}(\mathrm{z}, \mathrm{w})$ is constructed from the $(n, 1)$th entry of the matrix product $\mathrm{w}^{-1}\mathrm{z}$:

$$\mathrm{S}(\mathrm{z}, \mathrm{w}) = \left[\mathrm{w}^{-1}\mathrm{z}\right]_{n1}^{u_1 - v_n} \equiv \left(\mathrm{w}^{-1}\mathrm{z}\right)_{n1}^{u_1 - v_n} \left(\bar{\mathrm{w}}^{-1}\bar{\mathrm{z}}\right)_{n1}^{\bar{u}_1 - \bar{v}_n}. \tag{6.7}$$

We verify that the operator $\mathbb{S}_n$ in (6.6) is indeed a solution of equation (6.5). For this, we calculate $\mathrm{S}\mathrm{L}_1(u_1, \dots, u_n)\mathrm{L}_2(v_1, \dots, v_n)\mathrm{S}^{-1}$, choosing the function $\mathrm{S}(\mathrm{z}, \mathrm{w})$ in the form ($\lambda = u_1 - v_n$, $\bar{\lambda} = \bar{u}_1 - \bar{v}_n$)

$$\mathrm{S}(\mathrm{z}, \mathrm{w}) = \left(\mathrm{w}^{-1}\mathrm{z}\right)_{n1}^{\lambda} \left(\bar{\mathrm{w}}^{-1}\bar{\mathrm{z}}\right)_{n1}^{\bar{\lambda}}. \tag{6.8}$$

Using the formulas

$$\mathrm{D}_{ki}^{z} \,:\, \mathrm{w}^{-1}\mathrm{z} \mapsto \mathrm{w}^{-1}\mathrm{z} \cdot \mathbf{e}_{ki}, \tag{6.9a}$$

$$\mathrm{D}_{ki}^{w} \,:\, \mathrm{w}^{-1}\mathrm{z} \mapsto -\mathbf{e}_{ki} \cdot \mathrm{w}^{-1}\mathrm{z}, \tag{6.9b}$$

which follow from (5.21), it is easy to obtain the following relations:

(6.10a)

$$\sum_{k>i} \mathbf{e}_{ik} \mathrm{D}_{ki}^{z} \,:\, \left(\mathrm{w}^{-1}\mathrm{z}\right)_{n1} \mapsto \sum_{k>1} \mathbf{e}_{1k} \left(\mathrm{w}^{-1}\mathrm{z}\right)_{nk} = -\left(\mathrm{w}^{-1}\mathrm{z}\right)_{n1} \mathbf{e}_{11} + \sum_{k\geq 1} \mathbf{e}_{1k} \left(\mathrm{w}^{-1}\mathrm{z}\right)_{nk},$$

(6.10b)

$$\sum_{k>i} \mathbf{e}_{ik} \mathrm{D}_{ki}^{w} \,:\, \left(\mathrm{w}^{-1}\mathrm{z}\right)_{n1} \mapsto -\sum_{k<n} \mathbf{e}_{kn} \left(\mathrm{w}^{-1}\mathrm{z}\right)_{k1} = \left(\mathrm{w}^{-1}\mathrm{z}\right)_{n1} \mathbf{e}_{nn} - \sum_{k\leq n} \mathbf{e}_{kn} \left(\mathrm{w}^{-1}\mathrm{z}\right)_{k1}.$$

Thus, we immediately arrive at the following transformation formulas for the matrices L_1 and L_2:

$$\mathrm{S}(\mathrm{z}, \mathrm{w})\, (\mathrm{L}_1)_{ik}\, \mathrm{S}^{-1}(\mathrm{z}, \mathrm{w}) = \Big(\mathrm{z}\,(\mathrm{u} - \lambda\, \mathbf{e}_{11} - \mathrm{D})\mathrm{z}^{-1}\Big)_{ik} + \frac{\lambda}{(\mathrm{w}^{-1}\mathrm{z})_{n1}}\, z_{i1} \left(\mathrm{w}^{-1}\right)_{nk}, \tag{6.11a}$$

$$\mathrm{S}(\mathrm{z}, \mathrm{w})\, (\mathrm{L}_2)_{ik}\, \mathrm{S}^{-1}(\mathrm{z}, \mathrm{w}) = \Big(\mathrm{w}\,(\mathrm{v} + \lambda\, \mathbf{e}_{nn} - \mathrm{D})\mathrm{w}^{-1}\Big)_{ik} - \frac{\lambda}{(\mathrm{w}^{-1}\mathrm{z})_{n1}}\, z_{i1} \left(\mathrm{w}^{-1}\right)_{nk}. \tag{6.11b}$$

The matrices $\mathrm{u} = \sum_k u_k \mathbf{e}_{kk}$ and $\mathrm{v} = \sum_k v_k \mathbf{e}_{kk}$ are diagonal. If $\lambda = u_1 - v_n$, then the first terms in the transformed L-operators (6.11) change to $\mathrm{L}_1(v_n, u_2, \dots, u_n)$ and $\mathrm{L}_2(v_1, \dots, v_{n-1}, u_1)$, respectively. It remains to multiply the operators (6.11) and represent the resulting expression in the form $\mathrm{L}_1(v_n, u_2, \dots, u_n)\mathrm{L}_2(v_1, \dots, v_{n-1}, u_1)$ with the help of the formulas

$$\sum_i \left(\mathrm{w}^{-1}\right)_{ni} \mathrm{L}_2(v_1, \dots, v_{n-1}, u_1)_{ik} = u_1 \cdot \left(\mathrm{w}^{-1}\right)_{nk},$$

$$\sum_k \mathrm{L}_1(v_n, u_2, \dots, u_n)_{ik}\, z_{k1} = v_n \cdot z_{i1},$$

which can easily be obtained from (6.2) by recalling that the matrix D (see (5.26)) is upper triangular.

As an example, we present the explicit form of the operators $\mathbb{S}_k$ in the case of the groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(3,\mathbb{C})$. For $\mathrm{SL}(2,\mathbb{C})$, there are only three operators, and they are fairly simple:

$$\mathbb{S}_1 = [i\partial_{w_{21}}]^{v_2-v_1}; \quad \mathbb{S}_2 = [z_{21}-w_{21}]^{u_1-v_2}; \quad \mathbb{S}_3 = [i\partial_{z_{21}}]^{u_2-u_1}. \tag{6.12}$$

For the group $\mathrm{SL}(3,\mathbb{C})$, we have five operators, and they are more complicated:

$$\begin{aligned} &\mathbb{S}_1 = [i\,(\partial_{w_{21}} + w_{32}\partial_{w_{31}})]^{v_2-v_1}\,; \quad \mathbb{S}_2 = [i\partial_{w_{32}}]^{v_3-v_2}; \\ &\mathbb{S}_3 = [z_{31} - w_{31} - w_{32}\,(z_{21}-w_{21})]^{u_1-v_3}\,; \\ &\mathbb{S}_4 = [i\,(\partial_{z_{21}} + z_{32}\partial_{z_{31}})]^{u_2-u_1}\,; \quad \mathbb{S}_5 = [i\partial_{z_{32}}]^{u_3-u_2}. \end{aligned} \tag{6.13}$$

If the rank of the algebra grows, the formulas become more and more cumbersome.

We constructed the complete set $\mathbb{S}_k(\mathbf{u})$ of generators of the permutation group $\mathfrak{S}_{2n}$ of $2n$ parameters $(v_1,\dots,v_n,u_1,\dots,u_n)$. Now we check that they satisfy the required relations.

Proposition 11. *We have*

$$\mathbb{S}_k(s_k\mathbf{u})\mathbb{S}_k(\mathbf{u}) = \mathbb{1}, \tag{6.14a}$$

$$\mathbb{S}_i(s_k\mathbf{u})\mathbb{S}_k(\mathbf{u}) = \mathbb{S}_k(s_i\mathbf{u})\mathbb{S}_i(\mathbf{u}), \quad |i-k|>1, \tag{6.14b}$$

$$\mathbb{S}_k(s_{k+1}s_k\mathbf{u})\mathbb{S}_{k+1}(s_k\mathbf{u})\mathbb{S}_k(\mathbf{u}) = \mathbb{S}_{k+1}(s_ks_{k+1}\mathbf{u})\mathbb{S}_k(s_{k+1}\mathbf{u})\mathbb{S}_{k+1}(\mathbf{u}). \tag{6.14c}$$

Proof. Only the relations involving the new generator $\mathbb{S}_n$ need a proof, because all others are relations for intertwining operators, which we proved earlier. The first relation $\mathbb{S}_n\mathbb{S}_n = \mathbb{1}$ reduces to the identity $(\mathrm{w}^{-1}\mathrm{z})_{n1}^{v_n-u_1}\cdot(\mathrm{w}^{-1}\mathrm{z})_{n1}^{u_1-v_n} = 1$.

The second relation $\mathbb{S}_n\mathbb{S}_k = \mathbb{S}_k\mathbb{S}_n$, $|n-k|>1$, reduces to one of the following identities, depending on the value of k:

$$(\mathrm{w}^{-1}\mathrm{z})_{n1}^{u_1-v_n}\cdot(\mathrm{D}_k^w)^{v_{k+1}-v_k} = (\mathrm{D}_k^w)^{v_{k+1}-v_k}\cdot(\mathrm{w}^{-1}\mathrm{z})_{n1}^{u_1-v_n} \quad \text{for} \quad k=1,\dots,n-1,$$

$$(\mathrm{w}^{-1}\mathrm{z})_{n1}^{u_1-v_n}\cdot(\mathrm{D}_k^z)^{u_{k+1}-u_k} = (\mathrm{D}_k^z)^{u_{k+1}-u_k}\cdot(\mathrm{w}^{-1}\mathrm{z})_{n1}^{u_1-v_n} \quad \text{for} \quad k=n+1,\dots,2n-1,$$

which are implied by the following easily proved formulas:

$$\mathrm{D}_k^z : (\mathrm{w}^{-1}\mathrm{z})_{n1} \mapsto 0; \quad \mathrm{D}_i^w : (\mathrm{w}^{-1}\mathrm{z})_{n1} \mapsto 0 \ \text{ for } \ k>1 \ \text{ and } \ i<n-1.$$

Equation (6.14c) is equivalent to the star-triangle relation. The operator $\mathbb{S}_n$ is involved in the following relations of type (6.14c):

$$\mathbb{S}_n(b)\,\mathbb{S}_{n+1}(a+b)\,\mathbb{S}_n(a) = \mathbb{S}_{n+1}(a)\,\mathbb{S}_n(a+b)\,\mathbb{S}_{n+1}(b), \tag{6.15a}$$

$$\mathbb{S}_n(b)\,\mathbb{S}_{n-1}(a+b)\,\mathbb{S}_n(a) = \mathbb{S}_{n-1}(a)\,\mathbb{S}_n(a+b)\,\mathbb{S}_{n-1}(b), \tag{6.15b}$$

where the dependence on parameters is indicated explicitly, $\mathbb{S}_{n+1}(a) = [i\mathrm{D}_1^z]^a$, $\mathbb{S}_{n-1}(a) = \left[i\mathrm{D}_{n-1}^w\right]^a$, and $\mathbb{S}_n(a)$ is the operator of multiplication by the function $(\mathrm{w}^{-1}\mathrm{z})_{n1}^a\ (\bar{\mathrm{w}}^{-1}\bar{\mathrm{z}})_{n1}^{\bar{a}}$.

After the change (5.41) of variables, we transform the operator $\mathbb{S}_{n+1}(a)$ into the operator $\mathbb{S}_{n+1}(a) = [i\partial_x]^a$, where $x = x_{21}$. The function $(\mathrm{w}^{-1}\mathrm{z})_{n1}$ is linear in x:

$$(\mathrm{w}^{-1}\mathrm{z})_{n1} = Ax + B = A(x-x_0), \quad x_0 = -A^{-1}B,$$

where the functions A and B depend on the other variables, so that they can be regarded as constants. As in the case of intertwining operators, we see that equation (6.15a) is equivalent to the identity

$$[x-x_0]^b\,[i\partial]^{a+b}\,[x-x_0]^a = [i\partial]^a\,[x-x_0]^{a+b}\,[i\partial]^b,$$

which is a concise operator restatement of the star-triangle relation. Equation (6.15b) is proved similarly.

The operators $\mathbb{S}_k$ form a complete set of generators of the permutation group that acts on the $2n$ parameters involved in the matrix product $\mathrm{L}_1(u_1,\dots,u_n)\mathrm{L}_2(v_1,\dots,v_n)$.

An arbitrary element of the group in question can be represented as a word whose letters are symbols $\mathbb{S}_k$. Such a representation is not unique in the sense that one and the same permutation can be represented in different ways. However, since the generators satisfy (6.14), every two words that represent the same permutation can be transformed one into the other by relations (6.14). In the next section, we consider a specific representation of the operator $\check{\mathrm{R}}$ in terms of the operators $\mathrm{R}_k(\mathbf{u})$. This representation turns out to be convenient in the construction of the Baxter Q-operators and is a natural generalization of the representation of $\check{\mathrm{R}}$ in terms of $\mathrm{R}_1(\mathbf{u})$ and $\mathrm{R}_2(\mathbf{u})$ in the case of the group $\mathrm{SL}(2,\mathbb{C})$.

In the same way, we can consider a product of an arbitrary number of Lax matrices and construct the generators of the group $\mathfrak{S}_{nN}$, the permutation group on $n\cdot N$ parameters for the product of N L-operators satisfying relations (6.14). As an example, we consider the product $\mathrm{L}_1(u_1,\dots,u_n)\mathrm{L}_2(v_1,\dots,v_n)\mathrm{L}_3(w_1,\dots,w_n)$ of three L-operators because, in this case, the Yang–Baxter equation (3.1) arises naturally. We combine all parameters in one set in the following order: $(w_1,\dots,w_n,v_1,\dots,v_n,u_1,\dots,u_n)$. Then, $3n-1$ elementary permutations acting on $(w_1,\dots,w_n,v_1,\dots,v_n,u_1,\dots,u_n)$ give a complete set of generators. Among them, again, the $3n-3$ permutations that act separately on $w_1,\dots,w_n$, $v_1,\dots,v_n$, and $u_1,\dots,u_n$ can be constructed from intertwining operators,

$$(\overbrace{w_1,\dots,w_n}^{\mathbb{S}_1,\dots,\mathbb{S}_{n-1}},\ \overbrace{v_1,\dots,v_n}^{\mathbb{S}_{n+1},\dots,\mathbb{S}_{2n-1}},\ \overbrace{u_1,\dots,u_n}^{\mathbb{S}_{2n+1},\dots,\mathbb{S}_{3n-1}}),$$

$$\mathbb{U}_k=\begin{cases}\mathbb{1}\otimes\mathbb{1}\otimes\mathrm{S}_k, & k=1,\dots,n-1,\\ \mathbb{1}\otimes\mathrm{S}_{k-n}\otimes\mathbb{1}, & k=n+1,\dots,2n-1,\\ \mathrm{S}_{k-2n}\otimes\mathbb{1}\otimes\mathbb{1}, & k=2n+1,\dots,3n-1.\end{cases}$$

The remaining operators $\mathbb{S}_n$ and $\mathbb{S}_{2n}$, which interchange w_n with v_1 and v_n with u_1, respectively,

$$(w_1,\dots,\overbrace{w_n,v_1}^{\mathbb{S}_n},\dots,v_n,\dots,\overbrace{v_n,u_1}^{\mathbb{S}_{2n}},\dots,u_n),$$

are constructed like (6.6) and (6.7). The generators $\mathbb{S}_k$ allow us to construct all elements of the given permutation group on $3n$ parameters and, in particular, the two elements that perform the following specific permutations:

$$(w_1,\dots,w_n,v_1,\dots,v_n,u_1,\dots,u_n)\xrightarrow{\check{\mathrm{R}}_{12}(u-v)}(w_1,\dots,w_n,u_1,\dots,u_n,v_1,\dots,v_n),$$

$$(w_1,\dots,w_n,v_1,\dots,v_n,u_1,\dots,u_n)\xrightarrow{\check{\mathrm{R}}_{23}(v-w)}(v_1,\dots,v_n,w_1,\dots,w_n,u_1,\dots,u_n).$$

It is easily seen that there are two representations of the operator performing the following permutation:

$$\begin{aligned}&(w_1,\dots,w_n,v_1,\dots,v_n,u_1,\dots,u_n)\\&\qquad\xrightarrow{\check{\mathrm{R}}_{12}(v-w)\check{\mathrm{R}}_{23}(u-w)\check{\mathrm{R}}_{12}(u-v)}(u_1,\dots,u_n,v_1,\dots,v_n,w_1,\dots,w_n),\\&(w_1,\dots,w_n,v_1,\dots,v_n,u_1,\dots,u_n)\\&\qquad\xrightarrow{\check{\mathrm{R}}_{23}(u-v)\check{\mathrm{R}}_{12}(u-w)\check{\mathrm{R}}_{23}(v-w)}(u_1,\dots,u_n,v_1,\dots,v_n,w_1,\dots,w_n).\end{aligned}$$

The operators $\check{\mathrm{R}}_{12}\check{\mathrm{R}}_{23}\check{\mathrm{R}}_{12}$ and $\check{\mathrm{R}}_{12}\check{\mathrm{R}}_{23}\check{\mathrm{R}}_{12}$ are constructed from the generators $\mathbb{S}_k$, and one representation is transformed into the other with the help of (6.14), because both operators represent one and the same element of the permutation group on $3n$ parameters. As a consequence, we see that the operators $\check{\mathrm{R}}_{12}$ and $\check{\mathrm{R}}_{23}$ satisfy the Yang–Baxter relation

$$\check{\mathrm{R}}_{23}(u-v)\check{\mathrm{R}}_{12}(u-w)\check{\mathrm{R}}_{23}(v-w)=\check{\mathrm{R}}_{12}(v-w)\check{\mathrm{R}}_{23}(u-w)\check{\mathrm{R}}_{12}(u-v). \tag{6.16}$$

□

6.2. The operators R_k. In the present subsection, we find a certain canonical representation for the operator $\check{\mathrm{R}}$,

$$(v_1,\dots,v_n,u_1,\dots,u_n)\xrightarrow{\check{\mathrm{R}}(u-v)}(u_1,\dots,u_n,v_1,\dots,v_n),$$

which turns out to be convenient in the construction of the Baxter Q-operator.

We represent a compound permutation performed by the operator $\check{\mathrm{R}}$ as a product of n permutations each of which interchanges only the parameters u_k and v_k in the product of L-operators. Each of these permutations is performed by the operator R_k,

$$(u_1,\dots,u_k,\dots,v_k,\dots,v_n)\xrightarrow{\mathrm{R}_k}(u_1,\dots,v_k,\dots,u_k,\dots,v_n),$$

and thus, the defining equation for the operator R_k has the form

$$\begin{aligned}\mathrm{R}_k\,\mathrm{L}_1(u_1,\dots,u_k,\dots,u_n)\,\mathrm{L}_2(v_1,\dots,v_k,\dots,v_n)\\ =\mathrm{L}_1(u_1,\dots,v_k,\dots,u_n)\,\mathrm{L}_2(v_1,\dots,u_k,\dots,v_n)\,\mathrm{R}_k.\end{aligned}\tag{6.17}$$

If necessary, we explicitly indicate the space in which the operator R_k acts exactly in the same way as in the case of the operator R_{ij}, and move the index k upward: R_{ij}^k. Thus, the operator $\check{\mathrm{R}}_{12}$ can be written as the product of n operators R_k:

$$\check{\mathrm{R}}_{12}(u-v)=\mathrm{R}_{12}^1\,\mathrm{R}_{12}^2\cdots\mathrm{R}_{12}^n.\tag{6.18}$$

Consider the following explicit representation of the operator R_k in terms of the generators $\mathbb{S}_k$:

$$\mathrm{R}_k=(\mathbb{S}_{n+k-1}\cdots\mathbb{S}_{n+1})\,(\mathbb{S}_k\cdots\mathbb{S}_{n-1})\,\mathbb{S}_n\,(\mathbb{S}_{n-1}\cdots\mathbb{S}_k)\,(\mathbb{S}_{n+1}\cdots\mathbb{S}_{n+k-1})\tag{6.19}$$

(to simplify notation, we do not indicate the arguments of the operators $\mathbb{S}_k$). For the proof, it suffices to check that the sequence of elementary permutations in (6.19) leads to the permutation of the parameters u_k and v_k in the product of two L-operators, while the other parameters remain at their positions. At the first step, the operator $\mathbb{S}_{n+1}\cdots\mathbb{S}_{n+k-1}$ performs the cyclic permutation of parameters

$$u_1\,u_2\dots\,u_{k-1}\,u_k\mapsto u_k\,u_1\dots\,u_{k-2}\,u_{k-1},$$

so that the parameter u_k appears at the first position. At step 2, the operator $\mathbb{S}_{n-1}\cdots\mathbb{S}_k$ performs the cyclic permutation

$$v_k\,v_{k+1}\dots\,v_{n-1}\,v_n\mapsto v_{k+1}\dots\,v_{n-1}\,v_n\,v_k$$

of the parameter set, so that the parameter v_k appears at the last position. Now, the parameters u_k and v_k are at the required positions, and the operator $\mathbb{S}_n$ interchanges them. It remains to return them to their own positions, so that, finally, the parameters u_k and v_k become interchanged. Schematically, the complete set of permutations looks like this (we underline the set of parameters on which the permutation is performed):

$$\begin{aligned}&(\dots v_k\dots v_n\,\underline{u_1\dots u_k}\dots)\xrightarrow{\mathbb{S}_{n+1}\cdots\mathbb{S}_{n+k-1}}(\dots\underline{v_k\dots v_n}\,u_k\,u_1\dots u_{k-1}\dots)\\ &\xrightarrow{\mathbb{S}_{n-1}\cdots\mathbb{S}_k}(\dots v_{k+1}\dots v_n\,\underline{v_k\,u_k}\,u_1\dots u_{k-1}\dots)\xrightarrow{\mathbb{S}_n}(\dots\underline{v_{k+1}\dots v_n\,u_k}\,v_k\,u_1\dots u_{k-1}\dots)\\ &\xrightarrow{\mathbb{S}_k\cdots\mathbb{S}_{n-1}}(\dots u_k\dots v_n\,\underline{v_k\,u_1\dots u_{k-1}}\dots)\xrightarrow{\mathbb{S}_{n+k-1}\cdots\mathbb{S}_{n+1}}(\dots u_k\dots v_n\,u_1\dots v_k\dots).\end{aligned}$$

We have presented explicit formulas in the two simplest cases. In the case of $\mathrm{SL}(2,\mathbb{C})$, there are two operators R_1 and R_2 that are constructed from the three elementary building blocks $\mathbb{S}_1$, $\mathbb{S}_2$, and $\mathbb{S}_3$:

$$\begin{aligned}\mathrm{R}_1\,(u_1-v_1;v_2-v_1)&=\mathbb{S}_1\mathbb{S}_2\mathbb{S}_1=[i\partial_{w_{21}}]^{u_1-v_2}[z_{21}-w_{21}]^{u_1-v_1}[i\partial_{w_{21}}]^{v_2-v_1},\\ \mathrm{R}_2\,(u_2-v_2;u_2-u_1)&=\mathbb{S}_3\mathbb{S}_2\mathbb{S}_3=[i\partial_{z_{21}}]^{u_1-v_2}[z_{21}-w_{21}]^{u_2-v_2}[i\partial_{z_{21}}]^{u_2-u_1}.\end{aligned}$$

In the case of $\mathrm{SL}(3,\mathbb{C})$, there are three operators R_k constructed from the five operators $\mathbb{S}_i$:

$$\begin{aligned}
\mathrm{R}_1(u_1-v_1;v_3-v_1,v_2-v_1) &= \mathbb{S}_1\mathbb{S}_2\mathbb{S}_3\mathbb{S}_2\mathbb{S}_1 = \left[i\left(\partial_{w_{21}}+w_{32}\partial_{w_{31}}\right)\right]^{u_1-v_2}\left[i\partial_{w_{32}}\right]^{u_1-v_3}\\
&\quad\times\left[z_{31}-w_{31}-w_{32}\left(z_{21}-w_{21}\right)\right]^{u_1-v_1}\left[i\partial_{w_{32}}\right]^{v_3-v_1}\left[i\left(\partial_{w_{21}}+w_{32}\partial_{w_{31}}\right)\right]^{v_2-v_1},\\
\mathrm{R}_2(u_2-v_2;v_3-v_2;u_2-u_1) &= \mathbb{S}_2\mathbb{S}_4\mathbb{S}_3\mathbb{S}_4\mathbb{S}_2 = \left[i\partial_{w_{32}}\right]^{u_2-v_3}\left[i\left(\partial_{z_{21}}+z_{32}\partial_{z_{31}}\right)\right]^{u_1-v_2}\\
&\quad\times\left[z_{31}-w_{31}-w_{32}\left(z_{21}-w_{21}\right)\right]^{u_2-v_2}\left[i\left(\partial_{z_{21}}+z_{32}\partial_{z_{31}}\right)\right]^{u_2-u_1}\left[i\partial_{w_{32}}\right]^{v_3-v_2},\\
\mathrm{R}_3(u_3-v_3;u_3-u_1,u_3-u_2) &= \mathbb{S}_5\mathbb{S}_4\mathbb{S}_3\mathbb{S}_4\mathbb{S}_5 = \left[i\partial_{z_{32}}\right]^{u_2-v_3}\left[i\left(\partial_{z_{21}}+z_{32}\partial_{z_{31}}\right)\right]^{u_1-v_3}\\
&\quad\times\left[z_{31}-w_{31}-w_{32}\left(z_{21}-w_{21}\right)\right]^{u_3-v_3}\left[i\left(\partial_{z_{21}}+z_{32}\partial_{z_{31}}\right)\right]^{u_3-u_1}\left[i\partial_{z_{32}}\right]^{u_3-u_2}.
\end{aligned}$$

We have presented these expressions as an illustration. As the algebra rank grows, the explicit formulas become more and more cumbersome.

To clearly show the dependence of the operator R_k on the parameters u_i and v_k, we indicate the parameter combination on which each of the operators $\mathbb{S}_j$ depends:

$$\begin{aligned}
\mathrm{R}_k = {}&(\overbrace{\mathbb{S}_{n+k-1}}^{u_{k-1}-v_k}\cdots\overbrace{\mathbb{S}_{n+1}}^{u_1-v_k})\,(\overbrace{\mathbb{S}_k}^{u_k-v_{k+1}}\cdots\overbrace{\mathbb{S}_{n-1}}^{u_k-v_n})\,\overbrace{\mathbb{S}_n}^{u_k-v_k}\\
&\times(\overbrace{\mathbb{S}_{n-1}}^{v_n-v_k}\cdots\overbrace{\mathbb{S}_k}^{v_{k+1}-v_k})\,(\overbrace{\mathbb{S}_{n+1}}^{u_k-u_1}\cdots\overbrace{\mathbb{S}_{n+k-1}}^{u_k-u_{k-1}}).
\end{aligned}\tag{6.20}$$

It is convenient to regard the difference u_k-v_k as the spectral parameter on which the operator $\mathrm{R}_k=\mathrm{R}_k(u_k-v_k)$ depends. Thus, the above formula can be written as follows ($\lambda=u_k-v_k$):

$$\begin{aligned}
\mathrm{R}_k(\lambda) = {}&(\overbrace{\mathbb{S}_{n+k-1}}^{\lambda-u_{k,k-1}}\cdots\overbrace{\mathbb{S}_{n+1}}^{\lambda-u_{k,1}})\,(\overbrace{\mathbb{S}_k}^{\lambda-v_{k+1,k}}\cdots\overbrace{\mathbb{S}_{n-1}}^{\lambda-v_{n,k}})\,\overbrace{\mathbb{S}_n}^{\lambda}\\
&\times(\overbrace{\mathbb{S}_{n-1}}^{v_{n,k}}\cdots\overbrace{\mathbb{S}_k}^{v_{k+1,k}})\,(\overbrace{\mathbb{S}_{n+1}}^{u_{k,1}}\cdots\overbrace{\mathbb{S}_{n+k-1}}^{u_{k,k-1}}),
\end{aligned}\tag{6.21}$$

where $u_{i,k}=u_i-u_k$ and $v_{i,k}=v_i-v_k$.

- $\mathrm{R}_k(u_k-v_k)$ as intertwining operator.

The operator $\mathrm{R}_k(u_k-v_k)$ interchanges the parameters u_k and v_k in the product of Lax matrices; i.e., the following transformation of parameters is performed in the initial Lax matrices: $u_k\to v_k=u_k-(u_k-v_k)$ and $v_k\to u_k=v_k+(u_k-v_k)$. At first sight, this transformation corresponds to the transformations $\sigma_k\to\sigma_k+(u_k-v_k)$ and $\rho_k\to\rho_k-(u_k-v_k)$ of the representation parameters, However, the sum of all representation parameters must be fixed. Therefore, the proper parameter transformation must be accompanied by a common translation so that the sum remain fixed ($\lambda=u_k-v_k$):

$$\begin{aligned}
&\sigma_k\to\sigma_k'=\sigma_k+\lambda-\frac{\lambda}{n},\quad \sigma_i\to\sigma_i'=\sigma_i-\frac{\lambda}{n},\\
&\rho_k\to\rho_k'=\rho_k-\lambda+\frac{\lambda}{n},\quad \rho_i\to\rho_i'=\rho_i+\frac{\lambda}{n},\quad i\neq k.
\end{aligned}\tag{6.22}$$

Thus, we obtain the following intertwining relations for the operators $\mathrm{R}_k(\lambda)$:

$$\mathrm{R}_k(\lambda)\;\mathrm{T}^{\boldsymbol{\sigma}}\otimes\mathrm{T}^{\boldsymbol{\rho}}=\mathrm{T}^{\boldsymbol{\sigma}'}\otimes\mathrm{T}^{\boldsymbol{\rho}'}\,\mathrm{R}_k(\lambda),\tag{6.23}$$

where the representation parameters on the right-hand side are determined by formulas (6.22), or equivalently,

$$\begin{aligned}
&\sigma_{k-1,k}'=\sigma_{k-1,k}-\lambda,\quad \sigma_{k,k+1}'=\sigma_{k,k+1}+\lambda,\quad \sigma_{i,i+1}'=\sigma_{i,i+1},\quad i\neq k,\\
&\rho_{k-1,k}'=\rho_{k-1,k}+\lambda,\quad \rho_{k,k+1}'=\rho_{k,k+1}-\lambda,\quad \rho_{i,i+1}'=\rho_{i,i+1},\quad i\neq k.
\end{aligned}\tag{6.24}$$

If $u_k = v_k$, then the operator R_k performs no permutation. Therefore, it is natural to expect that if $\lambda = u_k - v_k = 0$, then the operator R_k is the identity, $\mathrm{R}_k(0) = \mathbb{1}$. From (6.21), it can easily be seen that this is actually the case. All parameters indicated in formula (6.21) are exponents. For the operator $\mathbb{S}_n$, this is the power to which the function $\gamma(\mathrm{z},\mathrm{w})$ is raised, and, for the other operators $\mathbb{S}_k$, this is the power to which the right translation generator is raised. If an exponent is zero, then the corresponding operator is the identity, and for the inverse operator the index differs in sign. Therefore, if $\lambda = 0$, then the operator $\mathbb{S}_n$ is the identity, and the operators to the left and to the right of $\mathbb{S}_n$ are inverse to each other.

Proposition 12. *We list the properties of the operator* $\mathrm{R}_k(\lambda)$ *concerning the dependence on the parameters and the variables* z *and* w.

- *The operator* $\mathrm{R}_k(\lambda)$ *depends only on the parameters* $u_1,\dots,u_k$ *and* $v_k,\dots,v_n$:

$$\mathrm{R}_k(\lambda) = \mathrm{R}_k(\lambda; v_n - v_k,\dots,v_{k+1} - v_k; u_k - u_1,\dots,u_k - u_{k-1}). \tag{6.25}$$

- *The operator* $\mathrm{R}_k(\lambda)$ *commutes with the operators* $\mathbb{S}_i$ *for* $i \le k-2$ *and* $i \ge n+k+1$:

$$\mathrm{R}_k\mathbb{S}_i = \mathbb{S}_i\mathrm{R}_k, \quad i = 1,2,\dots,k-2, n+k+1, n+k+2,\dots,2n-1. \tag{6.26}$$

- *The operator* $\mathrm{R}_k(\lambda)$ *commutes with the variables of the matrix* w *located in the columns with indices* $1,2,\dots,k-1$ *and with the variables of the matrix* z *located in the columns with indices* $k,k+1,\dots,n-1$,

$$\mathrm{R}_k w_{ij} = w_{ij}\mathrm{R}_k, \;\; j = 1,2,\dots,k-1; \quad \mathrm{R}_k z_{ij} = z_{ij}\mathrm{R}_k, \;\; j = k,k+1,\dots,n-1. \tag{6.27}$$

All these properties of the operator $\mathrm{R}_k(\lambda)$ follow directly from formula (6.21) and the explicit expressions for the generators D_j (see (5.23)) from which the operators $\mathbb{S}_i$ are constructed.

Dropping the explicit indication to the parameters and spaces in which the operators act, we can represent the factorization formula (6.18) for an $\check{\mathrm{R}}$-operator in the following form:

$$\check{\mathrm{R}}_{12}(u-v) = \mathrm{R}^1_{12}(u_1 - v_1)\mathrm{R}^2_{12}(u_2 - v_2)\cdots\mathrm{R}^n_{12}(u_n - v_n)\,. \tag{6.28}$$

The concise notation $\mathrm{R}_k(\lambda)$ assumes that the parameters on which the operator $\mathrm{R}_k(\lambda)$ depends are uniquely determined by the product of Lax matrices on which the operator acts. To avoid ambiguity, we explain this by restoring all parameters in the last formula:

$$\begin{aligned}\check{\mathrm{R}}(u-v) = {} & \mathrm{R}_1(u_1 - v_1; u_2 - v_1,\dots,u_n - v_1)\\ & \times \mathrm{R}_2(u_2 - v_2; u_3 - v_2,\dots,u_n - v_2; u_2 - u_1)\\ & \times \mathrm{R}_3(u_3 - v_3; u_4 - v_3,\dots,u_n - v_3; u_3 - u_1, u_3 - u_2)\\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ & \times \mathrm{R}_n(u_n - v_n; u_n - u_1,\dots,u_n - u_{n-1}).\end{aligned} \tag{6.29}$$

The operator R_n performs the permutation $u_n \rightleftarrows v_n$. Therefore, the next operator R_{n-1} acts on the product of the Lax matrices with u_n and v_n interchanged, and we must change the parameters in the expression for this operator accordingly, i.e., $u_n \rightleftarrows v_n$. The operator R_{n-1} performs the permutation $u_{n-1} \rightleftarrows v_{n-1}$. Therefore, the next operator R_{n-2} acts on the product of the Lax matrices with u_{n-1}, u_n and v_{n-1}, v_n interchanged, etc. The operator $\mathrm{R}_k(\lambda_k)(\lambda_k = u_k - v_k)$ depends on the parameters $u_1,\dots,u_k$ and $v_k,\dots,v_n$, but, when its turn comes, the parameters $v_{k+1},\dots,v_n$ in the product of Lax matrices become $u_{k+1},\dots,u_n$, and therefore, in $\mathrm{R}_k(\lambda_k)$ only the dependence on v_k remains. Thus, for each operator R_k in (6.28), the dependence on the v-parameters reduces

to a dependence on v_k, and it is convenient to merge it into the spectral parameter λ_k:

(6.30)
$$\begin{aligned}\check{\mathrm{R}}(u-v) &= \mathrm{R}_1(\lambda_1;\lambda_1+u_{21},\dots,\lambda_1+u_{n1})\mathrm{R}_2(\lambda_2;\lambda_2+u_{32},\dots,\lambda_2+u_{n2};u_{21})\\ &\quad\times \mathrm{R}_3(\lambda_3;\lambda_3+u_{43},\dots,\lambda_3+u_{n3};u_{31},u_{32})\cdots\mathrm{R}_n(\lambda_n;u_{n1},\dots,u_{n,n-1}).\end{aligned}$$

Proposition 13. *The operators* R_j *satisfy the following relations*:

$$\mathrm{R}^k_{12}(u_k-v_k)\,\mathrm{R}^j_{23}(v_j-w_j)=\mathrm{R}^j_{23}(v_j-w_j)\mathrm{R}^k_{12}(u_k-v_k),\quad j\neq k, \tag{6.31}$$

$$\begin{aligned}&\mathrm{R}^k_{12}(v_k-w_k)\,\mathrm{R}^k_{23}(u_k-w_k)\,\mathrm{R}^k_{12}(u_k-v_k)\\ &\qquad=\mathrm{R}^k_{23}(u_k-v_k)\,\mathrm{R}^k_{12}(u_k-w_k)\,\mathrm{R}^k_{23}(v_k-w_k),\end{aligned} \tag{6.32}$$

$$\mathrm{R}^k_{12}(u_k-v_k)\,\mathrm{R}^i_{12}(u_i-v_i)=\mathrm{R}^i_{12}(u_i-v_i)\,\mathrm{R}^k_{12}(u_k-v_k). \tag{6.33}$$

To prove the first two relations, it suffices to check that the operators on both sides perform the following equal permutations of parameters in the product

$$\mathrm{L}_1(u_1,\dots,u_n)\mathrm{L}_2(v_1,\dots,v_n)\mathrm{L}_3(w_1,\dots,w_n) \tag{6.34}$$

of Lax matrices:

$$\begin{aligned}&(\dots w_k\dots w_j\dots v_k\dots v_j\dots u_k\dots u_j\dots)\\ &\qquad\xrightarrow{\mathrm{R}^k_{12}\,\mathrm{R}^j_{23}=\mathrm{R}^j_{23}\,\mathrm{R}^k_{12}}(\dots w_k\dots v_j\dots u_k\dots w_j\dots v_k\dots u_j\dots),\\ &(\dots w_k\dots v_k\dots u_k\dots)\xrightarrow{\mathrm{R}^k_{12}\,\mathrm{R}^k_{23}\,\mathrm{R}^k_{12}=\mathrm{R}^k_{23}\,\mathrm{R}^k_{12}\,\mathrm{R}^k_{23}}(\dots u_k\dots v_k\dots w_k\dots).\end{aligned}$$

The latter relation means that if we interchange two "disjoint" pairs of parameters in the product $\mathrm{L}_1(u_1,\dots,u_n)\ \mathrm{L}_2(v_1,\dots,v_n)$, then the result does not depend on which pair of parameters is interchanged first,

$$(\dots v_k\dots v_i\dots u_k\dots u_i\dots)\xrightarrow{\mathrm{R}^k_{12}\mathrm{R}^i_{12}=\mathrm{R}^i_{12}\mathrm{R}^k_{12}}(\dots u_k\dots u_i\dots v_k\dots v_i\dots).$$

Thus, the operators $\mathrm{R}_k(u_k-v_k)$ in the expression (6.28) for the R-matrix can be arranged in an arbitrary order.

In the preceding section it was mentioned that the Yang–Baxter relation (3.7) for the operator $\check{\mathrm{R}}$ is a consequence of formulas (6.14) for the operators $\mathbb{S}_j$, the elementary building blocks from which the operator $\check{\mathrm{R}}$ is constructed. In the present section, we use more complicated building blocks, the compound operators R_k, built from the operators $\mathbb{S}_j$ as elementary blocks, to construct the operator $\check{\mathrm{R}}$. For the operators R_k, relations (6.31)–(6.33) play the same role as formulas (6.14) do for the operators $\mathbb{S}_j$. We prove that relations (6.31)–(6.33) ensure the Yang–Baxter relations

$$\check{\mathrm{R}}_{23}(u-v)\check{\mathrm{R}}_{12}(u-w)\check{\mathrm{R}}_{23}(v-w)=\check{\mathrm{R}}_{12}(v-w)\check{\mathrm{R}}_{23}(u-w)\check{\mathrm{R}}_{12}(u-v) \tag{6.35}$$

for the operator $\check{\mathrm{R}}$,

$$\check{\mathrm{R}}(u-v)=\mathrm{R}_1(u_1-v_1)\mathrm{R}_2(u_2-v_2)\cdots\mathrm{R}_n(u_n-v_n)\,. \tag{6.36}$$

We substitute the decomposition (6.36) for the operator $\check{\mathrm{R}}_{ik}$ in (6.35). Using (6.31) and (6.33), we permute the operators R_i so as to arrange them in the following order: first, the operators R^1_{ik}, next, the operators R^2_{ik}, etc. After that, (6.35) takes the form

$$\begin{aligned}&\mathrm{R}^1_{23}(u_1-v_1)\mathrm{R}^1_{12}(u_1-w_1)\mathrm{R}^1_{23}(v_1-w_1)\cdots\mathrm{R}^n_{23}(u_n-v_n)\mathrm{R}^n_{12}(u_n-w_n)\mathrm{R}^n_{23}(v_n-w_n)\\ &\quad=\mathrm{R}^1_{12}(v_1-w_1)\mathrm{R}^1_{23}(u_1-w_1)\mathrm{R}^1_{12}(u_1-v_1)\cdots\mathrm{R}^n_{12}(v_n-w_n)\mathrm{R}^n_{23}(u_n-w_n)\mathrm{R}^n_{12}(u_n-v_n).\end{aligned}$$

It remains to use (6.32) to verify that the two sides in (6.32) are equal.

We note that the construction of the general $\mathrm{SL}(n,\mathbb{C})$-invariant solution of the Yang–Baxter equation is considerably more complicated in comparison with the case of the

group $\mathrm{SL}(2,\mathbb{C})$, though the main principle remains the same. The basic building blocks are intertwining operators, and the "assembling rules" are determined by a permutation group. In the first part, the exposition was sufficiently detailed. Therefore, in the second part, we consider only the properties of the operators R_k that are used in the sequel for constructing the transfer matrices and Q-operators.

§7. Verma modules and finite-dimensional representations of $\mathrm{SL}(n,\mathbb{C})$

The operators $\mathbb{S}_k$, which are elementary building blocks, are well defined on the function space on which the principal series representation of the group $\mathrm{SL}(n,\mathbb{C})$ is realized. For other types of irreducible representations, the situation is more complicated.

In the case where the parameters $\sigma_{k,k+1}$ and $\bar{\sigma}_{k,k+1}$ are positive integers, the representation space of the principal nonunitary series of the group $\mathrm{SL}(n,\mathbb{C})$ has a finite-dimensional invariant subspace of polynomials that are solutions of system (5.14). If we restrict ourselves to representations of the Lie algebra $\mathrm{sl}(n,\mathbb{C})$, then, at a point in general position, any representation in the space $\mathbb{C}[\mathrm{z},\bar{\mathrm{z}}]$ of polynomials in z_{ik} and $\bar{z}_{ik}$ is irreducible. At a singular point, we obtain a finite-dimensional representation of the Lie algebra, which extends to a finite-dimensional representation of the Lie group. Thus, the Verma modules, which are infinite-dimensional representations of the Lie algebra $\mathrm{sl}(n,\mathbb{C})$ in the space of polynomials, occupy an intermediate level between the principal series representations and finite-dimensional representations of the group $\mathrm{SL}(n,\mathbb{C})$.

In this case, the operators $\mathbb{S}_k$ need additional regularization. Therefore, it is convenient to combine them into bigger building blocks, the operators R_k, which are already well defined on the space of polynomials. In the case of the group $\mathrm{SL}(2,\mathbb{C})$, we presented various formulas to describe the action of the operators R_k on the space of polynomials. Formula (4.8) describing the action of an operator on the generating function of the basis monomials admits a natural generalization to the case of the group $\mathrm{SL}(n,\mathbb{C})$.

7.1. Verma modules. To fix the notation, we give the definition of a Verma module for the Lie algebra $\mathrm{sl}(n)$. The Lie algebra $\mathrm{sl}(n)$ consists of $(n\times n)$-matrices with zero trace and contains n^2-1 independent generators, which are conveniently constructed from the matrices $\mathbf{e}_{ik}$,

$$(\mathbf{e}_{ik})_{nm}=\delta_{in}\delta_{km},\quad [\mathbf{e}_{ik},\mathbf{e}_{nm}]=\delta_{kn}\mathbf{e}_{im}-\delta_{im}\mathbf{e}_{nk}. \tag{7.1}$$

The Cartan subalgebra consists of the diagonal matrices with zero trace. To each matrix in the Cartan subalgebra, we assign a vector in the n-dimensional space,

$$\mathbf{h}=\begin{pmatrix} h_1 & 0 & 0 & \dots & 0\\ 0 & h_2 & 0 & \dots & 0\\ 0 & 0 & h_3 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & h_n\end{pmatrix}\leftrightarrow \mathrm{h}=(h_1,h_2,\dots,h_n),\ \ h_1+h_2+\dots+h_n=0.$$

From (7.1), it is seen that the matrices $\mathbf{e}_{ik}$ $(i<k)$ are eigenvectors of the adjoint action of the Cartan algebra,

$$[\mathbf{h},\mathbf{e}_{ik}]=(\alpha_{ik},\mathrm{h})\cdot\mathbf{e}_{ik},\quad (\alpha_{ik},\mathrm{h})=h_i-h_k,$$

and correspond to the positive root vectors $\alpha_{ik}=(\dots,1,\dots,-1,\dots)$ with two nonzero components. The negative roots $-\alpha_{ik}$ correspond to the matrices $\mathbf{e}_{ik}$ $(i>k)$. The vectors $\alpha_k\equiv\alpha_{k,k+1}$ form the set of simple roots,

$$\alpha_1=(1,-1,0,\dots,0),\quad \alpha_2=(0,1,-1,0,\dots,0),\dots,\alpha_{n-1}=(0,\dots,0,1,-1).$$

In the Chevalley basis,

$$\mathbf{e}_k \equiv \mathbf{e}_{k,k+1}, \quad \mathbf{f}_k \equiv \mathbf{e}_{k+1,k}, \quad \mathbf{h}_k \equiv \mathbf{e}_{k,k} - \mathbf{e}_{k+1,k+1},$$

the commutation relations can be written in a concise form with the help of the Cartan matrix

$$\mathrm{A}_{ik} = (\alpha_i, \alpha_k) = 2\delta_{ik} - \delta_{i,k+1} - \delta_{i+1,k},$$
$$[\mathbf{h}_i, \mathbf{h}_k] = 0, \quad [\mathbf{h}_i, \mathbf{e}_k] = \mathrm{A}_{ik}\mathbf{e}_k, \quad [\mathbf{h}_i, \mathbf{f}_k] = -\mathrm{A}_{ik}\mathbf{e}_k, \quad [\mathbf{e}_i, \mathbf{f}_k] = \delta_{ik}\mathbf{h}_k.$$

The basis generators satisfy the Serre relations

$$[\mathrm{ad}\,(\mathbf{e}_i)]^{1-\mathrm{A}_{ik}}\,\mathbf{e}_k = 0; \quad [\mathrm{ad}\,(\mathbf{f}_i)]^{1-\mathrm{A}_{ik}}\,\mathbf{f}_k = 0,$$

and the remaining generators are the commutators of the basis ones.

The Verma module $\mathrm{V}_{\lambda_1\ldots\lambda_{n-1}}$ is a free $\mathrm{sl}(n)$-module generated by the lowest weight vector $|0\rangle$:

$$\mathbf{h}_k|0\rangle = \lambda_k \cdot |0\rangle, \quad \lambda_k \in \mathbb{C}; \quad \mathbf{f}_k|0\rangle = 0.$$

The parameters λ_k are called the Dynkin indices of the representation in question. The Verma module $\mathrm{V}_{\lambda_1\ldots\lambda_{n-1}}$ is irreducible in the case where the $\lambda_k \in \mathbb{C}$ are in general position. The points for which at least one of the numbers λ_k is equal to $1 - n_k$ with $n_k \in \mathbb{N}$ are exceptional. For such values of λ_k the Verma module has additional invariant subspaces.

In the realization under consideration, the lowering generators of the Chevalley basis of the Lie algebra $\mathrm{sl}(n, \mathbb{C})$ have the following form:

$$\mathbf{f}_k = \mathrm{E}_{k+1,k} = -\sum_{m=1}^{k} z_{km}\,\frac{\partial}{\partial z_{k+1,m}}.$$

A lowest weight vector must be annihilated by the generators $\mathbf{f}_k$: $\mathbf{f}_k|0\rangle = 0$. The corresponding function $\Phi(\mathrm{z}) \leftrightarrow |0\rangle$ satisfying all these equations is constant, $\Phi(\mathrm{z}) = 1$. To find the parameters λ_k, we must calculate the eigenvalues of the generators $\mathbf{h}_k$ corresponding the eigenvector $|0\rangle \leftrightarrow \Phi(\mathrm{z}) = 1$. In the realization in question, we have $\mathbf{h}_k = 1 - \sigma_{k,k+1} + \cdots$, where the contributions omitted contain derivatives; therefore, the Dynkin indices look like this:

$$\mathbf{h}_k|0\rangle = (1 - \sigma_{k,k+1}) \cdot |0\rangle \to \lambda_k = 1 - \sigma_{k,k+1}.$$

Applying the raising operators $\mathbf{e}_k$ to a lowest weight vector, we obtain a Verma module. At a point in general position, we have the space of polynomials $\mathbb{C}[\mathrm{z}] = \mathbb{C}[z_{ik}]$ $(i < k)$ in $\frac{n(n-1)}{2}$ variables. Since the holomorphic generators $\mathbf{h}_k, \mathbf{e}_k, \mathbf{f}_k$ commute with the antiholomorphic generators $\bar{\mathbf{h}}_k, \bar{\mathbf{e}}_k, \bar{\mathbf{f}}_k$, we see that the holomorphic and antiholomorphic sectors separate, $\mathrm{V}_{\lambda_1,\bar{\lambda}_1,\ldots,\lambda_{n-1},\bar{\lambda}_{n-1}} = \mathrm{V}_{\lambda_1,\ldots,\lambda_{n-1}} \otimes \mathrm{V}_{\bar{\lambda}_1,\ldots,\bar{\lambda}_{n-1}}$. We assume that the parameters λ_k and $\bar{\lambda}_k$ are independent because, for polynomials, there is no need to impose the integrality restriction to the difference $\sigma_{k,k+1} - \bar{\sigma}_{k,k+1}$. Usually, the basis ω_k dual to the simple roots α_i is introduced:

$$(\alpha_i, \omega_k) = \delta_{ik}, \quad i, k = 1, \ldots, n-1.$$

The transition matrix for these bases is the Cartan matrix

$$\alpha_i = \sum_k \mathrm{A}_{ik}\omega_k; \quad (\alpha_i, \alpha_k) = \mathrm{A}_{ik}, \quad (\omega_i, \omega_k) = \mathrm{A}^{-1}_{ik}.$$

The Verma module is characterized by the weight vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ in the basis ω_k,

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_{n-1}\omega_{n-1}.$$

The action of the Weil group on the weight vectors is defined as follows:

$$w \cdot \boldsymbol{\lambda} \equiv w(\boldsymbol{\lambda} - \boldsymbol{\rho}) + \boldsymbol{\rho},$$

where $\boldsymbol{\rho} = \omega_1 + \omega_2 + \cdots + \omega_{n-1}$ is a Weil vector and $w(\boldsymbol{\lambda})$ is the usual action of the Weil group by reflections,

$$\mathrm{s}_\alpha(\boldsymbol{\lambda}) = \boldsymbol{\lambda} - (\alpha, \boldsymbol{\lambda})\alpha.$$

We obtain the following formula for the elements $\mathrm{s}_k \equiv \mathrm{s}_{\alpha_k}$ of the Weil group that correspond to the reflections with respect to the simple roots α_k,

$$\mathrm{s}_k \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda} - (\alpha_k, \boldsymbol{\lambda} - \boldsymbol{\rho})\alpha_k = \boldsymbol{\lambda} - (\lambda_k - 1)\sum_p \mathrm{A}_{kp}\omega_p, \quad \lambda_i \xrightarrow{\mathrm{s}_k} \lambda_i - (\lambda_k - 1)\mathrm{A}_{ki}.$$

Everything becomes more visual if we represent the vectors in the n-dimensional space by using the embedding of the weight vectors into the n-dimensional space, as was done at the beginning. In this case, the formulas become simpler:

$$\begin{aligned}
&\alpha_k = \mathbf{e}_k - \mathbf{e}_{k+1}, \quad \omega_k = \mathbf{e}_1 + \cdots + \mathbf{e}_k - \frac{k}{n} \cdot (\mathbf{e}_1 + \cdots + \mathbf{e}_n)\,,\\
&\boldsymbol{\lambda} = \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_{n-1}\omega_{n-1} = (\ell_1 - c)\mathbf{e}_1 + (\ell_2 - c)\mathbf{e}_2 + \cdots + (\ell_n - c)\mathbf{e}_n,\\
&\ell_k = \lambda_k + \lambda_{k+1} + \cdots + \lambda_{n-1}, \quad c = \frac{1}{n} \cdot (\lambda_1 + 2\lambda_2 + \cdots + (n-1)\lambda_{n-1})\,.
\end{aligned}$$

Substituting $\lambda_k = 1 - \sigma_k + \sigma_{k+1}$, we obtain $\ell_k = n - k - \sigma_k + \sigma_n$, $c = \sigma_n$, whence

$$\boldsymbol{\lambda} = (n - 1 - \sigma_1, n - 2 - \sigma_2, \ldots, -\sigma_n).$$

In this notation, the action of the Weil group has the form

$$\mathrm{s}_k \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda} - (\lambda_k - 1)\,(\mathbf{e}_k - \mathbf{e}_{k+1}) = \boldsymbol{\lambda} + (\sigma_k - \sigma_{k+1})\,(\mathbf{e}_k - \mathbf{e}_{k+1})$$

and reduces to a simple permutation of parameters, $\sigma_k \rightleftarrows \sigma_{k+1}$. We note that, most frequently, to characterize a Verma module, instead of a weight vector the vector $\boldsymbol{\lambda} - \boldsymbol{\rho}$ is used, obtained by translation by the Weil vector. In the sequel, we shall characterize a representation by the vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ as before, because, up to the sign and the translation by a constant vector, $\boldsymbol{\sigma}$ coincides with the standard weight vector translated by the Weil vector,

$$\boldsymbol{\lambda} - \boldsymbol{\rho} = -\boldsymbol{\sigma} + \frac{n-1}{2}\,(1, 1, \ldots, 1)\,.$$

7.2. The action of the operator R_k on the generating function. In Subsection 4.1, we considered different representations for the operators R_k acting in the space of polynomials. Apparently, the most natural representation is (4.8). First, this representation has a clear group meaning because it is a direct consequence of the intertwining relations (4.12). Second, our experience shows that formula (4.8) gives rise to simple proofs of the required relations for the operators R_k. Finally, and most importantly, formula (4.8) for the action of the operators R_k on the generating function of basis monomials has a simple generalization to the case of the algebra $\mathrm{sl}(n)$.

In the present subsection, we obtain the corresponding representation for the operator R_k: analyzing the restriction of the integral operator R_k obtained for the continuous series to the space of polynomials, we find a generalization of formula (4.8).

First, we consider a known case, in which we represent the formula in a convenient form and then give a proof that can be generalized easily. Substituting in (4.8) the required parameters, $a = u_1 - v_2 + 1$ for both operators, $b = v_1 - v_2 + 1$ for R_1, and

$b = u_1 - u_2 + 1$ for R_2, we obtain the following formulas for the action of the operators on the generating function:

$$
\begin{aligned}
&\mathrm{R}_1 \,:\, (1-x_1z_1)^{u_2-u_1-1}(1-x_2z_2)^{v_2-v_1-1} \\
&\qquad\qquad \mapsto (1-x_1z_1)^{u_2-u_1-1}(1-x_2z_2)^{v_2-u_1-1}(1-x_2z_1)^{u_1-v_1}, \\
&\mathrm{R}_2 \,:\, (1-x_1z_1)^{u_2-u_1-1}(1-x_2z_2)^{v_2-v_1-1} \\
&\qquad\qquad \mapsto (1-x_1z_1)^{v_2-u_1-1}(1-x_2z_2)^{v_2-v_1-1}(1-x_1z_2)^{u_2-v_2}.
\end{aligned}
\tag{7.2}
$$

In this formula, the operators R_1 and R_2 are the restrictions to the space of holomorphic polynomials of the corresponding "big" operators acting in the principal series representation spaces. Calculating the derivative $\partial_{x_1}^n \partial_{x_2}^m$ of both sides of (7.2) at $x_1 = x_2 = 0$, we arrive at formula (4.7) for the action of operators on an arbitrary monomial $z_1^n z_2^m$.

An analog of formulas (7.2) for the restrictions of "big" operators to the space of polynomials depending on z_1, z_2 and $\bar{z}_1, \bar{z}_2$ has almost the same form. We only need to replace the parentheses by brackets, $(1-a)^\lambda \to [1-a]^\lambda \equiv (1-a)^\lambda(1-\bar{a})^{\bar{\lambda}}$. In the sequel, we consider this more general case.

First, we clarify the meaning of the generating function. To characterize a representation, we need the difference of σ_1 and σ_2, and it is convenient to express this difference in terms of the parameters u_1 and u_2 in the L-operator, $\sigma_{12} = \sigma_1 - \sigma_2 = u_2 - u_1$. We write the action of the representation operator on a function in the following convenient form:

$$
\mathrm{T}_g^{u_1,u_2}\,\Phi(z) = [d-bz]^{u_2-u_1-1} \cdot \Phi\left(\frac{-c+az}{d-bz}\right), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
$$

so that

$$
\begin{aligned}
\mathrm{T}_{g_1}^{u_1,u_2} \,&:\, 1 \mapsto [1-x_1z_1]^{u_2-u_1-1}, \quad g_1 = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \\
\mathrm{T}_{g_2}^{v_1,v_2} \,&:\, 1 \mapsto [1-x_2z_2]^{v_2-v_1-1}, \quad g_2 = \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

We see that the generating function arises as a result of the action of the group transformation on the identity,

$$
\mathrm{T}_{g_1}^{u_1,u_2} \otimes \mathrm{T}_{g_2}^{v_1,v_2} \,:\, 1 \mapsto [1-x_1z_1]^{u_2-u_1-1}[1-x_2z_2]^{v_2-v_1-1}.
$$

By direct calculation, we can verify that the analogs of formulas (7.2) for the restrictions of "big" operators to the space of polynomials depending on z_1, z_2 and on $\bar{z}_1, \bar{z}_2$ can be represented in the following concise form:

$$
\begin{aligned}
\mathrm{R}_1 \,&:\, \mathrm{T}_{g_1}^{u_1,u_2} \otimes \mathrm{T}_{g_2}^{v_1,v_2} \cdot 1 \mapsto \mathrm{T}^{v_1,u_2}(g_1) \otimes \mathrm{T}^{u_1,v_2}(g_2)\ [1+x_{12}z_1]^{u_1-v_1}, \\
\mathrm{R}_2 \,&:\, \mathrm{T}_{g_1}^{u_1,u_2} \otimes \mathrm{T}_{g_2}^{v_1,v_2} \cdot 1 \mapsto \mathrm{T}^{u_1,v_2}(g_1) \otimes \mathrm{T}^{v_1,u_2}(g_2)\ [1+x_{12}z_2]^{u_2-v_2}.
\end{aligned}
\tag{7.3}
$$

We use (6.19) to present a proof of the formula for R_1:

$$
\mathrm{R}_1 = \mathrm{N}^{-1} \cdot [z_{12}]^{v_2-v_1}\,[i\partial_2]^{u_1-v_1}\,[z_{12}]^{u_1-v_2} = \mathrm{N}^{-1} \cdot [i\partial_2]^{u_1-v_2}\,[z_{12}]^{u_1-v_1}\,[i\partial_2]^{v_2-v_1}.
$$

We recall that the normalization N is chosen so as to ensure the condition $\mathrm{R}_1 : 1 \mapsto 1$. To prove (7.3), we move the operators T in the product $\mathrm{R}_1\mathrm{T}_{g_1} \otimes \mathrm{T}_{g_2}$ so that they be located to the left of R_1. First, we use the property of the intertwining operator that interchanges v_1 and v_2,

$$
[i\partial_2]^{v_2-v_1}\,\mathrm{T}_{g_1}^{u_1,u_2} \otimes \mathrm{T}_{g_2}^{v_1,v_2} = \mathrm{T}_{g_1}^{u_1,u_2} \otimes \mathrm{T}_{g_2}^{v_2,v_1}\,[i\partial_2]^{v_2-v_1}.
$$

At the next step, when the operators T are moved to the left, we make the change $u_1 \rightleftarrows v_1$ and also modify the function $z_{12} \to z_{12} - x_{12}z_1z_2$, obtaining

$$
[z_{12}]^{u_1-v_1}\,\mathrm{T}_{g_1}^{u_1,u_2} \otimes \mathrm{T}_{g_2}^{v_2,v_1} = \mathrm{T}_{g_1}^{v_1,u_2} \otimes \mathrm{T}_{g_2}^{v_2,u_1}\,[z_{12} - x_{12}z_1z_2]^{u_1-v_1}.
$$

The latter formula can easily be verified by direct calculation. A more conceptual proof is naturally stated in terms of matrices. We postpone it, because it is more convenient to start with the general case.

Next, as before, we use a property of the intertwining operator that interchanges u_1 and v_2:

$$[i\partial_2]^{u_1-v_2}\,\mathrm{T}_{g_1}^{v_1,u_2}\otimes\mathrm{T}_{g_2}^{v_2,u_1}=\mathrm{T}_{g_1}^{v_1,u_2}\otimes\mathrm{T}_{g_2}^{u_1,v_2}\,[i\partial_2]^{u_1-v_2}.$$

Once the operators T have been moved to the left, it remains to apply the operator that arises on the right to the identity,

$$\mathrm{N}^{-1}\cdot[i\partial_2]^{u_1-v_2}\,[z_{12}-x_{12}z_1z_2]^{u_1-v_1}\,[i\partial_2]^{v_2-v_1}\;:\;1\mapsto[1+x_{12}z_1]^{u_1-v_1}\,.$$

Everything becomes simpler after the scale transformation $z_2\to z_2\cdot(1+x_{12}z_1)^{-1}$:

$$\begin{aligned}&\mathrm{N}^{-1}\cdot[i\partial_2]^{u_1-v_2}\,[z_{12}-x_{12}z_1z_2]^{u_1-v_1}\,[i\partial_2]^{v_2-v_1}\cdot 1\\&\qquad=[1+x_{12}z_1]^{u_1-v_1}\cdot\underline{\mathrm{N}^{-1}\cdot[i\partial_2]^{u_1-v_2}\,[z_{12}]^{u_1-v_1}\,[i\partial_2]^{v_2-v_1}\cdot 1}.\end{aligned}$$

The underlined expression is equal to the identity, because it is the result of the action of R_1 on the identity. It is important that, for the above calculation, we only need to know the coefficient of z_2 in the expression $z_{12}-x_{12}z_1z_2$, because this is the coefficient that determines the scale transformation.

Now, we proceed to generalization. To characterize the representation, we need the differences $\sigma_{k,k+1}=\sigma_k-\sigma_{k+1}=u_{k+1}-u_k$, and

$$\mathrm{T}_g^{u_1,u_2,\dots,u_n}\,\Phi(\mathrm{z})=[\Delta_1]^{u_2-u_1-1}[\Delta_2]^{u_3-u_2-1}\cdots[\Delta_{n-1}]^{u_n-u_{n-1}-1}\cdot\Phi(\mathrm{z}')\,.$$

The generating function is generalized in the following natural way:

$$\mathrm{T}_{g_1}^{u_1,u_2,\dots,u_n}\otimes\mathrm{T}_{g_2}^{v_1,v_2,\dots,v_n}\,\Phi(\mathrm{z},\mathrm{w}),$$

where we must take $\Phi(\mathrm{z},\mathrm{w})=1$; two upper triangular matrices with 1s on the diagonal will play the role of g_1 and g_2.

Proposition 14. *The operator* R_k *takes polynomials to polynomials, and the result of the action of* R_k *on the generating function has the form*

$$\begin{aligned}\mathrm{R}_k:&\,\mathrm{T}_{g_1}^{u_1,\dots,u_k,\dots,u_n}\otimes\mathrm{T}_{g_2}^{v_1,\dots,v_k,\dots,v_n}\cdot 1\\&\mapsto\mathrm{T}_{g_1}^{u_1,\dots,v_k,\dots,u_n}\otimes\mathrm{T}_{g_2}^{v_1,\dots,u_k,\dots,v_n}\cdot\left[\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\right]_{kk}^{u_k-v_k}\,.\end{aligned}\tag{7.4}$$

In the proof of this formula, we can follow the proof given in the case of $\mathrm{SL}(2,\mathbb{C})$ step by step. We use representation (6.19) for the operator R_k,

$$\mathrm{R}_k=\mathrm{N}^{-1}\cdot(\mathbb{S}_{n+k-1}\cdots\mathbb{S}_{n+1})\,(\mathbb{S}_k\cdots\mathbb{S}_{n-1})\,\mathbb{S}_n\,(\mathbb{S}_{n-1}\cdots\mathbb{S}_k)\,(\mathbb{S}_{n+1}\cdots\mathbb{S}_{n+k-1}).$$

The normalization N is chosen so as to ensure that $\mathrm{R}_k:1\mapsto 1$. We recall that, in this formula, $\mathbb{S}_n=\left[\mathrm{w}^{-1}\mathrm{z}\right]_{n1}^{u_k-v_k}$ and the other operators are the intertwining operators that provide the necessary permutations of the parameters.

Again, we move the operators T in the product $\mathrm{R}_1\mathrm{T}_{g_1}\otimes\mathrm{T}_{g_2}$ so that they will be located to the left of R_1. At the first step, the action of the intertwining operators $(\mathbb{S}_{n-1}\cdots\mathbb{S}_k)\,(\mathbb{S}_{n+1}\cdots\mathbb{S}_{n+k-1})$ yields the following permutation of the representation parameters:

$$\begin{aligned}u_1\,u_2\dots u_{k-1}\,u_k\,u_{k+1}\dots u_n&\longmapsto u_k\,u_1\dots u_{k-2}\,u_{k-1}\,u_{k+1}\dots u_n,\\u_1\dots u_{k-1}\,v_k\,v_{k+1}\dots v_{n-1}\,v_n&\longmapsto u_1\dots u_{k-1}\,v_{k+1}\dots v_{n-1}\,v_n\,v_k.\end{aligned}$$

At the next step, when the operators T are moved to the left, the representation parameters are changed, $u_k\rightleftarrows v_k$, and, second, the function is changed, $\left[\mathrm{w}^{-1}\mathrm{z}\right]_{n1}\to\left[\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\right]_{n1}$. This yields the equation

$$\left[\mathrm{w}^{-1}\mathrm{z}\right]_{n1}^{u_k-v_k}\,\mathrm{T}_{g_1}^{u_k,\dots}\otimes\mathrm{T}_{g_2}^{\dots,v_k}=\mathrm{T}_{g_1}^{v_k,\dots}\otimes\mathrm{T}_{g_2}^{\dots,u_k}\,\left[\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\right]_{n1}^{u_k-v_k}\,.$$

Under the transformations $\mathrm{z} \to g_1^{-1}\mathrm{z} = \mathrm{z}' \cdot \mathrm{h}_1$ and $\mathrm{w} \to g_2^{-1}\mathrm{w} = \mathrm{w}' \cdot \mathrm{h}_2$, the function on the right is transformed as follows:

$$\begin{aligned}\left[\mathrm{w}^{-1} g_2^{-1} g_1 \mathrm{z}\right]_{n1}^{u_k - v_k} &\to \left[\mathrm{w}'^{-1} g_2^{-1} g_1 \mathrm{z}'\right]_{n1}^{u_k - v_k} \\ &= \left[\mathrm{h}_2 \mathrm{w}^{-1} \mathrm{z} \mathrm{h}_1^{-1}\right]_{n1}^{u_k - v_k} = [\mathrm{h}_2]_{nn}^{u_k - v_k} \left[\mathrm{w}^{-1}\mathrm{z}\right]_{n1}^{u_k - v_k} \left[\mathrm{h}_1^{-1}\right]_{11}^{u_k - v_k}.\end{aligned}$$

The latter equation follows from the fact that the matrix $\mathrm{w}^{-1}\,\mathrm{z}$ is lower triangular and the matrices h_1^{-1} and h_2 are upper triangular. Thus, the required function $\left[\mathrm{w}^{-1}\mathrm{z}\right]_{n1}^{u_k - v_k}$ on the left-hand side of the equation can be recovered, and the factors $[\mathrm{h}_2]_{nn}^{u_k - v_k}$ and $[\mathrm{h}_1]_{11}^{v_k - u_k}$ yield the above-mentioned changes of the representation parameters.

At the third step, the intertwining operators $(\mathbb{S}_{n+k-1} \cdots \mathbb{S}_{n+1})\,(\mathbb{S}_k \cdots \mathbb{S}_{n-1})$ provide the inverse permutation of the representation parameters. Thus, eventually, we obtain the same parameters with which we started, but with u_k and v_k interchanged.

After the operator T has been moved to the left, it remains to find the action on the identity of the operator that occurs on the right:

$$\begin{aligned}\mathrm{N}^{-1} \cdot (\mathbb{S}_{n+k-1} \cdots \mathbb{S}_{n+1})\,(\mathbb{S}_k \cdots \mathbb{S}_{n-1})\,\left[\mathrm{w}^{-1} g_2^{-1} g_1 \mathrm{z}\right]_{n1}^{u_k - v_k} \\ \times (\mathbb{S}_{n-1} \cdots \mathbb{S}_k)\,(\mathbb{S}_{n+1} \cdots \mathbb{S}_{n+k-1}),\end{aligned}$$

We prove that this operator maps the identity to a simple function, namely,

$$1 \mapsto \left[\mathrm{w}^{-1} g_2^{-1} g_1 \mathrm{z}\right]_{kk}^{u_k - v_k}.$$

We represent this operator in a more explicit form by indicating the dependence on the parameters:

$$\begin{aligned}&\mathrm{N}^{-1} \cdot (\overbrace{\mathbb{S}_{n+k-1}}^{u_{k-1}-v_k} \cdots \overbrace{\mathbb{S}_{n+1}}^{u_1-v_k})\,(\overbrace{\mathbb{S}_k}^{u_k-v_{k+1}} \cdots \overbrace{\mathbb{S}_{n-1}}^{u_k-v_n})\Phi(\mathrm{z};\mathrm{w})\,(\overbrace{\mathbb{S}_{n-1}}^{v_n-v_k} \cdots \overbrace{\mathbb{S}_k}^{v_{k+1}-v_k})\,(\overbrace{\mathbb{S}_{n+1}}^{u_k-u_1} \cdots \overbrace{\mathbb{S}_{n+k-1}}^{u_k-u_{k-1}}), \\ &\Phi(\mathrm{z};\mathrm{w}) = \left[\mathrm{w}^{-1} g_2^{-1} g_1 \mathrm{z}\right]_{n1}^{u_k - v_k}.\end{aligned}$$

Using the formulas

$$\begin{aligned}\mathbb{S}_k\,\Phi(\mathrm{z};\mathrm{w}) &= [i\mathrm{D}_k^{\mathrm{w}}]^{\lambda}\;\Phi(\mathrm{z};\mathrm{w}) = [i\partial_y]^{\lambda}\,\mathrm{e}^{\,y\,\mathrm{D}_k^{\mathrm{w}}}\;\Phi(\mathrm{z};\mathrm{w})\Big|_{y=0} \\ &= [i\partial_y]^{\lambda}\Phi\big(\mathrm{z};\mathrm{w}\,(\mathbb{1} + y\,\mathbf{e}_{k+1,k})\big)\,\mathrm{e}^{\,y\,\mathrm{D}_k^{\mathrm{w}}}\Big|_{y=0}, \\ \mathbb{S}_{n+k}\,\Phi(\mathrm{z};\mathrm{w}) &= [i\mathrm{D}_k^{\mathrm{z}}]^{\lambda}\;\Phi(\mathrm{z};\mathrm{w}) = [i\partial_x]^{\lambda}\,\mathrm{e}^{\,x\,\mathrm{D}_k^{\mathrm{z}}}\;\Phi(\mathrm{z};\mathrm{w})\Big|_{x=0} \\ &= [i\partial_x]^{\lambda}\Phi\big(\mathrm{z}\,(\mathbb{1} + x\,\mathbf{e}_{k+1,k});\mathrm{w}\big)\,\mathrm{e}^{\,x\,\mathrm{D}_k^{\mathrm{z}}}\Big|_{x=0},\end{aligned}$$

we reshape the expression to the required form step by step. After the first step, we obtain

$$\begin{aligned}&\overbrace{\mathbb{S}_{n-1}}^{u_k-v_n}\;\Phi(\mathrm{z};\mathrm{w})\overbrace{\mathbb{S}_{n-1}}^{v_n-v_k} \\ &\quad= [i\partial_{y_{n-1}}]^{u_k-v_n}\;\Phi\big(\mathrm{z};\mathrm{w}\,(\mathbb{1} + y_{n-1}\,\mathbf{e}_{n,n-1})\big)\mathrm{e}^{\,y_{n-1}\,\mathrm{D}_{n-1}^{\mathrm{w}}}\,[i\mathrm{D}_{n-1}^{\mathrm{w}}]^{v_k-v_n}\Big|_{y=0} \\ &\quad= [i\partial_{y_{n-1}}]^{u_k-v_n}\;\Phi\big(\mathrm{z};\mathrm{w}\,(\mathbb{1} + y_{n-1}\,\mathbf{e}_{n,n-1})\big)[i\partial_{y_{n-1}}]^{v_k-v_n}\mathrm{e}^{\,y_{n-1}\,\mathrm{D}_{n-1}^{\mathrm{w}}}\Big|_{y=0}.\end{aligned}$$

Acting similarly, after all necessary transformations we obtain the following representation for the operation in question:

$$\begin{aligned}
&\mathrm{N}^{-1}\cdot(\overbrace{[i\partial_{x_{k-1}}]}^{u_{k-1}-v_k}\cdots\overbrace{[i\partial_{x_1}]}^{u_1-v_k})\,(\overbrace{[i\partial_{y_k}]}^{u_k-v_{k+1}}\cdots\overbrace{[i\partial_{y_{n-1}}]}^{u_k-v_n})\\
&\qquad\times\Phi'(\mathrm{z};\mathrm{w})\,(\overbrace{[i\partial_{y_{n-1}}]}^{v_n-v_k}\cdots\overbrace{[i\partial_{y_k}]}^{v_{k+1}-v_k})\,(\overbrace{[i\partial_{x_1}]}^{u_k-u_1}\cdots\overbrace{[i\partial_{x_{k-1}}]}^{u_k-u_{k-1}})\\
&\qquad\times\mathrm{e}^{\,y_{n-1}\,\mathrm{D}^{\mathrm{w}}_{n-1}}\cdots\mathrm{e}^{\,y_k\,\mathrm{D}^{\mathrm{w}}_k}\,\mathrm{e}^{\,x_1\,\mathrm{D}^{\mathrm{z}}_1}\cdots\mathrm{e}^{\,x_{k-1}\,\mathrm{D}^{\mathrm{z}}_{k-1}}\Big|_{x=y=0},
\end{aligned}$$

where the function $\Phi'(\mathrm{z};\mathrm{w})$ differs from $\Phi(\mathrm{z};\mathrm{w})$ only in the following transformation of the matrix arguments:

$$\begin{aligned}
\mathrm{z}&\to\mathrm{z}\big(\mathbb{1}+x_1\,\mathbf{e}_{2,1}\big)\cdots\big(\mathbb{1}+x_{k-1}\,\mathbf{e}_{k,k-1}\big);\\
\mathrm{w}&\to\mathrm{w}\big(\mathbb{1}+y_{n-1}\,\mathbf{e}_{n,n-1}\big)\cdots\big(\mathbb{1}+y_k\,\mathbf{e}_{k+1,k}\big).
\end{aligned}$$

It remains to calculate how this operator acts on the identity. Here, we have two considerable simplifications. First, the rightmost operator constructed from exponents disappears, because it takes the identity to itself.

Second, as was mentioned above, when calculating the contribution of the operator that acts on the variable y_{n-1}, only the coefficient of y_{n-1} in $\Phi'(\mathrm{z};\mathrm{w})$ is important; next we need the coefficient of y_{n-2}, etc. As a result, we have the following reduction in the defining matrix:

$$\begin{aligned}
&\big(\mathbb{1}-y_k\,\mathbf{e}_{k+1,k}\big)\cdots\big(\mathbb{1}-y_{n-1}\,\mathbf{e}_{n,n-1}\big)\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\big(\mathbb{1}+x_1\,\mathbf{e}_{2,1}\big)\cdots\big(\mathbb{1}+x_{k-1}\,\mathbf{e}_{k,k-1}\big)\\
&\qquad\longrightarrow(-)^{n-k+1}y_k\cdots y_{n-1}x_1\cdots x_{k-1}\cdot\mathbf{e}_{k+1,k}\cdots\mathbf{e}_{n,n-1}\,\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\,\mathbf{e}_{2,1}\cdots\mathbf{e}_{k,k-1}.
\end{aligned}$$

After these simplifications, everything splits into the product of n rank one operators considered above, whence

$$\big(\mathbf{e}_{k+1,k}\cdots\mathbf{e}_{n,n-1}\,\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\,\mathbf{e}_{2,1}\cdots\mathbf{e}_{k,k-1}\big)_{n1}^{u_k-v_k}=\big(\mathrm{w}^{-1}g_2^{-1}g_1\mathrm{z}\big)_{kk}^{u_k-v_k}.$$

Thus, we obtain the result stated above for the action of the operator R_k on the generating function of the basis monomials.

The defining relations for the operators R_k and their properties, as listed in Subsection 6.2, remain valid also for the operators R_k restricted to the space of polynomials. The proofs of the required formulas can be obtained by considering the restrictions of the corresponding formulas for continuous series to the space of polynomials. Moreover, formula (7.4) can be regarded as an independent definition of the operator R_k acting on the space of polynomials. The defining relations for the operators R_k and all their properties can be obtained directly from the definition (7.4) in the same way as we did it in the case of an algebra of rank 1. We do not list the corresponding formulas.

Passage to finite-dimensional representations is performed as in the case of an algebra of rank 1. The operators R_k do not map the finite-dimensional representation space into itself, but, for the resulting R-matrix, the representation space is invariant. The explicit formula for the entire R-matrix (an analog of (4.21)) turns out to be too cumbersome in the general case of the algebra $\mathrm{sl}(n)$.

§8. Conclusion

Now, we can describe the relationship between the factorization of the solution of the Yang–Baxter equation in the product of the operators R_k and the factorization of the transfer-matrix in the product of Q-operators.

The general picture looks like this. The main object is the R-matrix, a linear operator that depends on the spectral parameter u and acts in the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2$. The operator R is a solution of the Yang–Baxter equation

$$\mathrm{R}_{12}(u)\mathrm{R}_{13}(u+v)\mathrm{R}_{23}(v) = \mathrm{R}_{23}(v)\mathrm{R}_{13}(u+v)\mathrm{R}_{12}(u).$$

All operators act in the general tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$. The index ik shows that the operator R_{ik} acts nontrivially in the tensor product $\mathbb{V}_i \otimes \mathbb{V}_k$. The spaces $\mathbb{V}_k$ are representation spaces for the group $\mathrm{SL}(n,\mathbb{C})$ or the Lie algebra $\mathrm{sl}(n)$. A representation $\mathbb{V}_{\boldsymbol{\sigma}}$ is parametrized by a collection $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of n numbers satisfying the additional condition $\sigma_1 + \sigma_2 + \dots + \sigma_n = \frac{n(n-1)}{2}$. The operator $\mathrm{R}_{12} = \mathrm{R}_{\boldsymbol{\sigma}\boldsymbol{\rho}}$ acts in the product $\mathbb{V}_{\boldsymbol{\sigma}} \otimes \mathbb{V}_{\boldsymbol{\rho}}$. The Yang–Baxter equation implies the following simpler defining relation for the operator R:

$$\mathrm{R}_{12}(u-v)\mathrm{L}_1(u)\mathrm{L}_2(v) = \mathrm{L}_2(v)\mathrm{L}_1(u)\mathrm{R}_{12}(u-v). \tag{8.1}$$

In the defining relation, the first L-operator $\mathrm{L}_1(u)$ acts in the tensor product $\mathbb{C}^n \otimes \mathbb{V}_{\boldsymbol{\sigma}}$ and does not depend on the parameters $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$, where $u_i = u - \sigma_i$, and the second L-operator $\mathrm{L}_2(v)$ acts in the tensor product $\mathbb{C}^n \otimes \mathbb{V}_{\boldsymbol{\rho}}$ and does not depend on the parameters $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$, where $v_i = v - \rho_i$. If we extract the permutation operator $\mathrm{P}_{12} : \mathbb{V}_1 \otimes \mathbb{V}_2 \to \mathbb{V}_2 \otimes \mathbb{V}_1$ from R_{12}, then the defining equation for the operator $\check{\mathrm{R}} = \mathrm{P} \cdot \mathrm{R}$ will take the form

$$\check{\mathrm{R}} \cdot \mathrm{L}_1(\boldsymbol{u})\mathrm{L}_2(\boldsymbol{v}) = \mathrm{L}_1(\boldsymbol{v})\mathrm{L}_2(\boldsymbol{u}) \cdot \check{\mathrm{R}}. \tag{8.2}$$

Thus, the operator $\check{\mathrm{R}}$ permutes the entire set $\boldsymbol{u}$ of parameters of the first L-operator with the entire set $\boldsymbol{v}$ of parameters of the second L-operator. Such a permutation can be decomposed into a product of elementary permutations. Each elementary permutation performed by the operator $\check{\mathrm{R}}_k$ interchanges only the parameters u_k and v_k, fixing the remaining parameters in the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. We introduce the notation $\boldsymbol{u}_k = (u_1, \dots, u_{k-1}, v_k, u_{k+1}, \dots, u_n)$ for a vector in which all components but the kth coincide with the components of the vector $\boldsymbol{u}$, and at the kth position u_k is replaced by v_k. The vector $\boldsymbol{v}_k$ is defined similarly. The operators $\check{\mathrm{R}}_k$ are solutions of the corresponding defining equations,

$$\check{\mathrm{R}}_k \cdot \mathrm{L}_1(\boldsymbol{u})\mathrm{L}_2(\boldsymbol{v}) = \mathrm{L}_1(\boldsymbol{u}_k)\mathrm{L}_2(\boldsymbol{v}_k) \cdot \check{\mathrm{R}}_k. \tag{8.3}$$

The general R-matrix decomposes into a product of simpler operators R_k:

$$\mathrm{R}_{\boldsymbol{\sigma}\boldsymbol{\rho}}(u-v) = \mathrm{R}_1(u_1 - v_1) \cdot \mathrm{PR}_2(u_2 - v_2) \cdot \mathrm{PR}_3(u_3 - v_3) \cdots \mathrm{PR}_n(u_n - v_n), \tag{8.4}$$

where, in the operators $\mathrm{R}_k(u_k - v_k) = \mathrm{P}\check{\mathrm{R}}_k(u_k - v_k)$, we explicitly indicate only the dependence on the combinations $u_k - v_k$ including the spectral parameter $u - v$, and suppress the dependence on the parameters that do not contain $u - v$. If the parameters coincide, $u_k = v_k$, there is no permutation in the defining equation of the operator $\check{\mathrm{R}}_k(u_k - v_k)$. Therefore, $\check{\mathrm{R}}_k(0) = \mathbb{1}$. If all but the distinguished components u_k and v_k of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ coincide, then, in the expression for the operator $\mathrm{R}_{\boldsymbol{\sigma}\boldsymbol{\rho}}(u-v)$, only the operator $\mathrm{R}_k(u_k - v_k)$ survives in the product. We fix the vector $\boldsymbol{\sigma}$ and put $v = 0$ for simplicity. In the second representation, the parameters $\boldsymbol{\rho}$ can be chosen arbitrarily. For $\rho_i = \sigma_i - u$, $i \neq k$, and $\rho_k = \sigma_k + (n-1)u$, we obtain $u_i = v_i (v = 0)$ for $i \neq k$. Thus, there exist n points of degeneration,

$$\boldsymbol{\rho}_k = (\sigma_1 - u, \sigma_2 - u, \dots, \sigma_k + u \cdot (n-1), \dots, \sigma_n - u)\,,$$

at which the operator $\mathrm{R}_{\boldsymbol{\sigma}\boldsymbol{\rho}}$ transforms into the operator R_k,

$$\mathrm{R}_{\boldsymbol{\sigma}\boldsymbol{\rho}_k}(u) = \mathrm{R}_k\left(u \cdot (n-1)\right).$$

The R-matrix $\mathrm{R}_{12}(u)$ is used as an elementary building block in the construction of the transfer matrix:

$$\mathsf{T}_{\boldsymbol{\rho}}(u) = \mathrm{tr}_{\mathbb{V}_0}\, \mathrm{R}_{10}(u) \cdots \mathrm{R}_{N0}(u), \tag{8.5}$$

where the trace is calculated with respect to the auxiliary space $\mathbb{V}_0 = \mathbb{V}_{\boldsymbol{\rho}}$. The transfer matrix acts in the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_N$ of representations. We restrict our consideration to a homogeneous spin chain, where the representations $\mathbb{V}_k$ are parametrized by one and the same vector $\boldsymbol{\sigma}$, $\mathbb{V}_k = \mathbb{V}_{\boldsymbol{\sigma}}$. Since the operator $\mathrm{R}_{12}(u)$ is a solution of the Yang–Baxter equation, the operators $\mathsf{T}_{\boldsymbol{\rho}}(u)$ commute with each other. It turns out that the properties of the operator $\mathrm{R}(u)$ are inherited by the operator $\mathsf{T}_{\boldsymbol{\rho}}(u)$.

- Like the operator $\mathrm{R}(u)$, the operator $\mathsf{T}_{\boldsymbol{\rho}}(u)$ can be represented as a product of simpler commuting Q-operators:

$$\mathsf{T}_{\boldsymbol{\rho}}(u) = \mathrm{Q}_1(u+\rho_1) \cdot \mathcal{P}\mathrm{Q}_2(u+\rho_2) \cdot \mathcal{P}\mathrm{Q}_3(u+\rho_3) \cdots \mathcal{P}\mathrm{Q}_n(u+\rho_n), \tag{8.6}$$

  where $\mathcal{P}$ is the cyclic translation operator $\mathcal{P}, \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_N \to \mathbb{V}_2 \otimes \mathbb{V}_3 \otimes \cdots \otimes \mathbb{V}_N \otimes \mathbb{V}_1$.
- Every operator $\mathrm{Q}_k(u)$ is constructed from the operators $\mathrm{R}^k_{i0}(u) : \mathbb{V}_i \otimes \mathbb{V}_0 \to \mathbb{V}_i \otimes \mathbb{V}_0$:

$$\mathrm{Q}_k(u) = \mathrm{tr}\, \mathrm{R}^k_{10}(u) \cdot \mathrm{R}^k_{20}(u) \cdots \mathrm{R}^k_{N0}(u), \tag{8.7}$$

  where the lower indices indicate the spaces in which the operators R^k_{i0} act.
- Moreover, in the same way as the operator $\mathrm{R}_{\boldsymbol{\sigma}\boldsymbol{\rho}}$ turns into the operator R_k at specific points $\boldsymbol{\rho}_k$, the transfer matrix turns into one of the Q-operators for a specific choice of the parameters $\boldsymbol{\rho}_k$ in the auxiliary space,

$$\mathsf{T}_{\boldsymbol{\rho}_k}(u) = \mathrm{Q}_k\left(nu + \sigma_k\right). \tag{8.8}$$

In an oncoming paper, we shall prove these statements and obtain equations satisfied by the operators Q_k.

At specific integral points where the parameters involved in $\boldsymbol{\rho}$ are chosen so that the differences $\rho_{k,k+1}$ are positive integers, a finite-dimensional representation splits off from the infinite-dimensional representation $\mathbb{V}_{\boldsymbol{\rho}}$ [49]. The transfer matrix $\mathrm{t}_{\boldsymbol{\rho}}(u)$ for which the auxiliary space is a finite-dimensional representation can be expressed in terms of the transfer matrices $\mathsf{T}_{\boldsymbol{\rho}}(u)$:

$$\mathrm{t}_{\boldsymbol{\rho}}(u) = \sum_{\mathrm{P}} (-)^{\mathrm{sign}(\mathrm{P})} \cdot \mathsf{T}_{\mathrm{P}\boldsymbol{\rho}}(u), \tag{8.9}$$

where $\mathrm{P}\boldsymbol{\rho} = (\rho_{k_1}, \rho_{k_2}, \ldots, \rho_{k_n})$ is an arbitrary permutation of the numbers $(\rho_1, \rho_2, \ldots, \rho_n)$. Summation is taken over all permutations, and $\mathrm{sign}(\mathrm{P})$ is the parity of P. Now, if we decompose $\mathsf{T}_{\boldsymbol{\rho}}(u)$ into a product of Q-operators, we obtain a representation of the transfer matrix $\mathrm{t}_{\boldsymbol{\rho}}(u)$ in the following form:

$$\mathcal{P}^{1-n} \cdot \mathrm{t}_{\boldsymbol{\rho}}(u) = \begin{vmatrix} \mathrm{Q}_1(u+\rho_1) & \mathrm{Q}_2(u+\rho_1) & \cdots & \mathrm{Q}_n(u+\rho_1) \\ \mathrm{Q}_1(u+\rho_2) & \mathrm{Q}_2(u+\rho_2) & \cdots & \mathrm{Q}_n(u+\rho_2) \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Q}_1(u+\rho_n) & \mathrm{Q}_2(u+\rho_n) & \cdots & \mathrm{Q}_n(u+\rho_n) \end{vmatrix}. \tag{8.10}$$

This representation is a source of general relations among Q-operators and transfer matrices with different finite-dimensional auxiliary spaces.

Acknowledgments. We want to thank P. P. Kulish, M. A. Semenov-Tyan-Shanskii, and V. O. Tarasov for their interest in our work, numerous discussions, and critical remarks.

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Received 19/NOV/2008

Translated by B. M. BEKKER