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ON THE ZEROS OF ENTIRE FUNCTIONS WITH A MAJORANT OF INFINITE ORDER

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To our Teacher Viktor Petrovich Havin on the occasion of his 75th birthday

Abstract. The class of entire functions f satisfying $\ln|f(z)| \le C_f\lambda(|z|)$, $z \in \mathbb{C}$, where λ is a majorant mentioned in the title, admits a zero set $\{z_j\}$ such that $\{|z_j|\}$ is a uniqueness set for this class.

Introduction

Let $\mathbb{C}$ denote the complex plane, let $H(\mathbb{C})$ be the set of all entire functions, and let λ be a monotone increasing, positive function on $\mathbb{R}_+ = (0;+\infty)$; we put $\varphi(x) = \ln\lambda(x)$, $x \in \mathbb{R}_+$.

We introduce the following function classes:

$$H(\lambda,+\infty) = \{f \in H(\mathbb{C}) : \ln|f(z)| \le C_f \cdot \lambda(|z|),\ z \in \mathbb{C}\}, \tag{1}$$

$$\widetilde{H}(\lambda,+\infty) = \{f \in H(\mathbb{C}) : \ln|f(z)| \le A_f \cdot \lambda(B_f \cdot |z|),\ z \in \mathbb{C}\}. \tag{2}$$

Here and in what follows, A_f, B_f, C_f are arbitrary positive numbers depending on the function f only. If $\lambda \in C^1(\mathbb{R}_+)$ and $\alpha_\lambda = \lim_{x\to+\infty} x\frac{\lambda'(x)}{\lambda(x)} < +\infty$, then it is not hard to show that the classes $H(\lambda,+\infty)$ and $\widetilde{H}(\lambda,+\infty)$ coincide. However, this is no longer true if $\alpha_\lambda = +\infty$, for example, in the case where

$$\lambda(t) = \underbrace{\exp\exp\cdots\exp}_{n}(t^\rho), \qquad t \in \mathbb{R}_+, \quad \rho \in \mathbb{R}_+.$$

If $\lambda(t) = t^\rho$, $t \in \mathbb{R}_+$, $\rho \in \mathbb{R}_+$, then $H(\lambda,+\infty)$ coincides with the class of functions of finite order and normal type; we denote this class by $H(\rho,+\infty)$.

For $f \in H(\mathbb{C})$, we denote $Z_f \overset{\text{def}}{=} \{z \in \mathbb{C} : f(z) = 0\}$. The following property of the zeros of functions of class $H(\rho,+\infty)$ is well known: a sequence $\{z_k\}_{k=1}^{+\infty}$ can be represented as $Z = Z_f$ with $f \in H(\rho,+\infty)$, $f \not\equiv 0$, $\rho \notin N$, $\rho > 0$, if and only if

$$n(r) = \operatorname{card}\{z_k : |z_k| \le r\} \le C_f \cdot r^\rho. \tag{3}$$

For $\rho \in \mathbb{N}$, besides (3), the following Lindelöf condition arises (see [1]): there exists $M > 0$ such that

$$\Bigl|\sum_{|z_k|\le r} \frac{1}{z_k^\rho}\Bigr| \le M, \quad r \in \mathbb{R}_+. \tag{4}$$

By using (4), it is not difficult to construct a sequence $\{z_k\}_{k=1}^{+\infty} = Z_f$, $f \not\equiv 0$, $f \in H(\rho,+\infty)$, $\rho \in \mathbb{N}$, such that any $g \in H(\rho,+\infty)$ with $Z_g = \widetilde{Z}_f$, where $\widetilde{Z}_f = \{|z_k|\}_{k=1}^{+\infty}$,

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satisfies $g(z) = 0$, $z \in \mathbb{C}$; we mention $\left\{k^{\frac{1}{\rho}} e^{\frac{i\pi k}{\rho}}\right\}_{k=1}^{+\infty}$ as an example of such a sequence. In other words, the set $\widetilde{Z}_f$ cannot coincide with Z_g for any $g \in H(\rho, +\infty)$, $\rho \in \mathbb{N}$, $g(z) \not\equiv 0$. Thus, the distribution of the arguments of $\{z_k\}_{k=1}^{+\infty}$ is also of importance for the representation $Z = Z_g$, rather then only the growth of the function $n(r)$.

Definition. We say that a set X of entire functions satisfies the *Lindelöf condition* if there exists $f \in X$, $f \not\equiv 0$, $Z_f = \{z_k\}_{k=1}^{+\infty}$, such that if $g \in X$, $Z_g \supseteq \widetilde{Z}_f$, where $\widetilde{Z}_f = \{|z_k|\}_{k=1}^{+\infty}$, then $g(z) = 0$ for all $z \in \mathbb{C}$.

The above remarks imply that the class $H(\rho, +\infty)$ with $\rho \in \mathbb{N}$ satisfies the Lindelöf condition, while for $\rho \notin N$ it does not. The following question arises naturally: what happens for the other λ, e.g., for $\lambda(t) = \underbrace{\exp \cdots \exp}_{n}(t^\rho)$, $t, \rho \in \mathbb{R}_+$?

In this paper we show that if $\alpha_\lambda = +\infty$, then the class $H(\lambda, +\infty)$ satisfies the Lindelöf condition and the class $\widetilde{H}(\lambda, +\infty)$ does not. In the case where $\alpha_\lambda = +\infty$ we obtain a description of the zero sets of functions in $\widetilde{H}(\lambda, +\infty)$. This description is in terms of $\{|z_k|\}_{k=1}^{+\infty}$, i.e., in terms of $n(r)$. At the same time, Theorem 1 shows that the zero sets of functions of class $H(\lambda, +\infty)$ fail to admit a similar description; here the distribution of the arguments of the zeros becomes important.

There are many publications devoted to properties of the root sets for functions in $\widetilde{H}(\lambda, +\infty)$; we mention [2, 3] and the references therein. The method applied in those papers is based on Fourier series, and the results do not even characterize the functions λ for which the zero sets can be described in terms of the counting function $n(r)$ only.

§1. Formulation of the main results and proof of auxiliary statements

In what follows, a monotone increasing positive function λ of class $C^{(1)}(\mathbb{R}_+)$ is called a *weight function* if $\alpha_\lambda = +\infty$.

Theorem 1. *Let λ be a weight function of class $C^{(2)}(\mathbb{R}_+)$. Suppose that the function $\psi(x) = \ln \lambda(e^x)$ is convex on $\mathbb{R}_+$ and that*

$$\frac{\psi''(x)}{(\psi'(x))^{2-\delta}} = O(1), \quad x \to +\infty, \tag{5}$$

for some $0 < \delta < 1$. Then the class $H(\lambda, +\infty)$ satisfies the Lindelöf condition.

Theorem 2. *Let λ be a weight function of class $C^{(1)}(\mathbb{R}_+)$. Suppose that the function $\varphi(x) = \ln \lambda(x)$ is convex on $\mathbb{R}_+$. The following statements are equivalent*:

1) *a sequence of complex numbers $\{z_k\}_{k=1}^{+\infty}$ is representable in the form Z_f for some $f \in \widetilde{H}(\lambda, +\infty)$, $f \not\equiv 0$;*

2) *there exists a positive number C such that*

$$\sum_{k=1}^{+\infty} \exp(-\varphi(C|z_k|)) < +\infty.$$

In Theorem 2 it is assumed that $\lambda \in C^{(1)}(\mathbb{R}_+)$. However, the claim remains valid under the assumption that $\lambda \in C(\mathbb{R}_+)$. Indeed, it is easily seen that if $\lambda \in C(\mathbb{R}_+)$ and $\lambda_*(r) = \int_r^{er} \frac{\lambda(t)}{t}\, dt$, then $\lambda(r) \le \lambda_*(r) \le \lambda(er)$, while $\widetilde{H}(\lambda, +\infty) = \widetilde{H}(\lambda_*, +\infty)$.

The proof of the above theorems is based on several auxiliary statements.

Lemma 1 (see [4])**.** *Let z_n be a sequence of complex numbers converging to infinity and satisfying*

$$\sum_{n=1}^{+\infty} \frac{1}{|z_n|^{p_n+1}} < +\infty.$$

Then the Weierstrass product

$$B(z,z_n,p_n)=\prod_{n=1}^{+\infty}\Big(1-\frac{z}{z_n}\Big)\cdot\exp\bigg(\frac{z}{z_n}+\frac{1}{2}\Big(\frac{z}{z_n}\Big)^2+\dots+\frac{1}{p_n}\Big(\frac{z}{z_n}\Big)^{p_n}\bigg)$$

converges in $\mathbb{C}$ *and obeys the estimate*

$$\ln|B(z,z_n,p_n)|\le\sum_{n=1}^{+\infty}\Big|\frac{z}{z_n}\Big|^{p_n+1},\qquad z\in\mathbb{C}.$$

Lemma 2 (see [5])**.** *Suppose* $\lambda\in C^{(1)}(\mathbb{R}_+)$, $\lambda(x)>0$, $x\in\mathbb{R}_+$, *and* $x\frac{\lambda'(x)}{\lambda(x)}\nearrow+\infty$ *as* $x\to+\infty$. *Then there exists a function* $f\in H(\lambda,+\infty)$ *with the following properties*:

$$\text{a)}\lim_{r\to+\infty}\frac{\ln M(r,f)}{\lambda(r)}=1,\quad \text{b)}\lim_{r\to+\infty}\frac{T(r,f)}{\lambda(r)}=1,\quad \text{c)}\lim_{r\to+\infty}\frac{N(r,f)}{\lambda(r)}=1,$$

where, as usual, $M(r,f)=\max_{|z|\le r}|f(z)|$,

$$T(r,f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln^+|f(re^{i\varphi})|\,d\varphi,$$

and $N(r,f)$ *is the Nevanlinna counting function* (*see* [1]):

$$N(r,f)=\int_0^r\frac{n(t)-n(0)}{t}\,dt,$$

$n(0)$ *being the order of the zero of* f *at* $z=0$.

In the next lemma we estimate the number of real zeros of a function of class $H(\lambda,+\infty)$ on the segment $[0;r]$; this lemma will play an important role in the proof of Theorem 1.

Lemma 3. *Let* λ *be as in Theorem* 1. *If* $g\in H(\lambda,+\infty)$ *and* $n_R(r,g)$ *is the number of positive* (*real*) *zeros of* g *on the segment* $[0;r]$, *then* $n_R(g,r)\le C_g\cdot\lambda(r)$, $C_g>0$, $r\in\mathbb{R}_+$.

Proof. Let $\Omega=\{t+i\tau\,:\,|\tau|<s(t)\}$, $t\in\mathbb{R}_+$, where $s(t)=\frac{\pi}{\psi'(t)}$ and $\psi(t)=\ln\lambda(e^t)$; there is no loss of generality in assuming that $\psi'(0)>3$. Obviously, if $f\in H(\lambda,+\infty)$, then $\ln|f(e^z)|\le C_f\cdot\lambda(e^{\operatorname{Re}z})$, $z\in\mathbb{C}$.

Let $s(t)=\frac{\lambda(e^t)}{\pi\cdot\lambda'(e^t)\cdot e^t}$, $t\in\mathbb{R}_+$; we put $\Omega_+=\{t+i\tau\,:\,t>0,0<\tau<s(t)\}$ and $\Pi_+=\{u+iv\,:\,0<v<\frac{\pi}{2}\}$. Let $\mathrm{Z}_+(\zeta)=\mathrm{Z}(t+i\tau)$ be a conformal mapping onto the strip Π_+ such that $Z_+(iS(0))=\frac{i\pi}{2}$, $Z_+(0)=0$, $\lim_{t\to+\infty}Z_+(t)=+\infty$. By the Riemann–Schwarz principle, the function $\mathrm{Z}_+(\zeta)$ admits an analytic continuation Z to the domain

$$\Omega=\big\{\zeta=t+i\tau\,:\,|\tau|<s(|t|),t\in\mathbb{R}\big\},$$

Z maps Ω onto the strip $\Pi=\big\{u+iv\,:\,-\frac{\pi}{2}<v<\frac{\pi}{2}\big\}$, and the function $\mathrm{Z}_+(t)=\mathrm{Z}(t)$, $t\in\mathbb{R}_+$, is strictly monotone increasing on $[0;+\infty)$.

By the Warschawski theorem (the assumptions of it are fulfilled by (5)), we have

$$\begin{aligned}u=\operatorname{Re}\mathrm{Z}(\sigma+i\tau)&=\frac{\pi}{2}\cdot\int_0^\sigma\frac{dt}{s(t)}+O(1)=\frac{\pi}{2}\cdot\frac{1}{\pi}\int_0^\sigma\frac{\lambda'(e^t)\cdot e^t}{\lambda(e^t)}\,dt+O(1)\\&=\frac{1}{2}\cdot\int_0^\sigma d\ln\lambda(e^t)+O(1)=\frac{1}{2}\cdot\ln\lambda(e^\sigma)+O(1),\quad\sigma\to+\infty.\end{aligned}$$

Consequently,

$$\ln\lambda(e^\sigma)=2\cdot\operatorname{Re}\mathrm{Z}(\sigma+i\tau)+O(1),\tag{6}$$

where $u+iv=\mathrm{Z}(\sigma+i\tau)\in\Pi$.

We map the right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ conformally onto Π. Then, if $f \in H(\lambda, +\infty)$, then the function $F(z) = f\big(e^{\mathrm{Z}^{-1}(\ln z)}\big)$ is holomorphic in $\mathbb{C}_+$, and

$$\ln|F(z)| = \ln|f(e^{\mathrm{Z}^{-1}(\ln z)})| \le C_f \cdot \lambda(e^{\operatorname{Re}\mathrm{Z}^{-1}(\ln z)}), \tag{7}$$

where Z^{-1} is the function inverse to Z.

Using (6), we see that if $\sigma + i\tau = \mathrm{Z}^{-1}(\ln z)$, $\mathrm{Z}(\sigma + i\tau) = \ln z = \ln|z| + i \cdot \arg z$, where $|\arg z| < \frac{\pi}{2}$, then $\mathrm{Z}^{-1}(\ln z) = \mathrm{Z}^{-1}(\ln|z| + i \cdot \arg z) = \sigma + i\tau \in \Omega$. Therefore, $\ln|z| = \operatorname{Re}\mathrm{Z}(\mathrm{Z}^{-1}(\ln z))$, but, by (6),

$$\frac{1}{2} \cdot \ln\lambda(e^{\sigma}) = \operatorname{Re}\mathrm{Z}(\sigma + i\tau) + O(1) = \ln|z| + O(1) \quad (\sigma \to +\infty).$$

Thus, using (7), finally we obtain

$$\ln|F(z)| = \ln|f(e^{\mathrm{Z}^{-1}(\ln z)})| \le C_f \cdot \lambda(e^{\operatorname{Re}\mathrm{Z}^{-1}(\ln z)}) \le \widetilde{C_f} \cdot \lambda(e^{\sigma}) = C_f \cdot |z|^2.$$

Consequently, $\ln|F(z)| \le C_f \cdot |z|^2$, $z \in \mathbb{C}_+$, and the function $F(z)$ is holomorphic in $\mathbb{C}_+$.

The results pertaining to the zeros of such functions (see [8]) show that the number $\widetilde{n}_R(t)$ of real zeros of F on the segment $[1; t]$, $t > 1$, satisfies

$$\widetilde{n}_R(t) \le A_f \cdot t^2. \tag{8}$$

On the other hand, if $F(\rho_k) = 0$, $k = 1, 2, \dots$, then $f\big(e^{\mathrm{Z}^{-1}(\ln \rho_k)}\big) = 0$, $k = 1, 2, \dots$. This means that $e^{\mathrm{Z}^{-1}(\ln \rho_k)} = r_k$, where the r_k, $k = 1, 2, \dots$, are real zeros of f. Since $\mathrm{Z}^{-1}(\ln \rho_k) = \ln r_k$, we have

$$\ln \rho_k = \mathrm{Z}(\ln r_k). \tag{9}$$

Therefore, $\widetilde{n}_R(t) = \operatorname{card}\{\rho_k : \rho_k \le t\} \le C_f \cdot t^2$. Since Z^{-1} grows monotonically on $\mathbb{R}_+$, estimate (8) implies that

$$n_R(t) = \operatorname{card}\{\rho_k : \mathrm{Z}^{-1}(\ln \rho_k) \le \mathrm{Z}^{-1}(\ln t)\} \le A_f \cdot t^2.$$

Now, by (9), we have

$$\widetilde{n}_R(t) = \operatorname{card}\{r_k : \ln r_k \le \mathrm{Z}^{-1}(\ln t)\} \le C_f \cdot t^2$$

or

$$\widetilde{n}_R(t) = \operatorname{card}\{r_k : r_k \le e^{\mathrm{Z}^{-1}(\ln t)}\} \le A_f \cdot t^2.$$

Now, we put $Z^{-1}(\ln t) = \rho$, $t \in \mathbb{R}_+$; then $t^2 = 2\mathrm{Z}(\rho)$. Therefore,

$$\widetilde{n}_R(\rho) = \operatorname{card}\{r_k : r_k \le e^{\rho}\} \le A_f \cdot e^{2\mathrm{Z}(\rho)}, \quad \rho \in \mathbb{R}_+.$$

Recalling (6), we see that the last estimate implies that

$$\widetilde{n}_R(\rho) = \operatorname{card}\{r_k : r_k \le e^{\rho}\} \le A_f \cdot \lambda(e^{\rho}) \cdot e^{O(1)} = C_f \cdot \lambda(e^{\rho}).$$

Putting $e^{\rho} = r$, $r > 0$, finally we get

$$n_R(r) \le C_f \cdot \lambda(r),$$

which proves the lemma. □

Lemma 4. *If $\varphi \in C^1(R_+)$ is convex on $\mathbb{R}_+$ and $\varphi(0) = 0$, then*

$$C\varphi(x) \le \varphi(Cx)$$

for any number $C \ge 1$.

Proof. Obviously, $\varphi(x)=\int_0^x \varphi'(u)\,du+\varphi(0)=\int_0^x \varphi'(u)\,du$, so that $C\varphi(x) = C\int_0^x \varphi'(u)\,du$. Changing the variables $Cu = t$, we obtain

$$C\varphi(x) = \int_0^{Cx} \varphi'\Big(\frac{t}{C}\Big)\,dt, \quad C \geq 1.$$

Since $\varphi'(x)$ is a monotone increasing function, we have $\varphi'\big(\frac{t}{C}\big) \leq \varphi'(t)$ for all t; consequently, $\int_0^{Cx} \varphi'\big(\frac{t}{C}\big)\,dt \leq \int_0^{Cx} \varphi'(t)\,dt$, i.e., $C\varphi(x) \leq \varphi(Cx)$. □

Lemma 5 (see [9])**.** *Let $\varphi(x)$ be a positive measurable function bounded on every segment $[a;b] \subset \mathbb{R}_+$. For each $K > 1$, there exists a sequence $x_n \to +\infty$ such that for all x with $x_n \leq x \leq x_n + \frac{1}{[\ln^+ \varphi(x_n)]^K} + \frac{1}{\varphi(x_n)}$ we have*

$$\varphi(x) \leq K \cdot \varphi(x_n).$$

Lemma 6. *Let $Z_f = \{z_k\}_{k=1}^{+\infty}$, $f \in \widetilde{H}(\lambda, +\infty)$, $f \not\equiv 0$. Then $n(r) \leq \frac{A_f}{\ln 2} \cdot \lambda(2B_f \cdot r)$, where A_f and B_f are as in* (2).

Proof. Suppose $f \in \widetilde{H}(\lambda, +\infty)$, $f(z_k) = 0$, $k = 1, 2, \ldots$, $f \not\equiv 0$. Without loss of generality, we may assume that $f(0) = 1$. The Jensen identity yields

$$\begin{aligned}\int_0^r \frac{n(t)}{t}\,dt &= \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\varphi})|\,d\varphi \leq \frac{1}{2\pi}\int_0^{2\pi} A_f\lambda(B_f r)\,d\varphi \\ &= \frac{1}{2\pi}\cdot 2\pi \cdot A_f\lambda(B_f r) = A_f \cdot \lambda(B_f r),\end{aligned}$$

where $0 < r < +\infty$. Since $n(r)$ grows monotonically, we deduce that

$$n(r)\cdot \ln 2 \leq \int_r^{2r} \frac{n(t)}{t}\,dt \leq A_f \cdot \lambda(2B_f \cdot r),$$

whence $n(r) \leq \frac{A_f}{\ln 2} \cdot \lambda(2B_f \cdot r)$, as required. □

§2. Proof of the main results

First, we prove Theorem 1.

Let f be the function of class $H(\lambda, +\infty)$ constructed in Lemma 2, let $\{z_k\}_{k=1}^{+\infty}$ be the set of zeros of f, let $n(r) = \{\operatorname{card} z_k \,:\, |z_k| < r\}$, and let

$$N(r) = \int_0^r \frac{n(t)}{t}\,dt.$$

Without loss of generality, we assume that $f(0) = 1$. By Lemma 2, we have $N(r) \geq C_0 \cdot \lambda(r)$, $r \geq 1$, for some $C_0 > 0$. Suppose that $g \in H(\lambda, +\infty)$, $g(|z_k|) = 0$, $k = 1, 2, \ldots$, and $g(z) \not\equiv 0$. Then, by Lemma 3, we have $n_R(r) = n(r)$, where n_R is the set of all positive roots of g on the segment $[0; r]$, and $n_R(r) = n(r) \leq C_1 \cdot \lambda(r)$, where $C_1 > 0$, $r \in \mathbb{R}_+$. We put $\varphi(r) = \frac{N'_+(r)}{N(r)}$, $r \in \mathbb{R}_+$, where $N'_+(r)$ is the right derivative of the function N. Let $\{x_n\}_{n=1}^{+\infty}$ be the sequence built in Lemma 5 starting with φ. Applying that lemma and the identity

$$\ln\frac{N(x'_n)}{N(x_n)} = \int_{x_n}^{x'_n} \frac{N'_+(t)}{N(t)}\,dt,$$

where $x'_n = x_n + \frac{1}{\varphi(x_n)}$, we obtain the inequality

$$\ln\frac{N(x'_n)}{N(x_n)} \leq K \cdot \varphi(x_n)\cdot(x'_n - x_n) = K \cdot \varphi(x_n)\cdot\frac{1}{\varphi(x_n)} = K,$$

i.e., $N(x_n') \le e^K \cdot N(x_n)$. Now, since λ is convex, we get

$$\begin{aligned}\lambda'(x_n) &\le \frac{1}{x_n' - x_n} \cdot \int_{x_n}^{x_n'} \lambda'(t)\,dt \le \frac{1}{x_n' - x_n} \cdot [\lambda(x_n') - \lambda(x_n)] \\ &\le \frac{\lambda(x_n')}{x_n' - x_n} \le C \cdot \varphi(x_n) \cdot N(x_n').\end{aligned}$$

Then $\lambda'(x_n) \le C \cdot e^K \cdot N(x_n) \cdot \varphi(x_n)$, i.e., $\lambda'(x_n) \le C \cdot e^K \cdot N(x_n) \cdot \frac{N_+'(x_n)}{N(x_n)}$, whence $\lambda'(x_n) \le C \cdot e^K \cdot N_+'(x_n)$ (note that $x_n \to +\infty$ as $n \to +\infty$).

Therefore, $\lambda'(x_n) \le C \cdot e^K \cdot \frac{n(x_n)}{x_n} \le C \cdot e^K \cdot \frac{\lambda(x_n)}{x_n}$, so that $\frac{\lambda'(x_n)}{\lambda(x_n)} \cdot x_n \le C \cdot e^K < +\infty$, which contradicts the assumptions of the theorem. This proves Theorem 1.

Now we start the proof of Theorem 2. We check the implication (1) $\Rightarrow$ (2).

Suppose $f \in \widetilde{H}(\lambda, +\infty)$, $f(z_k) = 0$, $k = 1, 2, \dots$, $f \not\equiv 0$. Without loss of generality, we may assume that $\lambda(0) = 1$, i.e., $\varphi(0) = 0$. Lemma 1 shows that if $Z = Z_f$, $Z = \{z_k\}_{k=1}^{+\infty}$ (we assume that $0 < |z_1| \le |z_2| \le \dots \le |z_k| \le \dots$), then $n(r) \le A \cdot \lambda(B \cdot r)$, i.e.,

$$n(r) \le A \cdot \exp \varphi(B \cdot r), \quad r \in \mathbb{R}_+, \ A \in \mathbb{R}_+, \ B \in \mathbb{R}_+. \tag{10}$$

We fix $C > 0$. For $R > 0$ we have

$$I_R = \sum_{|z_k| \le R} \exp(-\varphi(C|z_k|)) = \int_0^R \exp(-\varphi(Ct))\,dn(t).$$

Integration by parts yields the formula

$$\begin{aligned} I_R &= n(t) \cdot \exp(-\varphi(Ct))\Big|_0^R + \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct)\,dt \\ &= n(R) \cdot \exp(-\varphi(CR)) + C \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct)\,dt. \end{aligned} \tag{11}$$

Using (10) and choosing $C > B_f$, we see that $I_R \le A_f \cdot \exp(\varphi(B_f r) - \varphi(Cr)) + C \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct)\,dt$. Since $\varphi(t)$ is a monotone increasing function on $\mathbb{R}_+$, we have

$$I_R \le A_f + C \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct)\,dt \le A_f + C \int_0^R \exp(\varphi(B_f t) - \varphi(Ct)) \cdot \varphi'(Ct).$$

We take C so large that $\varphi(B_f x) \le \frac{\varphi(Cx)}{2}$, $x \in \mathbb{R}_+$; then

$$\begin{aligned} I_R &\le A_f + A_0 \cdot \int_0^R \exp\Big(\frac{-\varphi(Cx)}{2}\Big) \cdot \varphi'(Cx)\,dx \\ &= A_f + \frac{A_f}{2} \cdot \int_0^R \exp\Big(\frac{-\varphi(Cx)}{2}\Big) \cdot d\Big(\frac{\varphi(Cx)}{2}\Big) \le \exp\Big(\frac{-\varphi(CR)}{2}\Big) \le \text{const}, \end{aligned}$$

i.e., $I_R \le \text{const}$ for all $R \in \mathbb{R}_+$. Therefore, $\sum_{k=1}^{+\infty} \exp(-\varphi(C|z_k|)) < +\infty$, which establishes the implication (1) $\Rightarrow$ (2).

Now we prove the reverse implication (2) $\Rightarrow$ (1). Let $\{z_k\}_{k=1}^{+\infty}$ be a sequence in $\mathbb{C}$ saisfying (10) and such that $1 < |z_1| \le |z_2| \le \dots \le |z_k| \le \dots$. There is no loss of generality in assuming that $C = 1$. We construct a function $f \in \widetilde{H}(\lambda, +\infty)$, $f \not\equiv 0$, such that $f(z_k) = 0$, $k = 1, 2, \dots$, and $f(z) \not\equiv 0$ for $z \ne z_k$. Put $p_n = \big[\frac{\varphi(|z_n|)}{\ln 2}\big]$, $n = 1, 2, \dots$, where $[a]$ stands for the integral part of a.

We prove that the infinite product

$$B(z,z_n)=\prod_{n=1}^{+\infty}\Big(1-\frac{z}{z_n}\Big)\cdot\exp\bigg(\frac{z}{z_n}+\frac{1}{2}\Big(\frac{z}{z_n}\Big)^2+\dots+\frac{1}{p_n}\Big(\frac{z}{z_n}\Big)^{p_n}\bigg)$$

converges and does not vanish identically, and $B\in\widetilde{H}(\lambda,+\infty)$. The estimate of Lemma 1 shows that

$$\ln|B(z,z_n)|\le\sum_{k=1}^{+\infty}\Big|\frac{z}{z_n}\Big|^{p_n+1},\quad z\in\mathbb{C}.$$

To estimate the sum obtained, we put $m_n(z)=\big|\frac{z}{z_n}\big|$, $z\in\mathbb{C}$, $n=1,2,\dots$. Then

$$I=\sum_{k=1}^{+\infty}\Big|\frac{z}{z_n}\Big|^{p_n+1}=\sum_{m_n(z)\le\frac12}\Big|\frac{z}{z_n}\Big|^{p_n+1}+\sum_{m_n(z)>\frac12}\Big|\frac{z}{z_n}\Big|^{p_n+1}=I_1+I_2.$$

We bound each sum separately. We have

$$\begin{aligned}I_1=\sum_{m_n(z)\le\frac12}\Big|\frac{z}{z_n}\Big|^{p_n+1}&\le\sum_{m_n(z)\le\frac12}\Big(\frac12\Big)^{p_n+1}=\sum_{m_n(z)\le\frac12}\exp(-(p_n+1)\cdot\ln 2)\\&<e\cdot\sum_{m_n(z)\le\frac12}\exp(-\varphi(|z_n|))\le C,\end{aligned}$$

because $\sum_{k=1}^{+\infty}\exp(-\varphi(|z_k|))<+\infty$.

Passing to the sum I_2, we have

$$\begin{aligned}I_2&=\sum_{m_n(z)>\frac12}\Big|\frac{z}{z_n}\Big|^{p_n+1}=\sum_{m_n(z)>\frac12}\exp\Big((p_n+1)\cdot\ln\Big|\frac{z}{z_n}\Big|\Big)\\&=\sum_{\frac12<m_n(z)<1}\exp\Big((p_n+1)\cdot\ln\Big|\frac{z}{z_n}\Big|\Big)+\sum_{m_n(z)\ge1}\exp\Big((p_n+1)\cdot\ln\Big|\frac{z}{z_n}\Big|\Big)\\&=I_2^1+I_2^2.\end{aligned}$$

Since $\ln\frac{|z|}{|z_n|}\ge0$ for $0<|z_n|\le|z|$, and $p_n+1\le\frac{\varphi(|z_n|)}{\ln 2}+1\le2\varphi(|z_n|)+1$, $n=1,2,\dots$, we get

$$\begin{aligned}I_2^2&\le\sum_{|z_n|\le|z|}\exp\Big((2\varphi(|z_n|)+1)\cdot\ln\Big|\frac{z}{z_n}\Big|\Big)\le|z|\cdot\sum_{|z_n|\le|z|}\exp\Big(2\varphi(|z_n|)\cdot\ln\Big|\frac{z}{z_n}\Big|\Big)\\&\le|z|\cdot\int_1^{|z|}\exp\bigg(2\varphi(t)\cdot\ln\frac{|z|}{t}\bigg)\,dn(t).\end{aligned}$$

Now we note that the convergence of the series $\sum_{n=1}^{+\infty}\exp(-\varphi(|z_n|))\le C<+\infty$ implies that $\sum_{|z_n|\le R}\exp(-\varphi(|z_n|))\le C$. Therefore, $\exp(-\varphi(R))\cdot n(R)\le C$. This results in the estimate

$$n(R)\le C\cdot\exp(\varphi(R)),\tag{12}$$

valid for all $R\in\mathbb{R}_+$. Using (12) and integrating by parts, we obtain

$$I_2^2\le|z|\cdot\bigg(n(|z|)-\int_1^{|z|}\Big(\exp(2\varphi(t))\cdot\ln\frac{|z|}{t}\Big)\cdot n(t)\cdot\Big(2\varphi'(t)\cdot\ln\frac{|z|}{t}-\frac{2\varphi(t)}{t}\Big)\bigg)\,dt.$$

Recall that φ is a monotone increasing function of class $C^{(1)}(\mathbb{R}_+)$, so that $\varphi'(t) \geq 0$, $t \in \mathbb{R}_+$, and $\ln \frac{|z|}{t} \geq 0$, $1 \leq t \leq |z|$. We continue the last-written estimate:

$$\begin{aligned} I_2^2 &\leq n(|z|) \cdot |z| + 2|z| \cdot \int_1^{|z|} \frac{n(t) \cdot \varphi(t)}{t} \cdot \exp\Big(2\varphi(t) \cdot \ln \frac{|z|}{t}\Big)\, dt \\ &\leq 3 \cdot n(|z|) \cdot |z| \cdot \varphi(|z|) \cdot \int_1^{|z|} \exp\Big(2\varphi(t) \cdot \ln \frac{|z|}{t}\Big)\, dt. \end{aligned} \tag{13}$$

We denote the last integral in (13) by $\widetilde{I}_2^2$. To estimate it, we put $\ln \frac{|z|}{t} = u$, obtaining

$$\widetilde{I}_2^2 \leq |z| \cdot \int_0^{\ln |z|} \exp(2\varphi(|z|e^{-u})u) \cdot e^{-u}\, du \leq |z| \cdot \int_0^{\ln |z|} \exp(2\varphi(|z|e^{-u})u)\, du. \tag{14}$$

Since φ is convex, we can bound the expression $\varphi(|z|e^{-u})u$ for $u \geq 1$ as follows:

$$\begin{aligned} \varphi(|z|e^{-u})u &= u\int_0^{|z|e^{-u}} \varphi'(t)\, dt = \int_0^{u|z|e^{-u}} \varphi'\Big(\frac{x}{u}\Big)\, dx \\ &\leq \int_0^{|z|} \varphi'\Big(\frac{x}{u}\Big)\, dx \leq \int_0^{|z|} \varphi'(x)\, dx = \varphi(|z|). \end{aligned}$$

Thus,

$$\varphi(|z|e^{-u})u \leq \varphi(|z|) \tag{15}$$

for all $u \geq 1$.

Obviously, if $0 \leq u \leq 1$, then

$$\varphi(|z|e^{-u})u \leq \varphi(|z|). \tag{16}$$

Consequently, $\varphi(|z|e^{-u})u \leq \varphi(|z|)$ for all $u \in \mathbb{R}_+$. Therefore, estimate (14) implies that

$$\widetilde{I}_2^2 \leq |z| \cdot \exp(2\varphi(|z|)) \cdot \ln |z|. \tag{17}$$

A combination of (14) and (15) yields

$$I_2^2 \leq 3 \cdot n(|z|) \cdot \varphi(|z|) \cdot |z|^2 \cdot \ln |z| \cdot \exp(2\varphi(|z|)).$$

Since φ is convex and $\varphi'(x)x \to +\infty$ as $x \to +\infty$, we can use Lemma 6 to obtain the final estimate:

$$I_2^2 \leq C_1 \cdot \exp(\varphi(C_2 \cdot |z|)). \tag{18}$$

The quantity I_2^1 can be estimated as follows:

$$I_2^1 = \sum_{\frac{1}{2} < m_n(z) < 1} \exp\Big((p_n + 1) \cdot \ln\Big|\frac{z}{z_n}\Big|\Big) \leq \sum_{m_n(z) > \frac{1}{2}} 1 \leq \sum_{|z_n| < 2|z|} 1 \leq n(2|z|).$$

Using (12), we conclude that

$$n(2|z|) \leq C \cdot \exp(\varphi(2|z|)). \tag{19}$$

To complete the proof of Theorem 2, now it suffices to combine estimates (18) and (19).

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