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OPTIMIZATION PROBLEMS RELATED TO THE JOHN UNIQUENESS THEOREM

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Abstract. The spectrum structure problem is considered for a distribution that is periodic in the mean and satisfies uniqueness conditions of John type. The solution of this problem is obtained for a wide class of distributions on arbitrary Riemannian two-point-homogeneous spaces.

§1. Introduction

The classical John uniqueness theorem (see [1] and also [2, Chapter 6]) states that if a function $f \in C^\infty(\mathbb{R}^n)$ with zero integrals over all spheres of a fixed radius r vanishes in some ball of radius r, then $f = 0$ in $\mathbb{R}^n$. If $n = 1$, then this statement is obviously valid also for $f \in C(\mathbb{R}^1)$ (the integral over a zero-dimensional sphere is understood as the sum of values of the function at the points of this sphere). However, for $n \geq 2$, the condition of infinite smoothness of f cannot be relaxed (see [1] for $n = 2, 3$ and [3, Part 2, Theorem 1.2] in the general case).

The John theorem has been refined and developed further in various directions [3]–[14].

First, its generalizations were studied for functions f satisfying the convolution equation $f * T = 0$, where T is a given distribution with compact support in $\mathbb{R}^n$. The solutions f were assumed to be equal to zero in the convex hull of the support of T. Moreover, meaningful problems and results arise even in the one-dimensional situation (see [3], [5, Appendix II], [9, Chapter 6, §F].

Second, analogs of the John theorem on Riemannian two-point-homogeneous spaces X were obtained in [12, 14]. This class of spaces arises naturally in differential geometry and is defined as the class of Riemannian manifolds possessing the following property (see [15, Chapter 1]): for any two pairs of points (x_1, x_2) and (y_1, y_2) in X that satisfy the relation $d(x_1, x_2) = d(y_1, y_2)$, where $d(\cdot, \cdot)$ is the distance on X, there exists an isometry of X that takes x_1 to y_1 and x_2 to y_2. The methods of [12]–[14] turned out to be rather useful in many problems related to periodicity in the mean, both on the spaces X and on other homogeneous spaces.

Third, some "spectral" analogs of the John theorem were proved for functions of finite smoothness. The essence of those theorems is that the greater is the order of smoothness of a function f satisfying conditions of John type, the greater is the number of zero terms in its Fourier expansion in spherical harmonics (see [12]–[14]).

Fourth, a deep relationship was found between the John theorem (and its analogs) and microlocal analysis, which is widely used in current investigations on partial differential equations; see [10] and [16, Chapter 8].

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Along with being of interest in themselves, the results mentioned above turned out to be important because of their numerous and significant applications in optimization problems of integral geometry, in the theory of gap series, as well as in studying various classes of functions periodic in the mean and their generalizations (see [3]).

New effects that emerged in "spectral" analogs of the John theorem have led to the natural problem of finding the exact dependence between the order of smoothness of a function f belonging to the above-mentioned classes and the number of zero Fourier coefficients in its expansion in spherical harmonics (see [3, Part 2, Chapter 4, Problems 4.3 and 4.17]).

The present paper is devoted to the solution of this problem on arbitrary two-point-homogeneous spaces.

In §2, the basic notation is outlined. Exact statements of the main results are presented in §3. The machinery necessary for their proof is developed in §§4, 5. In particular, in §5 we construct maps of a special kind that possess the transmutation property on compact two-point-homogeneous spaces. They are of independent interest and play an important role in a number of other problems related to functions periodic in the mean. Proofs of the main results are given in §6.

§2. Notation

In the paper, we use the following standard notation: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_+$ are, respectively, the sets of real, complex, and natural numbers and the sets of integers and nonnegative integers; $[\lambda]$ is the integral part of a number $\lambda \in \mathbb{R}$; $\overline{\lambda}$ is the complex conjugate to $\lambda \in \mathbb{C}$; $\binom{j}{k}$ is a binomial coefficient; Γ is the gamma-function; $(a)_j = \Gamma(a+j)/\Gamma(a)$ $(j \in \mathbb{Z}_+)$ is the Pochhammer symbol; J_ν is the Bessel function of the first kind with index ν; $F(a,b;c;z)$ is the Gauss hypergeometric function; $R_l^{(\alpha,\beta)}(t) = F(-l, l+\alpha+\beta+1; \alpha+1; (1-t)/2)$ $(l \in \mathbb{Z}_+)$ are normalized Jacobi polynomials.

For an entire function $f : \mathbb{C} \to \mathbb{C}$, we set $\mathcal{Z}(f) = \{z \in \mathbb{C} : f(z) = 0\}$. If $f \not\equiv 0$ and $\lambda \in \mathcal{Z}(f)$, we denote by $n_\lambda(f)$ the multiplicity of the zero λ.

Let T be a distribution with compact support in $\mathbb{R}^1$. Its Fourier transform is defined by the relation $\widehat{T}(z) = \langle T(x), e^{-ixz}\rangle$ $(z \in \mathbb{C})$. By the symbol $r(T)$ we denote the radius of the smallest closed ball in $\mathbb{R}^1$ that contains the support of T. We also put $r_0(T) = \inf\{r > 0 : T = 0 \text{ outside of } [-r, r]\}$.

Let X be a Riemannian two-point-homogeneous space. We need the following classes of functions and distributions on an open set $\mathcal{O}$ in X: $C^s(\mathcal{O})$ $(s \in \mathbb{Z}_+$ or $s = \infty)$ is the space of s times continuously differentiable functions; $\mathcal{E}(\mathcal{O}) = C^\infty(\mathcal{O})$; $\mathcal{D}(\mathcal{O})$ is the space of compactly supported functions infinitely differentiable in $\mathcal{O}$; $\mathrm{RA}(\mathcal{O})$ is the class of real-analytic functions; $L^p(\mathcal{O})$ and $L^{p,\mathrm{loc}}(\mathcal{O})$ $(p \geq 1)$ are the classes of functions p-integrable and p-locally integrable on $\mathcal{O}$ against the Riemannian measure $d\mu$; $W^s_{1,\mathrm{loc}}(\mathcal{O})$ is the set of functions $f \in L^{1,\mathrm{loc}}(\mathcal{O})$ such that $Df \in L^{1,\mathrm{loc}}(\mathcal{O})$ for any differential operator D on X of order not exceeding s; $\mathcal{D}'(\mathcal{O})$ and $\mathcal{E}'(\mathcal{O})$ are the spaces of distributions and compactly supported distributions on $\mathcal{O}$, respectively.

For any class $\mathfrak{W}(-R,R)$ of distributions on the interval $(-R,R) \subset \mathbb{R}^1$, we denote by $\mathfrak{W}_\natural(-R,R)$ the set of all even distributions belonging to $\mathfrak{W}(-R,R)$.

Now we define the class $\mathcal{M}^\nu(X)$ $(\nu \in \mathbb{Z})$. Let $\mathcal{M}^0(X)$ be the set of all complex-valued measures on X, and let $\mathcal{M}^1(X)$ be the set of all distributions $f \in \mathcal{D}'(X)$ such that $Df \in \mathcal{M}^0(X)$ for any differential operator D on X of order not exceeding one. For example, the surface delta-function of a sphere in X belongs to $\mathcal{M}^0(X)$, but the indicator function of a ball in X belongs to $\mathcal{M}^1(X)$. If $\nu > 1$ and ν is even (respectively, ν is odd), we denote by $\mathcal{M}^\nu(X)$ the set of all distributions $f \in \mathcal{D}'(X)$ for which $L^{[\nu/2]}f \in \mathcal{M}^0(X)$ (respectively, $L^{[\nu/2]}f \in \mathcal{M}^1(X)$), where L is the Laplace–Beltrami operator on X. Finally, if $\nu < 0$

and ν is even (respectively, ν is odd), we denote by $\mathcal{M}^\nu(X)$ the set of all distributions $f \in \mathcal{D}'(X)$ such that $f = P(L)u$ for some $u \in \mathcal{M}^0(X)$ (respectively, $u \in \mathcal{M}^1(X)$) and for a polynomial P of degree not exceeding $[(1-\nu)/2]$.

For a distribution $f \in \mathcal{D}'(\mathcal{O})$, $\bar{f}$ denotes complex conjugation, supp f stands for the support of f, and ord f for the order of f. The delta-function at a point $o \in X$ is denoted by δ_o. Finally, we denote by $\times$ the convolution of distributions on domains in X in the cases where it exists (see [15, Chapter 2, §5]). For the convolution of distributions on $\mathbb{R}^n$, we use the usual symbol "$*$".

§3. Statement of the main results

In accordance with [15, Chapter 1, §§4.2 and 4.3], the class of two-point-homogeneous spaces X consists of: (1) the real Euclidean spaces $\mathbb{R}^n$; (2) the hyperbolic spaces $\mathbb{H}^n_{\mathbb{K}}$ ($\mathbb{K}$ denotes the fields $\mathbb{R}$ or $\mathbb{C}$, or the quaternion division ring $\mathbb{Q}$); (3) the hyperbolic Cayley plane $\mathbb{H}^2_{\mathbb{C}a}$; (4) the Euclidean spheres $\mathbb{S}^n$; (5) the projective spaces $\mathbb{P}^n_{\mathbb{K}}$; (6) the projective Cayley plane $\mathbb{P}^2_{\mathbb{C}a}$.

Let $\mathfrak{X}_1$ be the class of noncompact spaces X different from $\mathbb{R}^n$, and let $\mathfrak{X}_2$ be the class of compact spaces X. For $X \in \mathfrak{X}_1$, we assume that the maximum of the sectional curvature of X is equal to -1, and for $X \in \mathfrak{X}_2$ we assume that the minimum of the sectional curvature of X is equal to 1. Moreover, the real dimension a_X of the space X is assumed to be at least 2. We define $\mathcal{X} = \{x \in X \,:\, d(o,x) < \operatorname{diam} X\}$, where o is a fixed point (origin) in X and

$$\operatorname{diam} X = \sup_{x,y \in X} d(x,y).$$

Then $\mathcal{X}$ can be regarded as a Riemannian manifold $(\mathcal{D}, ds^2)$, where the domain $\mathcal{D}$ and the Riemannian metric ds^2 are given in the following way (see [14], [17, §§19, 20]):

(1) $X = \mathbb{R}^n \,:\, \mathcal{D} = \mathbb{R}^n$, $ds^2 = |dx|^2$;

(2) $X = \mathbb{H}^n_{\mathbb{R}} \,:\, \mathcal{D} = \{x \in \mathbb{R}^n \,:\, |x| < 1\}$, $ds^2 = (1-|x|^2)^{-2}|dx|^2$;

(3) $X = \mathbb{H}^n_{\mathbb{C}} \,:\, \mathcal{D} = \{z \in \mathbb{C}^n \,:\, |z| < 1\}$, $ds^2 = (1-|z|^2)^{-1}|dz|^2 + (1-|z|^2)^{-2}\mathfrak{F}_1(z,dz)$, where

$$\mathfrak{F}_1(z,dz) = \sum_{i,j=1}^n \bar{z}_i z_j \, dz_i \, d\bar{z}_j;$$

(4) $X = \mathbb{H}^n_{\mathbb{Q}} \,:\, \mathcal{D} = \{z \in \mathbb{C}^{2n} \,:\, |z| < 1\}$ $ds^2 = (1-|z|^2)^{-1}|dz|^2 + (1-|z|^2)^{-2}\mathfrak{F}_2(z,dz)$, where

$$\begin{aligned}\mathfrak{F}_2(z,dz) = \sum_{i,j=1}^n \big(&\big(\bar{z}_i z_j + z_{n+i}\bar{z}_{n+j}\big)\, dz_i \, d\bar{z}_j + \big(\bar{z}_i z_{n+j} - z_{n+i}\bar{z}_j\big)\, dz_i \, d\bar{z}_{n+j} \\ &+ \big(\bar{z}_{n+i} z_j - z_i \bar{z}_{n+j}\big)\, dz_{n+i} \, d\bar{z}_j + \big(z_i \bar{z}_j + \bar{z}_{n+i} z_{n+j}\big)\, dz_{n+i} \, d\bar{z}_{n+j}\big);\end{aligned}$$

(5) $X = \mathbb{H}^2_{\mathbb{C}a} \,:\, \mathcal{D} = \{x \in \mathbb{R}^{16} \,:\, |x| < 1\}$, $ds^2 = \dfrac{|\,dx|^2}{1-|x|^2} + \dfrac{1}{2}\,\dfrac{\mathfrak{F}_3(x,dx)}{(1-|x|^2)^2}$, where

$$\mathfrak{F}_3(x,dx) = \sum_{i,j=1}^{16} \frac{\partial^2}{\partial y_i \partial y_j}\big(\Phi(x,y)\big)\, dx_i \, dx_j,$$

$$\begin{aligned}
\Phi(x,y) &= 2\big(p_1(x)p_1(y) + \cdots + p_8(x)p_8(y)\big) + p_9(x)p_9(y) + p_{10}(x)p_{10}(y),\\
p_1(x) &= x_1x_2 - x_3x_4 - x_5x_6 - x_7x_8 - x_9x_{10} - x_{11}x_{12} - x_{13}x_{14} - x_{15}x_{16},\\
p_2(x) &= x_1x_4 - x_9x_{12} - x_5x_8 - x_{13}x_{16} + x_3x_2 + x_{11}x_{10} + x_7x_6 + x_{15}x_{14},\\
p_3(x) &= x_1x_6 - x_9x_{14} + x_5x_2 + x_{13}x_{10} + x_3x_8 + x_{11}x_{16} - x_7x_4 - x_{15}x_{12},\\
p_4(x) &= x_1x_8 + x_9x_{16} + x_5x_4 - x_{13}x_{12} - x_3x_6 + x_{11}x_{14} + x_7x_2 - x_{15}x_{10},
\end{aligned}$$

$$
\begin{aligned}
p_5(x) &= x_1x_{10} + x_9x_2 + x_5x_{14} - x_{13}x_6 + x_3x_{12} - x_{11}x_4 - x_7x_{16} + x_{15}x_8,\\
p_6(x) &= x_1x_{12} + x_9x_4 - x_5x_{16} + x_{13}x_8 - x_3x_{10} + x_{11}x_2 - x_7x_{14} + x_{15}x_6,\\
p_7(x) &= x_1x_{14} + x_9x_6 - x_5x_{10} + x_{13}x_2 + x_3x_{16} - x_{11}x_8 + x_7x_{12} - x_{15}x_4,\\
p_8(x) &= x_1x_{16} - x_9x_8 + x_5x_{12} + x_{13}x_4 - x_3x_{14} - x_{11}x_6 + x_7x_{10} + x_{15}x_2,\\
p_9(x) &= x_1^2 + x_3^2 + \dots + x_{15}^2, \quad p_{10}(x) = x_2^2 + x_4^2 + \dots + x_{16}^2.
\end{aligned}
$$

We note that the polynomials $p_1, \dots, p_8$ and the form $\Phi(x, y)$ have a clear interpretation in terms of the Cayley numbers (see [14, §3]).

(6) $X = \mathbb{S}^n : \mathcal{D} = \mathbb{R}^n$, $ds^2 = (1 + |x|^2)^{-2}|dx|^2$;

(7) $X = \mathbb{P}^n_{\mathbb{R}} : \mathcal{D} = \mathbb{R}^n$,

$$
ds^2 = (1 + |x|^2)^{-1}|dx|^2 - (1 + |x|^2)^{-2} \sum_{i,j=1}^{n} x_i x_j \, dx_i \, dx_j;
$$

(8) $X = \mathbb{P}^n_{\mathbb{C}} : \mathcal{D} = \mathbb{C}^n$, $ds^2 = (1 + |z|^2)^{-1}|dz|^2 - (1 + |z|^2)^{-2}\mathfrak{F}_1(z, dz)$;

(9) $X = \mathbb{P}^n_{\mathbb{Q}} : \mathcal{D} = \mathbb{C}^{2n}$, $ds^2 = (1 + |z|^2)^{-1}|dz|^2 - (1 + |z|^2)^{-2}\mathfrak{F}_2(z, dz)$;

(10) $X = \mathbb{P}^2_{\mathbb{C}a} : \mathcal{D} = \mathbb{R}^{16}$, $ds^2 = (1 + |x|^2)^{-1}|dx|^2 - 2^{-1}(1 + |x|^2)^{-2}\mathfrak{F}_3(x, dx)$.

In the models given above, the distance on X is defined by the relation

$$
d(0, x) = \begin{cases} |x|, & X = \mathbb{R}^n, \\ \operatorname{arth} |x|, & X \in \mathfrak{X}_1, \\ \arctan |x|, & X \in \mathfrak{X}_2, \end{cases} \tag{3.1}
$$

and by the invariance condition for d with respect to the isometry group G of the space X. Relation (3.1) shows that the geodesic ball $B_R = \{x \in X : d(0, x) < R\}$ coincides with the open Euclidean ball in $\mathbb{R}^{a_X}$ centered at the origin and having the corresponding radius. Henceforth we assume that $0 < R \le \operatorname{diam} X$.

Let $\mathfrak{X}_3$ be the class of two-point-continuous spaces of constant curvature, i.e., $\mathfrak{X}_3 = \{\mathbb{R}^n, \mathbb{H}^n_{\mathbb{R}}, \mathbb{S}^n, \mathbb{P}^n_{\mathbb{R}}\}$. We choose $k \in \mathbb{Z}_+$ and $m \in \{0, \dots, M_X(k)\}$, where

$$
M_X(k) = \begin{cases} 0, & X \in \mathfrak{X}_3, \\ [k/2], & X \notin \mathfrak{X}_3. \end{cases}
$$

We set

$$
\varepsilon_X = \begin{cases} -1, & X \in \mathfrak{X}_1, \\ 1, & X \in \mathfrak{X}_2, \end{cases}
$$

and $\beta_X = n/2 - 1, -1/2, 0, 1, 3$, respectively, in each of the following five cases: (1) $X = \mathbb{H}^n_{\mathbb{R}}$, $X = \mathbb{S}^n$; (2) $X = \mathbb{P}^n_{\mathbb{R}}$; (3) $X = \mathbb{H}^n_{\mathbb{C}}$, $X = \mathbb{P}^n_{\mathbb{C}}$; (4) $X = \mathbb{H}^n_{\mathbb{Q}}$, $X = \mathbb{P}^n_{\mathbb{Q}}$; (5) $X = \mathbb{H}^2_{\mathbb{C}a}$, $X = \mathbb{P}^2_{\mathbb{C}a}$. We define $\mathcal{H}^{k,m}_X = \mathcal{H}^k_{a_X}$ for $X \in \mathfrak{X}_3$, and

$$
\mathcal{H}^{k,m}_X = \{f \in \mathcal{H}^k_{a_X} : (Lf)(x) = 4\varepsilon_X(m - \beta_X)(k - m)(1 + \varepsilon_X|x|^2)f(x)\}
$$

for $X \notin \mathfrak{X}_3$, where $\mathcal{H}^k_{a_X}$ is the space of homogeneous harmonic polynomials of degree k in $\mathbb{R}^{a_X}$. Let $\mathrm{O}(a_X)$ denote the orthogonal group in $\mathbb{R}^{a_X}$. Upon identifying $\mathcal{H}^{k,m}_X$ with the space of restrictions of its elements to the sphere $\mathbb{S}^{a_X - 1} = \{x \in \mathbb{R}^{a_X} : |x| = 1\}$, $\mathcal{H}^{k,m}_X$ becomes an invariant subspace of the quasiregular representation $\mathfrak{T}(\tau)$ of the group $K = G \cap \mathrm{O}(a_X)$ on $L^2(\mathbb{S}^{a_X - 1})$. If $\mathfrak{T}^{k,m}(\tau)$ is the restriction of $\mathfrak{T}(\tau)$ to $\mathcal{H}^{k,m}_X$, then $\mathfrak{T}(\tau)$ is an orthogonal direct sum of pairwise nonequivalent irreducible unitary representations $\mathfrak{T}^{k,m}(\tau)$, $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_X(k)\}$ (see [14, §2]).

An arbitrary point $x \in \mathbb{R}^{a_X} \setminus \{0\}$ can be represented in the form $x = \varrho\sigma$, where $\varrho = |x|$, $\sigma = x/|x|$. Any function $f \in L^{1,\mathrm{loc}}(B_R)$ has a Fourier series of the form

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{m=0}^{M_X(k)} \sum_{j=1}^{d_X^{k,m}} f_{k,m,j}(\varrho) Y_j^{k,m}(\sigma), \tag{3.2}$$

where $d_X^{k,m} = \dim \mathcal{H}_X^{k,m}$, $\{Y_j^{k,m}\}$ is a fixed orthonormal basis in $\mathcal{H}_X^{k,m}$ relative to a surface measure $d\omega$ on $\mathbb{S}^{a_X-1}$, and

$$f_{k,m,j}(\varrho) = \int_{\mathbb{S}^{a_X-1}} f(\varrho\sigma) \overline{Y_j^{k,m}(\sigma)}\, d\omega(\sigma). \tag{3.3}$$

Let $\{t_{i,j}^{k,m}(\tau)\}$, $i, j \in \{1, \dots, d_X^{k,m}\}$, be the matrix of the representation $\mathfrak{T}^{k,m}(\tau)$ with respect to the basis $\{Y_j^{k,m}\}$, and let $d\tau$ be the Haar measure of total mass 1. The series (3.2) can be extended to distributions $f \in \mathcal{D}'(B_R)$ in the following way:

$$f \sim \sum_{k=0}^{\infty} \sum_{m=0}^{M_X(k)} \sum_{j=1}^{d_X^{k,m}} f^{k,m,j}, \tag{3.4}$$

where the series (3.4) converges to f in $\mathcal{D}'(B_R)$ and the distribution $f^{k,m,j}$ acts on the space $\mathcal{D}(B_R)$ by the rule

$$\begin{aligned} \langle f^{k,m,j}, \psi \rangle &= \left\langle f, d_X^{k,m} \int_K \psi(\tau^{-1}x) t_{j,j}^{k,m}(\tau)\, d\tau \right\rangle \\ &= \langle f, \overline{(\bar{\psi})_{k,m,j}(\varrho)}\, \overline{Y_j^{k,m}(\sigma)} \rangle, \quad \psi \in \mathcal{D}(B_R) \end{aligned} \tag{3.5}$$

(see the proof of formula (5.19) in [3, Part 1, §5.2] and of Proposition 2.7 in [18, Chapter 1]). For any set $\mathfrak{W}(B_R) \subset \mathcal{D}'(B_R)$, we put

$$\mathfrak{W}_{k,m,j}(B_R) = \left\{ f \in \mathfrak{W}(B_R) \,:\, f = f^{k,m,j} \right\}.$$

Note that $\mathfrak{W}_{0,0,1}(B_R) = \mathfrak{W}_\natural(B_R)$, where $\mathfrak{W}_\natural(B_R)$ is the set of all radial distributions $f \in \mathfrak{W}(B_R)$. It is easily seen that the support of a distribution $f \in \mathcal{D}'_{k,m,j}(B_R)$ is K-invariant. If $f \in \mathcal{E}'_{k,m,j}(B_R)$, we define $r(f) = \inf\{r > 0 \,:\, \operatorname{supp} f \subset \overline{B_r}\}$, where $\overline{B_r} = \{x \in X \colon d(0,x) \le r\}$.

Suppose that $T \in \mathcal{E}'_\natural(\mathcal{X})$, $R > r(T)$, and $s \in \mathbb{Z}_+ \cup \{\infty\}$. Consider the classes

$$\begin{aligned} \mathcal{D}'_T(B_R) &= \{f \in \mathcal{D}'(B_R) \,:\, f \times T = 0 \ \text{ in } \ B_{R-r(T)}\}, \\ C_T^s(B_R) &= (\mathcal{D}'_T \cap C^s)(B_R), \\ \mathrm{RA}_T(B_R) &= (\mathcal{D}'_T \cap \mathrm{RA})(B_R). \end{aligned}$$

The relation

$$(f \times T)^{k,m,j} = f^{k,m,j} \times T \ \text{ in } \ B_{R-r(T)} \tag{3.6}$$

(see (3.5)) shows that a distribution f from $\mathcal{D}'(B_R)$ belongs to $\mathcal{D}'_T(B_R)$ if and only if $f^{k,m,j} \in \mathcal{D}'_T(B_R)$ for all k, m, j.

Theorem 3.1. *Suppose $\nu, s \in \mathbb{Z}$, $s \ge \max\{0, 2[(1-\nu)/2]\}$, and $T \in \mathcal{E}'_\natural(\mathcal{X}) \cap \mathcal{M}^\nu(X)$. Also, suppose that $R > r(T) > 0$, $f \in \mathcal{D}'_T(B_R)$, and*

$$f = 0 \ \text{ in } \ B_{r(T)}. \tag{3.7}$$

Then the following statements are valid.

(i) *If $f \in W^s_{1,\mathrm{loc}}(B_R)$, then $f^{k,m,j} = 0$ in B_R for all $0 \le k \le s + \nu + 1$, $m \in \{0, \dots, M_X(k)\}$, $j \in \{1, \dots, d_X^{k,m}\}$.*

(ii) *If* $f \in C^s(B_R)$, *then* $f^{k,m,j} = 0$ *in* B_R *for all* $0 \le k \le s+\nu+2$, $m \in \{0,\dots,M_X(k)\}$, $j \in \{1,\dots,d_X^{k,m}\}$.

Now several remarks are in order. First, every distribution T from $\mathcal{E}'_\natural(\mathcal{X})$ lies in $\mathcal{M}^\nu(X)$ for some $\nu \in \mathbb{Z}$ (see Propositions 5.2, 5.4, and 5.5 and Remark 5.2 in §5). Next, generally speaking, the radius $r(T)$ in the condition (3.7) cannot be reduced; see [12, Theorem 1]. Moreover, the dependence between the order of smoothness of a distribution f and the set of zero coefficients in its Fourier expansion is also sharp (see Theorem 3.2 below).

Theorem 3.1 easily implies the following statement.

Corollary 3.1. *Suppose* $T \in \mathcal{E}'_\natural(\mathcal{X})$, $R > r(T) > 0$, $f \in \mathcal{D}'_T(B_R)$, *and* $f = 0$ *in* $B_{r(T)}$. *Then the following assertions are valid*:

(i) *if* $f = 0$ *in* $B_{r(T)+\varepsilon}$ *for some* $\varepsilon > 0$, *then* $f = 0$ *in* B_R;
(ii) *if* $f \in C^\infty_T(B_R)$, *then* $f = 0$ *in* B_R;
(iii) *if* $T \in \mathcal{D}_\natural(\mathcal{X})$, *then* $f = 0$ *in* B_R.

Corollary 3.1 is new for compact spaces X. For Euclidean and hyperbolic spaces, this result was obtained earlier in [3, Part 3] and [12], where the noncompactness of X was used essentially.

In the following theorem, $\widetilde{T}$ denotes the spherical transform of a distribution $T \in \mathcal{E}'_\natural(\mathcal{X})$ (see the definition (5.2) in §5).

Theorem 3.2. *Suppose* $\nu, s \in \mathbb{Z}$, $s \ge \max\{0, 2[(1-\nu)/2]\}$, $k \in \mathbb{Z}_+$, $m \in \{0,\dots,M_X(k)\}$, $j \in \{1,\dots,d_X^{k,m}\}$. *Also, suppose that* $T \in \mathcal{E}'_\natural(\mathcal{X}) \cap \mathcal{M}^\nu(X)$, $r = r(T) > 0$ *and that, for some* $c_1 \in \mathbb{C} \setminus \{0\}$, $c_2 > -\frac{1}{2} - \frac{a_X}{2} - \nu$, $c_3 \in \mathbb{R}^1$, *the following asymptotic relation holds true*:

$$\widetilde{T}(z) = c_1 z^{c_2} \cos(rz + c_3) + O(|z|^{c_2-1} e^{r|\operatorname{Im} z|}) \quad as \quad z \to \infty, \ \operatorname{Re} z \ge 0. \tag{3.8}$$

Then for any $k > s+\nu+1$ ($k > s+\nu+2$) *there exists a nonzero* $f \in (W^s_{1,\mathrm{loc}} \cap \mathcal{D}'_{k,m,j})(\mathcal{X})$ ($f \in C^s_{k,m,j}(\mathcal{X})$, *respectively*) *that satisfies* (3.7) *and lies in* $\mathcal{D}'_T(\mathcal{X})$.

We note that condition (3.8) is fulfilled for a wide class of distributions $T \in \mathcal{M}^\nu(X)$. In many cases, this condition is verified easily by using known methods of asymptotic expansions [19, Chapter 2, Theorem 10.2]. In particular, it is satisfied if T is the indicator function of a ball or the surface delta-function of a sphere in X (see [13, 14]). In these cases, Theorem 3.2 holds true for $\nu = 1$ and $\nu = 0$, respectively.

The result below shows that Theorem 3.2 fails if we do not assume that condition (3.8) is fulfilled.

Theorem 3.3. *For any* $\nu \in \mathbb{Z}$, *there exists* $T \in \mathcal{E}'_\natural(\mathcal{X}) \cap \mathcal{M}^\nu(X)$ *satisfying the following conditions*:

(1) $T \notin \mathcal{M}^{\nu+1}(X)$ *and* $r(T) > 0$;
(2) *if* $R > r(T)$ *and a distribution* $f \in \mathcal{D}'_T(B_R)$ *satisfies* (3.7), *then* $f = 0$. *In particular,* $f^{k,m,j} = 0$ *in* B_R *for all* k, m, j.

The main results of the present paper were announced by the authors in [20]. On Euclidean spaces, analogs of Theorems 3.1 and 3.2 for some other classes of distributions were obtained in [11] by different methods; these methods do not work in the general case.

§4. Auxiliary constructions

4.1. Infinitesimal operators. In this subsection, we assume that $X \ne \mathbb{R}^n$ unless otherwise stipulated. In the expressions $\pm$ and $\mp$ below, the upper sign corresponds to

the compact case and the lower sign to the noncompact case. For $1 \le j \le a_X$, we define differential operators A_j on $\mathcal{X}$ in the following way:

$X = \mathbb{S}^n$, $\mathbb{H}^n_{\mathbb{R}}$:

$$A_j = (1 \mp |x|^2)\frac{\partial}{\partial x_j} \pm 2x_j \sum_{l=1}^{n} x_l \frac{\partial}{\partial x_l}; \tag{4.1}$$

$X = \mathbb{P}^n_{\mathbb{R}}$:

$$A_j = \frac{\partial}{\partial x_j} + x_j \sum_{l=1}^{n} x_l \frac{\partial}{\partial x_l}; \tag{4.2}$$

$X = \mathbb{P}^n_{\mathbb{C}}$, $\mathbb{H}^n_{\mathbb{C}}$:

$$\begin{aligned} A_j &= \frac{\partial}{\partial z_j} \pm \bar{z}_j \sum_{l=1}^{n} \bar{z}_l \frac{\partial}{\partial \bar{z}_l}, \quad 1 \le j \le n, \\ A_j &= \frac{\partial}{\partial \bar{z}_{j-n}} \pm z_{j-n} \sum_{l=1}^{n} z_l \frac{\partial}{\partial z_l}, \quad n+1 \le j \le 2n; \end{aligned} \tag{4.3}$$

$X = \mathbb{P}^n_{\mathbb{Q}}$, $\mathbb{H}^n_{\mathbb{Q}}$:

(4.4)

$$\begin{aligned} A_j &= \frac{\partial}{\partial z_j} \pm \bar{z}_j \sum_{l=1}^{2n} \bar{z}_l \frac{\partial}{\partial \bar{z}_l} \mp z_{j+n} \sum_{l=1}^{n} \left(\bar{z}_{n+l} \frac{\partial}{\partial z_l} - \bar{z}_l \frac{\partial}{\partial z_{n+l}} \right), \quad 1 \le j \le n, \\ A_j &= \frac{\partial}{\partial z_j} \pm \bar{z}_j \sum_{l=1}^{2n} \bar{z}_l \frac{\partial}{\partial \bar{z}_l} \mp z_{j-n} \sum_{l=1}^{n} \left(\bar{z}_l \frac{\partial}{\partial z_{n+l}} - \bar{z}_{n+l} \frac{\partial}{\partial z_l} \right), \quad n+1 \le j \le 2n, \\ A_j &= \frac{\partial}{\partial \bar{z}_{j-2n}} \pm z_{j-2n} \sum_{l=1}^{2n} z_l \frac{\partial}{\partial z_l} \mp \bar{z}_{j-n} \sum_{l=1}^{n} \left(z_{n+l} \frac{\partial}{\partial \bar{z}_l} - z_l \frac{\partial}{\partial \bar{z}_{n+l}} \right), \quad 2n+1 \le j \le 3n, \\ A_j &= \frac{\partial}{\partial \bar{z}_{j-2n}} \pm z_{j-2n} \sum_{l=1}^{2n} z_l \frac{\partial}{\partial z_l} \mp \bar{z}_{j-3n} \sum_{l=1}^{n} \left(z_l \frac{\partial}{\partial \bar{z}_{n+l}} - z_{n+l} \frac{\partial}{\partial \bar{z}_l} \right), \quad 3n+1 \le j \le 4n; \end{aligned}$$

$X = \mathbb{P}^2_{\mathbb{C}a}$, $\mathbb{H}^2_{\mathbb{C}a}$:

$$\begin{aligned} A_j &= \left(1 \mp \sum_{l=1}^{8} x_{2l-1}^2\right) \frac{\partial}{\partial x_j} \mp \varepsilon_j \sum_{l=1}^{8} P_{l,\lambda_j}(x) \frac{\partial}{\partial x_{2l}} \pm 2x_j \sum_{l=1}^{16} x_l \frac{\partial}{\partial x_l}, \\ &\quad j \in \{1, 3, \ldots, 15\}, \\ A_j &= \left(1 \mp \sum_{l=1}^{8} x_{2l}^2\right) \frac{\partial}{\partial x_j} \mp \varepsilon_{j-1} \sum_{l=1}^{8} Q_{l,\lambda_{j-1}}(x) \frac{\partial}{\partial x_{2l-1}} \pm 2x_j \sum_{l=1}^{16} x_l \frac{\partial}{\partial x_l}, \\ &\quad j \in \{2, 4, \ldots, 16\}. \end{aligned} \tag{4.5}$$

Here $\varepsilon_1 = 1$, $\varepsilon_j = -1$, $j \in \{3, 5, \ldots, 15\}$, $\lambda_1 = 0$, $\lambda_3 = 4$, $\lambda_5 = 2$, $\lambda_7 = 6$, $\lambda_9 = 1$, $\lambda_{11} = 5$, $\lambda_{13} = 3$, $\lambda_{15} = 7$, and the polynomials $P_{l,s} : \mathbb{R}^{16} \to \mathbb{R}^1$, $Q_{l,s} : \mathbb{R}^{16} \to \mathbb{R}^1$,

$0 \le s \le 7$ are defined by the formulas

$$\begin{aligned}
&\mathbf{i}_s(p_1(x) + p_5(x)\mathbf{i}_1 + p_3(x)\mathbf{i}_2 + p_7(x)\mathbf{i}_3 + p_2(x)\mathbf{i}_4 + p_6(x)\mathbf{i}_5 + p_4(x)\mathbf{i}_6 + p_8(x)\mathbf{i}_7)\\
&\quad = P_{1,s}(x) + P_{5,s}(x)\mathbf{i}_1 + P_{3,s}(x)\mathbf{i}_2 + P_{7,s}(x)\mathbf{i}_3 + P_{2,s}(x)\mathbf{i}_4\\
&\qquad\qquad + P_{6,s}(x)\mathbf{i}_5 + P_{4,s}(x)\mathbf{i}_6 + P_{8,s}(x)\mathbf{i}_7,\\
&(p_1(x) + p_5(x)\mathbf{i}_1 + p_3(x)\mathbf{i}_2 + p_7(x)\mathbf{i}_3 + p_2(x)\mathbf{i}_4 + p_6(x)\mathbf{i}_5 + p_4(x)\mathbf{i}_6 + p_8(x)\mathbf{i}_7)\mathbf{i}_8\\
&\quad = Q_{1,s}(x) + Q_{5,s}(x)\mathbf{i}_1 + Q_{3,s}(x)\mathbf{i}_2 + Q_{7,s}(x)\mathbf{i}_3 + Q_{2,s}(x)\mathbf{i}_4\\
&\qquad\qquad + Q_{6,s}(x)\mathbf{i}_5 + Q_{4,s}(x)\mathbf{i}_6 + Q_{8,s}(x)\mathbf{i}_7,
\end{aligned}$$

where $\mathbf{i}_0, \mathbf{i}_1, \dots, \mathbf{i}_7$ is the standard basis in the algebra of octaves (see [14, §3]) and the A_j are infinitesimal operators corresponding to involutive isometrics of the spaces X (see [14, the proof of Lemma 7]).

Next, we set

$$\mathcal{N}_X(k) = \begin{cases} (k+1)/2, & X = \mathbb{P}^n_{\mathbb{R}},\\ k, & X \ne \mathbb{P}^n_{\mathbb{R}},\end{cases}$$
$$\rho_X = \alpha_X + \beta_X + 1,$$

where $\alpha_X = -1 + a_X/2$ and β_X is as defined in §3. We note that the Riemannian measure on $\mathcal{X}$ has the form

$$d\mu(x) = \begin{cases} (1-|x|^2)^{-\rho_X-1}\,dx, & X \in \mathfrak{X}_1,\\ (1+|x|^2)^{-\rho_X-1}\,dx, & X \in \mathfrak{X}_2\end{cases} \tag{4.6}$$

(dx is the Lebesgue measure in $\mathbb{R}^{a_X}$), and the area of a sphere of radius r in X is equal to

$$A_X(r) = \begin{cases} b_X(\sinh r)^{2\alpha_X+1}(\cosh r)^{2\beta_X+1}, & X \in \mathfrak{X}_1,\\ b_X(\sin r)^{2\alpha_X+1}(\cos r)^{2\beta_X+1}, & X \in \mathfrak{X}_2,\end{cases} \tag{4.7}$$

where

$$b_X = \int_{\mathbb{S}^{a_X-1}} d\omega = \frac{2\pi^{a_X/2}}{\Gamma(a_X/2)}. \tag{4.8}$$

We also define a differential operator $D(\alpha,\beta)$ by

$$(D(\alpha,\beta)\varphi)(\varrho) = \frac{(1\pm\varrho^2)^{\beta+1}}{\varrho^\alpha}\frac{d}{d\varrho}\left(\frac{\varrho^\alpha}{(1\pm\varrho^2)^\beta}\varphi(\varrho)\right). \tag{4.9}$$

Proposition 4.1. *Suppose $T \in \mathcal{E}'_\natural(\mathcal{X})$, $R > r(T)$, $k \in \mathbb{Z}_+$, and $m \in \{0,\dots,M_X(k)\}$. Also, suppose $\varphi(\varrho)Y(\sigma) \in C^s_T(B_R)$ for some $Y \in \mathcal{H}^{k,m}_X \setminus \{0\}$ and $s \in \mathbb{N}$. Then:*

(i) *the function $\varphi(\varrho)Y^{k,m}_j(\sigma)$ lies in $C^s_T(B_R)$ for all $j \in \{1,\dots,d^{k,m}_X\}$, and a similar result holds true for the class $(\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(B_R)$;*

(ii) *the function $(D(-k, m+1-\mathcal{N}_X(k+1))\varphi)(\varrho)Y^{k+1,m}_j(\sigma)$ belongs to $C^{s-1}_T(B_R)$ for all $j \in \{1,\dots,d^{k+1,m}_X\}$;*

(iii) *if $m \le M_X(k+1)-1$, then $(D(-k,\beta_X-m)\varphi)(\varrho)Y^{k+1,m+1}_j(\sigma) \in C^{s-1}_T(B_R)$ for all $j \in \{1,\dots,d^{k+1,m+1}_X\}$;*

(iv) *if $k \ge 1$ and $m \le M_X(k-1)$, then*

$$(D(k+2\alpha_X, \mathcal{N}_X(k)+\rho_X-1-m)\varphi)(\varrho)Y^{k-1,m}_j(\sigma) \in C^{s-1}_T(B_R)$$

for all $j \in \{1,\dots,d^{k-1,m}_X\}$;

(v) *if $m \ge 1$, then $(D(k+2\alpha_X, \alpha_X+m)\varphi)(\varrho)Y^{k-1,m-1}_j(\sigma) \in C^{s-1}_T(B_R)$ for all $j \in \{1,\dots,d^{k-1,m-1}_X\}$.*

Proof. This can be proved by a minor modification of the proofs of Lemmas 6–8 in [14]. □

Proposition 4.2. *Suppose that* $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $R > r(T)$, $k \in \mathbb{Z}_+$, *and that* $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(B_R)$ *and* $f(x) = u(\varrho)Y(\sigma)$ *for some* $Y \in \mathcal{H}_n^k \setminus \{0\}$. *Then*:

(i) *the function* $u(\varrho)Y_j^k(\sigma)$ *belongs to* $(\mathcal{D}'_T \cap L^{1,\mathrm{loc}})(B_R)$ *for all* $j \in \{1, \dots, d(n,k)\}$;

(ii) *if* $f \in C_T^1(B_R)$, *then*

$$\left(u'(\varrho) - \frac{ku(\varrho)}{\varrho}\right) Y_j^{k+1}(\sigma) \in C_T(B_R)$$

for all $j \in \{1, \dots, d(n,k+1)\}$;

(iii) *if* $f \in C_T^1(B_R)$ *and* $k \geq 1$, *then*

$$\left(u'(\varrho) + \frac{n+k-2}{\rho} u(\varrho)\right) Y_j^{k-1}(\sigma) \in C_T(B_R)$$

for all $j \in \{1, \dots, d(n,k-1)\}$.

The proof of Proposition 4.2 can be found in [3, Part 1, §5.4].

4.2. Generalized spherical harmonics. A spherical harmonic on X is a radial eigenfunction of the operator L. Below we need some generalizations of these functions, which we introduce separately for $\mathbb{R}^n$, for compact spaces X, and for hyperbolic spaces.

In all three cases, we assume that $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_X(k)\}$, $j \in \{1, \dots, d_X^{k,m}\}$, $x = \varrho\sigma \in \mathcal{X} \setminus \{0\}$, $\lambda \in \mathbb{C}$, $\eta \in \mathbb{Z}_+$, $\varkappa = \eta$ if $\lambda \neq 0$, and $\varkappa = 2\eta$ if $\lambda = 0$.

For $X = \mathbb{R}^n$ we define

$$\Phi_{\lambda,\eta,k,m,j}(x) = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\sqrt{b_X}\,\varrho^k \left(\frac{d}{dz}\right)^{\varkappa} \left(\frac{J_{\frac{n}{2}+k-1}(z\varrho)}{(z\varrho)^{n/2+k-1}}\right)\Bigg|_{z=\lambda} Y_j^{k,m}(\sigma).$$

Let X be compact. We set

$$\Phi_{\lambda,\eta,k,m,j}(x) = \sqrt{b_X}\left(\frac{d}{dz}\right)^{\varkappa} \times \Big(\varrho^k(1+\varrho^2)^{m+1-\mathcal{N}_X(k+1)} F(a(z), b(z); c; \varrho^2/(1+\varrho^2))\Big)\Big|_{z=\lambda} Y_j^{k,m}(\sigma),$$

where

$$\begin{aligned} a(z) &= \nu_X(z) + \mathcal{N}_X(k+1) - m - 1, \\ b(z) &= \nu_X(-z) + \mathcal{N}_X(k+1) - m - 1, \\ c &= k + \alpha_X + 1, \quad \nu_X(z) = (\rho_X + z)/2. \end{aligned} \tag{4.10}$$

Finally, using the above notation, we define $\Phi_{\lambda,\eta,k,m,j}$ for hyperbolic spaces X by the formula

$$\Phi_{\lambda,\eta,k,m,j}(x) = \sqrt{b_X}\left(\frac{d}{dz}\right)^{\varkappa}\left(\varrho^k(1-\varrho^2)^{m-k} F\left(a(-iz), b(-iz); c; \frac{\varrho^2}{\varrho^2-1}\right)\right)\Bigg|_{z=\lambda} Y_j^{k,m}(\sigma).$$

It is easily seen that the functions $\Phi_{\lambda,\eta,k,m,j}$ admit a continuous extension to the point $x = 0$, becoming real-analytic functions on $\mathcal{X}$ and

$$(L + \lambda^2 - \varepsilon_X \rho_X^2)\Phi_{\lambda,0,k,m,j} = 0, \tag{4.11}$$

where $\varepsilon_X \rho_X^2$ is assumed to be zero for $X = \mathbb{R}^n$ (see [15, Introduction, Lemma 3.6] and the proof of formula (41) in [14]). If $N \in \mathbb{Z}_+$ and $r \in (0, \operatorname{diam} X)$, then

$$\|\Phi_{\lambda,0,k,m,j}\|_{C^N(\overline{B_r})} \leq c(1+|\lambda|)^{N-k} e^{r|\operatorname{Im}\lambda|}, \tag{4.12}$$

where the constant $c > 0$ does not depend on λ (see [15, Introduction, Lemma 3.6] and the proof of estimate (45) in [14]).

Other properties of $\Phi_{\lambda,0,k,m,j}$ (differentiation formulas, integral representations, asymptotic estimates, and so on) are given in [3, Part 1, Chapter 5.3], [14, §4], [13].

In conclusion, we consider the relationship between $\Phi_{\lambda,0,k,m,j}$ on compact spaces X and the Jacobi polynomials.

For $l \in \mathbb{Z}_+$, we set

$$\lambda_l = 2l + \alpha + \beta + 1, \quad \mu_l = \frac{2\lambda_l}{b_X l!} \frac{\Gamma(\alpha + l + 1)\Gamma(\lambda_l - l)}{\Gamma(\beta + l + 1)(\Gamma(\alpha + 1))^2}, \tag{4.13}$$

where

$$\alpha = \alpha_X + k, \quad \beta = \beta_X + 2\mathcal{N}_X(k+1) - k - 2m - 2.$$

We note that

$$\mu_l = O(l^{2\alpha+1}) \ \text{ as } \ l \to \infty \tag{4.14}$$

and

$$\Phi_{\lambda_l,0,k,m,j}(p) = \sqrt{b_X}\varrho^k(1+\varrho^2)^{m+1-\mathcal{N}_X(k+1)} R_l^{(\alpha,\beta)}\left((1-\varrho^2)/(1+\varrho^2)\right) Y_j^{k,m}(\sigma). \tag{4.15}$$

Proposition 4.3. *The collection of functions $\{\Phi_{\lambda_l,0,k,m,j}\}_{l=0}^{\infty}$ forms an orthogonal basis in the space $L^2_{k,m,j}(X)$. Moreover,*

$$\int_X |\Phi_{\lambda_l,0,k,m,j}(x)|^2 \, d\mu(x) = 1/\mu_l.$$

Proof. The mapping

$$g \mapsto G(x) = \varrho^k(1+\varrho^2)^{m+1-\mathcal{N}_X(k+1)} g((1-\varrho^2)/(1+\varrho^2)) Y_j^{k,m}(\sigma)$$

is an isomorphism of the space $L^2((-1,1), 2^{-\alpha-\beta-2}(1-t)^\alpha(1+t)^\beta dt)$ onto $L^2_{k,m,j}(X)$, because, by (4.6),

$$\begin{aligned}
\int_X |G(x)|^2 \, d\mu(x) &= \int_{\mathbb{R}^a X} \varrho^{2k}(1+\varrho^2)^{2m-2\mathcal{N}_X(k+1)-\alpha_X-\beta_X} \left| g\left(\frac{1-\varrho^2}{1+\varrho^2}\right)\right|^2 |Y_j^{k,m}(\sigma)|^2 \, dx \\
&= \int_0^\infty \varrho^{2\alpha+1}(1+\varrho^2)^{-\alpha-\beta-2} \left| g\left(\frac{1-\varrho^2}{1+\varrho^2}\right)\right|^2 d\varrho \\
&= 2^{-\alpha-\beta-2} \int_{-1}^1 (1-t)^\alpha(1+t)^\beta |g(t)|^2 \, dt.
\end{aligned}$$

Since the polynomials $R_l^{(\alpha,\beta)}$, $l \in \mathbb{Z}_+$, form an orthogonal basis in the space $L^2((-1,1), (1-t)^\alpha(1+t)^\beta dt)$ and

$$\int_{-1}^1 (1-t)^\alpha(1+t)^\beta (R_l^{(\alpha,\beta)}(t))^2 \, dt = \frac{2^{\alpha+\beta+2}}{\mu_l b_X}$$

(see [21, 10.8 (3) and 10.8 (4)]), we obtain the required statement. □

§5. Transmutation operators related to expansions in Jacobi polynomials

5.1. Analogs of the spherical transformation. For $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$, we set

$$\mathcal{F}_j^{k,m}(f)(\lambda) = \left\langle f, \overline{\Phi_{\bar\lambda,0,k,m,j}} \right\rangle, \quad \lambda \in \mathbb{C}. \tag{5.1}$$

Then $\mathcal{F}_j^{k,m}(f)$ is an even entire function of the variable λ. If $f \in \mathcal{E}'_\natural(\mathcal{X})$, then we shall write $\widetilde{f}(\lambda)$ instead of $\mathcal{F}_1^{0,0}(f)(\lambda)$, i.e.,

$$\widetilde{f}(\lambda) = \langle f, \Phi_{\lambda,0,0,0,1} \rangle. \tag{5.2}$$

For Euclidean and hyperbolic spaces, $\widetilde{f}$ coincides with the spherical transform of the distribution f (see [15, Chapter 4]). If X is compact, then $\widetilde{f}$ is an analytic continuation of the discrete Fourier–Jacobi transform (see (4.15)).

We shall need the basic properties of $\mathcal{F}_j^{k,m}$ on compact two-point-homogeneous spaces X. This condition, imposed on X, will be assumed throughout in this subsection.

Proposition 5.1. *Suppose $T \in \mathcal{E}'_\natural(\mathcal{X})$, $R \in (r(T), \pi/2]$, $f \in \mathcal{D}'(B_R)$, and*

$$Lf = (\rho_X^2 - \lambda^2)f$$

for some $\lambda \in \mathbb{C}$. Then

$$f \times T = \widetilde{T}(\lambda)f \tag{5.3}$$

in the ball $B_{R-r(T)}$.

Proof. This statement is obtained by standard arguments (see, e.g., [15, Chapter 4, §2, the proof of Proposition 2.4]). □

Corollary 5.1. *Let $T \in \mathcal{E}'_\natural(\mathcal{X})$, and let*

$$n(\lambda, T) = \begin{cases} n_\lambda(\widetilde{T}) - 1 & \text{if} \quad \lambda \in \mathcal{Z}(\widetilde{T}) \setminus \{0\}, \\ n_\lambda(\widetilde{T})/2 - 1 & \text{if} \quad \lambda = 0 \in \mathcal{Z}(\widetilde{T}). \end{cases}$$

Then $\Phi_{\lambda,\eta,k,m,j} \in \mathrm{RA}_T(\mathcal{X})$ for all $\lambda \in \mathcal{Z}(\widetilde{T})$, $\eta \in \{0, \dots, n(\lambda, T)\}$.

Proof. Using (4.11) and (5.3), we obtain the required statement for $\eta = 0$. The general case follows from this one by differentiation (see the definition of $\Phi_{\lambda,\eta,k,m,j}$). □

Proposition 5.2. *Suppose $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$, $T \in \mathcal{E}'_\natural(\mathcal{X})$, and $r(f) + r(T) < \pi/2$. Then*

$$\mathcal{F}_j^{k,m}(f \times T)(\lambda) = \mathcal{F}_j^{k,m}(f)(\lambda)\widetilde{T}(\lambda). \tag{5.4}$$

In particular,

$$\mathcal{F}_j^{k,m}(P(L)f)(\lambda) = P(\rho_X^2 - \lambda^2)\mathcal{F}_j^{k,m}(f)(\lambda) \tag{5.5}$$

for any polynomial P.

Proof. By (4.11) and (5.3), we have

$$\left\langle f \times T, \overline{\Phi_{\bar{\lambda},0,k,m,j}} \right\rangle = \left\langle f, \overline{\Phi_{\bar{\lambda},0,k,m,j}} \times T \right\rangle = \widetilde{T}(\lambda) \left\langle f, \overline{\Phi_{\bar{\lambda},0,k,m,j}} \right\rangle,$$

which implies (5.4). Substituting $T = P(L)\delta_0$ in (5.4), we get (5.5). □

Proposition 5.3. *The transformation $\mathcal{F}_j^{k,m}$ is injective on $\mathcal{E}'_{k,m,j}(\mathcal{X})$.*

Proof. Suppose $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$ and $\mathcal{F}_j^{k,m}(f) = 0$. We take $\psi \in \mathcal{D}_\natural(\mathcal{X})$ such that $r(\psi) < \pi/2 - r(f)$. The following integral representation is valid for $\theta \in (0, \pi/2)$:

$$\Phi_{\lambda,0,k,m}(\tan\theta) = \frac{1}{\sqrt{b_X}} \int_0^\theta \cos(\lambda t) Q_{X,k,m}(t, \theta)\, dt, \tag{5.6}$$

where

$$\begin{aligned}
\Phi_{\lambda,0,k,m}(\varrho) &= \varrho^k(1+\varrho^2)^{m+1-\mathcal{N}_X(k+1)} F\left(a(\lambda), b(\lambda); c; \frac{\varrho^2}{1+\varrho^2}\right),\\
Q_{X,k,m}(t,\theta) &= \frac{2^{c-1/2}\Gamma(c)\sqrt{b_X}}{\sqrt{\pi}\,\Gamma(c-1/2)}(\sin\theta)^{-k-2\alpha_X}\\
&\qquad \times (\cos\theta)^{-\beta_{\mathcal{X}}-1/2}(\cos t-\cos\theta)^{c-3/2} v_{X,k,m}\left(\frac{\cos\theta-\cos t}{2\cos\theta}\right),\\
v_{X,k,m}(z) &= F\left(\frac{1}{2}+\beta_X+k-2m, \frac{1}{2}-\beta_X-2\mathcal{N}_X(k)+k+2m; c-\frac{1}{2}; z\right)
\end{aligned} \tag{5.7}$$

(see (4.10) and [14, formula (17)]). Using (5.4), (5.1) and (5.6), we obtain

$$\begin{aligned}
0 &= \mathcal{F}_j^{k,m}(f\times\psi)(\lambda)\\
&= (b_X)^{-1}\int_0^{\pi/2}\cos(\lambda t)\int_t^{\pi/2} A_X(\theta)(f\times\psi)_{k,m,j}(\tan\theta)Q_{X,k,m}(t,\theta)\,d\theta\,dt,
\end{aligned}$$

whence

$$\int_t^{\pi/2} A_X(\theta)(f\times\psi)_{k,m,j}(\tan\theta)Q_{X,k,m}(t,\theta)\,d\theta = 0$$

for all $t\in(0,\pi/2)$. Then the Titchmarsh convolution theorem implies that $f\times\psi=0$ (see Theorem 4.3.3 in [16, Chapter 4] in the one-dimensional case). Since ψ is arbitrary, f is zero. □

Proposition 5.4. (i) *Let $f\in(C^s\cap\mathcal{E}'_{k,m,j})(\mathcal{X})$ for some $s\in\mathbb{Z}_+$. Then*

$$|\mathcal{F}_j^{k,m}(f)(\lambda)| \le \gamma\frac{e^{r(f)|\operatorname{Im}\lambda|}}{(1+|\lambda|)^{s+k}}, \quad \lambda\in\mathbb{C}, \tag{5.8}$$

where the constant γ does not depend on λ.

(ii) *If $f\in\mathcal{E}'_{k,m,j}(\mathcal{X})$, and for some $s\in\mathbb{Z}_+$ we have*

$$\mathcal{F}_j^{k,m}(f)(\lambda_l) = O(l^{-s-k-2\alpha_X-3})$$

as $l\to+\infty$, where λ_l is defined in (4.13), then $f\in C^s(\mathcal{X})$.

Proof. First we prove statement (i). The definition of $\mathcal{F}_j^{k,m}$ and (4.6) show that

$$\mathcal{F}_j^{k,m}(f)(\lambda) = \sqrt{b_X}\int_0^\infty \frac{\varrho^{2\alpha_X+1}}{(1+\varrho^2)^{\rho_X+1}} f_{k,m,j}(\varrho)\Phi_{\lambda,0,k,m}(\varrho)\,d\varrho. \tag{5.9}$$

We set

$$D_1 = D(k+1+2\alpha_X, \mathcal{N}_X(k+1)+\rho_X-m-1), \quad D_2 = D(-k, m+1-\mathcal{N}_X(k+1)),$$

where D is defined by (4.9). Integrating in (5.9) by parts and using differentiation formulas for $\Phi_{\lambda,0,k,m}(\varrho)$ (see [14, (36)–(39)]), we deduce that

$$\begin{aligned}
\mathcal{F}_j^{k,m}(f)(\lambda) = æ\int_0^{\tan r(f)} &\frac{\varrho^{2\alpha_X+1}}{(1+\varrho^2)^{\rho_X+1}}(D_2^{s-2[s/2]}(D_1D_2)^{[s/2]}f_{k,m,j})(\varrho)\\
&\times \Phi_{\lambda,0,k+s-2[s/2],m}(\varrho)\,d\varrho,
\end{aligned} \tag{5.10}$$

where

$$æ = \frac{\sqrt{b_X}}{((\rho_X+2(\mathcal{N}_X(k+1)-m-1))^2-\lambda^2)^{[s/2]}}\left(\frac{-1}{2(k+\alpha_X+1)}\right)^{s-2[s/2]}.$$

Relation (5.10) and estimate (4.12) imply (5.8). Part (ii) easily follows from Proposition 4.3, (4.14), and (4.12). □

The following statement is an analog of the Paley–Wiener theorem for the transformation $\mathcal{F}_j^{k,m}$.

Proposition 5.5. (i) *Suppose* $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$ *and* $\operatorname{supp} f \subset \overline{B_r}$. *Then*

$$|\mathcal{F}_j^{k,m}(f)(\lambda)| \le c_1(1+|\lambda|)^{c_2} e^{r|\operatorname{Im}\lambda|} \quad \text{for all} \quad \lambda \in \mathbb{C}, \tag{5.11}$$

where $c_1, c_2 > 0$ *do not depend on* λ. *Conversely, for any even entire function* $w(\lambda)$ *satisfying an estimate of the form* (5.11) *with some* $r \in [0, \pi/2)$, *there exists a distribution* $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$ *such that*

$$\operatorname{supp} f \subset \overline{B_r} \quad \text{and} \quad \mathcal{F}_j^{k,m}(f) = w. \tag{5.12}$$

(ii) *If* $f \in \mathcal{D}_{k,m,j}(\mathcal{X})$ *and* $\operatorname{supp} f \subset \overline{B_r}$, *then for any* $N \in \mathbb{Z}_+$ *there exists a constant* $c_N > 0$ *such that*

$$|\mathcal{F}_j^{k,m}(f)(\lambda)| \le c_N(1+\lambda)^{-N} e^{r|\operatorname{Im}\lambda|} \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{5.13}$$

Conversely, for every even entire function $w(\lambda)$ *satisfying an estimate of the form* (5.13) *with some* $r \in [0, \pi/2)$ *and all* $N \in \mathbb{Z}_+$, *there exists a function* $f \in \mathcal{D}_{k,m,j}(\mathcal{X})$ *for which conditions* (5.12) *are fulfilled.*

Proof. (i) From (4.12) and the definition of the order of a distribution, we see that for any $\varepsilon > 0$ there exists $æ_\varepsilon > 0$ $\varkappa_\varepsilon > 0$ such that

$$|\mathcal{F}_j^{k,m}(f)(\lambda)| \le æ_\varepsilon e^{(r+\varepsilon)|\operatorname{Im}\lambda|}(1+|\lambda|)^{\operatorname{ord} f - k}, \qquad \lambda \in \mathbb{C}. \tag{5.14}$$

Using (5.14) and the Phragmén–Lindelöf principle, we obtain (5.11) (see, e.g., the proof of Lemma 4.3 in [22, Chapter 3]). We prove the converse statement. First, we consider the case where w has finitely many zeros. In this situation, w is an even polynomial, by the Hadamard factorization theorem. We write w in the form $w(\lambda) = c(\lambda^2 - z_1^2)\dots(\lambda^2 - z_l^2)$. Let $Y_j^{k,m}(\partial)$ be the differential operator associated with the polynomial $Y_j^{k,m}(x) = \varrho^k Y_j^{k,m}(\sigma)$. For $g \in C^k(B_\varepsilon)$, $\varepsilon \in (0, \pi/2)$, we have

$$Y_j^{k,m}(\partial)(|x|^{2N} g(x))(0) = 0, \quad N \in \mathbb{N}. \tag{5.15}$$

From (4.8) and (5.15), we conclude that

$$\mathcal{F}_j^{k,m}(Y_j^{k,m}(\partial)^* \delta_0)(\lambda) = \frac{2^k(\alpha_X+1)_k}{\sqrt{b_X}}, \tag{5.16}$$

where $Y_j^{k,m}(\partial)^*$ is the operator adjoint to $Y_j^{k,m}(\partial)$. This relation and (5.5) imply that conditions (5.12) are fulfilled for the distribution $f = cP_1(L)Y_j^{k,m}(\partial)^*\delta_0$, where

$$P_1(t) = \frac{\sqrt{b_X}}{2^k(\alpha_X+1)_k}(-t+\rho_X^2 - z_1^2)\cdots(-t+\rho_X^2 - z_l^2).$$

Now, suppose w has infinitely many zeros, and let c_2 be the constant in estimate (5.11) for the function w and $s = 2\alpha_X$. We choose a natural number $l \ge (s + c_2 + 6)/2$ and consider the even entire function

$$W(\lambda) = \frac{w(\lambda)}{(\lambda^2 - z_1^2)\cdots(\lambda^2 - z_l^2)},$$

where $z_1, \dots, z_l \in \mathcal{Z}(w)$. By the Paley–Wiener theorem for the one-dimensional Fourier transformation, there exists an even function $\varphi \in C^{s+2}(\mathbb{R}^1)$ such that $\operatorname{supp}\varphi \subset [-r, r]$ and

$$W(\lambda) = \int_0^r \varphi(t)\cos(\lambda t)\,dt, \quad \lambda \in \mathbb{C}.$$

Using (5.2), (5.6), and the method of solving the integral Abel equation (for example, see [15, Chapter 1, the proof of Theorem 2.6]), we can easily find an integrable radial function h on $\mathcal{X}$ for which

$$\operatorname{supp} h \subset \overline{B_r} \quad \text{and} \quad \widetilde{h}(\lambda) = W(\lambda). \tag{5.17}$$

Relations (5.17), (5.16) and Proposition 5.2 imply that the distribution $f = Y_j^{k,m}(\partial)^*\delta_0 \times P_1(L)h$ satisfies (5.12). This proves (i). Statement (ii) is a straightforward consequence of (i) and Proposition 5.4. □

Remark 5.1. Propositions 5.2 and 5.4 and the proof of Proposition 5.5 (i) show that the constant c_2 in (5.11) is related to $\operatorname{ord} f$ in the following way: (i) the estimate

$$\mathcal{F}_j^{k,m}(f)(\lambda) = O((1+|\lambda|)^{\operatorname{ord} f - k} e^{r(f)|\operatorname{Im}\lambda|}), \quad \lambda \in \mathbb{C}, \tag{5.18}$$

is valid;

(ii) inequality (5.11) implies that $\operatorname{ord} f \le \max\{0, 2[(c_2 + k + 2\alpha_X + 5)/2]\}$. Finally, we give an inversion formula for the transformation $\mathcal{F}_j^{k,m}$.

Proposition 5.6. *Suppose $f \in (\mathcal{E}'_{k,m,j} \cap C^s)(\mathcal{X})$ for some $s \ge 2\alpha_X + 3$. Then*

$$\mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l) = O(l^{k+2\alpha_X+1-s}) \quad \text{as} \quad l \to +\infty, \tag{5.19}$$

and

$$f = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l) \Phi_{\lambda_l,0,k,m,j} \tag{5.20}$$

in $\mathcal{X}$, where the λ_l and μ_l are as in (4.13).

Proof. Estimate (5.19) follows from (4.14) and (5.8). To prove (5.20), it suffices to apply Proposition 4.3 and (4.12). □

Remark 5.2. For noncompact two-point-homogeneous spaces X, there are appropriate analogs of the above properties of the transformation $\mathcal{F}_j^{k,m}$. They can be obtained from the general theory of the Fourier–Helgason transformation on X (see [15, Chapter 4], [18, Chapter 3]).

5.2. The maps $\mathfrak{A}_{k,m,j}$. In this subsection, we still assume that the space X is compact. The maps $\mathfrak{A}_{k,m,j}$ constructed below make it possible to reduce a series of problems for convolution equations on X to the one-dimensional case.

Suppose $k \in \mathbb{Z}_+$, $m \in \{0, \dots, M_X(k)\}$, and $j \in \{1, \dots, d_X^{k,m}\}$. For $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$, we set (see (4.13))

$$\mathfrak{A}_{k,m,j}(f)(t) = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l) \cos(\lambda_l t), \quad t \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big). \tag{5.21}$$

By using (4.14), (5.18), and (5.21), it is easy to show that $\mathfrak{A}_{k,m,j}(f)$ is an element of the space $\mathcal{D}'_\natural(-\frac{\pi}{2}, \frac{\pi}{2})$.

Let $T \in \mathcal{E}'_\natural(\mathcal{X})$. Proposition 5.5 and the one-dimensional Paley–Wiener theorem allow us to introduce $\Lambda(T) \in \mathcal{E}'_\natural(-\frac{\pi}{2}, \frac{\pi}{2})$ by the relation

$$\widehat{\Lambda(T)}(\lambda) = \widetilde{T}(\lambda), \quad \lambda \in \mathbb{C}. \tag{5.22}$$

The correspondence $\Lambda : T \to \Lambda(T)$ is a bijection of $\mathcal{E}'_\natural(\mathcal{X})$ onto $\mathcal{E}'_\natural(-\frac{\pi}{2}, \frac{\pi}{2})$, and

$$r(\Lambda(T)) = r(T). \tag{5.23}$$

Lemma 5.1. (i) *Suppose* $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$, $T \in \mathcal{E}'_\natural(\mathcal{X})$, *and* $r(f) + r(T) < \pi/2$. *Then the following relation is valid on the interval* $\big(r(T) - \frac{\pi}{2}, \frac{\pi}{2} - r(T)\big)$:

$$\mathfrak{A}_{k,m,j}(f \times T) = \mathfrak{A}_{k,m,j}(f) * \Lambda(T). \tag{5.24}$$

(ii) *Suppose* $f \in \Big(\mathcal{E}'_{k,m,j} \cap C^{2\alpha_X+k+3+N}\Big)(\mathcal{X})$ *for some* $N \in \mathbb{Z}_+$. *Then* $\mathfrak{A}_{k,m,j}(f) \in C^N_\natural\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$, *and for* $\theta \in \big(0, \frac{\pi}{2}\big)$ *we have*

$$f_{k,m,j}(\tan\theta) = \int_0^\theta \mathfrak{A}_{k,m,j}(f)(t) Q_{X,k,m}(t,\theta)\, dt, \tag{5.25}$$

where the function $Q_{X,k,m}(t,\theta)$ *is as given in* (5.7).

(iii) *If* $f \in \mathcal{E}'_{k,m,j}(\mathcal{X})$ *and* $r \in (0, \pi/2]$, *then* $f = 0$ *in* B_r *if and only if* $\mathfrak{A}_{k,m,j}(f) = 0$ *on* $(-r, r)$.

Proof. Part (i) follows immediately from (5.21), (5.22), and Proposition 5.2. We prove (ii). Estimate (5.19) and the definition of $\mathfrak{A}_{k,m,j}$ show that $\mathfrak{A}_{k,m,j}(f) \in C^N_\natural\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ and

$$\int_0^\theta \mathfrak{A}_{k,m,j}(f)(t) Q_{X,k,m}(t,\theta) dt = \sum_{l=0}^\infty \mu_l \mathcal{F}^{k,m}_j(f)(\lambda_l) \int_0^\theta \cos(\lambda_l t) Q_{X,k,m}(t,\theta)\, dt.$$

Now Proposition 5.6 and relation (5.6) imply part (ii). If $f \in \mathcal{D}_{k,m,j}(\mathcal{X})$, statement (iii) is an easy consequence of part (ii). In the general case, (iii) is obtained with the help of the standard smoothing by convolutions with delta-shaped sequences (see (5.21) and (5.23)). □

We extend the operator $\mathfrak{A}_{k,m,j}$ to the space $\mathcal{D}'_{k,m,j}(B_R)$, $R \in (0, \pi/2]$. Let $f \in \mathcal{D}'_{k,m,j}(B_R)$. We set

$$\langle \mathfrak{A}_{k,m,j}(f), \psi\rangle = \langle \mathfrak{A}_{k,m,j}(f\eta), \psi\rangle, \quad \psi \in \mathcal{D}(-R, R), \tag{5.26}$$

where $\eta \in \mathcal{D}_\natural(B_R)$ and $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. By Lemma 5.1 (iii), relation (5.26) determines $\mathfrak{A}_{k,m,j}(f)$ as a distribution in $\mathcal{D}'_\natural(-R, R)$ and

$$\mathfrak{A}_{k,m,j}\Big(f\big|_{B_r}\Big) = \mathfrak{A}_{k,m,j}(f)\big|_{(-r,r)}$$

for any $r \in (0, R]$.

Theorem 5.1. *For* $R \in (0, \pi/2]$, $N \in \mathbb{Z}_+$, *and* $\nu = 2\alpha_X + k + 3 + N$, *the following statements hold true.*

(i) *Let* $f \in \mathcal{D}'_{k,m,j}(B_R)$ *and* $r \in (0, R]$. *Then* $f = 0$ *in* B_r *if and only if* $\mathfrak{A}_{k,m,j}(f) = 0$ *on* $(-r, r)$.

(ii) *If* $f \in C^\nu_{k,m,j}(B_R)$, *then* $\mathfrak{A}_{k,m,j}(f) \in C^N_\natural(-R, R)$, *and* (5.25) *is valid for* $\theta \in (0, R)$.

(iii) *The mapping* $\mathfrak{A}_{k,m,j}$ *is continuous from* $\mathcal{D}'_{k,m,j}(B_R)$ *to* $\mathcal{D}'_\natural(-R, R)$ *and from* $C^\nu_{k,m,j}(B_R)$ *to* $C^N_\natural(-R, R)$.

(iv) *For* $\lambda \in \mathbb{C}$ *and* $\mu \in \mathbb{Z}_+$,

$$\mathfrak{A}_{k,m,j}(\Phi_{\lambda,\mu,k,m,j}) = u_{\lambda,\mu}, \tag{5.27}$$

where

$$u_{\lambda,\mu}(t) = \begin{cases} \frac{(it)^\mu e^{i\lambda t} + (-it)^\mu e^{-i\lambda t}}{2} & \text{if} \quad \lambda \neq 0, \\ (-1)^\mu t^{2\mu} & \text{if} \quad \lambda = 0. \end{cases}$$

(v) *Suppose* $f \in \mathcal{D}'_{k,m,j}(B_R)$, $T \in \mathcal{E}'_\natural(\mathcal{X})$, *and* $r(T) < R$. *Then* (5.24) *is fulfilled on the interval* $(r(T) - R, R - r(T))$. *In particular, for any polynomial* P *we have the following transmutation relation*:

$$\mathfrak{A}_{k,m,j}(P(L)f) = P\left(\frac{d^2}{dt^2} + \rho_X^2\right) \mathfrak{A}_{k,m,j}(f). \tag{5.28}$$

Proof. Using the definition of $\mathfrak{A}_{k,m,j}$ on $\mathcal{D}'_{k,m,j}(B_R)$ and Lemma 5.1 (ii), (iii), we obtain (i) and (ii). Now we prove (iii). First, suppose $f_q \in \mathcal{D}'_{k,m,j}(B_R)$, $q = 1, 2, \ldots$, and $f_q \to 0$ in $\mathcal{D}'(B_R)$ as $q \to +\infty$. We take $\psi \in \mathcal{D}(-R, R)$ and choose $\eta \in \mathcal{D}_\natural(B_R)$ so that $\eta = 1$ in $B_{r_0(\psi)+\varepsilon}$ for some $\varepsilon \in (0, R - r_0(\psi))$. Set $\eta_l(x) = \eta(x)\,\overline{\Phi_{\lambda_l,0,k,m,j}}(x)$. Then the series

$$\sum_{l=0}^{\infty} \mu_l \int_{-R}^{R} \psi(t)\cos(\lambda_l t)\,dt\,\eta_l(x)$$

converges in $\mathcal{D}(B_R)$ (see (4.12)). Denoting its sum by $\varphi(x)$, we have

$$\langle \mathfrak{A}_{k,m,j}(f_q), \psi\rangle = \sum_{l=0}^{\infty} \mu_l \int_{-R}^{R} \psi(t)\cos(\lambda_l t)\,dt\,\langle f_q, \eta_l\rangle = \langle f_q, \varphi\rangle,$$

whence $\mathfrak{A}_{k,m,j}(f_q) \to 0$ in $\mathcal{D}'(-R, R)$. Now, suppose $f_q \in C^\nu_{k,m,j}(B_R)$, $q = 1, 2, \ldots$, and $f_q \to 0$ in $C^\nu(B_R)$. We fix $r \in (0, R)$ and choose $\eta \in \mathcal{D}_\natural(B_R)$ so that $\eta = 1$ in $B_{r+\varepsilon}$ for some $\varepsilon \in (0, R - r)$. By (5.21), (4.11), (4.12), and the symmetry of the operator L, we conclude that

$$\|\mathfrak{A}_{k,m,j}(\eta f_q)\|_{C^N[-r,r]} \le c\|\eta f_q\|_{C^\nu(E)}, \tag{5.29}$$

where $E = \operatorname{supp}\eta$ and the constant $c > 0$ does not depend on q. Estimate (5.29) and statement (i) imply that

$$\lim_{q\to+\infty} \|\mathfrak{A}_{k,m,j}(f_q)\|_{C^N[-r,r]} = 0.$$

Thus, $\mathfrak{A}_{k,m,j}(f_q) \to 0$ in $C^N(-R, R)$, which completes the proof of (iii). Next, by (5.25) and (5.6),

$$\int_0^\theta (\mathfrak{A}_{k,,m,j}(\Phi_{\lambda,0,k,m,j})(t) - \cos(\lambda t))Q_{X,k,m}(t, \theta)\,dt = 0$$

for all $\theta \in (0, R)$. Therefore

$$\mathfrak{A}_{k,m,j}(\Phi_{\lambda,0,k,m,j})(t) = \cos(\lambda t), \quad t \in (-R, R) \tag{5.30}$$

(see the proof of Proposition 5.3). Differentiating (5.30) with respect to λ, we obtain (iv). Finally, in (v) we may assume that $f \in \mathcal{D}_{k,m,j}(B_R)$. Moreover,

$$f \times T = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l)\widetilde{T}(\lambda_l)\Phi_{\lambda_l,0,k,m,j}, \tag{5.31}$$

where the series converges in $C^\infty(\mathcal{X})$ (see Propositions 5.1 and 5.6). Applying the operator $\mathfrak{A}_{k,m,j}$ to (5.31) and taking (5.22) and (5.27) into account, we have

$$\mathfrak{A}_{k,m,j}(f \times T)(t) = \sum_{l=0}^{\infty} \mu_l \mathcal{F}_j^{k,m}(f)(\lambda_l)\widehat{\Lambda(T)}(\lambda_l)\cos(\lambda_l t). \tag{5.32}$$

Comparing $\mathfrak{A}_{k,m,j}(f) * \Lambda(T)$ with (5.32), we get (5.24) on the interval $(r(T) - R, R - r(T))$. Relation (5.28) is a special case of (5.24) for $T = P(L)\delta_0$. □

§6. Proof of the main results

To prove Theorem 3.1, we need two lemmas.

Lemma 6.1. *Suppose $T \in \mathcal{M}^0_\natural(X)$, $R > r(T) > 0$, $f \in (\mathcal{D}'_T \cap L^{1,\mathrm{loc}}_\natural)(B_R)$, and $f = 0$ in $B_{r(T)}$. Then $f = 0$ in B_R.*

Proof. Case 1: X is noncompact. By the condition imposed on f, there exists $g_1 \in L^{1,\mathrm{loc}}[0,R)$ such that $f(x) = g_1(d(0,x))$ in B_R. We define a sequence $f_q \in L^{1,\mathrm{loc}}_\natural(B_R)$, $q = 1, 2, \dots$, in the following way: $f_q(x) = g_q(d(0,x))$, where

$$g_q(t) = \int_0^t \frac{1}{A_X(\xi)} \int_0^\xi g_{q-1}(\eta) A_X(\eta)\, d\eta\, d\xi, \quad t \in [0,R), \tag{6.1}$$

for $q \geq 2$. Relation (6.1) implies that $f_q = 0$ in $B_{r(T)}$ and $f_{q+1} \in C^{2q-1}_\natural(B_R)$. Moreover,

$$Lf_{q+1} = f_q \ \text{ in } \ B_R, \tag{6.2}$$

because the radial part of the Laplace–Beltrami operator L is equal to

$$L^{\mathrm{rad}} = \frac{\partial^2}{\partial r^2} + \frac{A'_X(r)}{A_X(r)} \frac{\partial}{\partial r} \tag{6.3}$$

(see [15, Chapter 2, Proposition 5.26]). We prove that $f_q \times T = 0$ in $B_{R-r(T)}$. For $q = 1$, this follows from the assumptions of the lemma. Suppose $f_q \times T = 0$ in $B_{R-r(T)}$ for some $q \in \mathbb{N}$. Convolving the two sides of (6.2) with T, we obtain

$$L(f_{q+1} \times T) = 0. \tag{6.4}$$

Since $T \in \mathcal{M}^0(X)$, we have

$$f_{q+1} \times T \in C_\natural(B_{R-r(T)}). \tag{6.5}$$

Moreover,

$$(f_{q+1} \times T)(0) = 0, \tag{6.6}$$

because $f_{q+1} = 0$ in $B_{r(T)}$. Relations (6.3)–(6.6) show that $f_{q+1} \times T = 0$ in $B_{R-r(T)}$. Thus, by induction, $f_q \in \mathcal{D}'_T(B_R)$ for all $q \in \mathbb{N}$. Taking the smoothness of f_{q+1} into account and using [3, Part 3, Lemma 2.8], [12, remark after Lemma 8], we see that $f_q = 0$ in B_R for all sufficiently large q. Combining this with (6.2), we conclude that $f = L^{q-1} f_q = 0$ in B_R.

Case 2: X is compact. There exists a function $\psi \in (\mathcal{E}'_\natural \cap C^3)(\mathcal{X})$ such that $r(\psi) = r(T)$ and $T = P(L)\psi$ for some polynomial P (see Propositions 5.2, 5.4, and 5.5). From (5.22) and (5.8) it follows that $\Lambda(\psi) \in (\mathcal{E}'_\natural \cap C)(-\pi/2, \pi/2)$. The argument adduced in case 1 allows us to assume that $f \in C^q_T(B_R)$, where $q = 2 \deg P + 2\alpha_X + 4$. We set $F = \mathfrak{A}_{0,0,1}(P(L)f)$. Then, by Theorem 5.1 (i), (ii), (v), the distribution F belongs to $C_\natural(-R,R)$, $F = 0$ on the interval $(-r(\psi), r(\psi))$, and $F * \Lambda(\psi) = 0$ on $(r(\psi) - R, R - r(\psi))$. Now, to complete the proof it suffices to use [3, Part 3, Theorem 1.1 (2)], Theorem 5.1 (i), and the ellipticity of the operator $P(L)$. □

Lemma 6.2. *Let $T \in \mathcal{M}^1_\natural(X)$, and let $R > r(T) > 0$. Suppose a function $f \in C_T(B_R)$ possesses the following properties*:

1) *f has the form $f(x) = \varphi(\varrho) Y(\sigma)$, where $Y \in \mathcal{H}^{k,m}_X$ for some $k \in \{0,1,2,3\}$, $m \in \{0, \dots, M_X(k)\}$;*
2) *$f = 0$ in $B_{r(T)}$.*

Then $f = 0$ in B_R.

Proof. If $k = 0$, then Lemma 6.2 is an immediate consequence of Lemma 6.1. Let $k \in \{1,2,3\}$. First, we consider the case of $X = \mathbb{R}^n$. Without loss of generality, we may assume that $f(x) = \varphi(\varrho)(\sigma_1 + i\sigma_2)^k$ (see Proposition 4.2 (i)). We set $U_k(x) = u_k(\varrho)$,

$x \in B_R$, where

$$u_1(\varrho) = \int_0^\varrho \varphi(\xi)\, d\xi, \tag{6.7}$$

$$u_2(\varrho) = \int_0^\varrho \eta \int_0^\eta \frac{\varphi(\xi)}{\xi}\, d\xi\, d\eta, \tag{6.8}$$

$$u_3(\varrho) = \int_0^\varrho \zeta \int_0^\zeta \eta \int_0^\eta \frac{\varphi(\xi)}{\xi^2}\, d\xi\, d\eta\, d\zeta. \tag{6.9}$$

We define

$$V_k(x) = (U_k * T)(x), \quad x \in B_{R-r(T)}. \tag{6.10}$$

Relations (6.7)–(6.9) show that $V_k \in C^k_\natural(B_{R-r(T)})$ and

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)^k V_k = 0.$$

This yields

$$V_k(x) = \sum_{s=0}^{k-1} c_s \varrho^{2s}, \quad c_s \in \mathbb{C}. \tag{6.11}$$

Since $U_k \in C^k(B_R)$, $U_k = 0$ in $B_{r(T)}$, and $T \in \mathcal{M}^1_\natural(\mathbb{R}^n)$, we obtain

$$(L^s V_k)(0) = 0, \quad s \in \{0, \dots, k-1\}.$$

These relations and (6.11) imply that $V_k = 0$. Then, by Lemma 6.1, we have $U_k = 0$ (see (6.10)). Now relations (6.7)–(6.9) show that $f = 0$ in B_R.

Now, let X be a compact space. For $\varkappa \in \{1, 2, 3\}$ and $\mu \in \{0, \dots, M_X(\varkappa)\}$, we put $U_{\varkappa,\mu}(x) = u_{\varkappa,\mu}(\varrho)$, $p \in B_R$, where

$$u_{1,0}(\varrho) = \int_0^\varrho \frac{\varphi(\xi)}{1+\xi^2}\, d\xi, \tag{6.12}$$

$$u_{2,0}(\varrho) = \int_0^\varrho \frac{\eta}{(1+\eta^2)^{\mathcal{N}_X(2)}} \int_0^\eta \frac{(1+\xi^2)^{\mathcal{N}_X(2)-2}\varphi(\xi)}{\xi}\, d\xi\, d\eta, \tag{6.13}$$

$$u_{2,1}(\varrho) = \int_0^\varrho \eta(1+\eta^2)^{\beta_X - 1} \int_0^\eta \frac{\varphi(\xi)}{\xi(1+\xi^2)^{\beta_X+1}}\, d\xi\, d\eta, \tag{6.14}$$

$$\begin{aligned} u_{3,0}(\varrho) = \int_0^\varrho \frac{\zeta}{(1+\zeta^2)^{\mathcal{N}_X(2)}} \int_0^\zeta \frac{\eta}{(1+\eta^2)^{\mathcal{N}_X(3)-\mathcal{N}_X(2)+1}} \\ \times \int_0^\eta \frac{(1+\xi^2)^{\mathcal{N}_X(3)-2}\varphi(\xi)}{\xi^2}\, d\xi\, d\eta\, d\zeta, \end{aligned} \tag{6.15}$$

$$u_{3,1}(\varrho) = \int_0^\varrho \frac{\zeta}{(1+\zeta^2)^2} \int_0^\zeta \eta(1+\eta^2)^{\beta_X} \int_0^\eta \frac{\varphi(\xi)}{\xi^2(1+\xi^2)^{\beta_X+1}}\, d\xi\, d\eta\, d\zeta. \tag{6.16}$$

We define

$$V_{\varkappa,\mu}(x) = (U_{\varkappa,\mu} \times T)(x), \quad x \in B_{R-r(T)}. \tag{6.17}$$

Clearly, $V_{\varkappa,\mu} \in C^\varkappa_\natural(B_{R-r(T)})$. We prove that

$$V_{k,m}(x) = 0, \quad x \in B_{R-r(T)}. \tag{6.18}$$

For each of the spaces X, we have the following relations (see (4.1)–(4.5)):
$X = \mathbb{S}^n$ or $X = \mathbb{P}^n_{\mathbb{R}}$:

$$(A_1 + iA_2)^\varkappa V_{\varkappa,0} = ((A_1 + iA_2)^\varkappa U_{\varkappa,0}) \times T; \tag{6.19}$$

$X = \mathbb{P}^n_{\mathbb{C}}$:

$$A_2^{\varkappa} V_{\varkappa,0} = (A_2^{\varkappa} U_{\varkappa,0}) \times T, \tag{6.20}$$

$$A_2 A_{n+1} V_{2,1} = (A_2 A_{n+1} U_{2,1}) \times T, \tag{6.21}$$

$$A_{n+1} A_2^2 V_{3,1} = (A_{n+1} A_2^2 U_{3,1}) \times T; \tag{6.22}$$

$X = \mathbb{P}^n_{\mathbb{Q}}$:

$$A_1^{\varkappa} V_{\varkappa,0} = (A_1^{\varkappa} U_{\varkappa,0}) \times T, \tag{6.23}$$

$$(A_{3n+2} A_1 - A_2 A_{3n+1}) V_{2,1} = ((A_{3n+2} A_1 - A_2 A_{3n+1}) U_{2,1}) \times T, \tag{6.24}$$

$$(A_{3n+2} A_1^2 - A_2 A_{3n+1} A_1) V_{3,1} = ((A_{3n+2} A_1^2 - A_2 A_{3n+1} A_1) U_{3,1}) \times T; \tag{6.25}$$

$X = \mathbb{P}^2_{\mathbb{C}_a}$:

$$(A_1 + iA_2)^{\varkappa} V_{\varkappa,\mu} = ((A_1 + iA_2)^{\varkappa} U_{\varkappa,\mu}) \times T. \tag{6.26}$$

We set $v_{\varkappa,\mu}(\varrho) = V_{\varkappa,\mu}(x) \mid_{|x|=\varrho}$. With the help of (4.1)–(4.5), (6.12)–(6.16) and [14, formulas (57), (58), (61)], we can write (6.19)–(6.26) in the following form:

$X = \mathbb{S}^n$ or $X = \mathbb{P}^n_{\mathbb{R}}$:

$$\begin{aligned}(D(-\varkappa+1, -\mathcal{N}_X(\varkappa)) D(-\varkappa+2, 1-\mathcal{N}_X(\varkappa-1)) \cdots D(0,0) v_{\varkappa,0})(\varrho) \frac{(x_1+ix_2)^{\varkappa}}{\varrho^{\varkappa}} \\ = \left(\varphi(\varrho) \frac{(x_1+ix_2)^{\varkappa}}{\varrho^{\varkappa}}\right) \times T;\end{aligned}$$

$X = \mathbb{P}^n_{\mathbb{C}}$:

$$\begin{aligned}&(D(-\varkappa+1, -\varkappa+1) D(-\varkappa+2, -\varkappa+2) \cdots D(0,0) v_{\varkappa,0})(\varrho) \frac{\bar{z}_2^{\varkappa}}{\varrho^{\varkappa}} = \left(\varphi(\varrho) \frac{\bar{z}_2^{\varkappa}}{\varrho^{\varkappa}}\right) \times T, \\ &(D(-1,0) D(0,0) v_{2,1})(\varrho) \frac{z_1 \bar{z}_2}{\varrho^2} = \left(\varphi(\varrho) \frac{z_1 \bar{z}_2}{\varrho^2}\right) \times T, \\ &(D(-2,0) D(-1,-1) D(0,0) v_{3,1})(\varrho) \frac{z_1 \bar{z}_2^2}{\varrho^3} = \left(\varphi(\varrho) \frac{z_1 \bar{z}_2^2}{\varrho^3}\right) \times T;\end{aligned}$$

$X = \mathbb{P}^n_{\mathbb{Q}}$:

$$\begin{aligned}&(D(-\varkappa+1, -\varkappa+1) D(-\varkappa+2, -\varkappa+2) \cdots D(0,0) v_{\varkappa,0})(\varrho) P_{\varkappa,0}(\sigma) = (\varphi(\varrho) P_{\varkappa,0}(\sigma)) \times T, \\ &(D(-1,1) D(0,0) v_{2,1})(\varrho) P_{2,1}(\sigma) = (\varphi(\varrho) P_{2,1}(\sigma)) \times T, \\ &(D(-2,1) D(-1,-1) D(0,0) v_{3,1})(\varrho) P_{3,1}(\sigma) = (\varphi(\varrho) P_{3,1}(\sigma)) \times T,\end{aligned}$$

where $P_{k,m}(z) = \bar{z}_1^{k-2m} (\bar{z}_1 z_{n+2} - \bar{z}_2 z_{n+1})^m \in \mathcal{H}_X^{k,m}$;

$X = \mathbb{P}^2_{\mathbb{C}a}$:

$$(D(0,0)v_{1,0})(\varrho)R_{1,0}(\sigma) = (\varphi(\varrho)R_{1,0}(\sigma)) \times T,$$

$$\begin{aligned}
&(D(-1,1)D(0,0)v_{2,0})(\varrho)R_{2,0}(\sigma) + \frac{3}{4}(D(-1,3)D(0,0)v_{2,0})(\varrho)R_{2,1}(\sigma)\\
&\quad = \Big(\varphi(\varrho)R_{2,0}(\sigma) + \frac{3}{4}(D(-1,3)D(0,0)u_{2,0})(\varrho)R_{2,1}(\sigma)\Big) \times T,\\
&(D(-2,2)D(-1,-1)D(0,0)v_{3,0})(\varrho)R_{3,0}(\sigma) + (D(-2,3)D(-1,-1)D(0,0)v_{3,0})(\varrho)R_{3,1}(\sigma)\\
&\quad = \Big(\varphi(\varrho)R_{3,0}(\sigma) + (D(-2,3)D(-1,-1)D(0,0)u_{3,0})(\varrho)R_{3,1}(\sigma)\Big) \times T,\\
&(D(-1,1)D(0,0)v_{2,1})(\varrho)R_{2,0}(\sigma) + \frac{3}{4}(D(-1,3)D(0,0)v_{2,1})(\varrho)R_{2,1}(\sigma)\\
&\quad = \Big((D(-1,1)D(0,0)u_{2,1})(\varrho)R_{2,0}(\sigma) + \frac{3}{4}\varphi(\varrho)R_{2,1}(\sigma)\Big) \times T,\\
&(D(-2,2)D(-1,-1)D(0,0)v_{3,1})(\varrho)R_{3,0}(\sigma) + (D(-2,3)D(-1,-1)D(0,0)v_{3,1})(\varrho)R_{3,1}(\sigma)\\
&\quad = ((D(-2,2)D(-1,-1)D(0,0)u_{3,1})(\varrho)R_{3,0}(\sigma) + \varphi(\varrho)R_{3,1}(\sigma)) \times T,
\end{aligned}$$

where

$$\begin{aligned}
R_{k,m}(x) = &\sum_{l=0}^{[k/2]-m} \left(-\frac{1}{4}\right)^l (l+1)(l+2)\binom{k-2m-l+2}{l+2}(x_1+ix_2)^{k-2m-2l}\\
&\times (p_9(x) - p_{10}(x) + 2ip_1(x))^{m+l} \in \mathcal{H}_X^{k,m},
\end{aligned}$$

and $p_1(x)$, $p_9(x)$, and $p_{10}(x)$ are defined in §3. Since $\varphi(\varrho)Y(\sigma) \in \mathcal{D}'_T(B_R)$, these relations yield

$$D(-\varkappa+1, 1-\mathcal{N}_X(\varkappa))D(-\varkappa+2, 1-\mathcal{N}_X(\varkappa-1))\cdots D(0,0)v_{k,m} = 0$$

if $k = \varkappa$ and $m = 0$,

$$D(-1,\beta_X)D(0,0)v_{k,m} = 0$$

if $k = 2$ and $m = 1$, and

$$D(-2,\beta_X)D(-1,1)D(0,0)v_{k,m} = 0$$

if $k = 3$ and $m = 1$ (see Proposition 4.1 (i)). Consequently,

$$v_{k,m}(\varrho) = c_1, \quad k = 1, \ m = 0, \tag{6.27}$$

$$v_{k,m}(\varrho) = \frac{c_1}{(1+\varrho^2)^{\mathcal{N}_X(2)-1}} + c_2 \quad k = 2, \ m = 0, \tag{6.28}$$

$$v_{k,m}(\varrho) = \frac{c_1}{(1+\varrho^2)^{\mathcal{N}_X(3)-1}} + \frac{c_2}{(1+\varrho^2)^{\mathcal{N}_X(2)-1}} + c_3, \quad k = 3, \ m = 0, \tag{6.29}$$

$$v_{k,m}(\varrho) = \begin{cases} c_1(1+\varrho^2)^{\beta_X} + c_2 & \text{if } k = 2, \ m = 1, \ \beta_X \neq 0,\\ c_1 \ln(1+\varrho^2) + c_2 & \text{if } k = 2, \ m = 1, \ \beta_X = 0, \end{cases} \tag{6.30}$$

$$v_{k,m}(\varrho) = \begin{cases} c_1(1+\varrho^2)^{\beta_X} + \frac{c_2}{1+\varrho^2} + c_3 & \text{if } k = 3, \ m = 1, \ \beta_X \neq 0,\\ c_1 \ln(1+\varrho^2) + \frac{c_2}{1+\varrho^2} + c_3 & \text{if } k = 3, \ m = 1, \ \beta_X = 0, \end{cases} \tag{6.31}$$

where c_1, c_2, c_3 are complex constants. Next, we apply L^j, $j \in \{0,\dots,k-1\}$, to (6.17) with $\varkappa = k$ and $\mu = m$. Since $U_{k,m} \in C^k(B_R)$, $U_{k,m} = 0$ in $B_{r(T)}$, and $T \in \mathcal{M}^1_\natural(X)$, we have

$$(L^j V_{k,m})(0) = 0, \quad j \in \{0,\dots,k-1\}. \tag{6.32}$$

If $k = 1$, then (6.32) and (6.27) imply that $V_{1,0} = 0$. Suppose $k \geq 2$. Using the relations

$$L((1+\varrho^2)^{-c}) = -4c(\rho_X + c)(1+\varrho^2)^{-c} + 4c(\beta_X + c)(1+\varrho^2)^{-c+1}, \quad c \in \mathbb{R}^1,$$
$$L(\ln(1+\varrho^2)) = 4\rho_X - 4\beta_X(1+\varrho^2)$$

(see (6.3) and (4.7)), from (6.28)–(6.32) we conclude that

$$v_{k,m}(\varrho) = 0 \quad \text{for} \quad \varrho \in [0, \tan(R - r(T))).$$

Thus, (6.18) is established. By (6.18) and Lemma 6.1, $U_{k,m} = 0$ in B_R, but then $f = 0$ in B_R.

Finally, for hyperbolic spaces X the arguments are in perfect analogy with the compact case. Thus, Lemma 6.2 is proved. □

Corollary 6.1. *Suppose $T \in \mathcal{M}^1_\natural(X)$, $R > r(T) > 0$, $f \in C^s_T(B_R)$, and $f = 0$ in $B_{r(T)}$. Then $f^{k,m,j} = 0$ in B_R for $0 \leq k \leq s+3$, $0 \leq m \leq M_X(k)$, $1 \leq j \leq d^{k,m}_X$.*

Proof. Let $X \neq \mathbb{R}^n$. If $s = 0$, the required result is a consequence of Lemma 6.2 (see Proposition 4.1 (i) and (3.3)). Assuming that Corollary 6.1 is valid for all $0 \leq s \leq l-1$ for some $l \in \mathbb{N}$, we prove it for $s = l$. Let $f \in C^l_T(B_R)$ and $f = 0$ in $B_{r(T)}$. We take $k \in \{0, \dots, l+3\}$, $m \in \{0, \dots, M_X(k)\}$, and $j \in \{1, \dots, d^{k,m}_X\}$. By Proposition 4.1 (i) and relation (3.3), we have $f^{k,m,j} \in C^l_T(B_R)$ and $f^{k,m,j} = 0$ in $B_{r(T)}$. The inductive hypothesis yields $f^{k,m,j} = 0$ in B_R for $k \leq l+2$. Next, if $k = l+3$ and $m \leq M_X(l+2)$, then, by Proposition 4.1 (iii),

$$(D(l+3+2\alpha_X, \mathcal{N}_X(l+3) + \rho_X - 1 - m) f_{l+3,m,j})(\varrho) Y_1^{l+2,m}(\sigma) \in C^{l-1}_T(B_R).$$

Moreover,

$$(D(l+3+2\alpha_X, \mathcal{N}_X(l+3) + \rho_X - 1 - m) f_{l+3,m,j})(\varrho) = 0, \quad 0 \leq \varrho \leq \tan r(T).$$

Again with the help of the inductive hypothesis, we deduce that $f^{l+3,m,j} = 0$ in B_R. Similarly, Proposition 4.1 (iv) allows us to conclude that $f^{l+3,m,j} = 0$ in B_R for $M_X(l+2) < m \leq M_X(l+3)$, which completes the proof in the case where $X \neq \mathbb{R}^n$. For $X = \mathbb{R}^n$, Corollary 6.1 is proved in the same way with the help of Proposition 4.2. □

Proof of Theorem 3.1. The proofs of Lemma 6.2 and Corollary 6.1 show that Theorem 3.1 holds true for $\nu = 0, 1$. Suppose $\nu \in \mathbb{Z}$, $\nu < 0$. By the definition of $\mathcal{M}^\nu(X)$, we have $T = P(L)T_1$ for some $T_1 \in \mathcal{M}^0(X)$ and some polynomial P of degree not exceeding $[(1-\nu)/2]$. Moreover, if ν is odd, we may assume that $T_1 \in \mathcal{M}^1(X)$. Using (5.5), Proposition 5.5, and Remark 5.2, we see that $T_1 \in \mathcal{E}'_\natural(\mathcal{X})$ and $r(T_1) = r(T)$. Setting $F = P(L)(f^{k,m,j})$, we have $F \in \mathcal{D}'_{T_1}(B_R)$ and $F = 0$ in $B_{r(T_1)}$. Moreover, if $f \in f^s_{1,\mathrm{loc}}(B_R)$ ($f \in C^s(B_R)$), then $F \in W^{s-2[(1-\nu)/2]}_{1,\mathrm{loc}}(B_R)$ ($F \in C^{s-2[(1-\nu)/2]}(B_R)$, respectively). Taking into account the fact that $f^{k,m,j} = 0$ in $B_{r(T)}$ and using the above-proved results for $\nu = 0, 1$, we obtain Theorem 3.1 for $\nu < 0$.

Let $\nu > 0$. Then $L^{[\nu/2]}T \in \mathcal{M}^0(X)$. Moreover, if ν is odd, then we may assume that $L^{[\nu/2]}T \in \mathcal{M}^1(X)$. For any k, m, j, there exists $\Phi \in \mathcal{D}'_{k,m,j}(B_R)$ such that $L^{[\nu/2]}\Phi = f^{k,m,j}$ in B_R and $\Phi = 0$ in $B_{r(T)}$ (see [18, Chapter 3, proof of Theorem 11.2], [16, proof of Corollary 3.1.6], and also [14, formula (40)]). Moreover, if $f \in W^s_{1,\mathrm{loc}}(B_R)$ ($f \in C^s(B_R)$), then $\Phi \in W^{s+2[\nu/2]}_{1,\mathrm{loc}}(B_R)$ ($\Phi \in C^{s+2[\nu/2]}(B_R)$, respectively). Setting $T_2 = L^{[\nu/2]}T$, we see that $r(T_2) = r(T)$ and $\Phi \in \mathcal{D}'_{T_2}(B_R)$. As above, we obtain Theorem 3.1 for $\nu > 0$. □

Proof of Corollary 3.1. Statement (i) is easily derived from (ii) with the help of standard smoothing. To obtain (ii), it suffices to use Theorem 3.1 (ii). We prove (iii). Without loss of generality, we assume that the distribution f in the assumptions of Corollary 3.1

belongs to $\mathcal{D}'_{k,m,j}(B_R)$ (see (3.5) and (3.6)). Suppose $R_1 \in (r(T), R)$, $f_1 \in \mathcal{E}'_{k,m,j}(\mathcal{X})$, and $f_1 = f$ in B_{R_1}. Proposition 5.5 and its analog for noncompact X (see Remark 5.2) imply that there exists a function $F \in (C \cap \mathcal{E}'_{k,m,j})(\mathcal{X})$ such that $f_1 = P(L)F$ for an appropriate polynomial P. Then

$$F \times P(L)T = P(L)F \times T = f_1 \times T = 0$$

in $B_{R_1 - r(T)}$, whence $F = 0$ in B_{R_1} by Theorem 3.1 (ii). Now, since $R_1 \in (r(T), R)$ is arbitrary, we conclude that $f = 0$ in B_R. □

To prove Theorem 3.2, we need the following lemma.

Lemma 6.3. *Let w be an entire function satisfying the following conditions*:

(1) $\mathcal{Z}(w) \neq \varnothing$;

(2) *there exists a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers and two constants $R \in (0, \operatorname{diam} X)$ and $\gamma > 0$ such that*

$$|w(z)| > \frac{e^{R|\operatorname{Im} z|}}{|z|^{\gamma}} \quad \text{on the circles} \quad |z| = r_n. \tag{6.33}$$

Next, suppose that $g \in C(B_R \times \mathbb{C})$ is an entire function in the second variable for each fixed value of the first variable, and that for any $\psi \in \mathcal{D}(B_R)$ and any $N > 0$ the following inequality is valid:

$$\left| \int_{B_R} g(x,z)\psi(x)\, d\mu(x) \right| < c(2 + |z|)^{-N} e^{r_0(\psi)|\operatorname{Im} z|}, \quad z \in \mathbb{C}, \tag{6.34}$$

where $c > 0$ does not depend on z. Then if the series

$$\sum_{\lambda \in \mathcal{Z}(w)} \operatorname*{res}_{z=\lambda} \frac{g(x,z)}{w(z)} \tag{6.35}$$

converges in $\mathcal{D}'(B_R)$ to a distribution f, we have $f = 0$.

Proof. For $\psi \in \mathcal{D}(B_R)$ and $z \in \mathbb{C}$, we put

$$I_\psi(z) = \int_{B_R} g(x,z)\psi(x)\, d\mu(x), \quad S_{n,\psi} = \sum_{\substack{\lambda \in \mathcal{Z}(w) \\ |\lambda| < r_n}} \operatorname*{res}_{z=\lambda} \frac{I_\psi(z)}{w(z)}.$$

Then

$$S_{n,\psi} = \frac{1}{2\pi i} \int_{|z|=r_n} \frac{I_\psi(z)}{w(z)}\, dz.$$

With the help of this relation, from estimates (6.33) and (6.34) we deduce that $S_{n,\psi} \to 0$ as $n \to \infty$. Since ψ is arbitrary, this implies that $f = 0$ in B_R. □

Proof of Theorem 3.2. We set $w(z) = \widetilde{T}(z)$, $g(x,z) = z\Phi_{z,0,k,m,j}(x)$. We verify that the functions w and g satisfy the assumptions of Lemma 6.3. Since $r(T) > 0$, the set $\mathcal{Z}(w)$ is infinite. Estimate (6.33) with $\gamma = -c_2$, $R = r(T)$ and estimate (6.34) for any $R \in (0, \operatorname{diam} X)$ follow from (3.8), (4.11), and (4.12), by the symmetry of the operator L. We study the convergence of the series (6.35). Let

$$h_\lambda(x) = \operatorname*{res}_{z=\lambda} \frac{g(x,z)}{w(z)}, \quad n_\lambda = n_\lambda(w), \quad \lambda \in \mathcal{Z}(w).$$

The formula for computing the residue at a pole yields the following:

$$h_\lambda(x) = \sum_{\eta=0}^{n_\lambda - 1} \frac{1}{\eta!(n_\lambda - 1 - \eta)!} \left(\frac{z\,(z-\lambda)^{n_\lambda}}{w(z)} \right)^{(n_\lambda - 1 - \eta)} \Bigg|_{z=\lambda} \Phi_{\lambda,\eta,k,m,j}(x) \tag{6.36}$$

if $\lambda \neq 0$, and

$$h_0(x) = \sum_{\eta=0}^{\frac{n_0}{2}-1} \frac{1}{(2\eta)!(n_0-1-2\eta)!} \left(\frac{z^{n_0+1}}{w(z)}\right)^{(n_0-1-2\eta)} \Bigg|_{z=0} \Phi_{0,\eta,k,m,j}(x) \tag{6.37}$$

if $\lambda = 0 \in \mathcal{Z}(w)$. We enumerate the set $\{z \in \mathcal{Z}(w) : \operatorname{Re} z \geq 0,\, iz \notin (0,+\infty)\}$ in the order of increasing absolute values of its elements (if two absolute values are equal, the enumeration is arbitrary). We denote the resulting sequence by $\{z_n\}_{n=1}^{\infty}$. By (3.8), we have

$$c_1 z_n^{c_2} \cos(rz_n + c_3) = O(|z_n|^{c_2-1} e^{r|\operatorname{Im} z|}), \quad n \to \infty. \tag{6.38}$$

Estimate (6.38) shows that the sequence $\{\operatorname{Im} z_n\}_{n=1}^{\infty}$ is bounded, so that

$$\cos(rz_n + c_3) = O\left(\frac{1}{z_n}\right), \quad n \to \infty.$$

Then

$$rz_n + c_3 = \frac{\pi}{2} + \pi\zeta_n + \varepsilon_n \tag{6.39}$$

for some $\zeta_n \in \mathbb{Z}$ and $\varepsilon_n = O(1/z_n)$. Using formulas (6.39), (3.8) and the Rouché theorem, we can see that the sequence $\{z_n\}_{n=1}^{\infty}$ possesses the following properties: i) for all sufficiently large n, the zeros of z_n are simple; (ii) as $n \to \infty$, we have

$$z_n = \frac{\pi}{r}\left(n + l + \frac{1 - 2c_3/\pi}{2}\right) + O\left(\frac{1}{n}\right), \tag{6.40}$$

where the constant $l \in \mathbb{Z}$ does not depend on n. Next, let $\delta \in \left(0, \frac{\pi}{2}\right)$ be fixed. Relation (3.8) and the Cauchy integral formula imply that

$$\widetilde{T}'(z) = -c_1 r z^{c_2} \sin(rz + c_3) + O(|z|^{c_2-1} e^{r|\operatorname{Im} z|}) \quad \text{as} \quad z \to \infty, \quad |\arg z| \leq \delta. \tag{6.41}$$

Combining (6.40) and (6.41), we obtain

$$\widetilde{T}'(z_n) = (-1)^{n+l+1} c_1 \pi^{c_2} r^{1-c_2} n^{c_2} + O(n^{c_2-1}), \quad n \to \infty. \tag{6.42}$$

Using property (i) of the zeros z_n, relations (6.42), (6.34), and (6.36), and the fact that the function $g(x,z)/w'(z)$ is even, we see that the series (6.35) converges in $\mathcal{D}'(\mathcal{X})$ to a certain distribution f. Thus, we can apply Lemma 6.3 and conclude that $f = 0$ in $B_{r(T)}$. Moreover, $f \in \mathcal{D}'_T(\mathcal{X})$, by (6.36), (6.37), and Corollary 5.1. Next, on compact sets in $\mathcal{X}$ not containing zero, the function $\Phi_{z_n,0,k,m,j}(x)$ has asymptotic expansions uniform in x as $n \to \infty$, which are similar to the asymptotics of the Bessel functions (see [13, Lemma 8], [14, Corollary 2]). Therefore, f satisfies the smoothness conditions required in Theorem 3.2 (see (6.42), (6.40) and [23, §7.3.5 (ii)]). It remains to prove that f is nonzero. Let μ be a simple zero of w. Using Proposition 5.5 and Remark 5.2, we define $T_\mu \in \mathcal{E}'_\natural(\mathcal{X})$ by the relation

$$\widetilde{T}_\mu(z) = \frac{\widetilde{T}(z)}{z^2 - \mu^2}, \quad z \in \mathbb{C}.$$

Then, by Proposition 5.1, for any $\lambda \in \mathcal{Z}(w)$, $\lambda \neq \pm\mu$, we have $h_\lambda \times T_\mu(x) = 0$, $x \in B_{R_1}$, where $R_1 = \operatorname{diam} X - r(T)$. Similarly,

$$h_{\pm\mu} \times T_\mu(x) = \frac{1}{2}\Phi_{\mu,0,k,m,j}(x), \quad x \in B_{R_1}.$$

Hence, $f \times T_\mu = \Phi_{\mu,0,k,m,j}$ in B_{R_1}; in particular, $f \neq 0$. $\square$

Proof of Theorem 3.3. Suppose $T_1 \in \mathcal{D}_\natural(\mathcal{X})$, $r(T_1) > 0$. Let $T_2 \in \mathcal{E}'_\natural(\mathcal{X})$ be a distribution such that $T_2 \in \mathcal{M}^\nu(X)$, $T_2 \notin \mathcal{M}^{\nu+1}(X)$, and $r(T_2) < r(T_1)$. We check that $T = T_1 + T_2$ is the required distribution. The first assumption on T in Theorem 3.3 follows from the definition. Next, $R > r(T)$, $f \in \mathcal{D}'_T(B_R)$, and $f = 0$ in $B_{r(T)}$. Then $f \times T_1 = 0$ in $B_{R_1 - r(T)}$ if $r(T) < R_1 < 2r(T) - r(T_2)$ and $R_1 \le R$. Applying Corollary 3.1 (i), (iii), we conclude that $f = 0$ in B_R. Thus, T satisfies the second assumption of Theorem 3.3, which completes the proof. □

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