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ON THE UNIVERSAL WEIGHT FUNCTION FOR THE QUANTUM AFFINE ALGEBRA $U_q(\widehat{\mathfrak{gl}}_N)$

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Abstract. The investigation is continued of the universal weight function for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. Two recurrence relations are obtained for the universal weight function with the help of the method of projections. On the level of the evaluation representation of $U_q(\widehat{\mathfrak{gl}}_N)$, two recurrence relations are reproduced, which were calculated earlier for the off-shell Bethe vectors by combinatorial methods. One of the results of the paper is a description of two different but isomorphic currents or "new" realizations of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$, corresponding to two different Gauss decompositions of the fundamental L-operators.

§1. Introduction

The hierarchical (nested) Bethe ansatz was designed in [13] to construct the eigenvectors of the commuting integrals for quantum integrable models associated with the Lie algebra $\mathfrak{gl}_N$. It is based on the inductive procedure that relates $\mathfrak{gl}_N$ and $\mathfrak{gl}_{N-1}$ Bethe vectors. Since the Bethe vectors for models with $\mathfrak{gl}_2$ symmetry are known, this hierarchical procedure yields an implicit description of the Bethe vectors of the models with symmetry of higher rank.

If the parameters of these vectors satisfy the Bethe equations, then the corresponding vectors are eigenvectors of a commuting set of operators in some quantum integrable model. In the current paper, we consider Bethe vectors with free parameters, the so-called "off-shell Bethe vectors".

Further development of the off-shell Bethe vectors theory was achieved in [17], where they were presented as particular matrix elements of monodromy operators. This construction was used in [18] to obtain explicit formulas for the off-shell Bethe vectors on the tensor product of evaluation modules of $U_q(\widehat{\mathfrak{gl}}_N)$. Two different recurrence relations for the off-shell Bethe vectors on the evaluation $U_q(\widehat{\mathfrak{gl}}_N)$-modules were obtained in [18]. Iteration of these two relations allows us to obtain different explicit formulas for the off-shell Bethe vectors (see examples (2.18) and (2.20) below). The existence of two types of recurrence relations in the nested Bethe ansatz is a consequence of two different ways of embedding $U_q(\widehat{\mathfrak{gl}}_{N-1})$ in $U_q(\widehat{\mathfrak{gl}}_N)$. When these algebras are realized in terms of L-operators, the L-operator of $U_q(\widehat{\mathfrak{gl}}_{N-1})$ can be placed either into the top-left or into the down-right corner of the $U_q(\widehat{\mathfrak{gl}}_N)$ L-operator. Due to applications to the theory of quantized Knizhnik–Zamolodchikov equations, the off-shell Bethe vectors are called *weight functions*. We use both names for these objects. We say that a weight function is *universal* if it is defined in an arbitrary $U_q(\widehat{\mathfrak{gl}}_N)$-module generated by an arbitrary singular vector.

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An alternative approach to the construction of off-shell Bethe vectors for an arbitrary quantum affine algebra was developed in [8]. This approach involves the current realization of the corresponding symmetry algebra or a "new" realization of the quantum double of the Yangians and quantum affine algebras [2]. Due to the existence of two isomorphic [3] realizations of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ in terms of L-operators [16] and in terms of currents [2], it is possible to compare two different methods of the Bethe vectors construction in this case. Two different types of Borel subalgebras are related to these two realizations of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$.

It was conjectured in [11] that the projections onto the intersection of different type Borel subalgebras for the product of Drinfeld currents coincide with the off-shell Bethe vectors [17]. The background for this conjecture was the observation that both quantities satisfy the same coproduct property [8]. It was proved in [12] that calculation of the projection of the product of currents gives nested recurrence relations for the off-shell Bethe vectors similar to those obtained in [18] at the level of the tensor product of the evaluation $U_q(\widehat{\mathfrak{gl}}_N)$-modules. This leads to the conclusion that the projection method yields the universal off-shell Bethe vectors for an arbitrary $U_q(\widehat{\mathfrak{gl}}_N)$-module generated by a singular weight vector. In [12], only one type of the recurrence relations that lead to a formula of type (2.18) was considered. Here we generalize the results of [12] and also prove that, in order to get both types of recurrence relations, one needs to use two different isomorphic current realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. The origin of these two different realizations lies in two possibilities for introducing the Gauss decompositions of L-operators (see (3.1)–(3.3) and (3.4)–(3.6) below).

The paper is organized as follows. §2 serves as a reminder of the L-operator realization of $U_q(\widehat{\mathfrak{gl}}_N)$ and the Tarasov–Varchenko construction of the off-shell Bethe vectors in terms of matrix entries of these L-operators [17]. In §3 two different Gauss decompositions are introduced, together with the corresponding current realizations of $U_q(\widehat{\mathfrak{gl}}_N)$. Here we introduce the current Borel subalgebras and describe the projections onto the intersections of the standard and current Borel subalgebras. §4 is devoted to the calculation of the projections for the product of currents. Here, the main result is Theorem 1, which presents the universal weight functions as sums over the ordered products of projections of the simple and composed root currents. Using the fact that the projections of the composed root currents coincide with the Gauss coordinates of L-operators, in §5 we give explicit expressions for the universal weight vectors in terms of the matrix entries of L-operators, generalizing formulas of [18]. The main result of the paper is formulated in the form of Theorem 3 in §6. The paper contains two Appendices. Appendix A is devoted to reformulation of the Serre relations in the form of commutation relations between composed currents. We need these relations to prove Proposition B.1, which is the most difficult technical result of the paper. This proposition, formulated and proved in Appendix B, describes the ordering of the currents and the negative projections of them.

§2. Tarasov–Varchenko construction

2.1. $U_q(\widehat{\mathfrak{gl}}_N)$ in the L-operator formalism. Let $\mathrm{E}_{ij} \in \mathrm{End}(\mathbb{C}^N)$ be a matrix with the only nonzero entry equal to 1 at the intersection of the ith row and jth column. Consider a complex parameter q not equal to zero or a root of unity, and let $\mathrm{R}(u,v) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[[v/u]]$,

$$\begin{aligned}\mathrm{R}(u,v) = \sum_{1\le i\le N} \mathrm{E}_{ii}\otimes \mathrm{E}_{ii} &+ \frac{u-v}{qu-q^{-1}v} \sum_{1\le i<j\le N} (\mathrm{E}_{ii}\otimes \mathrm{E}_{jj} + \mathrm{E}_{jj}\otimes \mathrm{E}_{ii}) \\ &+ \frac{q-q^{-1}}{qu-q^{-1}v} \sum_{1\le i<j\le N} (u\mathrm{E}_{ij}\otimes \mathrm{E}_{ji} + v\mathrm{E}_{ji}\otimes \mathrm{E}_{ij}),\end{aligned} \tag{2.1}$$

which is the trigonometric R-matrix associated with the vector representation of $\mathfrak{gl}_N$. It satisfies the quantum Yang–Baxter equation

$$R_{12}(u_1,u_2)R_{13}(u_1,u_3)R_{23}(u_2,u_3) = R_{23}(u_2,u_3)R_{13}(u_1,u_3)R_{12}(u_1,u_2). \tag{2.2}$$

The algebra $U_q(\widehat{\mathfrak{gl}}_N)$ (with the zero central charge and the gradation operator dropped out) is an associative algebra with the unit generated by the modes $\mathrm{L}^{\pm}_{i,j}[\pm k]$, $k \geq 0$, $1 \leq i,j \leq N$ of the L-operators[1] $\mathrm{L}^{\pm}(z) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{N} \mathrm{E}_{ij} \otimes \mathrm{L}^{\pm}_{i,j}[\pm k] z^{\mp k}$, which satisfy the relations

$$\begin{aligned} &\mathrm{R}(u,v)\cdot(\mathrm{L}^{\pm}(u)\otimes \mathbf{1})\cdot(\mathbf{1}\otimes \mathrm{L}^{\pm}(v)) = (\mathbf{1}\otimes \mathrm{L}^{\pm}(v))\cdot(\mathrm{L}^{\pm}(u)\otimes \mathbf{1})\cdot \mathrm{R}(u,v),\\ &\mathrm{R}(u,v)\cdot(\mathrm{L}^{+}(u)\otimes \mathbf{1})\cdot(\mathbf{1}\otimes \mathrm{L}^{-}(v)) = (\mathbf{1}\otimes \mathrm{L}^{-}(v))\cdot(\mathrm{L}^{+}(u)\otimes \mathbf{1})\cdot \mathrm{R}(u,v), \end{aligned} \tag{2.3}$$

$$\mathrm{L}^{+}_{j,i}[0] = \mathrm{L}^{-}_{i,j}[0] = 0, \qquad \mathrm{L}^{+}_{k,k}[0]\mathrm{L}^{-}_{k,k}[0] = 1, \qquad 1 \leq i < j \leq N, \quad 1 \leq k \leq N. \tag{2.4}$$

The subalgebras formed by the modes $\mathrm{L}^{\pm}[n]$ of the L-operators $\mathrm{L}^{\pm}(t)$ are the standard Borel subalgebras $U_q(\mathfrak{b}^{\pm}) \subset U_q(\widehat{\mathfrak{gl}}_N)$. These Borel subalgebras are Hopf subalgebras of $U_q(\widehat{\mathfrak{gl}}_N)$. The coalgebraic structure of these subalgebras is given by the formulas

$$\Delta\big(\mathrm{L}^{\pm}_{i,j}(u)\big) = \sum_{k=1}^{N} \mathrm{L}^{\pm}_{k,j}(u) \otimes \mathrm{L}^{\pm}_{i,k}(u). \tag{2.5}$$

2.2. Evaluation homomorphism of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ onto $U_q(\mathfrak{gl}_N)$. Let $\mathsf{E}_{a,a}$, $\mathsf{E}_{a,a+1}$ and $\mathsf{E}_{a+1,a}$ be Chevalley generators of the algebra $U_q(\mathfrak{gl}_N)$ that satisfy the commutation relations

$$\mathsf{E}_{a,a}\mathsf{E}_{b,c}\mathsf{E}_{a,a}^{-1} = q^{\delta_{ab}-\delta_{ac}}\mathsf{E}_{b,c}, \qquad [\mathsf{E}_{a,a+1},\mathsf{E}_{b+1,b}] = \delta_{ab}\frac{\mathsf{E}_{a,a}\mathsf{E}_{a+1,a+1}^{-1} - \mathsf{E}_{a,a}^{-1}\mathsf{E}_{a+1,a+1}}{q-q^{-1}},$$

and the Serre relations

$$\begin{aligned} &\mathsf{E}^2_{a\pm1,a}\mathsf{E}_{a,a\mp1} - (q+q^{-1})\mathsf{E}_{a\pm1,a}\mathsf{E}_{a,a\mp1}\mathsf{E}_{a\pm1,a} + \mathsf{E}_{a,a-1}\mathsf{E}^2_{a\pm1,a} = 0,\\ &\mathsf{E}^2_{a,a\pm1}\mathsf{E}_{a\mp1,a} - (q+q^{-1})\mathsf{E}_{a,a\pm1}\mathsf{E}_{a\mp1,a}\mathsf{E}_{a,a\pm1} + \mathsf{E}_{a\mp1,a}\mathsf{E}^2_{a,a\pm1} = 0. \end{aligned} \tag{2.6}$$

The evaluation homomorphism of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ onto $U_q(\mathfrak{gl}_N)$ is defined by

$$\mathcal{E}v_z\big(\mathrm{L}^{+}(u)\big) = \mathrm{L}^{+} - \frac{z}{u}\,\mathrm{L}^{-}, \quad \mathcal{E}v_z\big(\mathrm{L}^{-}(u)\big) = \mathrm{L}^{-} - \frac{u}{z}\,\mathrm{L}^{+}, \tag{2.7}$$

where

$$\mathrm{L}^{+} = \begin{pmatrix} \mathsf{E}_{1,1} & \nu\mathsf{E}_{2,1}\mathsf{E}_{2,2} & \cdots & \nu\mathsf{E}_{N,1}\mathsf{E}_{N,N}\\ & \ddots & \ddots & \vdots\\ & 0 & \mathsf{E}_{N-1,N-1} & \nu\mathsf{E}_{N,N-1}\mathsf{E}_{N,N}\\ & & & \mathsf{E}_{N,N} \end{pmatrix},$$

$$\mathrm{L}^{-} = \begin{pmatrix} \mathsf{E}_{1,1}^{-1} & & & \\ -\nu\mathsf{E}_{2,2}^{-1}\mathsf{E}_{1,2} & \mathsf{E}_{2,2}^{-1} & 0 & \\ \vdots & \ddots & \ddots & \\ -\nu\mathsf{E}_{N,N}^{-1}\mathsf{E}_{1,N} & \cdots & -\nu\mathsf{E}_{N,N}^{-1}\mathsf{E}_{N-1,N} & \mathsf{E}_{N,N}^{-1} \end{pmatrix},$$

$\nu = q - q^{-1}$, and the composed root generators are defined as follows:

$$\begin{aligned} &\mathsf{E}_{c,a} = \mathsf{E}_{c,b}\mathsf{E}_{b,a} - q^{-1}\mathsf{E}_{b,a}\mathsf{E}_{c,b},\\ &\mathsf{E}_{a,c} = \mathsf{E}_{a,b}\mathsf{E}_{b,c} - q\mathsf{E}_{b,c}\mathsf{E}_{a,b}, \qquad a < b < c. \end{aligned}$$

[1] In this paper, we use the R-matrix and L-operators of the paper [18]. A discussion on the relationship between the choices of L-operators in the papers [18] and [11] can be found in [12].

These formulas can be checked inductively by substitution of (2.7) in (2.3). The coproduct of the L-operators (2.5) determines the coproduct of Chevalley generators

$$\begin{aligned}\Delta \mathsf{E}_{a,a} &= \mathsf{E}_{a,a}\otimes \mathsf{E}_{a,a},\\ \Delta \mathsf{E}_{a,a+1} &= \mathsf{E}_{a,a+1}\otimes 1 + \mathsf{E}_{a,a}^{-1}\mathsf{E}_{a+1,a+1}\otimes \mathsf{E}_{a,a+1},\\ \Delta \mathsf{E}_{a+1,a} &= 1\otimes \mathsf{E}_{a+1,a} + \mathsf{E}_{a+1,a}\otimes \mathsf{E}_{a,a}\mathsf{E}_{a+1,a+1}^{-1}.\end{aligned}$$

2.3. Combinatorial formulas for off-shell Bethe vectors. We recall the construction of the off-shell Bethe vectors [17]. Suppose that the L-operator[2]

$$\mathrm{L}(z) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{N} \mathrm{E}_{ij}\otimes \mathrm{L}_{i,j}[k]z^{-k}$$

of the Borel subalgebra $U_q(\mathfrak{b}^+)$ of $U_q(\widehat{\mathfrak{gl}}_N)$ satisfies the Yang–Baxter commutation relation with an R-matrix $\mathrm{R}(u,v)$. We use the notation $\mathrm{L}^{(k)}(z)\in \big(\mathbb{C}^N\big)^{\otimes M}\otimes U_q(\mathfrak{b}^+)$ for the L-operator acting nontrivially on the kth tensor factor in the product $\big(\mathbb{C}^N\big)^{\otimes M}$, $1\le k\le M$. Consider the following series in M variables:

$$\mathbb{T}(u_1,\dots,u_M) = \mathrm{L}^{(1)}(u_1)\cdots \mathrm{L}^{(M)}(u_M)\cdot \mathbb{R}^{(M,\dots,1)}(u_M,\dots,u_1) \tag{2.8}$$

with coefficients in $\big(\mathrm{End}(\mathbb{C}^N)\big)^{\otimes M}\otimes U_q(\mathfrak{b}^+)$, where

$$\mathbb{R}^{(M,\dots,1)}(u_M,\dots,u_1) = \overleftarrow{\prod_{M\ge j>1}}\ \overleftarrow{\prod_{j>i\ge 1}} \mathrm{R}^{(ji)}(u_j,u_i). \tag{2.9}$$

In the ordered product (2.9) of R-matrices, the factor $\mathrm{R}^{(ji)}$ is to the left of the factor $\mathrm{R}^{(ml)}$ if $j>m$, or $j=m$ and $i>l$. Consider the set of variables

$$\bar{t}_{[\bar{n}]} = \left\{t_1^1,\dots,t_{n_1}^1;t_1^2,\dots,t_{n_2}^2;\dots\dots;t_1^{N-2},\dots,t_{n_{N-2}}^{N-2};t_1^{N-1},\dots,t_{n_{N-1}}^{N-1}\right\}. \tag{2.10}$$

As in [17], we put

$$\begin{aligned}\mathbb{B}(\bar{t}_{[\bar{n}]}) = &\prod_{1\le a<b\le N-1}\prod_{1\le j\le n_b}\prod_{1\le i\le n_a} \frac{qt_j^b - q^{-1}t_i^a}{t_j^b - t_i^a}(\mathrm{tr}_{(\mathbb{C}^N)^{\otimes|\bar{n}|}}\otimes \mathrm{id})\\ &\times\Big(\mathbb{T}(t_1^1,\dots,t_{n_1}^1;\dots;t_1^{N-1},\dots,t_{n_{N-1}}^{N-1})\mathrm{E}_{21}^{\otimes n_1}\otimes\cdots\otimes \mathrm{E}_{N\,N-1}^{\otimes n_{N-1}}\otimes 1\Big),\end{aligned} \tag{2.11}$$

where $|\bar{n}| = n_1+\cdots+n_{N-1}$. The element $\mathbb{T}(\bar{t}_{\bar{n}})$ in (2.11) is given by (2.8) with the following identification: $M=|\bar{n}|$ and $u_i = t^a_{i-n_1-\cdots-n_{a-1}}$ for $a=1,\dots,N-1$ and $n_1+\cdots+n_{a-1}<i\le n_1+\cdots+n_a$. The coefficients of $\mathbb{B}(\bar{t}_{\bar{n}})$ are elements of the Borel subalgebra $U_q(\mathfrak{b}^+)$.

For the collection of positive integers $\bar{n}=\{n_1,\dots,n_{N-1}\}$, we consider the direct product $S_{\bar{n}} = S_{n_1}\times\cdots\times S_{n_{N-1}}$ of symmetric groups. For any function $G(\bar{t}_{[\bar{n}]})$, we denote by

$$\mathrm{Sym}_{\bar{t}_{[\bar{n}]}}\, G(\bar{t}_{[\bar{n}]}) = \sum_{\sigma\in S_{\bar{n}}} G({}^{\sigma}\bar{t}_{[\bar{n}]}) \tag{2.12}$$

the symmetrization over groups of variables $\{t_1^a,\dots,t_{n_a}^a\}$ of type a, where

$${}^{\sigma}\bar{t}_{[\bar{n}]} = \{t^1_{\sigma^1(1)},\dots,t^1_{\sigma^1(n_1)};\dots;t^{N-1}_{\sigma^{N-1}(1)},\dots,t^{N-1}_{\sigma^{N-1}(n_{N-1})}\}. \tag{2.13}$$

[2]We omit the superscript + in this L-operator, because only the positive standard Borel subalgebra $U_q(\mathfrak{b}^+)$ will be considered here and below.

Let

$$\beta(\bar{t}_{[\bar{n}]}) = \prod_{a=1}^{N-1} \prod_{1\le \ell<\ell'\le n_a} \frac{q^{-1}t^a_\ell - qt^a_{\ell'}}{t^a_\ell - t^a_{\ell'}} \tag{2.14}$$

be a function of the formal variables t^a_ℓ. As in [18], by the q-symmetrization of an arbitrary function $G(\bar{t}_{[\bar{n}]})$ we mean

$$\overline{\mathrm{Sym}}^{(q)}_{\bar{t}_{[\bar{n}]}} G(\bar{t}_{[\bar{n}]}) = \mathrm{Sym}_{\bar{t}_{[\bar{n}]}} \big(\beta(\bar{t}_{[\bar{n}]}) \cdot G(\bar{t}_{[\bar{n}]})\big). \tag{2.15}$$

A vector v is called a *weight singular vector* if $\mathrm{L}^+_{i,j}[n]v = 0$, $i > j$, $n \ge 0$, and v is an eigenvector of the diagonal matrix entries $\mathrm{L}^+_{i,i}(z)$:

$$\mathrm{L}^+_{i,j}(z)v = 0, \quad i > j, \quad \mathrm{L}^+_{i,i}(z)v = \lambda_i(z)v, \quad i = 1, \dots, N. \tag{2.16}$$

For any $U_q(\widehat{\mathfrak{gl}}_N)$-module V with a singular vector v, we denote

$$\mathbb{B}_V(\bar{t}_{[\bar{n}]}) = \mathbb{B}(\bar{t}_{[\bar{n}]})v. \tag{2.17}$$

In [17, 18], the vector-valued function $\mathbb{B}_V(\bar{t}_{[\bar{n}]})$ was called the *universal off-shell Bethe vector*.

Let $M_\Lambda(z)$ be an evaluation module generated by a singular vector v such that $\mathsf{E}_{a,a}v = q^{\Lambda_a}v$. By analysis of the relations of the hierarchical Bethe ansatz [13], the authors of [18] obtained two recurrence relations for the off-shell Bethe vectors $\mathbb{B}_{M_\Lambda(z)}(\bar{t}_{[\bar{n}]})$. By iterating these relations, many equivalent formulas for these objects can be found in terms of the $U_q(\mathfrak{gl}_N)$ generators $\mathsf{E}_{a,b}$. Two extreme cases were presented in [18]. First, the off-shell Bethe vector can be written as

$$\begin{aligned}\mathbb{B}_{M_\Lambda(z)}(\bar{t}_{[\bar{n}]}) &= (q-q^{-1})^{\sum_{a=1}^{N-1} n_a} \sum_{[[\bar{s}]]} \Bigg(\Bigg(\overleftarrow{\prod_{N-1\ge b>a\ge 1}} \frac{q^{s^b_{a-1}(s^b_{a-1}-s^b_a)}}{[s^b_a - s^b_{a-1}]_q!} \check{\mathsf{E}}^{s^b_a - s^b_{a-1}}_{b+1,a}\Bigg)v \\ &\times \overline{\mathrm{Sym}}^{(q)}_{\bar{t}_{[\bar{n}]}} \Bigg(\prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s^b_a} \frac{q^{\Lambda_{a+1}} t^a_{\ell+\tilde{\mathbf{s}}^b_a} - q^{-\Lambda_{a+1}} z}{t^{a+1}_{\ell+\tilde{\mathbf{s}}^b_{a+1}} - t^a_{\ell+\tilde{\mathbf{s}}^b_a}} \prod_{\ell'=1}^{\ell+\tilde{\mathbf{s}}^b_{a+1}-1} \frac{qt^{a+1}_{\ell'} - q^{-1}t^a_{\ell+\tilde{\mathbf{s}}^b_a}}{t^{a+1}_{\ell'} - t^a_{\ell+\tilde{\mathbf{s}}^b_a}}\Bigg)\Bigg),\end{aligned} \tag{2.18}$$

where $\check{\mathsf{E}}_{b+1,a} = \mathsf{E}_{b+1,a}\mathsf{E}_{b+1,b+1}$ and the sum is taken over all possible collections of nonnegative integers $[[\bar{s}]] = \{s^j_i\}$ such that

$$0 = s^a_0 \le s^a_1 \le \dots \le s^a_a, \quad n_a = \sum_{b=a}^{N-1} s^b_a, \quad a = 1, \dots, N-1, \tag{2.19}$$

and $\tilde{\mathbf{s}}^b_a = s^a_a + s^{a+1}_a + \dots + s^{b-1}_a$. Second, the same off-shell Bethe vector $\mathbb{B}_{M_\Lambda(z)}(\bar{t}_{[\bar{n}]})$ has a different presentation

$$\begin{aligned}\mathbb{B}_{M_\Lambda(z)}(\bar{t}_{[\bar{n}]}) &= (q-q^{-1})^{\sum_{a=1}^{N-1} n_a} \sum_{[[\bar{m}]]} \Bigg(\Bigg(\overrightarrow{\prod_{1\le b\le a\le N-1}} \frac{q^{m^b_{a+1}(m^b_a - m^b_{a+1})}}{[m^b_a - m^b_{a+1}]_q!} \check{\mathsf{E}}^{m^b_a - m^b_{a+1}}_{a+1,b}\Bigg)v \\ &\times \overline{\mathrm{Sym}}^{(q)}_{\bar{t}_{[\bar{n}]}} \Bigg(\prod_{a=2}^{N-1}\prod_{b=1}^{a-1}\prod_{\ell=0}^{m^b_a-1} \frac{q^{\Lambda_a} t^a_{\mathbf{m}^b_a-\ell} - q^{-\Lambda_a} z}{t^{a-1}_{\mathbf{m}^b_{a-1}-\ell} - t^a_{\mathbf{m}^b_a-\ell}} \prod_{\ell'=\mathbf{m}^b_{a-1}-\ell+1}^{n_{a-1}} \frac{qt^a_{\mathbf{m}^b_a-\ell} - q^{-1}t^{a-1}_{\ell'}}{t^a_{\mathbf{m}^b_a-\ell} - t^{a-1}_{\ell'}}\Bigg)\Bigg),\end{aligned} \tag{2.20}$$

where $[[\bar{m}]] = \{m^j_i\}$ is a collection of nonnegative integers such that

$$m^a_a \ge m^a_{a+1} \ge \dots \ge m^a_{N-1} \ge m^a_N = 0, \quad n_a = \sum_{b=1}^{a} m^b_a, \quad a = 1, \dots, N-1, \tag{2.21}$$

and $\mathbf{m}^b_a = m^1_a + m^2_a + \dots + m^b_a$. The ordering of the product of the noncommutative entries in (2.18) is the same as in (2.9) and the ordering in (2.20) is inverse.

A more general formula for the universal off-shell Bethe vectors was obtained in [12] by using the current realization of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ and the method of projections introduced in [9] and developed in [8]. Formula (2.18) was obtained in the latter paper after specialization to the evaluation modules. Our goal in the present paper is to describe the recurrence relations for the universal off-shell Bethe vectors or universal weight functions in terms of the modes of the $U_q(\widehat{\mathfrak{gl}}_N)$ currents. For this, we need to introduce two different current realizations of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$.

§3. Different type Borel subalgebras

3.1. Two Gauss decompositions of L-operators. The relationship between the L-operator realization [16] of $U_q(\widehat{\mathfrak{gl}}_N)$ and its current realization [2] has been known since the paper [3] was published. The main distinction in these two realizations of the same algebra lies in the different choice of the Borel subalgebras and the corresponding coalgebraic structures. To build an isomorphism between two realizations, we need to consider the Gauss decomposition [3] of the L-operators and identify linear combinations of certain Gauss coordinates with the total currents of $U_q(\widehat{\mathfrak{gl}}_N)$ corresponding to the simple roots of $\mathfrak{gl}_N$. In order to construct the universal off-shell Bethe vectors in terms of modes of currents, one must consider the ordered product of simple root currents and calculate the projection of this product onto the intersection of the current and the standard Borel subalgebras in $U_q(\widehat{\mathfrak{gl}}_N)$. The corresponding current Borel subalgebras will be introduced in Subsection 3.3.

For the L-operators fixed by relations (2.3) and (2.4), we have two possibilities to introduce the Gauss coordinates $F^{\pm}_{j,i}(t)$, $E^{\pm}_{j,i}(t)$, $j>i$, and $k^{\pm}_c(t)$:

$$\mathrm{L}^{\pm}_{i,j}(t) = F^{\pm}_{j,i}(t)k^{+}_i(t) + \sum_{1\le m<i} F^{\pm}_{j,m}(t)k^{\pm}_m(t)E^{\pm}_{m,i}(t), \qquad i<j, \tag{3.1}$$

$$\mathrm{L}^{\pm}_{i,i}(t) = k^{\pm}_i(t) + \sum_{1\le m<i} F^{\pm}_{i,m}(t)k^{\pm}_m(t)E^{\pm}_{m,i}(t), \tag{3.2}$$

$$\mathrm{L}^{\pm}_{i,j}(t) = k^{\pm}_j(t)E^{\pm}_{j,i}(t) + \sum_{1\le m<j} F^{\pm}_{j,m}(t)k^{\pm}_m(t)E^{\pm}_{m,i}(t), \qquad i>j, \tag{3.3}$$

or

$$\mathrm{L}^{\pm}_{i,j}(t) = \widehat{\mathrm{F}}^{\pm}_{j,i}(t)\widehat{k}^{+}_j(t) + \sum_{j<m\le N} \widehat{\mathrm{F}}^{\pm}_{m,i}(t)\widehat{k}^{+}_m(t)\widehat{\mathrm{E}}^{\pm}_{j,m}(t), \qquad i<j, \tag{3.4}$$

$$\mathrm{L}^{\pm}_{i,i}(t) = \widehat{k}^{\pm}_i(t) + \sum_{i<m\le N} \widehat{\mathrm{F}}^{\pm}_{m,i}(t)\widehat{k}^{\pm}_m(t)\widehat{\mathrm{E}}^{\pm}_{i,m}(t), \tag{3.5}$$

$$\mathrm{L}^{\pm}_{i,j}(t) = \widehat{k}^{\pm}_i(t)\widehat{\mathrm{E}}^{\pm}_{j,i}(t) + \sum_{i<m\le N} \widehat{\mathrm{F}}^{\pm}_{m,i}(t)\widehat{k}^{\pm}_m(t)\widehat{\mathrm{E}}^{\pm}_{j,m}(t), \qquad i>j. \tag{3.6}$$

Formulas (3.1)–(3.6) can be inverted to express the Gauss coordinates in terms of the matrix entries of L-operators, as well as to express one set of Gauss coordinates in terms of the other. This is possible because

$$\mathrm{L}^{\pm}_{i,i}[0] = k^{\pm}_i[0] = \widehat{k}^{\pm}_i[0], \qquad k^{+}_i[0]\,k^{-}_i[0] = 1, \tag{3.7}$$

and we have the following mode expansion of the Gauss coordinates:

$$F^{\pm}_{j,i}(t) = \sum_{\substack{n\ge 0\\ n<0}} F^{\pm}_{j,i}[n]t^{-n}, \qquad E^{\pm}_{i,j}(t) = \sum_{\substack{n>0\\ n\le 0}} E^{\pm}_{i,j}[n]t^{-n}, \qquad i<j, \tag{3.8}$$

which follows from the mode expansion (2.4) of the L-operators. The same rules of mode expansion are valid for the other set of Gauss coordinates $\widehat{\mathrm{F}}^{\pm}_{j,i}(t)$ and $\widehat{\mathrm{E}}^{\pm}_{i,j}(t)$.

It is convenient to introduce the following notation for the expression of the Gauss coordinates in terms of the matrix entries of the L-operators [14]. Let T be a matrix of size $(n+m)\times(n+m)$ with noncommutative entries, presented in the form

$$T=\begin{pmatrix} A & B\\ C & D\end{pmatrix},$$

where A, B, C, and D are matrices of size $n\times n$, $n\times m$, $m\times n$, and $m\times m$, respectively. Supposing that A and D are invertible matrices, we introduce the following notation:

$$\begin{vmatrix} \boxed{A} & B\\ C & D\end{vmatrix} := A-BD^{-1}C, \qquad \begin{vmatrix} A & B\\ C & \boxed{D}\end{vmatrix} := \widetilde{D},$$

where $\widetilde{D}$ is an $(m\times m)$-matrix with the entries $\widetilde{D}_{ij}=D_{ij}-\sum_{k,l=1}^{n} B_{lj}(A^{-1})_{kl}C_{ik}$. Then the Gauss coordinates can be expressed through the L-operator entries as follows:

$$k_i^{\pm}(u)=\begin{vmatrix} \mathrm{L}_{11}^{\pm}(u) & \dots & \mathrm{L}_{1,i-1}^{\pm}(u) & \mathrm{L}_{1,i}^{\pm}(u)\\ \vdots & & \vdots & \vdots\\ \mathrm{L}_{i-1,1}^{\pm}(u) & \dots & \mathrm{L}_{i-1,i-1}^{\pm}(u) & \mathrm{L}_{i-1,i}^{\pm}(u)\\ \mathrm{L}_{i,1}^{\pm}(u) & \dots & \mathrm{L}_{i,i-1}^{\pm}(u) & \boxed{\mathrm{L}_{i,i}^{\pm}(u)}\end{vmatrix},$$

$$E_{i,j}^{\pm}(u)=k_j^{\pm}(u)^{-1}\begin{vmatrix} \mathrm{L}_{11}^{\pm}(u) & \dots & \mathrm{L}_{1,i-1}^{\pm}(u) & \mathrm{L}_{1,i}^{\pm}(u)\\ \vdots & & \vdots & \vdots\\ \mathrm{L}_{i-1,1}^{\pm}(u) & \dots & \mathrm{L}_{i-1,i-1}^{\pm}(u) & \mathrm{L}_{i-1,i}^{\pm}(u)\\ \mathrm{L}_{j,1}^{\pm}(u) & \dots & \mathrm{L}_{j,i-1}^{\pm}(u) & \boxed{\mathrm{L}_{j,i}^{\pm}(u)}\end{vmatrix}, \qquad i<j,$$

$$F_{i,j}^{\pm}(u)=\begin{vmatrix} \mathrm{L}_{11}^{\pm}(u) & \dots & \mathrm{L}_{1,j-1}^{\pm}(u) & \mathrm{L}_{1,i}^{\pm}(u)\\ \vdots & & \vdots & \vdots\\ \mathrm{L}_{j-1,1}^{\pm}(u) & \dots & \mathrm{L}_{j-1,j-1}^{\pm}(u) & \mathrm{L}_{j-1,i}^{\pm}(u)\\ \mathrm{L}_{j,1}^{\pm}(u) & \dots & \mathrm{L}_{j,j-1}^{\pm}(u) & \boxed{\mathrm{L}_{j,i}^{\pm}(u)}\end{vmatrix} k_j^{\pm}(u)^{-1}, \qquad i>j,$$

and

$$\widehat{k}_i^{\pm}(u)=\begin{vmatrix} \boxed{\mathrm{L}_{i,i}^{\pm}(u)} & \mathrm{L}_{i,i+1}^{\pm}(u) & \dots & \mathrm{L}_{i,N}^{\pm}(u)\\ \mathrm{L}_{i+1,i}^{\pm}(u) & \mathrm{L}_{i+1,i+1}^{\pm}(u) & \dots & \mathrm{L}_{i+1,N}^{\pm}(u)\\ \vdots & \vdots & & \vdots\\ \mathrm{L}_{N,i}^{\pm}(u) & \mathrm{L}_{N,i+1}^{\pm}(u) & \dots & \mathrm{L}_{N,N}^{\pm}(u)\end{vmatrix},$$

$$\widehat{\mathrm{E}}_{i,j}(u)=\widehat{k}_j^{\pm}(u)^{-1}\begin{vmatrix} \boxed{\mathrm{L}_{j,i}^{\pm}(u)} & \mathrm{L}_{j,j+1}^{\pm}(u) & \dots & \mathrm{L}_{j,N}^{\pm}(u)\\ \mathrm{L}_{j+1,i}^{\pm}(u) & \mathrm{L}_{j+1,j+1}^{\pm}(u) & \dots & \mathrm{L}_{j+1,N}^{\pm}(u)\\ \vdots & \vdots & & \vdots\\ \mathrm{L}_{N,i}^{\pm}(u) & \mathrm{L}_{N,j+1}^{\pm}(u) & \dots & \mathrm{L}_{N,N}^{\pm}(u)\end{vmatrix}, \qquad i<j,$$

$$\widehat{\mathrm{F}}_{i,j}^{\pm}(u)=\begin{vmatrix} \boxed{\mathrm{L}_{j,i}^{\pm}(u)} & \mathrm{L}_{j,i+1}^{\pm}(u) & \dots & \mathrm{L}_{j,N}^{\pm}(u)\\ \mathrm{L}_{i+1,i}^{\pm}(u) & \mathrm{L}_{i+1,i+1}^{\pm}(u) & \dots & \mathrm{L}_{i+1,N}^{\pm}(u)\\ \vdots & \vdots & & \vdots\\ \mathrm{L}_{N,i}^{\pm}(u) & \mathrm{L}_{N,i+1}^{\pm}(u) & \dots & \mathrm{L}_{N,N}^{\pm}(u)\end{vmatrix} \widehat{k}_i^{\pm}(u)^{-1}, \qquad i>j.$$

The Gauss decomposition (3.4)–(3.6) was used[3] in [12] in order to obtain a recurrence relation for the universal weight function (4.6) and to prove the conjecture of [11] that the constructions of the off-shell Bethe vectors by using the L-operator approach [17] and the method of projections [8] coincide for an arbitrary $U_q(\widehat{\mathfrak{gl}}_N)$-module generated by arbitrary singular vectors. In this paper, another Gauss decomposition (3.1)–(3.3) will be used in order to get an alternative recurrence relation for the universal weight function (4.5), which will lead to formula (2.20) for the off-shell Bethe vector.

Note also that two different Gauss decompositions are in agreement with two ways of embedding the smaller algebra $U_q(\widehat{\mathfrak{gl}}_{N-1})$ in $U_q(\widehat{\mathfrak{gl}}_N)$. For the decomposition (3.1)–(3.3), it is natural to consider the embedding in the upper left corner of the L-operator, while for the decomposition (3.4)–(3.6), it is natural to use the lower right corner of the L-operator. A result of the present paper is the observation that two different Gauss decompositions yield different isomorphic current realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$.

3.2. Current realizations of $U_q(\widehat{\mathfrak{gl}}_N)$. The arguments of the paper [3] lead to commutation relations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ with the zero central charge and the gradation operator dropped out. These relations are expressed in terms of the linear combinations of Gauss coordinates

$$F_i(t) = F^+_{i+1,i}(t) - F^-_{i+1,i}(t), \quad E_i(t) = E^+_{i,i+1}(t) - E^-_{i,i+1}(t), \tag{3.9}$$

and the "diagonal" Gauss coordinates $k_i^\pm(t)$. The generating series $F_i(t)$, $E_i(t)$ are named the *total currents*, and the diagonal Gauss coordinates $k^\pm(t)$ are the *Cartan currents*. The corresponding commutation relations in the current realization of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ look like this:

$$\begin{aligned}
&(qz - q^{-1}w)E_i(z)E_i(w) = E_i(w)E_i(z)(q^{-1}z - qw),\\
&(q^{-1}z - qw)E_i(z)E_{i+1}(w) = E_{i+1}(w)E_i(z)(z - w),\\
&k_i^\pm(z)E_i(w)\left(k_i^\pm(z)\right)^{-1} = \frac{z-w}{q^{-1}z - qw}E_i(w),\\
&k_{i+1}^\pm(z)E_i(w)\left(k_{i+1}^\pm(z)\right)^{-1} = \frac{z-w}{qz - q^{-1}w}E_i(w),\\
&k_i^\pm(z)E_j(w)\left(k_i^\pm(z)\right)^{-1} = E_j(w) \quad \text{if} \quad i \neq j, j+1,\\
&(q^{-1}z - qw)F_i(z)F_i(w) = F_i(w)F_i(z)(qz - q^{-1}w),\\
&(z - w)F_i(z)F_{i+1}(w) = F_{i+1}(w)F_i(z)(q^{-1}z - qw),\\
&k_i^\pm(z)F_i(w)\left(k_i^\pm(z)\right)^{-1} = \frac{q^{-1}z - qw}{z-w}F_i(w),\\
&k_{i+1}^\pm(z)F_i(w)\left(k_{i+1}^\pm(z)\right)^{-1} = \frac{qz - q^{-1}w}{z-w}F_i(w),\\
&k_i^\pm(z)F_j(w)\left(k_i^\pm(z)\right)^{-1} = F_j(w) \quad \text{if} \quad i \neq j, j+1,\\
&[E_i(z), F_j(w)] = \delta_{i,j}\delta(z/w)(q - q^{-1})\left(k^-_{i+1}(z)/k^-_i(z) - k^+_{i+1}(w)/k^+_i(w)\right);
\end{aligned} \tag{3.10}$$

these formulas should be viewed as relations in formal series. This means that each relation in (3.10) involves an infinite set of relations among elements of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$, obtained by equating the coefficients of the same powers of the formal parameters z

[3]The Gauss coordinates $\widehat{\mathrm{F}}^\pm_{b,a}(t)$, $\widehat{\mathrm{E}}^\pm_{a,b}(t)$, $a < b$, and $\widehat{k}^\pm_a(t)$ were denoted in [12] by $\widetilde{F}^\pm_{b,a}(t)$, $\widetilde{E}^\pm_{a,b}(t)$, and $k^\pm_a(t)$ (cf. §6 in [12]).

and w. The symbol $\delta(z)$ in the above relations means the formal series $\delta(z)=\sum_{n\in\mathbb{Z}} z^n$. Also, the total currents satisfy the Serre relations

$$\begin{aligned}\mathrm{Sym}_{z_1,z_2}(E_i(z_1)E_i(z_2)E_{i\pm1}(w)-(q+q^{-1})E_i(z_1)E_{i\pm1}(w)E_i(z_2)\\ +E_{i\pm1}(w)E_i(z_1)E_i(z_2))=0,\\ \mathrm{Sym}_{z_1,z_2}(F_i(z_1)F_i(z_2)F_{i\pm1}(w)-(q+q^{-1})F_i(z_1)F_{i\pm1}(w)F_i(z_2)\\ +F_{i\pm1}(w)F_i(z_1)F_i(z_2))=0.\end{aligned} \tag{3.11}$$

To obtain the commutation relations (3.10), it suffices to substitute (3.1)–(3.3) in (2.3). If we substitute in (2.3) the decomposition (3.4)–(3.6), then, instead of (3.10), we arrive at the following slightly different commutation relations for the total currents:

$$\widehat{F}_i(t)=\widehat{\mathrm{F}}^+_{i+1,i}(t)-\widehat{\mathrm{F}}^-_{i+1,i}(t),\quad \widehat{E}_i(t)=\widehat{\mathrm{E}}^+_{i,i+1}(t)-\widehat{\mathrm{E}}^-_{i,i+1}(t) \tag{3.12}$$

and the Cartan currents $\widehat{k}^{\pm}_i(t)$, used in the papers [11, 12]:

$$\begin{aligned}&(q^{-1}z-qw)\widehat{E}_i(z)\widehat{E}_i(w)=\widehat{E}_i(w)\widehat{E}_i(z)(qz-q^{-1}w),\\ &(z-w)\widehat{E}_i(z)\widehat{E}_{i+1}(w)=\widehat{E}_{i+1}(w)\widehat{E}_i(z)(q^{-1}z-qw),\\ &\widehat{k}^{\pm}_i(z)\widehat{E}_i(w)\big(\widehat{k}^{\pm}_i(z)\big)^{-1}=\frac{z-w}{q^{-1}z-qw}\widehat{E}_i(w),\\ &\widehat{k}^{\pm}_{i+1}(z)\widehat{E}_i(w)\big(\widehat{k}^{\pm}_{i+1}(z)\big)^{-1}=\frac{z-w}{qz-q^{-1}w}\widehat{E}_i(w),\\ &\widehat{k}^{\pm}_i(z)\widehat{E}_j(w)\big(\widehat{k}^{\pm}_i(z)\big)^{-1}=\widehat{E}_j(w)\quad\text{if}\quad i\neq j,j+1,\\ &(qz-q^{-1}w)\widehat{F}_i(z)\widehat{F}_i(w)=\widehat{F}_i(w)\widehat{F}_i(z)(q^{-1}z-qw),\\ &(q^{-1}z-qw)\widehat{F}_i(z)\widehat{F}_{i+1}(w)=\widehat{F}_{i+1}(w)\widehat{F}_i(z)(z-w),\\ &\widehat{k}^{\pm}_i(z)\widehat{F}_i(w)\big(\widehat{k}^{\pm}_i(z)\big)^{-1}=\frac{q^{-1}z-qw}{z-w}\widehat{F}_i(w),\\ &\widehat{k}^{\pm}_{i+1}(z)\widehat{F}_i(w)\big(\widehat{k}^{\pm}_{i+1}(z)\big)^{-1}=\frac{qz-q^{-1}w}{z-w}\widehat{F}_i(w),\\ &\widehat{k}^{\pm}_i(z)\widehat{F}_j(w)\big(\widehat{k}^{\pm}_i(z)\big)^{-1}=\widehat{F}_j(w)\quad\text{if}\quad i\neq j,j+1,\\ &[\widehat{E}_i(z),\widehat{F}_j(w)]=\delta_{i,j}\delta(z/w)(q-q^{-1})\Big(\widehat{k}^{+}_i(z)/\widehat{k}^{+}_{i+1}(z)-\widehat{k}^{-}_i(w)/\widehat{k}^{-}_{i+1}(w)\Big).\end{aligned} \tag{3.13}$$

The Serre relations for the currents $\widehat{F}_i(z)$ and $\widehat{F}_i(z)$ will have the form (3.11) with $E_i(z)$ and $F_i(z)$ replaced by $\widehat{E}_i(z)$ and $\widehat{F}_i(z)$.

These commutation relations describe two isomorphic current realizations of the same algebra $U_q(\widehat{\mathfrak{gl}}_N)$.

For any series $G(t)=\sum\limits_{m\in\mathbb{Z}} G[m]t^{-m}$, we denote

$$G(t)^{(+)}=\sum_{m>0} G[m]\,t^{-m},\qquad G(t)^{(-)}=-\sum_{m\le 0} G[m]\,t^{-m}.$$

The initial conditions (2.4) lead to the relations

$$F^{\pm}_{i+1,i}(z)=z\left(z^{-1}F_i(z)\right)^{(\pm)},\qquad E^{\pm}_{i,i+1}(z)=E_i(z)^{(\pm)} \tag{3.14}$$

and to the same type of relations between the Gauss coordinates $\widehat{\mathrm{E}}^{\pm}_{i,i+1}(t)$, $\widehat{\mathrm{F}}^{\pm}_{i+1,i}(t)$ and the currents $\widehat{E}_i(t)$, $\widehat{F}_i(t)$.

3.3. Borel subalgebras and their intersections. We consider two types of Borel subalgebras of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$. The standard Borel subalgebras $U_q(\mathfrak{b}^\pm) \subset U_q(\widehat{\mathfrak{gl}}_N)$ are generated by the modes of the L-operators $\mathrm{L}^+(z)$ and $\mathrm{L}^-(z)$, respectively. For the generators in these subalgebras we can use either the modes of the Gauss coordinates (3.1)–(3.3), $E^\pm_{i,i+1}(t)$, $F^\pm_{i+1,i}(t)$, $k^\pm_j(t)$, or the modes of the Gauss coordinates (3.4)–(3.6), $\widehat{\mathrm{E}}^\pm_{i,i+1}(t)$, $\widehat{\mathrm{F}}^\pm_{i+1,i}(t)$, $\widehat{k}^\pm_j(t)$.

Borel subalgebras of another type are related to the current realizations of $U_q(\widehat{\mathfrak{gl}}_N)$. First, we consider the current Borel subalgebras generated by the modes of the currents $E_i(t)$, $F_i(t)$, $k^\pm_j(t)$.

The Borel subalgebra $U_F \subset U_q(\widehat{\mathfrak{gl}}_N)$ is generated by the modes of the currents $F_i[n]$, $k^+_j[m]$, where $i = 1, \dots, N-1$, $j = 1, \dots, N$, $n \in \mathbb{Z}$, and $m \geq 0$. The Borel subalgebra $U_E \subset U_q(\widehat{\mathfrak{gl}}_N)$ is generated by the modes of the currents $E_i[n]$, $k^-_j[-m]$ with $i = 1, \dots, N-1$, $j = 1, \dots, N$, $n \in \mathbb{Z}$, and $m \geq 0$. Also, we consider the subalgebra $U'_F \subset U_F$ generated by the elements $F_i[n]$, $k^+_j[m]$, $i = 1, \dots, N-1$, $j = 1, \dots, N$, $n \in \mathbb{Z}$, $m > 0$, and the subalgebra $U'_E \subset U_E$ generated by the elements $E_i[n]$, $k^-_j[-m]$, $i = 1, \dots, N-1$, $j = 1, \dots, N$, $n \in \mathbb{Z}$, $m > 0$. Next, we shall be interested in the intersections

$$U^-_f = U'_F \cap U_q(\mathfrak{b}^-), \qquad U^+_F = U_F \cap U_q(\mathfrak{b}^+) \tag{3.15}$$

and describe the properties of the projections to these intersections. We call U_F and U_E *the current Borel subalgebras*.

In [2], the *current* Hopf structure for the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ was defined:

$$\begin{aligned} \Delta^{(D)}\big(E_i(z)\big) &= 1 \otimes E_i(z) + E_i(z) \otimes k^-_{i+1}(z)\big(k^-_i(z)\big)^{-1}, \\ \Delta^{(D)}\big(F_i(z)\big) &= F_i(z) \otimes 1 + k^+_{i+1}(z)\big(k^+_i(z)\big)^{-1} \otimes F_i(z), \\ \Delta^{(D)}\big(k^\pm_i(z)\big) &= k^\pm_i(z) \otimes k^\pm_i(z). \end{aligned} \tag{3.16}$$

With respect to the current Hopf structure, the current Borel subalgebras are Hopf subalgebras of $U_q(\widehat{\mathfrak{gl}}_N)$. We can check (see [8, 11]) that the intersections U^-_f and U^+_F are coideals with respect to the coproduct (3.16),

$$\Delta^{(D)}(U^+_F) \subset U^+_F \otimes U_q(\widehat{\mathfrak{gl}}_N), \qquad \Delta^{(D)}(U^-_f) \subset U_q(\widehat{\mathfrak{gl}}_N) \otimes U^-_f,$$

and that the multiplication m in $U_q(\widehat{\mathfrak{gl}}_N)$ induces an isomorphism

$$m : U^-_f \otimes U^+_F \to U_F$$

of vector spaces. According to the general theory as presented in [8], we can introduce the projection operators $P^+ : U_F \subset U_q(\widehat{\mathfrak{gl}}_N) \to U^+_F$ and $P^- : U_F \subset U_q(\widehat{\mathfrak{gl}}_N) \to U^-_f$ by the formulas

$$\begin{gathered} P^+(F_-F_+) = \varepsilon(F_-)F_+, \qquad P^-(F_-F_+) = F_-\varepsilon(F_+) \\ \text{for any} \qquad F_- \in U^-_f, \quad F_+ \in U^+_F. \end{gathered} \tag{3.17}$$

Here the mapping $\varepsilon : U_F \mapsto \mathbb{C}$ is the counit defined on the generators U_F by the rules $\varepsilon(F_i[n]) = 0$ and $\varepsilon(k^+_j[m]) = \delta_{m,0}$.

Let $\overline{U}_F$ denote the extension of the algebra U_F formed by infinite sums of the monomials that are ordered products $a_{i_1}[n_1] \cdots a_{i_k}[n_k]$ with $n_1 \leq \cdots \leq n_k$, where $a_{i_l}[n_l]$ is either $F_{i_l}[n_l]$ or $k^+_{i_l}[n_l]$. In [8], it was proved that

(1) the action of the projections (3.17) extends to the algebra $\overline{U}_F$;

(2) for any $F \in \overline{U}_F$ with $\Delta^{(D)}(F) = \sum_i F_i' \otimes F_i''$, we have

$$F = \sum_i P^-(F_i') \cdot P^+(F_i''). \tag{3.18}$$

In [11, 12], we used the current Borel subalgebras $\widehat{U}_F$, $\widehat{U}_E$ generated by the modes of the currents $\widehat{F}_i(t)$, $\widehat{E}_i(t)$, $\widehat{k}_j^{\pm}(t)$ in the same way as has been done above for U_F, U_E. These current Borel subalgebras are Hopf subalgebras of $U_q(\widehat{\mathfrak{gl}}_N)$ with a different coproduct

$$\begin{aligned} \widehat{\Delta}^{(D)}\big(\widehat{E}_i(z)\big) &= \widehat{E}_i(z) \otimes 1 + \widehat{k}_i^-(z)\big(\widehat{k}_{i+1}^-(z)\big)^{-1} \otimes \widehat{E}_i(z), \\ \widehat{\Delta}^{(D)}\big(\widehat{F}_i(z)\big) &= 1 \otimes \widehat{F}_i(z) + \widehat{F}_i(z) \otimes \widehat{k}_i^+(z)\big(\widehat{k}_{i+1}^+(z)\big)^{-1}, \\ \widehat{\Delta}^{(D)}\big(\widehat{k}_i^{\pm}(z)\big) &= \widehat{k}_i^{\pm}(z) \otimes \widehat{k}_i^{\pm}(z). \end{aligned} \tag{3.19}$$

The standard Borel subalgebras $U_q(\mathfrak{b}^{\pm})$ are determined by the modes of the Gauss coordinates $\widehat{\mathrm{E}}_{i,i+1}^{\pm}(t)$, $\widehat{\mathrm{F}}_{i+1,i}^{\pm}(t)$, $\widehat{k}_j^{\pm}(t)$, and their intersections with the current Borel subalgebras $\widehat{U}_F$ are given by the same formulas (3.15). By using the coproduct (3.19), it can be checked that the coalgebraic properties of these intersections are changed to

$$\widehat{\Delta}^{(D)}(\widehat{U}_F^+) \subset U_q(\widehat{\mathfrak{gl}}_N) \otimes \widehat{U}_F^+, \qquad \widehat{\Delta}^{(D)}(\widehat{U}_f^-) \subset \widehat{U}_f^- \otimes U_q(\widehat{\mathfrak{gl}}_N). \tag{3.20}$$

Let $\widehat{\overline{U}}_F$ denote the extension of the algebra $\widehat{U}_F$ formed by infinite sums of the monomials that are ordered products $a_{i_1}[n_1] \cdots a_{i_k}[n_k]$ with $n_1 \le \cdots \le n_k$, where $a_{i_l}[n_l]$ is either $\widehat{F}_{i_l}[n_l]$ or $\widehat{k}_{i_l}^+[n_l]$. The projections to the intersections (3.20) are defined by formulas similar to (3.17), but property (3.18) is changed to $F = \sum_i \widehat{P}^-(F_i'') \cdot \widehat{P}^+(F_i')$, where $\widehat{\Delta}^{(D)}(F) = \sum_i F_i' \otimes F_i''$.

The intersections of Borel subalgebras of different types U_F^+ and $\widehat{U}_F^+$ contain the subalgebras U_f^+ and $\widehat{U}_f^+$ that consist of the nonnegative modes of the currents $F_i(t)$ and $\widehat{F}_i(t)$, correspondingly. The subalgebras $U_f^{\pm}$ and $\widehat{U}_f^{\pm}$ will be used below for describing the normal ordering of products of currents.

§4. Universal weight function and projections

Here we use the same notation as in [12]. Let Π be the set $\{1, \dots, N-1\}$ of indices of the simple positive roots of $\mathfrak{gl}_N$. A finite collection $I = \{i_1, \dots, i_n\}$ equipped with a linear ordering $i_i \prec \cdots \prec i_n$ and a map $\iota : I \to \Pi$ are said to form an *ordered* Π-*multiset*. With each Π-ordered multiset $I = \{i_1, \dots, i_n\}$, we associate an ordered set of variables $\{t_i \mid i \in I\} = \{t_{i_1}, \dots, t_{i_n}\}$. Each element $i_k \in I$ and each variable t_{i_k} has its own "type": $\iota(i_k) \in \Pi$.

Our basic calculations are performed at the level of formal series associated with certain ordered multisets. For brevity, we often write series as rational functions in accordance with the following rule. Let $\{t_i \mid i \in I\} = \{t_{i_1}, \dots, t_{i_n}\}$ be the ordered set of variables associated with an ordered set $I = \{i_1 \prec i_2 \prec \cdots \prec i_n\}$, and let $g(t_i \mid i \in I)$ be a rational function. Then with $g(t_i \mid i \in I)$ we associate the Laurent series that is the expansion of $g(t_i \mid i \in I)$ in the region $|t_{i_1}| \ll |t_{i_2}| \ll \cdots \ll |t_{i_n}|$. If, for instance, $1 \prec 2$, then with the rational function $(t_1 - t_2)^{-1}$ we associate the series $-\sum_{k \ge 0} t_1^k t_2^{-k-1}$. On the contrary, for the ordering $2 \prec 1$, with the same rational function $(t_1 - t_2)^{-1}$ we associate the series $\sum_{k \ge 0} t_2^k t_1^{-k-1}$.

Let $\bar{l}$ and $\bar{r}$ be two collections of nonnegative integers such that

$$l_a \le r_a, \quad a = 1, \dots, N-1. \tag{4.1}$$

Denote by $[\bar{l},\bar{r}]$ the segments formed by the positive integers $\{l_a+1, l_a+2, \dots, r_a-1, r_a\}$, including r_a and excluding l_a. The length of each segment is equal to $r_a - l_a$.

For a given collection $[\bar{l},\bar{r}]$ of segments, we denote by $\bar{t}_{[\bar{l},\bar{r}]}$ the sets of variables

$$\bar{t}_{[\bar{l},\bar{r}]} = \{t^1_{l_1+1}, \dots, t^1_{r_1}; t^2_{l_2+1}, \dots, t^2_{r_2}; \dots; t^{N-1}_{l_{N-1}+1}, \dots, t^{N-1}_{r_{N-1}}\}. \tag{4.2}$$

The number of variables of type a is equal to $r_a - l_a$. In this notation, the set of variables (2.10) is $\bar{t}_{[\bar{n}]} \equiv \bar{t}_{[\bar{0},\bar{n}]}$. We shall regard (4.2) as a list of ordered variables corresponding to two ordered Π-multisets:

$$I = \{r_{N-1} \prec \dots \prec l_{N-1}+1 \prec \dots \prec r_2 \prec \dots \prec l_2+1 \prec r_1 \prec \dots \prec l_1+1\}$$

and

$$\widehat{I} = \{l_1+1 \prec \dots \prec r_1 \prec l_2+1 \prec \dots \prec r_2 \prec \cdot\cdot \prec l_{N-1}+1 \prec \dots \prec r_{N-1}\}.$$

For any $a = 1, \dots, N-1$, we denote by $\bar{t}^a_{[l_a,r_a]} = \{t^a_{l_a+1}, \dots, t^a_{r_a}\}$ the sets of variables corresponding to the segments $[l_a, r_a] = \{l_a+1, l_a+2, \dots, r_a\}$. All of the variables in $\bar{t}^a_{[l_a,r_a]}$ are of type a. For the segments $[l_a, r_a] = [0, n_a]$ we use the short notation $\bar{t}_{[\bar{0},\bar{n}]} \equiv \bar{t}_{[\bar{n}]}$ and $\bar{t}^a_{[0,n_a]} \equiv \bar{t}^a_{[n_a]}$.

For a collection of variables $\bar{t}_{[\bar{l},\bar{r}]}$, we consider two types of ordered products of currents:

$$\mathcal{F}(\bar{t}_{[\bar{l},\bar{r}]}) = \overrightarrow{\prod_{1\le a\le N-1}} \Big(\overrightarrow{\prod_{l_a<\ell\le r_a}} F_a(t^a_\ell)\Big) = F_1(t^1_{l_1+1}) \cdots F_1(t^1_{r_1}) \cdots F_{N-1}(t^{N-1}_{r_{N-1}}) \tag{4.3}$$

and

$$\widehat{\mathcal{F}}(\bar{t}_{[\bar{l},\bar{r}]}) = \overleftarrow{\prod_{N-1\ge a\ge 1}} \Big(\overleftarrow{\prod_{r_a\ge\ell>l_a}} \widehat{F}_a(t^a_\ell)\Big) = \widehat{F}_{N-1}(t^{N-1}_{r_{N-1}}) \cdots \widehat{F}_1(t^1_{r_1}) \cdots \widehat{F}_1(t^1_{l_1+1}), \tag{4.4}$$

where the series $F_a(t) \equiv F_{a+1,a}(t)$ and $\widehat{F}_a(t) \equiv \widehat{F}_{a+1,a}(t)$ are defined by (3.9) and (3.12), respectively. As particular cases, we have $\mathcal{F}(\bar{t}^a_{[\bar{l}_a,\bar{r}_a]}) = F_a(t^a_{l_a+1})F_a(t^a_{l_a+2}) \cdots F_a(t^a_{r_a})$ and $\widehat{\mathcal{F}}(\bar{t}^a_{[\bar{l}_a,\bar{r}_a]}) = \widehat{F}_a(t^a_{r_a}) \cdots \widehat{F}_a(t^a_{l_a+2})\widehat{F}_a(t^a_{l_a+1})$.

The symbols $\overleftarrow{\prod_a} A_a$ and $\overrightarrow{\prod_a} A_a$ mean ordered products of noncommutative entries A_a such that A_a is on the right (respectively, on the left) from A_b for $b > a$:

$$\overleftarrow{\prod_{j\ge a\ge i}} A_a = A_j\, A_{j-1} \cdots A_{i+1}\, A_i, \qquad \overrightarrow{\prod_{i\le a\le j}} A_a = A_i\, A_{i+1} \cdots A_{j-1}\, A_j.$$

In accordance with [5, 7, 8, 10], the product of the currents (4.3) is a formal series over the ratios t^b_k/t^c_l with $b > c$ and t^a_i/t^a_j with $i > j$, taking values in the completion $\overline{U}_F$. Similarly, the product of the currents (4.4) is a formal series over the ratios t^b_k/t^c_l with $b < c$ and t^a_i/t^a_j with $i < j$, taking values in the completion $\widehat{\overline{U}}_F$.

This means that such products have the same analytical structure as the rational functions in the variables $\bar{t}_{[\bar{l},\bar{r}]}$ determined by Π-multisets I and $\widehat{I}$, respectively. The domains of analyticity of the rational functions defined by the Π-multisets I and $\widehat{I}$ are different. The products (4.3) and (4.4) have poles for some values of the ratios t^b_k/t^c_l and t^a_i/t^a_j. The operator-valued coefficients at these poles take values in the completions $\overline{U}_F$ and $\widehat{\overline{U}}_F$ and can be identified with composed root currents (see [10, 12]). In what follows, we shall consider projections of the following products of currents:

$$\mathcal{W}^N(\bar{t}_{[\bar{n}]}) = P^+\Big(F_1(t^1_1) \cdots F_1(t^1_{n_1}) \ \cdots\ F_{N-1}(t^{N-1}_1) \cdots F_{N-1}(t^{N-1}_{n_{N-1}})\Big) \tag{4.5}$$

and

$$\widehat{\mathcal{W}}^N(\bar{t}_{[\bar{n}]}) = \widehat{P}^+\left(\widehat{F}_{N-1}(t^{N-1}_{n_{N-1}})\cdots\widehat{F}_{N-1}(t^{N-1}_1)\ \cdots\ \widehat{F}_1(t^1_{n_1})\cdots\widehat{F}_1(t^1_1)\right). \tag{4.6}$$

In [10], it was proved that these projections can be analytically continued from their domains. This allows us to compare the universal weight functions defined by the projections (4.5) and (4.6).

It was conjectured in [11] and then proved in [12] that the universal weight function can be identified with the projection (4.6). A method for computing this projection was proposed in [10] and was further developed in [12]. In this paper we shall calculate the universal weight function given by the projection (4.5).

For any weight singular vector v as in (2.16), the related weight functions

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\mathcal{W}^N(\bar{t}_{[\bar{n}]})\prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a} k^+_a(t^a_\ell)\, v \tag{4.7}$$

and

$$\widehat{\mathbf{w}}^N_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\ \widehat{\mathcal{W}}^N(\bar{t}_{[\bar{n}]})\prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a} \widehat{k}^+_{a+1}(t^a_\ell)\, v \tag{4.8}$$

are vector-valued functions with values in the $U_q(\widehat{\mathfrak{gl}}_N)$-module V generated by the singular vector v. In [11, 12] they were called modified weight functions or *universal off-shell Bethe vectors*.

We shall show that the analytic continuations of the universal off-shell Bethe vectors defined by the universal weight functions (4.6) and (4.5) coincide:

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \widehat{\mathbf{w}}^N_V(\bar{t}_{[\bar{n}]}). \tag{4.9}$$

This will follow from the fact, proved in [12], that

$$\widehat{\mathbf{w}}^N_V(\bar{t}_{[\bar{n}]}) = \mathbb{B}_V(\bar{t}_{[\bar{n}]})$$

for an arbitrary $U_q(\widehat{\mathfrak{gl}}_N)$-module V generated by a singular vector v. In this paper we prove that

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \mathbb{B}_V(\bar{t}_{[\bar{n}]}), \tag{4.10}$$

which implies (4.9).

To prove (4.10), we use the same arguments as in [12]. We calculate the projection (4.5), rewrite the corresponding universal off-shell Bethe vector (4.7) in terms of ordered products of matrix entries of L-operators acting on the singular vector v and show that this yields the expression (2.20) for this vector on the evaluation modules. This gives a formula that differs from that obtained in [12] for off-shell Bethe vectors. Using the result of [8], we can check that the universal off-shell Bethe vector (4.7) defined by the projection satisfies the same comultiplication properties as the vector $\mathbb{B}_V(\bar{t}_{[\bar{n}]})$. This allows us to construct the weight function $\mathbf{w}^N_{V_1\otimes V_2}$ from the weight functions $\mathbf{w}^N_{V_1}$ and $\mathbf{w}^N_{V_2}$ (see [11] for more details). In its turn, this means that (4.10) is true for an arbitrary tensor product of evaluation representations of $U_q(\widehat{\mathfrak{gl}}_N)$. Then the classical result [1] implies that (4.10) is true for every irreducible finite-dimensional $U_q(\widehat{\mathfrak{gl}}_N)$-module V generated by a singular vector v.

We conclude this subsection with outlining our strategy for the calculation of projections. To calculate (4.6), we shall use the same approach as in [12] for the calculation of (4.5). In [12] we separated all factors $\widehat{F}_a(t^a_i)$ with $a < N-1$ in (4.6) and applied the ordering procedure based on (4.19) to this product. In this paper we separate the factors $F_a(t^a_i)$ with $a > 1$ in (4.5) and also apply the ordering procedure. In both cases under the

total projection we get the symmetrization of a sum of terms of the form $x_i P^-(y_i)P^+(z_i)$ with rational coefficients. Here the x_i are expressed via the modes of $\widehat{F}_{N-1}(t)$, and the y_i, z_i via the modes of $\widehat{F}_a(t)$ with $a < N-1$ in the case of (4.6). In the case of (4.5), the x_i are expressed via the modes of $F_1(t)$, and y_i, z_i via the modes of $F_a(t)$ with $a > 1$. As in [12], we reorder the x_i and $P^-(y_i)$ in both cases. At this stage, composed currents of different types arise, collected in special products that were called strings in [10].

4.1. q-Symmetrization. Consider the permutation group S_n; its action on the formal series of n variables will be defined on the elementary transpositions $\sigma_{i,i+1}$ as follows:

$$\pi(\sigma_{i,i+1})G(u_1,\dots,u_i,u_{i+1},\dots,u_n) = \frac{qu_i - q^{-1}u_{i+1}}{q^{-1}u_i - qu_{i+1}} G(u_1,\dots,u_{i+1},u_i,\dots,u_n).$$

The q-depending factor in this formula is chosen so that the products $F_a(u_1)\cdots F_a(u_n)$ and $\widehat{F}_a(u_n)\cdots\widehat{F}_a(u_1)$ be invariant under this action. Summing the action over all groups of permutations, we obtain an operator $\overline{\mathrm{Sym}}_{\bar t} = \sum\limits_{\sigma\in S_n} \pi(\sigma)$ that acts as follows:

$$\overline{\mathrm{Sym}}_{\bar t}\, G(\bar t) = \sum_{\sigma\in S_n} \prod_{\substack{\ell<\ell' \\ \sigma(\ell)>\sigma(\ell')}} \frac{q - q^{-1}t_{\sigma(\ell)}/t_{\sigma(\ell')}}{q^{-1} - qt_{\sigma(\ell)}/t_{\sigma(\ell')}}\, G({}^{\sigma}\bar t),$$

where ${}^{\sigma}\bar t = \{t_{\sigma(1)},\dots,t_{\sigma(n)}\}$, and the product is taken over all pairs (ℓ,ℓ') such that $\ell < \ell'$ and $\sigma(\ell) > \sigma(\ell')$ simultaneously. The operator $\overline{\mathrm{Sym}}_u$ will also be called the *q-symmetrization*. This operation differs somewhat from the q-symmetrization (2.15); the exact relationship between them will be described below. The operator $\frac{1}{n!}\overline{\mathrm{Sym}}_u$ is the group average with respect to the action π, which implies that $\overline{\mathrm{Sym}}_u\overline{\mathrm{Sym}}_u(\cdot) = n!\overline{\mathrm{Sym}}_u(\cdot)$. It is easily seen that

$$\overline{\mathrm{Sym}}_{t_1,\dots,t_n} = \sum_{\sigma\in S_n^{(s)}} \pi(\sigma)\, \overline{\mathrm{Sym}}_{t_1,\dots,t_s}\overline{\mathrm{Sym}}_{t_{s+1},\dots,t_n}, \tag{4.11}$$

where $s\in[n,0]$ is fixed and the sum is taken over the subset

$$S_n^{(s)} = \{\sigma\in S_n \mid \sigma(1) < \ldots < \sigma(s),\ \sigma(s+1) < \ldots < \sigma(n)\}.$$

We denote by $S_{\bar l,\bar r} = S_{l_1,r_1}\times\cdots\times S_{l_{N-1},r_{N-1}}$ the direct product of the groups S_{l_a,r_a} that permute the integers $l_a+1,\dots,r_a$. The q-symmetrization over the entire set of variables $\bar t_{[\bar l,\bar r]}$ is defined by the formula

$$\overline{\mathrm{Sym}}_{\bar t_{[\bar l,\bar r]}}\, G(\bar t_{[\bar l,\bar r]}) = \sum_{\sigma\in S_{\bar l,\bar r}} \prod_{1\le a\le N-1} \prod_{\substack{\ell<\ell' \\ \sigma^a(\ell)>\sigma^a(\ell')}} \frac{qt^a_{\sigma^a(\ell')} - q^{-1}t^a_{\sigma^a(\ell)}}{q^{-1}t^a_{\sigma^a(\ell')} - qt^a_{\sigma^a(\ell)}}\, G({}^{\sigma}\bar t_{[\bar l,\bar r]}), \tag{4.12}$$

where

$${}^{\sigma}\bar t_{[\bar l,\bar r]} = \{t^1_{\sigma^1(l_1+1)},\dots,t^1_{\sigma^1(r_1)};\dots;t^{N-1}_{\sigma^{N-1}(l_{N-1}+1)},\dots,t^{N-1}_{\sigma^{N-1}(r_{N-1})}\}$$

for $\sigma = \sigma^1\times\cdots\times\sigma^{N-1}\in S_{\bar l,\bar r}$.

The q-symmetrization (4.12) is related to the q-symmetrization (2.15) in the following way:

$$\overline{\mathrm{Sym}}^{(q)}_{\bar t_{[\bar l,\bar r]}}\, G(\bar t_{[\bar l,\bar r]}) = \prod_{a=1}^{N-1} \prod_{l_a<\ell<\ell'\le r_a} \frac{q^{-1}t^a_\ell - qt^a_{\ell'}}{t^a_\ell - t^a_{\ell'}}\, \overline{\mathrm{Sym}}_{\bar t_{[\bar l,\bar r]}}\, G(\bar t_{[\bar r,\bar l]}). \tag{4.13}$$

We say that the series $Q(\bar{t}_{[\bar{l},\bar{r}]})$ is *q-symmetric* if it is invariant under the action π of each group S_{l_a,r_a}. For any q-symmetric series we have

$$\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{l},\bar{r}]}} Q(\bar{t}_{[\bar{l},\bar{r}]}) = \prod_{a=1}^{N-1} (r_a - l_a)!\, Q(\bar{t}_{[\bar{l},\bar{r}]}). \tag{4.14}$$

The q-symmetrization $Q(\bar{t}_{[\bar{l},\bar{r}]}) = \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{l},\bar{r}]}} G(\bar{t}_{[\bar{l},\bar{r}]})$ of any series $G(\bar{t}_{[\bar{l},\bar{r}]})$ is a q-symmetric series.

Let $G^{\mathrm{sym}}(u_1,\dots,u_n)$ be a symmetric series in n variables u_k, i.e., $G^{\mathrm{sym}}({}^{\sigma}\bar{u}) = G^{\mathrm{sym}}(\bar{u})$ for any $\sigma \in S_n$. Then the following property of q-symmetrization can be checked:

$$\frac{1}{n!}\,\overline{\mathrm{Sym}}_{\bar{u}}\big(\beta(\bar{u})^{-1} G^{\mathrm{sym}}(\bar{u})\big) = \frac{1}{[n]_q!}\,\overline{\mathrm{Sym}}_{\bar{u}}\big(G^{\mathrm{sym}}(\bar{u})\big), \tag{4.15}$$

where $\beta(\bar{u}) = \prod_{k<k'} \frac{q^{-1}u_k - q u_k}{u_k - u_{k'}}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, and $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$.

4.2. Ordering of products of currents. We say that any expression $\sum_i f_-^{(i)} \cdot f_+^{(i)}$, where $f_-^{(i)} \in U_f^-$ and $f_+^{(i)} \in U_f^+$, is (*normally*) *ordered*. Using the property (3.18) of projections, we can present any product of currents in a normally ordered form. Taking (4.11) into account, we obtain the following statement.

Proposition 4.1. *The following formal series identities are valid*:

$$\begin{aligned}\mathcal{F}(\bar{t}_{[\bar{l},\bar{r}]}) = {} & \sum_{0\le m_1\le r_1-l_1} \cdots \sum_{0\le m_{N-1}\le r_{N-1}-l_{N-1}} \prod_{1\le a\le N-1} \frac{1}{(m_a)!(r_a-l_a-m_a)!} \\ & \times \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{l},\bar{r}]}} \Big(Z_{\bar{m}}(\bar{t}_{[\bar{l},\bar{r}]}) P^-\big(\mathcal{F}(\bar{t}_{[\bar{l},\bar{l}+\bar{m}]})\big) \cdot P^+\big(\mathcal{F}(\bar{t}_{[\bar{l}+\bar{m},\bar{r}]})\big)\Big),\end{aligned} \tag{4.16}$$

where

$$Z_{\bar{m}}(\bar{t}_{[\bar{l},\bar{r}]}) = \prod_{a=1}^{N-2} \prod_{\substack{l_a+m_a<\ell\le r_a \\ l_{a+1}<\ell'\le l_{a+1}+m_{a+1}}} \frac{q^{-1} - q t_{\ell'}^{a+1}/t_\ell^a}{1 - t_{\ell'}^{a+1}/t_\ell^a}. \tag{4.17}$$

To prove this, it suffices to use the coproduct (3.16) and formula (3.18).

For a collections of variables $\bar{t}_{[\bar{l},\bar{r}]}$ and currents $\widehat{F}_i(t)$, consider the ordered product (4.3). We introduce the series

$$\widehat{Z}_{\bar{s}}(\bar{t}_{[\bar{l},\bar{r}]}) = \prod_{a=1}^{N-2} \prod_{\substack{r_a-s_a<\ell\le r_a \\ l_{a+1}<\ell'\le r_{a+1}-s_{a+1}}} \frac{q - q^{-1}\, t_\ell^a/t_{\ell'}^{a+1}}{1 - t_\ell^a/t_{\ell'}^{a+1}}. \tag{4.18}$$

Then $\widehat{\mathcal{F}}(\bar{t}_{[\bar{l},\bar{r}]})$ can be presented in the ordered form (see [12])

$$\begin{aligned}\widehat{\mathcal{F}}(\bar{t}_{[\bar{l},\bar{r}]}) = {} & \sum_{0\le s_{N-1}\le r_{N-1}-l_{N-1}} \cdots \sum_{0\le s_1\le r_1-l_1} \prod_{1\le a\le N-1} \frac{1}{(s_a)!(r_a-l_a-s_a)!} \\ & \times \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{l},\bar{r}]}} \Big(\widehat{Z}_{\bar{s}}(\bar{t}_{[\bar{l},\bar{r}]}) \widehat{P}^-\big(\widehat{F}(\bar{t}_{[\bar{r}-s,\bar{r}]})\big) \cdot \widehat{P}^+\big(\widehat{F}(\bar{t}_{[\bar{l},\bar{r}-\bar{s}]})\big)\Big).\end{aligned} \tag{4.19}$$

Note that if $\bar{m} + \bar{s} = \bar{r} - \bar{l}$, then the rational series (4.17) and (4.18) are expansions of the same rational function in different zones.

4.3. Composite currents and strings. We introduce the currents $F_{j,i}(t)$ for $i<j$, enumerated by the $N(N-1)/2$ positive roots of the algebra $\mathfrak{gl}_N$. The currents $F_{i+1,i}(t)$, $i=1,\dots,N-1$, are associated with simple roots and coincide with $F_i(t)$; see (3.9). The currents $F_{j,i}(t)$ can be defined inductively by using the formulas

$$F_{j,i}(t) \;=\; -\operatorname*{res}_{w=t} F_{a,i}(t)F_{j,a}(w)\,\frac{dw}{w} \;=\; \operatorname*{res}_{w=t} F_{a,i}(w)F_{j,a}(t)\,\frac{dw}{w} \tag{4.20}$$

for any $a=i+1,\dots,j-1$. The coefficients of the series $F_{j,i}(t)$ belong to the completion $\overline{U}_F$ of the algebra U_F. The currents defined by (4.20) will be called *composite currents*. Relations (4.20) are equivalent to the formal series identities

$$\begin{aligned} F_{j,i}(t) &= \oint F_{a,i}(t)F_{j,a}(w)\,\frac{dw}{w} - \oint \frac{q-q^{-1}t/w}{1-t/w}\,F_{j,a}(w)F_{a,i}(t)\,\frac{dw}{w},\\ F_{j,i}(t) &= \oint F_{a,i}(w)F_{j,a}(t)\,\frac{dw}{w} - \oint \frac{q-q^{-1}w/t}{1-w/t}\,F_{j,a}(t)F_{a,i}(w)\,\frac{dw}{w}. \end{aligned} \tag{4.21}$$

In (4.21), $\oint \frac{dw}{w} g(w) = g_0$ for any formal series $g(w)=\sum_{n\in\mathbb{Z}} g_n z^{-n}$. Using (3.10), we can calculate the residues in (4.20):

$$F_{j,i}(t) = (q-q^{-1})^{j-i-1}F_{j-1}(t)F_{j-2}(t)\cdots F_{i+1}(t)F_i(t). \tag{4.22}$$

Calculating the formal integrals in (4.21), we obtain the expressions

$$\begin{aligned} F_{j,i}(t) &= F_{a,i}(t)F_{j,a}[0] - qF_{j,a}[0]F_{a,i}(t) - (q-q^{-1})\sum_{k<0} F_{j,a}[k]\,F_{a,i}(t)\,t^{-k},\\ F_{j,i}(t) &= F_{a,i}[0]F_{j,a}(t) - q^{-1}F_{j,a}(t)F_{a,i}[0] - (q-q^{-1})\sum_{k\ge 0} F_{j,a}(t)\,F_{a,i}[k]\,t^{-k}, \end{aligned} \tag{4.23}$$

which are useful for the calculation of the projections of composite currents. By a similar procedure, the composite currents $\widehat{F}_{j,i}(t)$ associated with the currents (3.12) were defined in [12]:

$$\widehat{F}_{j,i}(t) = (q-q^{-1})^{j-i-1}\widehat{F}_i(t)\widehat{F}_{i+1}(t)\cdots\widehat{F}_{j-2}(t)\widehat{F}_{j-1}(t). \tag{4.24}$$

The analytic properties of products of composite currents, used in the paper, are presented in Appendix A. The analytic properties of the currents $\widehat{F}_{j,i}(t)$ are given in the Appendix A of the paper [12].

In the process of calculating the universal weight functions (4.6) and (4.5), some special products of composite currents arise [10]. First, we define strings for the currents $F_{j,i}(t)$. Consider a collection of segments $[\bar{l},\bar{r}]$ and the associated set of variables $\bar{t}_{[\bar{l},\bar{r}]}$. Let $j=\min(a)$ such that $r_b=l_b$ for $b=1,\dots,a-1$. Let $\bar{m}$ be a set of nonnegative integers such that

$$0\le m_a\le r_a-l_a,\quad a=j+1,\dots,N-1, \tag{4.25}$$

and the following admissibility conditions are satisfied:

$$r_j-l_j=m_j\ge m_{j+1}\ge\cdots\ge m_{N-2}\ge m_{N-1}\ge m_N=0. \tag{4.26}$$

We define a *string* as an ordered product of composite currents:

$$\mathcal{S}^j_{\bar{m}}(\bar{t}^j_{[l_j,r_j]}) = \overrightarrow{\prod_{j+1\le a\le N}}\Big(\overrightarrow{\prod_{r_j-m_{a-1}<\ell\le r_j-m_a}} F_{a,j}(t^j_\ell)\Big). \tag{4.27}$$

The string (4.27) depends only on the type j variables $\{t^j_{l_j+1},\dots,t^j_{r_j}\}$ corresponding to the segment $[l_j,r_j]$. The collection $\bar{m}$ of nonnegative integers satisfying the admissibility condition (4.26) splits the segment $[l_j,r_j]$ into $N-j$ subsegments $[r_j-m_{a-1},r_j-m_a]$

for $a = j+1, \dots, N$. This splitting determines the product of the composite currents $\mathcal{F}^j_{\overline{m}}(\overline{t}^j_{[l_j,r_j]})$ in the string (4.27).

Now we describe the construction of strings for the currents $\widehat{F}_{j,i}(t)$. Again, we consider a collection of segments $[\overline{l},\overline{r}]$ and the associated set of variables $\overline{t}_{[\overline{l},\overline{r}]}$. Let j be equal to $\max(a)$ such that $r_b = l_b$ for $b = a+1, \dots, N-1$. Let $\overline{s}$ be a set of nonnegative integers such that

$$r_a - l_a \geq s_a \geq 0, \quad a = 1, \dots, j, \tag{4.28}$$

and the following admissibility conditions are satisfied:

$$r_j - l_j = s_j \geq s_{j-1} \geq \dots \geq s_2 \geq s_1 \geq s_0 = 0. \tag{4.29}$$

We define the string associated with the currents $\widehat{F}_i(t)$ as an ordered product of composite currents

$$\widehat{\mathcal{S}}^j_{\overline{s}}(\overline{t}^j_{[l_j,r_j]}) = \overleftarrow{\prod_{j\geq a\geq 1}} \Big(\overleftarrow{\prod_{l_j+s_a\geq \ell > l_j+s_{a-1}}} \widehat{F}_{j+1,a}(t^j_\ell) \Big). \tag{4.30}$$

The string (4.30) also depends only on the type j variables $\{t^j_{l_j+1}, \dots, t^j_{r_j}\}$ corresponding to the segment $[r_j, l_j]$. The set $\overline{s}$ of nonnegative integers satisfying the admissibility condition (4.29) splits the segment $[r_j, l_j]$ into j subsegments $[l_j + s_{a-1}, l_j + s_a]$ for $a = 1, \dots, j$. This splitting determines the product of composite currents in the string (4.30). The projection of the string (4.30) was calculated in [12]. Explicit formulas for the projections of the strings (4.30) and (4.27) will be given below without proof.

For two sets of variables $\overline{u}$ and $\overline{v}$, we introduce the series

$$\begin{aligned} U(u_1,\dots,u_k;v_1,\dots,v_k) &= \prod_{1\leq i\leq k} \frac{v_i/u_i}{1-v_i/u_i} \prod_{1\leq m<n\leq k} \frac{q^{-1}-qv_m/u_n}{1-v_m/u_n}, \\ V(u_1,\dots,u_k;v_1,\dots,v_k) &= \prod_{1\leq i\leq k} \frac{v_i/u_i}{1-v_i/u_i} \prod_{1\leq n<m\leq k} \frac{q-q^{-1}v_n/u_m}{1-v_n/u_m}. \end{aligned} \tag{4.31}$$

For two collections $\overline{m}$ and $\overline{s}$ that satisfy the restrictions (4.25), (4.26) and (4.28), (4.29), we define a series depending on the variables $\overline{t}_{[\overline{l},\overline{l}+\overline{m}]}$ and $\overline{t}_{[\overline{r}-\overline{s},\overline{r}]}$:

$$Y(\overline{t}_{[\overline{l},\overline{l}+\overline{m}]}) = \prod_{a=j+1}^{N-1} U(t^{a-1}_{l_{a-1}+m_{a-1}-m_a+1}, \dots, t^{a-1}_{l_{a-1}+m_{a-1}}; t^a_{l_a+1}, \dots, t^a_{l_a+m_a}) \tag{4.32}$$

and

$$X(\overline{t}_{[\overline{r}-\overline{s},\overline{r}]}) = \prod_{a=1}^{j-1} V(t^{a+1}_{r_{a+1}-s_{a+1}+s_a}, \dots, t^{a+1}_{r_{a+1}-s_{a+1}+1}; t^a_{r_a}, \dots, t^a_{r_a-s_a+1}). \tag{4.33}$$

If $j = N-1$ in (4.32), then we set $Y(\cdot) = 1$, and if $j = 1$ in (4.33), we set $X(\cdot) = 1$.

4.4. Recurrence relation. In this subsection we formulate recurrence relations among universal weight functions. The universal weight functions $\mathcal{W}^N$ and $\widehat{\mathcal{W}}^N$ of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ are defined by relations (4.5) and (4.6). In what follows, we denote by $\mathcal{W}^{N-1}$ the universal weight function of $U_q(\widehat{\mathfrak{gl}}_{N-1})$ generated by the currents $F_i(t)$, $E_i(t)$, $i = 2, \dots, N-1$, and $k^\pm_j(t)$, $j = 2, \dots, N$. Similarly, $\widehat{\mathcal{W}}^{N-1}$ is the universal weight function of the algebra $U_q(\widehat{\mathfrak{gl}}_{N-1})$ generated by the currents $\widehat{F}_i(t)$, $\widehat{E}_i(t)$, $i = 1, \dots, N-2$, and $\widehat{k}^\pm_j(t)$, $j = 1, \dots, N-1$.

Let $\bar{n}$ be a set of nonnegative integers, $\bar{n} = \{n_1, \dots, n_{N-1}\}$. We claim that the universal weight functions $\mathcal{W}^{N-1}(\bar{t}_{[\bar{n}]})$ and $\widehat{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}]})$ satisfy the following recurrence relations.

Proposition 4.2. (i) *The universal weight function* (4.5) *satisfies the recurrence relation*

$$\begin{aligned}\mathcal{W}^N(\bar{t}_{[\bar{n}]}) = \sum_{\bar{m}'} \prod_{a=1}^{N-1} \frac{1}{(m_a - m_{a+1})!(n_a - m_a)!} \\ \times \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \big(Z_{\bar{m}'}(\bar{t}_{[\bar{n}']}) \cdot Y(\bar{t}_{[\bar{m}]}) \cdot P^+\big(\mathcal{S}^1_{\bar{m}}(\bar{t}^1_{[n_1]})\big) \cdot \mathcal{W}^{N-1}(\bar{t}_{[\bar{m}',\bar{n}']})\big),\end{aligned} \tag{4.34}$$

where the sum is taken over all collections $\bar{m}' = \{m_2, m_3, \dots, m_{N-1}\}$ *such that* $n_1 = m_1 \geq m_2 \geq \dots \geq m_{N-1} \geq m_N = 0$ *for* $0 \leq m_a \leq n_a$, $a = 2, \dots, N-1$, *and* $\bar{m} = \{m_1, m_2, \dots, m_{N-1}\}$, $\bar{n}' = \{n_2, \dots, n_{N-1}\}$.

(ii) [12] *The universal weight function* (4.6) *satisfies the recurrence relation*

$$\begin{aligned}\widehat{\mathcal{W}}^N(\bar{t}_{[\bar{n}]}) = \sum_{\bar{s}'} \prod_{a=1}^{N-1} \frac{1}{(s_a - s_{a-1})!(n_a - s_a)!} \\ \times \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \big(\widehat{Z}_{\bar{s}'}(\bar{t}_{[\bar{n}']}) \cdot X(\bar{t}_{[\bar{n}-\bar{s},\bar{n}]}) \cdot \widehat{P}^+\big(\widehat{\mathcal{S}}^{N-1}_{\bar{s}}(\bar{t}^{N-1}_{[n_{N-1}]})\big) \cdot \widehat{\mathcal{W}}^{N-1}(\bar{t}_{[\bar{n}'-\bar{s}']})\big),\end{aligned} \tag{4.35}$$

where the sum is taken over all collections $\bar{s}' = \{s_1, s_2, \dots, s_{N-2}\}$ *such that* $n_{N-1} = s_{N-1} \geq s_{N-2} \geq \dots \geq s_1 \geq s_0 = 0$ *for* $0 \leq s_a \leq n_a$, $a = 1, \dots, N-2$, *and* $\bar{s} = \{s_1, \dots, s_{N-2}, s_{N-1}\}$, $\bar{n}' = \{n_1, \dots, n_{N-2}\}$.

Proof. Relation (4.35) was proved in [12]. We sketch the proof of the recurrence relation (4.34). We have a decomposition $\mathcal{F}(\bar{t}_{[\bar{n}]}) = \mathcal{F}(\bar{t}^1_{[n_1]})\mathcal{F}(\bar{t}_{[\bar{n}']})$, where the first factor depends only on the type 1 variables and the second factor does not depend on variables of this type:

$$\begin{aligned}\mathcal{F}(\bar{t}^1_{[n_1]}) &= F_1(t^1_1) \cdots F_1(t^1_{n_1}), \\ \mathcal{F}(\bar{t}_{[\bar{n}']}) &= F_2(t^2_1) \cdots F_2(t^2_{n_2}) \cdots F_{N-1}(t^{N-1}_1) \cdots F_{N-1}(t^{N-1}_{n_{N-1}}).\end{aligned}$$

We apply the ordering procedure of Proposition 4.1 to the product $\mathcal{F}(\bar{t}_{[\bar{n}']})$ and substitute the result in (4.6):

$$\begin{aligned}\mathcal{W}^N(\bar{t}_{[\bar{n}]}) = \sum_{\bar{m}'} \prod_{a=2}^{N-1} \frac{1}{m_a!(n_a - m_a)!} \\ \times \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}']}} \Big(Z_{\bar{m}'}(\bar{t}_{[\bar{n}']}) P^+\big(\mathcal{F}(\bar{t}^1_{[m_1]}) P^-\big(\mathcal{F}(\bar{t}_{[\bar{m}']})\big)\big) \cdot \mathcal{W}^{N-1}(\bar{t}_{[\bar{m}',\bar{n}']})\Big).\end{aligned} \tag{4.36}$$

The sum is taken over all possible collections $\bar{m}' = \{m_2, \dots, m_{N-1}\}$ of nonnegative integers such that $m_a \leq n_a$, $a = 2, \dots, N-1$, and q-symmetrization is performed over the variables $\bar{t}_{[\bar{n}']}$.

The proof of (4.34) is based on the following lemma, which is proved in Appendix B. □

Lemma 4.3. *For any collection* $\bar{n} = \{n_1, \dots, n_{N-1}\}$ *of positive integers and any collections* $\bar{m} = \{m_1, m_2, \dots, m_{N-1}\}$, $\bar{m}' = \{m_2, \dots, m_{N-1}\}$ *of nonnegative integers such that* $m_a \leq n_a$ *for* $a = 2, \dots, N-1$ *and* $m_1 = n_1$, *we have*

$$\begin{aligned}&P^+\Big(\mathcal{F}(\bar{t}^1_{[m_1]}) P^-\big(\mathcal{F}(\bar{t}_{[\bar{m}']})\big)\Big) \\ &\quad = \prod_{a=1}^{N-1} \frac{1}{(m_a - m_{a+1})!} \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{m}]}} \Big(Y(\bar{t}_{[\bar{m}]}) \cdot P^+\big(\mathcal{S}^1_{\bar{m}}(\bar{t}^1_{[m_1]})\big)\Big)\end{aligned} \tag{4.37}$$

if $m_1 \geq m_2 \geq \cdots \geq m_{N-1} \geq m_N = 0$. *Otherwise the right-hand side of* (4.37) *is equal to zero.*

The series $Y(\bar{t}_{[\bar{m}]})$ is defined by (4.32) with $j = 1$ and $\bar{l} = \bar{0}$.

The definition (4.17) shows that the series $Z_{\bar{m}'}(\bar{t}_{[\bar{n}']})$ is symmetric with respect to permutations of the variables $t_{[\bar{m}]}$ of the same type, and the universal weight function $\mathcal{W}^{N-1}(\bar{t}_{[\bar{m}',\bar{n}']})$ does not depend on the variables $t_{[\bar{m}]}$. After substitution of (4.37) in (4.36), we can include the series $Z_{\bar{m}'}(\bar{t}_{[\bar{n}']})$ and $\mathcal{W}^{N-1}(\bar{t}_{[\bar{m}',\bar{n}']})$ in the q-symmetrization $\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{m}]}}(\cdot)$ and to use (4.14) in order to replace the double q-symmetrization by a single one:

$$\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}']}}\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{m}]}}(\,\cdot\,) = \prod_{a=2}^{N-1} m_a!\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}}(\,\cdot\,).$$

This proves Proposition 4.2. □

4.5. Iteration of recurrence relations. Consider two collections of nonnegative integers $[[\bar{m}]] = \{m_j^i\}$ and $[[\bar{s}]] = \{s_i^j\}$ for $1 \leq i \leq j \leq N-1$. We say that the collections $[[\bar{m}]]$ and $[[\bar{s}]]$ are $\bar{n}$-admissible if they satisfy the admissibility conditions (2.19) and (2.21), respectively. We also set $m_N^j = s_0^j = 0$ for $j = 1, \ldots, N-1$. The collections $[[\bar{m}]]$ and $[[\bar{s}]]$ can be visualized as triangular matrices:

$$[[\bar{m}]] = \begin{pmatrix} m_1^1 & m_2^1 & \ldots & m_{N-2}^1 & m_{N-1}^1 \\ 0 & m_2^2 & \ldots & m_{N-2}^2 & m_{N-1}^2 \\ & & \ddots & \vdots & \vdots \\ & 0 & & m_{N-2}^{N-2} & m_{N-1}^{N-2} \\ & & & & m_{N-1}^{N-1} \end{pmatrix} \quad \begin{matrix} 0 = m_N^1 \leq m_{N-1}^1 \leq \cdots \leq m_1^1, \\ 0 = m_N^2 \leq m_{N-1}^2 \leq \cdots \leq m_2^2, \\ \vdots \\ 0 = m_N^{N-2} \leq m_{N-1}^{N-2} \leq m_{N-2}^{N-2}, \\ 0 = m_N^{N-1} \leq m_{N-1}^{N-1}, \end{matrix} \tag{4.38}$$

and

$$[[\bar{s}]] = \begin{pmatrix} s_1^1 & & & & \\ s_1^2 & s_2^2 & & 0 & \\ \vdots & \vdots & \ddots & & \\ s_1^{N-2} & s_2^{N-2} & \ldots & s_{N-2}^{N-2} & \\ s_1^{N-1} & s_2^{N-1} & \ldots & s_{N-2}^{N-1} & s_{N-1}^{N-1} \end{pmatrix} \quad \begin{matrix} 0 = s_0^1 \leq s_1^1, \\ 0 = s_0^2 \leq s_1^2 \leq s_2^2, \\ \vdots \\ 0 = s_0^{N-2} \leq s_1^{N-2} \leq \cdots \leq s_{N-2}^{N-2}, \\ 0 = s_0^{N-1} \leq s_1^{N-1} \leq \cdots \leq s_{N-1}^{N-1}. \end{matrix} \tag{4.39}$$

Let $\bar{m}^j$ and $\bar{s}^j$, $j = 1, \ldots, N-1$, be the jth rows of the admissible matrices $[[\bar{m}]]$ and $[[\bar{s}]]$. We introduce the collection of vectors

$$\bar{\mathbf{m}}^j = \bar{m}^1 + \bar{m}^2 + \cdots + \bar{m}^{j-1} + \bar{m}^j, \qquad j = 1, \ldots, N-1, \tag{4.40}$$

$$\bar{\mathbf{s}}^j = \bar{s}^j + \bar{s}^{j+1} + \cdots + \bar{s}^{N-2} + \bar{s}^{N-1}, \qquad j = 1, \ldots, N-1, \tag{4.41}$$

with nonnegative integer components. Put $\bar{\mathbf{m}}^0 = \bar{0}$ and $\bar{\mathbf{s}}^N = \bar{0}$. Let $\mathbf{m}_a^j$ and $\mathbf{s}_a^j$ be the components of the vectors $\bar{\mathbf{m}}^j$ and $\bar{\mathbf{s}}^j$:

$$\bar{\mathbf{m}}^j = \{n_1, n_2, \ldots, n_j, m_{j+1}^1 + \cdots + m_{j+1}^j, \ldots, m_{N-1}^1 + \cdots + m_{N-1}^j\},$$
$$\bar{\mathbf{s}}^j = \{s_1^j + \cdots + s_1^{N-1}, \ldots, s_{j-1}^j + \cdots + s_{j-1}^{N-1}, n_j, \ldots, n_{N-2}, n_{N-1}\}.$$

Note that $\mathbf{m}_j^j - \mathbf{m}_j^{j-1} = m_j^j$ and, by the admissibility conditions (2.19) and (2.21), $\bar{\mathbf{m}}^{N-1} = \bar{\mathbf{s}}^1 = \bar{n}$.

Iteration of the recurrence relations (4.34) and (4.35) yields the following statement.

Theorem 1. (i) *The weight function* (4.5) *can be presented as a total q-symmetrization of the sum over all $\bar{n}$-admissible matrices* $[[\bar{m}]]$ *of the ordered products of projections of the strings* (4.27) *with rational coefficients*:

$$\mathcal{W}^N(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{m}]]} \Bigg(\prod_{b=1}^{N-1} \prod_{a=b}^{N-1} \frac{1}{(m_a^b - m_{a+1}^b)!} \prod_{j=1}^{N-3} Z_{\bar{m}^j}(\bar{t}_{[\bar{\mathbf{m}}^{j-1},\bar{n}]}) \times \prod_{j=1}^{N-2} Y(\bar{t}_{[\bar{\mathbf{m}}^{j-1},\bar{\mathbf{m}}^j]}) \overrightarrow{\prod_{1\le j\le N-1}} P^+\big(\mathcal{S}^j_{\bar{m}^j}(\bar{t}^j_{[\mathbf{m}^{j-1}_j,\mathbf{m}^j_j]})\big)\Bigg). \tag{4.42}$$

(ii) [12] *The weight function* (4.6) *can be presented as a total q-symmetrization of the sum over all $\bar{n}$ admissible matrices* $[[\bar{s}]]$ *of the ordered products of projections of the strings* (4.30) *with rational coefficients*:

$$\widehat{\mathcal{W}}^N(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]} \Bigg(\prod_{b=1}^{N-1} \prod_{a=1}^{b} \frac{1}{(s_a^b - s_{a-1}^b)!} \prod_{j=3}^{N-1} \widehat{Z}_{\bar{s}^j}(\bar{t}_{[\bar{n}-\bar{\mathbf{s}}^{j+1}]}) \times \prod_{j=2}^{N-1} X(\bar{t}_{[\bar{n}-\bar{\mathbf{s}}^j,\bar{n}-\bar{\mathbf{s}}^{j+1}]}) \overleftarrow{\prod_{N-1\ge j\ge 1}} \widehat{P}^+\big(\widehat{\mathcal{S}}^j_{\bar{s}^j}(\bar{t}^j_{[s^j_j]})\big)\Bigg). \tag{4.43}$$

The products of rational series in (4.42) and (4.43) can be written as a single multivariable series. Define

$$\begin{aligned}\mathcal{Y}_{[[\bar{m}]]}(\bar{t}_{[\bar{n}]}) &= \prod_{j=1}^{N-3} Z_{\bar{m}^j}(\bar{t}_{[\bar{\mathbf{m}}^{j-1},\bar{n}]}) \prod_{j=1}^{N-2} Y(\bar{t}_{[\bar{\mathbf{m}}^{j-1},\bar{\mathbf{m}}^j]}) \\ &= \prod_{a=2}^{N-1} \prod_{b=1}^{a-1} \prod_{\ell=0}^{m_a^b-1} \frac{t^a_{\mathbf{m}^b_a-\ell}/t^{a-1}_{\mathbf{m}^b_{a-1}-\ell}}{1-t^a_{\mathbf{m}^b_a-\ell}/t^{a-1}_{\mathbf{m}^b_{a-1}-\ell}} \prod_{\ell'=\mathbf{m}^b_{a-1}-\ell+1}^{n_{a-1}} \frac{q^{-1}-qt^a_{\mathbf{m}^b_a-\ell}/t^{a-1}_{\ell'}}{1-t^a_{\mathbf{m}^b_a-\ell}/t^{a-1}_{\ell'}}\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}\mathcal{X}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]}) &= \prod_{j=3}^{N-1} \widehat{Z}_{\bar{s}^j}(\bar{t}_{[\bar{n}-\bar{\mathbf{s}}^{j+1}]}) \prod_{j=2}^{N-1} X(\bar{t}_{[\bar{n}-\bar{\mathbf{s}}^j,\bar{n}-\bar{\mathbf{s}}^{j+1}]}) \\ &= \prod_{b=2}^{N-1} \prod_{a=1}^{b-1} \prod_{\ell=1}^{s_a^b} \frac{t^a_{\ell+n_a-\mathbf{s}^b_a}/t^{a+1}_{\ell+n_{a+1}-\mathbf{s}^b_{a+1}}}{1-t^a_{\ell+n_a-\mathbf{s}^b_a}/t^{a+1}_{\ell+n_{a+1}-\mathbf{s}^b_{a+1}}} \prod_{\ell'=1}^{\ell+n_{a+1}-\mathbf{s}^b_{a+1}-1} \frac{q-q^{-1}t^a_{\ell+n_a-\mathbf{s}^b_a}/t^{a+1}_{\ell'}}{1-t^a_{\ell+n_a-\mathbf{s}^b_a}/t^{a+1}_{\ell'}}.\end{aligned} \tag{4.45}$$

We can formulate the following corollary to Theorem 1.

Corollary 4.4. *The weight functions* (4.5) *and* (4.6) *can be presented in the following compact form*:

$$\mathcal{W}^N(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{m}]]} \Bigg(\frac{\mathcal{Y}_{[[\bar{m}]]}(\bar{t}_{[\bar{n}]})}{\prod_{a\ge b}(m_a^b - m_{a+1}^b)!} \overrightarrow{\prod_{1\le j\le N-1}} P^+\big(\mathcal{S}^j_{\bar{m}^j}(\bar{t}^j_{[\mathbf{m}^{j-1}_j,\mathbf{m}^j_j]})\big)\Bigg), \tag{4.46}$$

$$\widehat{\mathcal{W}}^N(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]} \Bigg(\frac{\mathcal{X}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]})}{\prod_{a\le b}(s_a^b - s_{a-1}^b)!} \overleftarrow{\prod_{N-1\ge j\ge 1}} \widehat{P}^+\big(\widehat{\mathcal{S}}^j_{\bar{s}^j}(\bar{t}^j_{[s^j_j]})\big)\Bigg). \tag{4.47}$$

Theorem 1 reduces the calculation of the universal weight function to the calculation of projections of strings.

4.6. Projection of composed currents and strings. In this subsection we formulate several statements about projections of strings. Their proofs can be found in the papers [10, 12]. First, we define two types of screening operators:

$$S_A\, B = B \cdot A - qA \cdot B, \quad \widehat{S}_A\, B = B \cdot A - q^{-1} A \cdot B. \tag{4.48}$$

We use these operators when A is a zero mode and B is a total current $F_i(t)$ or $\widehat{F}_i(t)$. In this case these operators can be related to the standard coproduct (2.5) via the associated adjoint action (see [12]).

The first relation in (4.23) for the currents $F_i(t)$ and a similar relation for the currents $\widehat{F}_i(t)$ yield

$$\begin{aligned} &P^+\big(F_{j,i}(t)\big) = S_{F_{j,j-1}[0]}\big(P^+(F_{j-1,i}(t))\big), \\ &\widehat{P}^+\big(\widehat{F}_{j,i}(t)\big) = \widehat{S}_{\widehat{F}_{i+1,i}[0]}\big(\widehat{P}^+(\widehat{F}_{j,i+1}(t))\big), \quad i < j-1. \end{aligned} \tag{4.49}$$

We have used the fact, proved in [10], that the screening operators and the projections commute. Iterating these relations, we can write the projections for the composite currents in terms of successive application of screening operators:

$$\begin{aligned} P^+\big(F_{j+1,i}(u)\big) &= S_j S_{j-1} \cdots S_{i+1}\big(P^+(F_i(u))\big) \\ &= S_j S_{j-1} \cdots S_{i+1}\Big(u\big(u^{-1}F_i(u)\big)^{(+)}\Big) = u\,\big(u^{-1} S_j S_{j-1} \cdots S_{i+1}\big(F_i(u)\big)\big)^{(+)}, \\ \widehat{P}^+\big(\widehat{F}_{j+1,i}(u)\big) &= \widehat{S}_i \widehat{S}_{i+1} \cdots \widehat{S}_{j-1}\big(\widehat{P}^+(\widehat{F}_j(u))\big) \\ &= \widehat{S}_i \widehat{S}_{i+1} \cdots \widehat{S}_{j-1}\Big(u\big(u^{-1}\widehat{F}_j(u)\big)^{(+)}\Big) = u\,\Big(u^{-1} \widehat{S}_i \widehat{S}_{i+1} \cdots \widehat{S}_{j-1}\big(\widehat{F}_j(u)\big)\Big)^{(+)}, \end{aligned}$$

where we put $S_i \equiv S_{F_i[0]}$ and $\widehat{S}_i \equiv \widehat{S}_{\widehat{F}_i[0]}$.

For a set $\{u_1, \dots, u_n\}$ of formal variables, consider the rational functions

$$\begin{aligned} \varphi_{u_m}(u; u_1, \dots, u_n) &= \prod_{k=1,\, k\neq m}^{n} \frac{u}{u_m}\, \frac{u-u_k}{u_m-u_k} \prod_{k=1}^{n} \frac{q^{-1}u_m - qu_k}{q^{-1}u - qu_k}, \\ \widehat{\varphi}_{u_m}(u; u_1, \dots, u_n) &= \prod_{k=1,\, k\neq m}^{n} \frac{u}{u_m}\, \frac{u-u_k}{u_m-u_k} \prod_{k=1}^{n} \frac{qu_m - q^{-1}u_k}{qu - q^{-1}u_k}, \end{aligned} \tag{4.50}$$

which satisfy the normalization conditions $\varphi_{u_m}(u_s; u_1, \dots, u_n) = \widehat{\varphi}_{u_m}(u_s; u_1, \dots, u_n) = \delta_{ms}$. We set

$$\begin{aligned} F_{j+1,i}(u; u_1, \dots, u_n) &= F_{j+1,i}(u) - \sum_{m=1}^{n} \varphi_{u_m}(u; u_1, \dots, u_n) F_{j+1,i}(u_m), \\ \widehat{F}_{j+1,i}(u; u_1, \dots, u_n) &= \widehat{F}_{j+1,i}(u) - \sum_{m=1}^{n} \widehat{\varphi}_{u_m}(u; u_1, \dots, u_n) \widehat{F}_{j+1,i}(u_m), \end{aligned} \tag{4.51}$$

for $1 \le i \le j < N$, and similarly

$$\begin{aligned} P^+\big(F_{j+1,i}(u; u_1, \dots, u_n)\big) &= S_j S_{j-1} \cdots S_{i+1}\big(F_i(u)^{(+)}\big) \\ &\quad - \sum_{m=1}^{n} \varphi_{u_m}(u; u_1, \dots, u_n) S_j S_{j-1} \cdots S_{i+1}\big(F_i(u_m)^{(+)}\big), \\ \widehat{P}^+\big(\widehat{F}_{j+1,i}(u; u_1, \dots, u_n)\big) &= \widehat{S}_i \widehat{S}_{i+1} \cdots \widehat{S}_{j-1}\big(\widehat{F}_j(u)^{(+)}\big) \\ &\quad - \sum_{m=1}^{n} \widehat{\varphi}_{u_m}(u; u_1, \dots, u_n) \widehat{S}_i \widehat{S}_{i+1} \cdots \widehat{S}_{j-1}\big(\widehat{F}_j(u_m)^{(+)}\big). \end{aligned} \tag{4.52}$$

In the same way as in [12, Appendix B], we can deduce the following formulas for both types of projections of strings:

$$
\begin{aligned}
P^+\big(\mathcal{S}^j_{\overline{m}}(\overline{t}^j_{[l_j,r_j]})\big) &= \overleftarrow{\prod_{N\ge a\ge j+1}}\Big(\overleftarrow{\prod_{r_j-m_a\ge\ell>r_j-m_{a-1}}} P^+\big(F_{a,j}(t^j_\ell;t^j_{\ell+1},\dots,t^j_{r_j})\big)\Big)\\
&\times\prod_{l_j<\ell<\ell'\le r_j}\frac{q^{-1}-qt^j_\ell/t^j_{\ell'}}{1-t^j_\ell/t^j_{\ell'}}\prod_{j+1\le a\le N}\Big(\prod_{r_j-m_{a-1}<\ell<\ell'\le r_j-m_a}\frac{1-t^j_\ell/t^j_{\ell'}}{q-q^{-1}t^j_\ell/t^j_{\ell'}}\Big),
\end{aligned}
\tag{4.53}
$$

$$
\begin{aligned}
\widehat{P}^+\big(\widehat{\mathcal{S}}^j_{\overline{s}}(\overline{t}^j_{[l_j,r_j]})\big) &= \overrightarrow{\prod_{1\le a\le j}}\Big(\overrightarrow{\prod_{l_j+s_{a-1}<\ell\le l_j+s_a}} \widehat{P}^+\big(\widehat{F}_{j+1,a}(t^j_\ell;t^j_{l_j+1},\dots,t^j_{\ell-1})\big)\Big)\\
&\times\prod_{l_j<\ell<\ell'\le r_j}\frac{q-q^{-1}t^j_{\ell'}/t^j_\ell}{1-t^j_{\ell'}/t^j_\ell}\prod_{1\le a\le j}\Big(\prod_{l_j+s_{a-1}<\ell<\ell'\le l_j+s_a}\frac{1-t^j_{\ell'}/t^j_\ell}{q^{-1}-qt^j_{\ell'}/t^j_\ell}\Big).
\end{aligned}
\tag{4.54}
$$

The commutation relations between the total and Cartan currents, combined with formulas (4.53) and (4.54), imply the following relations that are important for what follows:

$$
\begin{aligned}
P^+\big(\mathcal{S}^j_{\overline{s}}(\overline{t}^j_{[l_j,r_j]})\big)\prod_{\ell=l_j+1}^{r_j}k^+_j(t^j_\ell) &= \prod_{j+1\le a\le N}\Big(\prod_{r_j-m_{a-1}<\ell<\ell'\le r_j-m_a}\frac{1-t^j_\ell/t^j_{\ell'}}{q-q^{-1}t^j_\ell/t^j_{\ell'}}\Big)\\
&\times\overleftarrow{\prod_{N\ge a\ge j+1}}\Big(\overleftarrow{\prod_{r_j-m_a\ge\ell>r_j-m_{a-1}}} P^+\big(F_{a,j}(t^j_\ell)\big)k^+_j(t^j_\ell)\Big),
\end{aligned}
\tag{4.55}
$$

$$
\begin{aligned}
\widehat{P}^+\big(\widehat{\mathcal{S}}^j_{\overline{s}}(\overline{t}^j_{[l_j,r_j]})\big)\prod_{\ell=l_j+1}^{r_j}\widehat{k}^+_{j+1}(t^j_\ell) &= \prod_{1\le a\le j}\Big(\prod_{l_j+s_{a-1}<\ell<\ell'\le l_j+s_a}\frac{1-t^j_{\ell'}/t^j_\ell}{q^{-1}-qt^j_{\ell'}/t^j_\ell}\Big)\\
&\times\overrightarrow{\prod_{1\le a\le j}}\Big(\overrightarrow{\prod_{l_j+s_{a-1}<\ell\le l_j+s_a}} \widehat{P}^+\big(\widehat{F}_{j+1,a}(t^j_\ell)\big)\widehat{k}^+_{j+1}(t^j_\ell)\Big).
\end{aligned}
\tag{4.56}
$$

§5. Universal weight functions and L-operators

In this section we use the factorization formulas (4.55) and (4.56) to present the off-shell Bethe vectors (4.8) and (4.7) in terms of the L-operator's entries. First, we relate the projections of currents to the Gauss coordinates of L-operators. It can be proved [12] that for $i<j$ we have

$$
\begin{aligned}
P^+\big(F_{j,i}(t)\big) &= (q^{-1}-q)^{j-i-1}\,\mathrm{F}^+_{j,i}(t),\\
\widehat{P}^+\big(\widehat{F}_{j,i}(t)\big) &= (q-q^{-1})^{j-i-1}\,\widehat{\mathrm{F}}^+_{j,i}(t).
\end{aligned}
\tag{5.1}
$$

For $j=i+1$, this is one of the relations proved in [3]. For $i<j-1$, formulas (5.1) are implied by the following relations for the Gauss coordinates of the matrix entries of L-operators:

$$
\begin{aligned}
(q^{-1}-q)\,\mathrm{F}^+_{j,i}(t) &= S_{j-1}\big(\mathrm{F}^+_{j-1,i}(t)\big),\\
(q-q^{-1})\,\widehat{\mathrm{F}}^+_{j,i}(t) &= \widehat{S}_i\big(\widehat{\mathrm{F}}^+_{j,i+1}(t)\big).
\end{aligned}
$$

Lemma 5.1. *For any $c=1,\dots,N$, denote by I_c and $\widehat{I}_c$ the left ideals of $U_q(\widehat{\mathfrak{b}}^+)$ generated by the modes of $\mathrm{E}^+_{i,j}(u)$ with $i<j\le c$ and by the modes of $\widehat{\mathrm{E}}^+_{i,j}(u)$ with $c\le i<j$. We have the inclusions $0=I_1\subset I_2\subset\dots\subset I_N$ and $0=\widehat{I}_N\subset\widehat{I}_{N-1}\subset\dots\subset\widehat{I}_1$.*

(i) *If $a < b$, then*

$$\mathrm{L}^+_{a,b}(t) \equiv \mathrm{F}^+_{b,a}(t)k^+_a(t) \mod I_c, \qquad \mathrm{L}^+_{a,a}(t) \equiv k^+_a(t) \mod I_c, \quad a \le c,$$
$$\mathrm{L}^+_{a,b}(t) \equiv \widehat{\mathrm{F}}^+_{b,a}(t)\widehat{k}^+_b(t) \mod \widehat{I}_c, \qquad \mathrm{L}^+_{b,b}(t) \equiv \widehat{k}^+_b(t) \mod \widehat{I}_c, \quad b \ge c.$$

(ii) *The left ideal I_c is generated by the modes of $\mathrm{L}^+_{j,i}(u)$ with $i < j \le c$; the left ideal $\widehat{I}_c$ is generated by the modes of $\mathrm{L}^+_{j,i}(u)$ with $c \le i < j$; and $I_N = \widehat{I}_1$.*

(iii) *For any $a \ge c$, the modes of $\mathrm{L}^+_{c,a}(t)$ normalize the ideal I_c:*

$$I_c \cdot \mathrm{L}^+_{c,a}(t) \subset I_c.$$

For any $a \le c$, the modes of $\mathrm{L}^+_{a,c}(t)$ normalize the ideal $\widehat{I}_c$:

$$\widehat{I}_c \cdot \mathrm{L}^+_{a,c}(t) \subset \widehat{I}_c.$$

Proof. Statements (i) and (ii) follow from the Gauss decompositions (3.1)–(3.3) and (3.4)–(3.6). Statement (iii) follows from (ii) and the RLL-relations (2.3). □

Theorem 2. *For any $U_q(\widehat{\mathfrak{gl}}_N)$-module V with a weight singular vector v, we have the following formulas for the universal modified weight functions*:

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\ \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\overline{m}]]} \Bigg(\frac{(q^{-1}-q)^{\sum\limits_{b=1}^{N-1}(n_b - m^b_b)}}{\prod\limits_{a\ge b}^{N-1}(m^b_a - m^b_{a+1})!}\, \mathcal{Y}_{[[\overline{m}]]}(\bar{t}_{[\bar{n}]})$$
$$\times \prod_{1\le b\le N-1}^{\longrightarrow} \Bigg(\prod_{N-1\ge a\ge b}^{\longleftarrow} \prod_{\ell=n_b-m^b_a+1}^{n_b-m^b_{a+1}} \Big(\mathrm{L}^+_{b,a+1}(t^b_\ell) \prod_{\ell'=n_b-m^b_a+1}^{\ell-1} \frac{t^b_{\ell'} - t^b_\ell}{q^{-1}t^b_{\ell'} - qt^b_\ell} \Big)\Bigg) \prod_{\ell=1}^{n_b-m^b_b} \mathrm{L}^+_{b,b}(t^b_\ell)\Bigg) v \tag{5.2}$$

and

$$\widehat{\mathbf{w}}^N_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\ \overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]} \Bigg(\frac{(q-q^{-1})^{\sum\limits_{b=1}^{N-1}(n_b - s^b_b)}}{\prod\limits_{a\le b}^{N-1}(s^b_a - s^b_{a-1})!}\, \mathcal{X}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]})$$
$$\times \prod_{N-1\ge b\ge 1}^{\longleftarrow} \Bigg(\prod_{1\le a\le b}^{\longrightarrow} \prod_{\ell=s^b_{a-1}+1}^{s^b_a} \Big(\mathrm{L}^+_{a,b+1}(t^b_\ell) \prod_{\ell'=\ell+1}^{s^b_a} \frac{t^b_\ell - t^b_{\ell'}}{q^{-1}t^b_\ell - qt^b_{\ell'}} \Big)\Bigg) \prod_{\ell=s^b_b+1}^{n_b} \mathrm{L}^+_{b+1,b+1}(t^b_\ell)\Bigg) v \tag{5.3}$$

(*the ordering in the products over ℓ is not important because the entries of* L*-operators with equal matrix indices commute*).

Proof. Let V be a $U_q(\widehat{\mathfrak{gl}}_N)$-module with a weight singular vector v. Using relations (4.55), (4.56), (5.1) and Corollary 4.4, we can write the modified weight functions $\mathbf{w}^N_V(\bar{t}_{[\bar{n}]})$ and $\widehat{\mathbf{w}}^N_V(\bar{t}_{[\bar{n}]})$ in the following form:

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\,\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\overline{m}]]} \Bigg(\frac{(q^{-1}-q)^{\sum_{b=1}^{N-1}(n_b - m^b_b)}}{\prod\limits_{a\ge b}^{N-1}(m^b_a - m^b_{a+1})!} \mathcal{Y}_{[[\overline{m}]]}(\bar{t}_{[\bar{n}]}) \prod_{1\le b\le N-1}^{\longrightarrow} \Bigg(\prod_{N-1\ge a\ge b}^{\longleftarrow}$$
$$\times \prod_{n_b-m^b_{a+1}\ge \ell > n_b-m^b_a}^{\longleftarrow} \Big(F^+_{a+1,b}(t^b_\ell) k^+_b(t^b_\ell) \prod_{\ell'=n_b-m^b_a+1}^{\ell-1} \frac{t^b_{\ell'} - t^b_\ell}{q^{-1}t^b_{\ell'} - qt^b_\ell} \Big)\Bigg) \prod_{\ell=1}^{n_b-m^b_b} k^+_b(t^b_\ell)\Bigg) v \tag{5.4}$$

and

$$\widehat{\mathbf{w}}_V^N(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})\,\overline{\mathrm{Sym}}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]} \Bigg(\frac{(q-q^{-1})^{\sum_{b=1}^{N-1}(n_b-s_b^b)}}{\prod\limits_{a\le b}^{N-1}(s_a^b-s_{a-1}^b)!} \mathcal{X}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]}) \overleftarrow{\prod_{N-1\ge b\ge 1}} \Bigg(\overrightarrow{\prod_{1\le a\le b}} \tag{5.5}$$
$$\times \overrightarrow{\prod_{s_{a-1}^b<\ell\le s_a^b}} \Big(\widehat{\mathrm{F}}_{b+1,a}^+(t_\ell^b)\widehat{k}_{b+1}^+(t_\ell^b) \prod_{\ell'=\ell+1}^{s_a^b} \frac{t_\ell^b-t_{\ell'}^b}{q^{-1}t_\ell^b-qt_{\ell'}^b} \Big)\Bigg) \prod_{\ell=s_b^b+1}^{n_b} \widehat{k}_{b+1}^+(t_\ell^b)\Bigg) v,$$

where the series $\mathcal{Y}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]})$ and $\mathcal{X}_{[[\bar{s}]]}(\bar{t}_{[\bar{n}]})$ are given by (4.44) and (4.45), respectively. Then the statement of the theorem follows from (5.4), (5.5), and Lemma 5.1 in the same way as was done in [12]. □

Formulas (5.2) and (5.3) can be simplified by using the q-symmetrization (2.15). Denoting $\widetilde{\mathbf{s}}_a^b = n_a - \mathbf{s}_a^b = s_a^a + \cdots + s_a^{b-1}$, we formulate the following corollary to Theorem 2.

Corollary 5.2. *The off-shell Bethe vectors for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ can be written as*

$$\mathbf{w}_V^N(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}^{(q)}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{m}]]} \Bigg((q^{-1}-q)^{\sum\limits_{a=1}^{N-1}(n_a-m_a^a)} \prod_{a\le b}^{N-1} \frac{1}{[m_b^a-m_{b+1}^a]_q!} \tag{5.6}$$
$$\times \prod_{a=2}^{N-1}\prod_{b=1}^{a-1}\prod_{\ell=0}^{m_a^b-1} \frac{t^a_{\mathbf{m}_a^b-\ell}/t^{a-1}_{\mathbf{m}_{a-1}^b-\ell}}{1-t^a_{\mathbf{m}_a^b-\ell}/t^{a-1}_{\mathbf{m}_{a-1}^b-\ell}} \prod_{\ell'=\mathbf{m}_{a-1}^b-\ell+1}^{n_{a-1}} \frac{q^{-1}-qt^a_{\mathbf{m}_a^b-\ell}/t^{a-1}_{\ell'}}{1-t^a_{\mathbf{m}_a^b-\ell}/t^{a-1}_{\ell'}}$$
$$\times \overrightarrow{\prod_{1\le a\le N-1}} \Big(\overleftarrow{\prod_{N-1\ge b\ge a}} \prod_{\ell=n_a-m_b^a+1}^{n_a-m_{b+1}^a} \mathrm{L}_{a,b+1}^+(t_\ell^a)\Big) \prod_{\ell=1}^{n_a-m_a^a} \mathrm{L}_{a,a}^+(t_\ell^a)\Bigg) v$$

and

$$\widehat{\mathbf{w}}_V^N(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}^{(q)}_{\bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]} \Bigg((q-q^{-1})^{\sum\limits_{b=1}^{N-1}(n_b-s_b^b)} \prod_{a\le b}^{N-1} \frac{1}{[s_a^b-s_{a-1}^b]_q!} \tag{5.7}$$
$$\times \prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s_a^b} \frac{1}{1-t^a_{\ell+\widetilde{\mathbf{s}}_a^b}/t^{a+1}_{\ell+\widetilde{\mathbf{s}}_{a+1}^b}} \prod_{\ell'=1}^{\ell+\widetilde{\mathbf{s}}_{a+1}^b-1} \frac{q-q^{-1}t^a_{\ell+\widetilde{\mathbf{s}}_a^b}/t^{a+1}_{\ell'}}{1-t^a_{\ell+\widetilde{\mathbf{s}}_a^b}/t^{a+1}_{\ell'}}$$
$$\times \overleftarrow{\prod_{N-1\ge b\ge 1}} \Big(\overrightarrow{\prod_{1\le a\le b}} \Big(\prod_{\ell=s_{a-1}^b+1}^{s_a^b} \mathrm{L}_{a,b+1}^+(t_\ell^b)\Big) \prod_{\ell=s_b^b+1}^{n_b} \mathrm{L}_{b+1,b+1}^+(t_\ell^b)\Big)\Bigg) v.$$

Proof. We only prove (5.6), because (5.7) can be verified in the same way. Consider the right-hand side of (5.2). Under the total q-symmetrization sign, there is an expression symmetric in the variables $\{t_\ell^b\}$ for $n_b - m_a^b + 1 \le \ell \le n_b - m_{a+1}^b$, for $b = 1, \dots, N-1$ and $a = b, \dots, N-1$. This follows from the commutativity of the matrix entries of L-operators, $[\mathrm{L}_{a,b}^+(t), \mathrm{L}_{a,b}^+(t')] = 0$, and the explicit form of the series (4.44). The product $\prod_{\ell'=n_b-m_a^b+1}^{\ell-1} \frac{t_{\ell'}^b-t_\ell^b}{q^{-1}t_{\ell'}^b-qt_\ell^b}$ is inverse to the function $\beta(t^b_{[n_b-m_a^b,n_b-m_{a+1}^b]})$ defined in (2.14). Applying formulas (4.13) and (4.15) successively, and using the explicit form of the series (4.44), we obtain (5.6). □

§6. Relationship between two weight functions

In this section we calculate the image of the off-shell Bethe vector (5.6) for the evaluation homomorphism (2.7). Let M_Λ be a $U_q(\mathfrak{gl}_N)$-module generated by a vector v satisfying the conditions $\mathsf{E}_{a,a}\, v = q^{\Lambda_a}\, v$ and $\mathsf{E}_{a,b}\, v = 0$ for $a < b$. Then v is a singular weight vector of the evaluation $U_q(\widehat{\mathfrak{gl}}_N)$-module $M_\lambda(z)$. The action of the matrix entries of L-operators in this module is given by

$$\begin{aligned} &\mathcal{E}v_z\big(\mathrm{L}^+_{a,b+1}(t)\big) = (q-q^{-1})\mathsf{E}_{b+1,a}\mathsf{E}_{b+1,b+1} \equiv (q-q^{-1})\check{\mathsf{E}}_{b+1,a}, \\ &\mathcal{E}v_z\big(\mathrm{L}^+_{a,a}(t)\big)v = \big(q^{\Lambda_a} - q^{-\Lambda_a}\frac{z}{t}\big)v = \lambda_a(t)v. \end{aligned} \tag{6.1}$$

Proposition 6.1. *For any evaluation $U_q(\widehat{\mathfrak{gl}}_N)$-module $M_\lambda(z)$ with singular weight vector v, we have*

$$\mathbf{w}_{M_\lambda(z)}(\bar{t}_{[\bar{n}]}) = \mathbb{B}_{M_\lambda(z)}(\bar{t}_{[\bar{n}]}). \tag{6.2}$$

Proof. Substituting (6.1) in (5.6) and reordering the factors

$$\prod_{a=2}^{N-1}\prod_{\ell=1}^{n_a-m_a^a}\lambda_a(t^a_\ell) = \prod_{a=2}^{N-1}\prod_{b=1}^{a-1}\prod_{\ell=0}^{m_a^b-1}\lambda_a(t^a_{\mathbf{m}_a^b-\ell}), \tag{6.3}$$

we obtain

$$\begin{aligned} \mathbf{w}^N_{M_\lambda(z)}(\bar{t}_{[\bar{n}]}) = (q-q^{-1})^{\sum_{a=1}^{N-1} n_a} \sum_{[[\overline{m}]]} \Bigg(\Big(\overrightarrow{\prod_{1\le b\le N-1}} \overleftarrow{\prod_{N-1\ge a\ge b}} \frac{\check{\mathsf{E}}^{m_a^b-m_{a+1}^b}_{a+1,b}}{[m_a^b-m_{a+1}^b]_q!}\Big)v \\ \times \overline{\mathrm{Sym}}^{(q)}_{\bar{t}_{[\bar{n}]}} \prod_{a=2}^{N-1}\prod_{b=1}^{a-1}\prod_{\ell=0}^{m_a^b-1} \frac{q^{\Lambda_a}t^a_{\mathbf{m}_a^b-\ell} - q^{-\Lambda_a}z}{t^{a-1}_{\mathbf{m}_{a-1}^b-\ell} - t^a_{\mathbf{m}_a^b-\ell}} \prod_{\ell'=\mathbf{m}_{a-1}^b-\ell+1}^{n_{a-1}} \frac{qt^a_{\mathbf{m}_a^b-\ell} - q^{-1}t^{a-1}_{\ell'}}{t^a_{\mathbf{m}_a^b-\ell} - t^{a-1}_{\ell'}} \Bigg). \end{aligned} \tag{6.4}$$

Reordering the generators $\check{\mathsf{E}}_{a+1,b}$ with the help of the Serre relations (2.6) shows that the representation of the universal weight function on the $U_q(\widehat{\mathfrak{gl}}_N)$ evaluation module $M_\lambda(z)$, constructed with the help of the projections (6.4), is equal to the representation (2.20) obtained by the combinatorial methods of [18]. □

Denote by J the left ideal of $U_q(\widehat{\mathfrak{gl}}_N)$ defined in statement (ii) of Lemma 5.1: $J = I_N = \widehat{I}_1$. This ideal is generated by the modes of the entries of the L-operator $\mathrm{L}^+_{j,i}(t)$ with $1 \le i < j \le N$. We use the following statement, which was proved in Theorem 3 of [12].

Lemma 6.2. *Viewing $U_q(\widehat{\mathfrak{b}}^+)$ as an algebra over $\mathbb{C}[[(q-1)]]$, we let $\mathcal{A}$, $\mathcal{B} \in U_q(\widehat{\mathfrak{b}}^+)$. If for any singular weight vector v we have $\mathcal{A}\,v = \mathcal{B}\,v$, then*

$$\mathcal{A} \equiv \mathcal{B} \mod J.$$

Theorem 3. *For the universal and modified weight functions, the following statements are true.*

(i) *If V is an irreducible finite-dimensional $U_q(\widehat{\mathfrak{gl}}_N)$-module V with a singular vector v, then the following two weight functions are equal*:

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \mathbb{B}_V(\bar{t}_{[\bar{n}]}).$$

(ii)

$$\mathbf{w}^N_V(\bar{t}_{[\bar{n}]}) = \widehat{\mathbf{w}}^N_V(\bar{t}_{[\bar{n}]}).$$

(iii) *If* $U_q(\widehat{\mathfrak{b}}^+)$ *is viewed as an algebra over* $\mathbb{C}[[(q-1)]]$, *then*

$$\beta(\bar{t}_{[\bar{n}]})\,\mathcal{W}^N(\bar{t}_{[\bar{n}]})\prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a}k_a^+(t_\ell^a)\equiv\mathbb{B}_V(\bar{t}_{[\bar{n}]})\mod J.$$

(iv)

$$\mathcal{W}^N(\bar{t}_{[\bar{n}]})\prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a}k_a^+(t_\ell^a)\equiv\widehat{\mathcal{W}}^N(\bar{t}_{[\bar{n}]})\prod_{a=1}^{N-1}\prod_{\ell=1}^{n_a}\widehat{k}_{a+1}^+(t_\ell^a)\mod J.$$

Proof. (i) Since $\mathbf{w}_V^N(\bar{t}_{[\bar{n}]})$ and $\mathbb{B}_V(\bar{t}_{[\bar{n}]})$ have the same comultiplication properties (see [11]), from (6.2) it follows that, for any tensor product of the $U_q(\widehat{\mathfrak{gl}}_N)$ evaluation modules $1_{g(z_0)}\otimes M_{\lambda_1}(z_1)\otimes\cdots\otimes M_{\lambda_k(z_k)}$ and the one-dimensional module $1_{g(z_0)}$, where the action of the elements $L_{ii}(z)$ is given by multiplication by $g(z_0)$, we have

$$\mathbf{w}^N_{1_{g(z_0)}\otimes M_{\lambda_1}(z_1)\otimes\cdots\otimes M_{\lambda_k(z_k)}}(\bar{t}_{[\bar{n}]})=\mathbb{B}_{1_{g(z_0)}\otimes M_{\lambda_1}(z_1)\otimes\cdots\otimes M_{\lambda_k(z_k)}}(\bar{t}_{[\bar{n}]}).\tag{6.5}$$

The weight functions in question are trivial for one-dimensional modules, and (6.5) means that both constructions of the universal weight functions of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ coincide. Then we can apply the classical result of [1]: every irreducible finite-dimensional $U_q(\widehat{\mathfrak{gl}}_N)$-module with a singular vector v is isomorphic to a subquotient of a tensor product of one-dimensional modules and evaluation modules generated by the tensor product of their weight singular vectors. Under that isomorphism, the singular vector v corresponds to the image of the tensor product of the singular vectors mentioned.

(ii) This follows from (i) and the fact that $\widehat{\mathbf{w}}_V^N(\bar{t}_{[\bar{n}]})=\mathbb{B}_V(\bar{t}_{[\bar{n}]})$; see [12].

(iii) This follows from (i) and Lemma 6.2.

(iv) This follows from (ii) and Lemma 6.2. □

§A. Analytic properties of composite currents

In this appendix we reformulate the Serre relations in terms of composed currents. We start with the following properties of "regularized" products of the usual total currents [7]:

(i) $B_1(z,w)=(q^{-1}z-qw)F_i(z)F_i(w)$ vanish at $z=w$: $B_1(z,z)=0$;

(ii) $B_2(z_1,z_2,z_3)=(z_1-z_2)(qz_2-q^{-1}z_3)(q^{-1}z_1-qz_3)F_i(z_1)F_{i+1}(z_2)F_i(z_3)$ vanish at $z_1=z_2=q^{-2}z_3$: $B_2(z,z,q^2z)=0$;

(iii) $B_3(z_1,z_2,z_3)=(qz_1-q^{-1}z_2)(z_2-z_3)(q^{-1}z_1-qz_3)F_{i+1}(z_1)F_i(z_2)F_{i+1}(z_3)$ vanish at $z_1=z_2=q^2z_3$: $B_3(z,z,q^{-2}z)=0$.

Property (i) follows from the commutation relations (3.10) written as $B_1(z,w)=-B_1(w,z)$. Properties (ii) and (iii) can be obtained from the Serre relations (3.11) and the delta-function identities [7]. Note that the antisymmetry of $B_1(z,w)$ implies that $B_2(z,w,u)=-B_2(u,w,z)$ and $B_3(z,w,u)=-B_3(u,w,z)$.

We show how to deduce the commutation relation between $F_i(z)$ and the composite current $F_{i+2,i}(w)=u^{-1}(u-w)F_i(u)F_{i+1}(w)\Big|_{u=w}=(q^{-1}-q)F_{i+1}(w)F_i(w)$. Since

$$B_2(z,w,u)=-B_2(u,w,z),$$

we have[4]

$$F_i(z)F_{i+2,i}(w)=w^{-1}\frac{z^{-1}}{(1-w/z)}\,\frac{qz^{-1}}{(1-q^2w/z)}\,B_2(w,w,z).$$

On the other hand,

$$\frac{q^{-1}-qz/w}{1-z/w}\,F_{i+2,i}(w)F_i(z)=w^{-1}\frac{q^{-1}w^{-1}}{(1-q^{-2}z/w)}\,\frac{w^{-1}}{(1-z/w)}\,B_2(w,w,z).$$

[4]The rational function $\frac{1}{1-x}$ is understood as the formal series $\sum_{n\geq 0}x^n$.

Now, property (ii) means that

$$\frac{qz^{-1}}{1-q^2w/z}\,B_2(w,w,z) = -\frac{q^{-1}w^{-1}}{1-q^{-2}z/w}\,B_2(w,w,z).$$

The identity $B_2(z,z,z)=0$, which follows from (i), means that

$$\frac{z^{-1}}{1-w/z}\,B_2(w,w,z) = -\frac{w^{-1}}{1-z/w}\,B_2(w,w,z).$$

This proves the commutation relation

$$F_{i+1,i}(z)F_{i+2,i}(w) = \frac{q^{-1}-qz/w}{1-z/w}\;F_{i+2,i}(w)F_{i+1,i}(z). \tag{A.1}$$

Combining (A.1), the commutation relation between $F_{i+1,i}(z)$ and $F_{i,i-1}(w)$, and the definition $F_{i+2,i-1}(w) = (q^{-1}-q)F_{i+2,i}(w)F_{i,i-1}(w)$, we see that

$$F_{i+1,i}(z)F_{i+2,i-1}(w) = F_{i+2,i-1}(w)F_{i+1,i}(z). \tag{A.2}$$

Generalizing relations (A.1), (A.2), we obtain the following statement.

Proposition A.1. *For any $i>j>k>l$ we have the following commutation relations*:

$$F_{jk}(z)F_{ik}(w) = \frac{q^{-1}-qz/w}{1-z/w}\,F_{ik}(w)F_{jk}(z), \tag{A.3}$$

$$F_{ik}(z)F_{ij}(w) = \frac{q^{-1}-qz/w}{1-z/w}\,F_{ij}(w)F_{ik}(z), \tag{A.4}$$

$$F_{jk}(z)F_{il}(w) = F_{il}(w)F_{jk}(z), \tag{A.5}$$

$$\frac{q^{-1}-qw/z}{1-w/z}\,F_{ij}(z)F_{ij}(w) = \frac{q^{-1}-qz/w}{1-z/w}\,F_{ij}(w)F_{ij}(z). \tag{A.6}$$

§B. Proof of Lemma 4.3

Before proving this lemma, we formulate several preliminary propositions.

For any $j=1,\dots,N-1$, denote by U_j the subalgebra of $\overline{U}_f$ formed by the modes of $F_j(t),\dots,F_{N-1}(t)$. Let $U_j^\varepsilon = U_j\cap \mathrm{Ker}\,\varepsilon$ be the corresponding augmentation ideal. Let $\bar m_j = \{m_j, m_{j+1},\dots,m_{N-1}\}$ be a collection of nonnegative integers satisfying the admissibility conditions $m_j\ge m_{j+1}\ge\dots\ge m_{N-1}\ge m_N=0$, and let $\bar m_{j+1} = \{m_{j+1},\dots,m_{N-1}\}$.

Proposition B.1. *For the product of $\mathcal{F}(\bar t^j_{[m_j]})$ and $P^-\left(\mathcal{S}^{j+1}_{\bar m_{j+1}}(\bar t^{j+1}_{[m_{j+1}]})\right)$, we have*

$$\begin{aligned}\mathcal{F}(\bar t^j_{[m_j]})\cdot P^-\big(\mathcal{S}^{j+1}_{\bar m_{j+1}}(\bar t^{j+1}_{[m_{j+1}]})\big) &= \prod_{i=j}^{N-1}\frac{1}{(m_i-m_{i+1})!}\\ &\times \overline{\mathrm{Sym}}_{t^j_{[m_j]}}\Big(\mathcal{S}^j_{\bar m_j}(\bar t^j_{[m_j]})\prod_{a=j+1}^{N-1}\overline{\mathrm{Sym}}_{\bar t^{j+1}_{[m_{j+1}-m_a,m_{j+1}-m_{a+1}]}}\\ &\times \mathcal{U}(t^j_{m_j-m_{j+1}+1},\dots,t^j_{m_j};t^{j+1}_1,\dots,t^{j+1}_{m_{j+1}})\Big)\quad \mathrm{mod}\;\; P^-\big(\mathcal{U}^\varepsilon_{j+1}\big)\cdot\mathcal{U}_j,\end{aligned} \tag{B.1}$$

where the symbol $\prod_{a=j+1}^{N-1}\overline{\mathrm{Sym}}_{\bar t^{j+1}_{[m_{j+1}-m_a,m_{j+1}-m_{a+1}]}}$ is the composition of the corresponding q-symmetrizations $\overline{\mathrm{Sym}}_{\bar t^{j+1}_{[0,m_{j+1}-m_{j+2}]}}\cdots\overline{\mathrm{Sym}}_{\bar t^{j+1}_{[m_{j+1}-m_{N-1},m_{j+1}]}}$. If the admissibility conditions $m_j\ge m_{j+1}\ge\dots\ge m_{N-1}\ge 0$ fail, then the right-hand side of (B.1) *is zero modulo $P^-\big(\mathcal{U}^\varepsilon_{j+1}\big)\cdot\mathcal{U}_j$.*

Proof. Proposition B.1 can be proved along the same lines as in the Appendix C of [12]. In this proof, the reader should use the commutation relations collected in Appendix A of the present paper. □

Lemma B.2. *The rational series* $\overline{\mathrm{Sym}}_{\bar{t}^{j+2}_{[m_{j+2}]}} \cdots \overline{\mathrm{Sym}}_{\bar{t}^{N-2}_{[m_{N-2}]}} \overline{\mathrm{Sym}}_{\bar{t}^{N-1}_{[m_{N-1}]}} Y(\bar{t}_{\overline{m}_{j+1}})$ *is symmetric in each group of variables* $\{t^{j+1}_{m_{j+1}-m_a+1}, \dots, t^{j+1}_{m_{j+1}-m_{a+1}}\}$ *for* $a = j+1, \dots, N-1$.

Proof. This lemma can be proved by induction. The case of $j = N-3$ follows from the fact, proved in [10], that the q-symmetrization $\overline{\mathrm{Sym}}_{\bar{v}}\, U(\bar{u};\bar{v})$ is a symmetric series with respect to the variables $\bar{u}$. For $j < N-3$, induction follows from the same fact, the property (4.11) of q-symmetrization, and the formula

$$\begin{aligned} U(u_1,\dots,u_k;v_1,\dots,v_k) &= \Pi(u_{s+1},\dots,u_k;v_1,\dots,v_s) \\ &\quad \times U(u_1,\dots,u_s;v_1,\dots,v_s)U(u_{s+1},\dots,u_k;v_{s+1},\dots,v_k), \end{aligned} \tag{B.2}$$

where $0 \le s \le k$, and $\Pi(u_{s+1},\dots,u_k;v_1,\dots,v_s)$ is a symmetric function with respect to each set of variables $\{u_{s+1},\dots,u_k\}$ and $\{v_1,\dots,v_s\}$. □

Proposition B.3. *The formal series identity*

$$\begin{aligned} &\mathcal{F}(\bar{t}^{j}_{[m_j]}) \cdot \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_{j+1}]}} \big(Y(\bar{t}_{[\overline{m}_{j+1}]}) \cdot P^-\big(\mathcal{S}^{j+1}_{\overline{m}_{j+1}}(\bar{t}^{j+1}_{[m_{j+1}]})\big)\big) \\ &\quad = \frac{1}{(m_j - m_{j+1})!}\, \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_j]}} \big(Y(\bar{t}_{[\overline{m}_j]}) \cdot \mathcal{S}^{j}_{\overline{m}_j}(\bar{t}^{j}_{[m_j]})\big) \mod P^-\big(U^{\varepsilon}_{j+1}\big) \cdot U_j \end{aligned} \tag{B.3}$$

is valid if the admissibility conditions $m_j \ge m_{j+1} \ge \dots \ge m_{N-1} \ge 0$ *are satisfied; otherwise the right-hand side of* (B.3) *is zero modulo* $P^-\big(U^{\varepsilon}_{j+1}\big) \cdot U_j$.

Proof. Lemma B.2 implies the relation

$$\begin{aligned} &\overline{\mathrm{Sym}}_{\bar{t}^{j+1}_{[m_{j+1}]}} \Bigg(\overline{\mathrm{Sym}}_{\bar{t}^{j+2}_{[m_{j+2}]}} \cdots \overline{\mathrm{Sym}}_{\bar{t}^{N-1}_{[m_{N-1}]}} Y(\bar{t}_{\overline{m}_{j+1}}) \\ &\qquad\qquad \times \prod_{a=j+1}^{N-1} \overline{\mathrm{Sym}}_{\bar{t}^{j+1}_{[m_{j+1}-m_a, m_{j+1}-m_{a+1}]}} U\big(\bar{t}^{j}_{[m_j, m_j - m_{j+1}]}; \bar{t}^{j+1}_{[m_{j+1}]}\big) \Bigg) \\ &\quad = \prod_{a=j+1}^{N-1} (m_a - m_{a+1})! \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_{j+1}]}} Y(\bar{t}_{\overline{m}_j}). \end{aligned} \tag{B.4}$$

The q-symmetrization $\overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_{j+1}]}}$ on the left-hand side of (B.3) does not affect the formal series depending on the variables $\bar{t}^{j}_{[m_j]}$. This means that the left-hand side of (B.3) can be written in the form $\overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_{j+1}]}} \Big(Y(\bar{t}_{[\overline{m}_{j+1}]}) \cdot \mathcal{F}(\bar{t}^{j}_{[m_j]}) \cdot P^-\big(\mathcal{S}^{j+1}_{\overline{m}_{j+1}}(\bar{t}^{j+1}_{[m_{j+1}]})\big)\Big)$. Then, substituting (B.1) in this expression and using (B.4), we obtain the right-hand side of (B.3). □

Proposition B.4. *The projection* $P^-\big(\mathcal{F}(\bar{t}_{[\overline{m}_2]})\big)$ *can be presented as*

$$\prod_{a=2}^{N-1} \frac{1}{(m_a - m_{a+1})}\, \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_2]}} \Big(Y(\bar{t}_{[\overline{m}_2]})\, P^-\big(\mathcal{S}^{2}_{\overline{m}_2}(\bar{t}^{2}_{[m_2]})\big)\Big) \mod P^-\big(\mathcal{U}^{\varepsilon}_{3}\big) \cdot \mathcal{U}_2 \tag{B.5}$$

if the admissibility conditions $m_2 \ge m_3 \ge \dots \ge m_{N-2} \ge m_{N-1} \ge m_N = 0$ *are satisfied; otherwise it is zero modulo* $P^-\big(\mathcal{U}^{\varepsilon}_{3}\big) \cdot \mathcal{U}_2$.

Proof. Formula (B.5) is a particular case of the following more general formal series identity:

$$\begin{aligned} P^-\big(\mathcal{F}(\bar{t}_{[\overline{m}_2]})\big) = \prod_{a=j+1}^{N-1} \frac{1}{(m_a - m_{a+1})!} P^-\Big(\mathcal{F}(\bar{t}^2_{[m_2]}) \cdots \mathcal{F}(\bar{t}^{j-1}_{[m_{j-1}]}) \\ \times \mathcal{F}(\bar{t}^j_{[m_j]})\, \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_{j+1}]}}\Big(Y(\bar{t}_{[\overline{m}_{j+1}]}) \mathcal{S}^{j+1}_{\overline{m}_{j+1}}(\bar{t}^{j+1}_{[m_{j+1}]})\Big)\Big) \mod P^-\big(U^\varepsilon_{j+2}\big) \cdot U_2. \end{aligned} \tag{B.6}$$

We prove (B.6) by induction. This identity is true for $j = N-1$. Suppose it is valid for some $j \le N-1$. Using (3.18) and (3.17), we can replace the string $\mathcal{S}^{j+1}_{\overline{m}_{j+1}}(\bar{t}^{j+1}_{[m_{j+1}]})$ on the right-hand side of (B.6) by its negative projection

$$P^-\big(\mathcal{S}^{j+1}_{\overline{m}_{j+1}}(\bar{t}^{j+1}_{[m_{j+1}]})\big).$$

Now we use Proposition B.3 to obtain the right-hand side of (B.6) for $j-1$, employing also the fact that $P^-\big(\mathcal{F}(\bar{t}^2_{[m_2]}) \cdots \mathcal{F}(\bar{t}^{j-1}_{[m_{j-1}]}) P^-(U^\varepsilon_{j+1}) \cdot U_j\big) \subset P^-(U^\varepsilon_{j+1}) \cdot U_2$ and $P^-(U^\varepsilon_{j+2}) \cdot U_2 \subset P^-(U^\varepsilon_{j+1}) \cdot U_2$. If the admissibility condition fails for some $j \ge 2$, namely, $m_j < m_{j+1}$, then, by Proposition B.3, the product

$$\mathcal{F}(\bar{t}^j_{[m_j]})\, \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_{j+1}]}}\Big(Y(\bar{t}_{[\overline{m}_{j+1}]})\, \mathcal{S}^{j+1}_{\overline{m}_{j+1}}(\bar{t}^{j+1}_{[m_{j+1}]})\Big)$$

is zero modulo elements of the form $P^-(U^\varepsilon_{j+1}) \cdot U_j$. Since the modes of the currents $F_k[n]$, $k = 2, \dots, j-1$, commute with any element in U^ε_{j+1}, we conclude that in this case the right-hand side of (B.5) is zero modulo $P^-(U^\varepsilon_{j+1}) \cdot U_2 \subset P^-(U^\varepsilon_3) \cdot U_2$. □

Proof of Lemma 4.3. For calculating the projection $P^+\big(\mathcal{F}(\bar{t}^1_{[n_1]}) P^-\big(\mathcal{F}(\bar{t}_{[\overline{m}_2]})\big)\big)$, we substitute the expression for $P^-\big(\mathcal{F}(\bar{t}_{[\overline{m}_2]})\big)$ given by (B.5). Since any element of U^ε_3 commutes with the current $F_1(t)$, the elements of $P^-\big(U^\varepsilon_3\big) \cdot U_2$ contribute nothing:

$$P^+\Big(\mathcal{F}(\bar{t}^1_{[n_1]}) \cdot P^-\big(U^\varepsilon_3\big) \cdot U_2\Big) = 0.$$

Then we have

$$\prod_{a=2}^{N-1} \frac{1}{(m_a - m_{a+1})!} P^+\Big(\mathcal{F}(\bar{t}^1_{[n_1]})\, \overline{\mathrm{Sym}}_{\bar{t}_{[\overline{m}_2]}}\Big(Y(\bar{t}_{[\overline{m}_2]}) P^-\big(\mathcal{S}^2_{\overline{m}_2}(\bar{t}^2_{[m_2]})\big)\Big)\Big),$$

where $m_N = 0$. By Proposition B.3, the latter expression is nonzero only if $m_2 \le n_1$, and in this case it is equal to the right-hand side of (4.37). □

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