ON THE ZEROS OF ENTIRE FUNCTIONS WITH A MAJORANT OF INFINITE ORDER

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To our Teacher Viktor Petrovich Havin on the occasion of his 75th birthday

ABSTRACT. The class of entire functions \( f \) satisfying \( \ln |f(z)| \leq C_f \lambda(|z|) \), \( z \in \mathbb{C} \), where \( \lambda \) is a majorant mentioned in the title, admits a zero set \( \{z_j\} \) such that \( \{|z_j|\} \) is a uniqueness set for this class.

INTRODUCTION

Let \( \mathbb{C} \) denote the complex plane, let \( H(\mathbb{C}) \) be the set of all entire functions, and let \( \lambda \) be a monotone increasing, positive function on \( \mathbb{R}_+ = (0; +\infty) \); we put \( \varphi(x) = \ln \lambda(x) \), \( x \in \mathbb{R}_+ \).

We introduce the following function classes:

\[
\begin{align*}
(1) & \quad H(\lambda, +\infty) = \{ f \in H(\mathbb{C}) : \ln |f(z)| \leq C_f \cdot \lambda(|z|), \ z \in \mathbb{C} \}, \\
(2) & \quad \tilde{H}(\lambda, +\infty) = \{ f \in H(\mathbb{C}) : \ln |f(z)| \leq A_f \cdot \lambda(B_f \cdot |z|), \ z \in \mathbb{C} \}.
\end{align*}
\]

Here and in what follows, \( A_f, B_f, C_f \) are arbitrary positive numbers depending on the function \( f \) only. If \( \lambda \in C^1(\mathbb{R}_+) \) and \( \alpha_\lambda = \lim_{x \to +\infty} x^{\lambda(x) / (\lambda(x) + 1)} < +\infty \), then it is not hard to show that the classes \( H(\lambda, +\infty) \) and \( \tilde{H}(\lambda, +\infty) \) coincide. However, this is no longer true if \( \alpha_\lambda = +\infty \), for example, in the case where

\[
\lambda(t) = \exp \exp \cdots \exp (t^\rho), \quad t \in \mathbb{R}_+, \quad \rho \in \mathbb{R}_+.
\]

If \( \lambda(t) = t^\rho, t \in \mathbb{R}_+, \rho \in \mathbb{R}_+ \), then \( H(\lambda, +\infty) \) coincides with the class of functions of finite order and normal type; we denote this class by \( H(\rho, +\infty) \).

For \( f \in H(\mathbb{C}) \), we denote \( Z_f \) the set of \( f \) satisfying \( \ln |f(z)| \leq C_f \lambda(|z|) \), \( z \in \mathbb{C} \). The following property of the zeros of functions of class \( H(\rho, +\infty) \) is well known: a sequence \( \{z_k\}_{k=1}^{\infty} \) can be represented as \( Z = Z_f \) with \( f \in H(\rho, +\infty), f \not\equiv 0, \rho \not\in \mathbb{N}, \rho > 0 \), if and only if

\[
(3) \quad n(r) = \text{card}\{z_k : |z_k| \leq r\} \leq C_f \cdot r^\rho.
\]

For \( \rho \in \mathbb{N} \), besides (3), the following Lindelöf condition arises (see \( \Pi \)): there exists \( M > 0 \) such that

\[
(4) \quad \left| \sum_{|z_k| \leq r} \frac{1}{z_k} \right| \leq M, \quad r \in \mathbb{R}_+.
\]

By using (4), it is not difficult to construct a sequence \( \{z_k\}_{k=1}^{+\infty} = Z_f, f \not\equiv 0, f \in H(\rho, +\infty), \rho \in \mathbb{N} \), such that any \( g \in H(\rho, +\infty) \) with \( Z_g = \tilde{Z}_f \), where \( \tilde{Z}_f = \{|z_k|\}_{k=1}^{+\infty} \),

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\]

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satisfies \( g(z) = 0, \ z \in \mathbb{C} \); we mention \( \{ k^{\frac{1}{2}} e^{\frac{\pi i k}{3}} \}_{k=1}^{\infty} \) as an example of such a sequence.

In other words, the set \( \tilde{Z}_f \) cannot coincide with \( Z_g \) for any \( g \in H(\rho, +\infty), \ \rho \in \mathbb{N}, \ g(z) \neq 0. \) Thus, the distribution of the arguments of \( \{ z_k \}_{k=1}^{\infty} \) is also of importance for the representation \( Z = Z_g, \) rather then only the growth of the function \( n(r). \)

**Definition.** We say that a set \( X \) of entire functions satisfies the Lindelöf condition

if there exists \( f \in X, \ f \neq 0, \ Z_f = \{ z_k \}_{k=1}^{\infty}, \) such that if \( g \in X, \ Z_g \supseteq \tilde{Z}_f, \) where \( \tilde{Z}_f = \{ \{ z_k \}_{k=1}^{\infty} \}, \) then \( g(z) = 0 \) for all \( z \in \mathbb{C}. \)

The above remarks imply that the class \( H(\rho, +\infty) \) with \( \rho \in \mathbb{N} \) satisfies the Lindelöf condition, while for \( \rho \notin \mathbb{N} \) it does not. The following question arises naturally: what happens for the other \( \lambda, \) e.g., for \( \lambda(t) = \exp \cdots \exp(t^\rho), \ t, \rho \in \mathbb{R}_+. \)

In this paper we show that if \( \alpha_\lambda = +\infty, \) then the class \( H(\lambda, +\infty) \) satisfies the Lindelöf condition and the class \( \tilde{H}(\lambda, +\infty) \) does not. In the case where \( \alpha_\lambda = +\infty \) we obtain a description of the zero sets of functions in \( \tilde{H}(\lambda, +\infty). \) This description is in terms of \( \{ z_k \}_{k=1}^{\infty}, \) i.e., in terms of \( n(r). \) At the same time, Theorem 1 shows that the zero sets of functions of class \( H(\lambda, +\infty) \) fail to admit a similar description; here the distribution of the arguments of the zeros becomes important.

There are many publications devoted to properties of the root sets for functions in \( \tilde{H}(\lambda, +\infty); \) we mention \([2, 3]\) and the references therein. The method applied in those papers is based on Fourier series, and the results do not even characterize the functions \( \lambda \) for which the zero sets can be described in terms of the counting function \( n(r). \) only.

§ 1. Formulation of the main results and proof of auxiliary statements

In what follows, a monotone increasing positive function \( \lambda \) of class \( C^{(1)}(\mathbb{R}_+) \) is called a **weight function** if \( \alpha_\lambda = +\infty. \)

**Theorem 1.** Let \( \lambda \) be a weight function of class \( C^{(2)}(\mathbb{R}_+). \) Suppose that the function 

\[
\psi(x) = \ln \lambda(e^x)
\]

is convex on \( \mathbb{R}_+ \) and that

\[
\frac{\psi''(x)}{(\psi'(x))^{2-\delta}} = O(1), \quad x \to +\infty,
\]

for some \( 0 < \delta < 1. \) Then the class \( H(\lambda, +\infty) \) satisfies the Lindelöf condition.

**Theorem 2.** Let \( \lambda \) be a weight function of class \( C^{(1)}(\mathbb{R}_+). \) Suppose that the function

\[
\varphi(x) = \ln \lambda(x)
\]

is convex on \( \mathbb{R}_+. \) The following statements are equivalent:

1) a sequence of complex numbers \( \{ z_k \}_{k=1}^{\infty} \) is representable in the form \( Z_f \) for some \( f \in \tilde{H}(\lambda, +\infty), \ f \neq 0; \)

2) there exists a positive number \( C \) such that

\[
\sum_{k=1}^{+\infty} \exp(-\varphi(C|z_k|)) < +\infty.
\]

In Theorem 2 it is assumed that \( \lambda \in C^{(1)}(\mathbb{R}_+). \) However, the claim remains valid under the assumption that \( \lambda \in C(\mathbb{R}_+). \) Indeed, it is easily seen that if \( \lambda \in C(\mathbb{R}_+) \) and \( \lambda_\ast(r) = \int_{0}^{r} \frac{\lambda(t)}{t} \, dt, \) then \( \lambda(r) \leq \lambda_\ast(r) \leq \lambda(er), \) while \( \tilde{H}(\lambda, +\infty) = \tilde{H}(\lambda_\ast, +\infty). \)

The proof of the above theorems is based on several auxiliary statements.

**Lemma 1** (see \([4]\)). Let \( z_n \) be a sequence of complex numbers converging to infinity and satisfying

\[
\sum_{n=1}^{+\infty} \frac{1}{|z_n| p_n+1} < +\infty.
\]
Then the Weierstrass product

\[ B(z, z_n, p_n) = \prod_{n=1}^{+\infty} \left( 1 - \frac{z}{z_n} \right) \cdot \exp \left( \frac{z}{z_n} + \frac{1}{2} \left( \frac{z}{z_n} \right)^2 + \cdots + \frac{1}{p_n} \left( \frac{z}{z_n} \right)^{p_n} \right) \]

converges in \( \mathbb{C} \) and obeys the estimate

\[ \ln |B(z, z_n, p_n)| \leq \sum_{n=1}^{+\infty} \frac{z}{|z_n|^{p_n+1}}, \quad z \in \mathbb{C}. \]

**Lemma 2** (see [5]). Suppose \( \lambda \in C^{(1)}(\mathbb{R}_+), \lambda(x) > 0, x \in \mathbb{R}_+, \) and \( x^{\frac{\lambda(x)}{\lambda(x)}} \rightarrow +\infty \) as \( x \rightarrow +\infty \). Then there exists a function \( f \in H(\lambda, +\infty) \) with the following properties:

a) \( \lim_{r \rightarrow +\infty} \frac{\ln M(r, f)}{\lambda(r)} = 1, \)  
b) \( \lim_{r \rightarrow +\infty} \frac{T(r, f)}{\lambda(r)} = 1, \)  
c) \( \lim_{r \rightarrow +\infty} \frac{N(r, f)}{\lambda(r)} = 1, \)

where, as usual, \( M(r, f) = \max_{|z| \leq r} |f(z)|, \)

\[ T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(e^{i\varphi})| d\varphi, \]

and \( N(r, f) \) is the Nevanlinna counting function (see [1]):

\[ N(r, f) = \int_{0}^{r} \frac{n(t) - n(0)}{t} dt, \]

\( n(0) \) being the order of the zero of \( f \) at \( z = 0 \).

In the next lemma we estimate the number of real zeros of a function of class \( H(\lambda, +\infty) \) on the segment \([0; r]\); this lemma will play an important role in the proof of Theorem 1.

**Lemma 3.** Let \( \lambda \) be as in Theorem 1. If \( g \in H(\lambda, +\infty) \) and \( n_R(r, g) \) is the number of positive (real) zeros of \( g \) on the segment \([0; r]\), then \( n_R(g, r) \leq C_g \cdot \lambda(r), C_g > 0, r \in \mathbb{R}_+. \)

**Proof.** Let \( \Omega = \{ t + it : |\tau| < s(t) \}, t \in \mathbb{R}_+ \), where \( s(t) = \frac{\pi}{\lambda(t)} \) and \( \psi(t) = \ln \lambda(e^t) \); there is no loss of generality in assuming that \( \psi'(0) > 3 \). Obviously, if \( f \in H(\lambda, +\infty) \), then \( \ln |f(e^z)| \leq C_f \cdot \lambda(e^{Re z}), z \in \mathbb{C}. \)

Let \( s(t) = \frac{\lambda(e^t)}{\pi \cdot \lambda(e^t) \cdot e^t}, t \in \mathbb{R}_+ \); we put \( \Omega_+ = \{ t + it : t > 0, 0 < \tau < s(t) \} \) and \( \Pi_+ = \{ u + iv : 0 < v < \frac{\pi}{2} \} \). Let \( Z_+(\zeta) = Z(t + it) \) be a conformal mapping onto the strip \( \Pi_+ \) such that \( Z_+(iS(0)) = \frac{\pi}{2}, Z_+(0) = 0, \lim_{s(t) \rightarrow +\infty}(Z_+(t)) = +\infty \). By the Riemann–Schwarz principle, the function \( Z_+(\zeta) \) admits an analytic continuation \( Z \) to the domain

\[ \Omega = \{ \zeta = t + it : |\tau| < s(|t|), t \in \mathbb{R} \}, \]

\( Z \) maps \( \Omega \) onto the strip \( \Pi = \{ u + iv : -\frac{\pi}{2} < v < \frac{\pi}{2} \} \), and the function \( Z_+(t) = Z(t), t \in \mathbb{R}_+ \), is strictly monotone increasing on \([0; +\infty)\).

By the Warschawski theorem (the assumptions of it are fulfilled by (5)), we have

\[ u = \text{Re} Z(\sigma + i\tau) = \frac{\pi}{2} \cdot \int_{0}^{\sigma} \frac{d\tau}{s(t)} + O(1) = \frac{\pi}{2} \cdot \frac{1}{\pi} \int_{0}^{\sigma} \frac{\lambda(e^t) \cdot e^t}{\lambda(e^t)} dt + O(1) \]

\[ = \frac{1}{2} \cdot \int_{0}^{\sigma} d\ln \lambda(e^t) + O(1) = \frac{1}{2} \cdot \ln \lambda(e^\sigma) + O(1), \quad \sigma \rightarrow +\infty. \]

Consequently,

\[ \ln \lambda(e^\sigma) = 2 \cdot \text{Re} Z(\sigma + i\tau) + O(1), \]

where \( u + iv = Z(\sigma + i\tau) \in \Pi. \)
We map the right half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$ conformally onto $\Pi$. Then, if $f \in H(\lambda, +\infty)$, then the function $F(z) = f(e^{Z^{-1}(\ln z)})$ is holomorphic in $\mathbb{C}_+$, and

$$
(7) \quad \ln |F(z)| = \ln |f(e^{Z^{-1}(\ln z)})| \leq C_f \cdot \lambda(e^{\Re Z^{-1}(\ln z)}),
$$

where $Z^{-1}$ is the function inverse to $Z$.

Using (6), we see that if $\sigma + i\tau = Z^{-1}(\ln z)$, $Z(\sigma + i\tau) = \ln z = \ln |z| + i \cdot \arg z$, where $|\arg z| < \frac{\pi}{2}$, then $Z^{-1}(\ln z) = Z^{-1}(\ln |z| + i \cdot \arg z) = \sigma + i\tau \in \Omega$. Therefore, $\ln |z| = \Re Z(Z^{-1}(\ln z))$, but, by (6),

$$
\frac{1}{2} \cdot \ln \lambda(e^\sigma) = \Re Z(\sigma + i\tau) + O(1) = \ln |z| + O(1) \quad (\sigma \to +\infty).
$$

Thus, using (7), finally we obtain

$$
\ln |F(z)| = \ln |f(e^{Z^{-1}(\ln z)})| \leq C_f \cdot \lambda(e^{\Re Z^{-1}(\ln z)}) \leq \tilde{C}_f \cdot \lambda(e^\sigma) = C_f \cdot |z|^2.
$$

Consequently, $\ln |F(z)| \leq C_f \cdot |z|^2$, $z \in \mathbb{C}_+$, and the function $F(z)$ is holomorphic in $\mathbb{C}_+$.

The results pertaining to the zeros of such functions (see [3]) show that the number $\bar{n}_R(t)$ of real zeros of $F$ on the segment $[1; t]$, $t > 1$, satisfies

$$
(8) \quad \bar{n}_R(t) = A_f \cdot t^2.
$$

On the other hand, if $F(\rho_k) = 0$, $k = 1, 2, \ldots$, then $f(e^{Z^{-1}(\ln \rho_k)}) = 0$, $k = 1, 2, \ldots$. This means that $e^{Z^{-1}(\ln \rho_k)} = r_k$, where the $r_k$, $k = 1, 2, \ldots$, are real zeros of $f$. Since $Z^{-1}(\ln \rho_k) = \ln r_k$, we have

$$
(9) \quad \ln \rho_k = Z(\ln r_k).
$$

Therefore, $\bar{n}_R(t) = \text{card}\{\rho_k : \rho_k \leq t\} \leq C_f \cdot t^2$. Since $Z^{-1}$ grows monotonically on $\mathbb{R}_+$, estimate (8) implies that

$$
\bar{n}_R(t) = \text{card}\{\rho_k : Z^{-1}(\ln \rho_k) \leq Z^{-1}(\ln t)\} \leq A_f \cdot t^2.
$$

Now, by (9), we have

$$
\bar{n}_R(t) = \text{card}\{r_k : \ln r_k \leq Z^{-1}(\ln t)\} \leq C_f \cdot t^2
$$

or

$$
\bar{n}_R(t) = \text{card}\{r_k : r_k \leq e^{Z^{-1}(\ln t)}\} \leq A_f \cdot t^2.
$$

Now, we put $Z^{-1}(\ln t) = \rho$, $t \in \mathbb{R}_+$; then $t^2 = 2Z(\rho)$. Therefore,

$$
\bar{n}_R(\rho) = \text{card}\{r_k : r_k \leq e^\rho\} \leq A_f \cdot e^{2Z(\rho)}, \quad \rho \in \mathbb{R}_+.
$$

Recalling (6), we see that the last estimate implies that

$$
\bar{n}_R(\rho) = \text{card}\{r_k : r_k \leq e^\rho\} \leq A_f \cdot \lambda(e^\rho) \cdot e^{O(1)} = C_f \cdot \lambda(e^\rho).
$$

Putting $e^\rho = r$, $r > 0$, finally we get

$$
n_R(r) \leq C_f \cdot \lambda(r),
$$

which proves the lemma.

□

**Lemma 4.** If $\varphi \in C^1(\mathbb{R}_+)$ is convex on $\mathbb{R}_+$ and $\varphi(0) = 0$, then

$$
C \varphi(x) \leq \varphi(Cx)
$$

for any number $C \geq 1$.  

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Proof. Obviously, \( \varphi(x) = \int_a^x \varphi'(u) du + \varphi(0) = \int_0^x \varphi'(u) du \), so that \( C \varphi(x) = C \int_0^x \varphi'(u) du \).

Changing the variables \( C u = t \), we obtain

\[
C \varphi(x) = \int_0^{C x} \varphi' \left( \frac{t}{C} \right) dt, \quad C \geq 1.
\]

Since \( \varphi'(x) \) is a monotone increasing function, we have \( \varphi' \left( \frac{t}{C} \right) \leq \varphi'(t) \) for all \( t \); consequently,

\[
\int_0^{C x} \varphi' \left( \frac{t}{C} \right) dt \leq \int_0^x \varphi'(t) dt, \quad \text{i.e., } C \varphi(x) \leq \varphi(Cx).
\]

\( \square \)

**Lemma 5** (see [9]). Let \( \varphi(x) \) be a positive measurable function bounded on every segment \([a;b] \subset \mathbb{R}_+\). For each \( K > 1 \), there exists a sequence \( x_n \to +\infty \) such that for all \( x \) with \( x_n \leq x \leq x_n + \frac{1}{\ln \varphi(x_n)} + \frac{1}{\varphi(x_n)} \) we have

\[
\varphi(x) \leq K \cdot \varphi(x_n).
\]

**Lemma 6.** Let \( Z_f \) be \( \{z_k\}_{k=1}^{+\infty} \), \( f \in \mathcal{H}(\lambda, +\infty) \), \( f \not\equiv 0 \). Then \( n(r) \leq A_f \cdot \lambda(2B_f \cdot r) \), where \( A_f \) and \( B_f \) are as in (2).

Proof. Suppose \( f \in \mathcal{H}(\lambda, +\infty) \), \( f(z_k) = 0 \), \( k = 1, 2, \ldots \), \( f \not\equiv 0 \). Without loss of generality, we may assume that \( f(0) = 1 \). The Jensen identity yields

\[
\int_0^r \frac{n(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\varphi})| d\varphi \leq \frac{1}{2\pi} \int_0^{2\pi} A_f \lambda(B_f r) d\varphi = \frac{1}{2\pi} 2\pi \cdot A_f \lambda(B_f r) = A_f \cdot \lambda(B_f r),
\]

where \( 0 < r < +\infty \). Since \( n(r) \) grows monotonically, we deduce that

\[
n(r) \cdot \ln 2 \leq \int_r^{2r} \frac{n(t)}{t} dt \leq A_f \cdot \lambda(2B_f \cdot r),
\]

whence \( n(r) \leq A_f \cdot \lambda(2B_f \cdot r) \), as required. \( \square \)

§2. \textbf{Proof of the main results}

First, we prove Theorem 1.

Let \( f \) be the function of class \( \mathcal{H}(\lambda, +\infty) \) constructed in Lemma 2, let \( \{z_k\}_{k=1}^{+\infty} \) be the set of zeros of \( f \), let \( n(r) = \{ \text{card } z_k : |z_k| < r \} \), and let

\[
N(r) = \int_0^r \frac{n(t)}{t} dt.
\]

Without loss of generality, we assume that \( f(0) = 1 \). By Lemma 2, we have \( N(r) \geq C_0 \cdot \lambda(r) \), \( r \geq 1 \), for some \( C_0 > 0 \). Suppose that \( g \in \mathcal{H}(\lambda, +\infty) \), \( g(|z_k|) = 0 \), \( k = 1, 2, \ldots \), and \( g(z) \not\equiv 0 \). Then, by Lemma 3, we have \( n_R(r) = n(r) \), where \( n_R \) is the set of all positive roots of \( g \) on the segment \([0;r] \), and \( n_R(r) = n(r) \leq C_1 \cdot \lambda(r) \), where \( C_1 > 0 \), \( r \in \mathbb{R}_+ \). We put \( \varphi(r) = \frac{N'_+(r)}{N(r)} \cdot r \in \mathbb{R}_+ \), where \( N'_+(r) \) is the right derivative of the function \( N \). Let \( \{x_n\}_{n=1}^{+\infty} \) be the sequence built in Lemma 5 starting with \( \varphi \). Applying that lemma and the identity

\[
\ln \frac{N(x'_n)}{N(x_n)} = \int_{x_n}^{x'_n} \frac{N'(t)}{N(t)} dt,
\]

where \( x'_n = x_n + \frac{1}{\varphi(x_n)} \), we obtain the inequality

\[
\ln \frac{N(x'_n)}{N(x_n)} \leq K \cdot \varphi(x_n) \cdot (x'_n - x_n) = K \cdot \varphi(x_n) \cdot \frac{1}{\varphi(x_n)} = K,
\]
We fix generality in assuming that $f$ satisfies (10) and such that $1 = 0$.

Integration by parts yields the formula

$$
\int_{x_n}^{x_n'} \lambda'(t) dt \leq \frac{1}{x_n' - x_n} \int_{x_n}^{x_n'} \lambda'(t) dt.
$$

Then $\lambda'(x_n) \leq C \cdot e^K \cdot N(x_n) \cdot \varphi(x_n)$, i.e., $\lambda'(x_n) \leq C \cdot e^K \cdot N(x_n) \cdot \frac{N'(x_n)}{N(x_n)}$, whence $\lambda'(x_n) \leq C \cdot e^K \cdot N'(x_n)$ (note that $x_n \to +\infty$ as $n \to +\infty$).

Therefore, $\lambda'(x_n) \leq C \cdot e^K \cdot n(x_n)$, so that $\lambda'(x_n) = x_n \leq C \cdot e^K < +\infty$, which contradicts the assumptions of the theorem. This proves Theorem 1.

Now we start the proof of Theorem 2. We check the implication (1) $\Rightarrow$ (2).

Suppose $f \in \tilde{H}(\lambda, +\infty), f(z_k) = 0, k = 1, 2, \ldots, f \neq 0$. Without loss of generality, we may assume that $\lambda(0) = 1$, i.e., $\varphi(0) = 0$. Lemma 1 shows that if $Z = Z_f$, $Z = \{z_k\}_{k=1}^{+\infty}$ (we assume that $0 < |z_1| \leq |z_2| \leq \cdots \leq |z_k| \leq \cdots$), then $n(r) \leq A \cdot \lambda(B \cdot r)$, i.e.,

$$
n(r) \leq A \cdot \exp(B \cdot r), \quad r \in \mathbb{R}_+, \quad A \in \mathbb{R}_+, \quad B \in \mathbb{R}_+.
$$

We fix $C > 0$. For $R > 0$ we have

$$
I_R = \sum_{|z_k| \leq R} \exp(-\varphi(C|z_k|)) = \int_0^R \exp(-\varphi(Ct)) dt.
$$

Integration by parts yields the formula

$$
I_R = n(t) \cdot \exp(-\varphi(Ct)) \bigg|_0^R + \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct) dt
$$

$$
= n(R) \cdot \exp(-\varphi(CR)) + C \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct) dt.
$$

Using (10) and choosing $C > B_f$, we see that $I_R \leq A_f \cdot \exp(\varphi(B_f r) - \varphi(Cr)) + C \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct) dt$. Since $\varphi(t)$ is a monotone increasing function on $\mathbb{R}_+$, we have

$$
I_R \leq A_f + C \int_0^R n(t) \cdot \exp(-\varphi(Ct)) \cdot \varphi'(Ct) dt \leq A_f + C \int_0^R \exp(\varphi(B_f t) - \varphi(Ct)) \cdot \varphi'(Ct).
$$

We take $C$ so large that $\varphi(B_f x) \leq \frac{\varphi(Cx)}{2}, \quad x \in \mathbb{R}_+$; then

$$
I_R \leq A_f + A_0 \cdot \int_0^R \exp\left(-\frac{\varphi(Cx)}{2}\right) \cdot \varphi'(Ct) dx
$$

$$
= A_f + A_0 \cdot \frac{A_f}{2} \cdot \int_0^R \exp\left(-\frac{\varphi(Cx)}{2}\right) \cdot d\left(\frac{\varphi(Cx)}{2}\right) \leq \exp\left(-\frac{\varphi(CR)}{2}\right) \leq \text{const},
$$

i.e., $I_R \leq \text{const}$ for all $R \in \mathbb{R}_+$. Therefore, $\sum_{k=1}^{+\infty} \exp(-\varphi(C|z_k|)) < +\infty$, which establishes the implication (1) $\Rightarrow$ (2).

Now we prove the reverse implication (2) $\Rightarrow$ (1). Let $\{z_k\}_{k=1}^{+\infty}$ be a sequence in $\mathbb{C}$ satisfying (10) and such that $1 < |z_1| \leq |z_2| \leq \cdots \leq |z_k| \leq \cdots$. There is no loss of generality in assuming that $C = 1$. We construct a function $f \in \tilde{H}(\lambda, +\infty), f \neq 0$, such that $f(z_k) = 0, k = 1, 2, \ldots$, and $f(z) \neq 0$ for $z \neq z_k$. Put $n_n = \lceil \frac{\varphi(|z_n|)}{ln} \rceil, n = 1, 2, \ldots$, where $[a]$ stands for the integral part of $a$. 

We prove that the infinite product
\[ B(z, z_n) = \prod_{n=1}^{+\infty} \left( 1 - \frac{z}{z_n} \right) \cdot \exp \left( \frac{z}{z_n} + \frac{1}{2} \left( \frac{z}{z_n} \right)^2 + \cdots + \frac{1}{p_n} \left( \frac{z}{z_n} \right)^{p_n} \right) \]
converges and does not vanish identically, and \( B \in \bar{H}(\lambda, +\infty) \). The estimate of Lemma 1 shows that
\[ \ln |B(z, z_n)| \leq \sum_{k=1}^{+\infty} \frac{z^{p_n+1}}{|z_n|}, \quad z \in \mathbb{C}. \]
To estimate the sum obtained, we put \( m_n(z) = \frac{z}{|z_n|}, \quad z \in \mathbb{C}, \quad n = 1, 2, \ldots \). Then
\[ I = \sum_{k=1}^{+\infty} \frac{z}{|z_n|}^{p_n+1} = \sum_{m_n(z) \leq \frac{1}{2}} \frac{z}{|z_n|}^{p_n+1} + \sum_{m_n(z) > \frac{1}{2}} \frac{z}{|z_n|}^{p_n+1} = I_1 + I_2. \]
We bound each sum separately. We have
\[ I_1 = \sum_{m_n(z) \leq \frac{1}{2}} \frac{z}{|z_n|}^{p_n+1} \leq \sum_{m_n(z) \leq \frac{1}{2}} \left( \frac{1}{2} \right)^{p_n+1} \exp(-(p_n + 1) \cdot \ln 2) \]
\[ < e \cdot \sum_{m_n(z) \leq \frac{1}{2}} \exp(-\varphi(|z_n|)) \leq C, \]
because \( \sum_{k=1}^{+\infty} \exp(-\varphi(|z_k|)) < +\infty \).
Passing to the sum \( I_2 \), we have
\[ I_2 = \sum_{m_n(z) > \frac{1}{2}} \frac{z}{|z_n|}^{p_n+1} = \sum_{m_n(z) > \frac{1}{2}} \exp \left( (p_n + 1) \cdot \ln \left| \frac{z}{z_n} \right| \right) \]
\[ = \sum_{\frac{1}{2} < m_n(z) < 1} \exp \left( (p_n + 1) \cdot \ln \left| \frac{z}{z_n} \right| \right) + \sum_{m_n(z) \geq 1} \exp \left( (p_n + 1) \cdot \ln \left| \frac{z}{z_n} \right| \right) \]
\[ = I_2^1 + I_2^2. \]
Since \( \frac{|z|}{|z_n|} \geq 0 \) for \( 0 < |z_n| \leq |z| \), and \( p_n + 1 \leq \frac{\varphi(|z_n|)}{\ln 2} + 1 \leq 2\varphi(|z_n|) + 1, n = 1, 2, \ldots \), we get
\[ I_2^2 \leq \sum_{|z_n| \leq |z|} \exp \left( (2\varphi(|z_n|) + 1) \cdot \ln \left| \frac{z}{z_n} \right| \right) \leq |z| \sum_{|z_n| \leq |z|} \exp \left( 2\varphi(|z_n|) \cdot \ln \left| \frac{z}{z_n} \right| \right) \]
\[ \leq |z| \int_1^{[\frac{|z|}{t}]} \exp \left( 2\varphi(t) \cdot \ln \frac{|z|}{t} \right) \, d\nu(t). \]
Now we note that the convergence of the series \( \sum_{n=1}^{+\infty} \exp(-\varphi(|z_n|)) \leq C < +\infty \) implies that \( \sum_{|z_n| \leq R} \exp(-\varphi(|z_n|)) \leq C \). Therefore, \( \exp(-\varphi(R)) \cdot n(R) \leq C \). This results in the estimate
\[ (12) \quad n(R) \leq C \cdot \exp(\varphi(R)), \]
valid for all \( R \in \mathbb{R}_+ \). Using (12) and integrating by parts, we obtain
\[ I_2^2 \leq |z| \left( n(|z|) - \int_1^{[\frac{|z|}{t}]} \left( \exp(2\varphi(t)) \cdot \ln \frac{|z|}{t} \right) \cdot n(t) \cdot \left( 2\varphi(t) \cdot \ln \frac{|z|}{t} - \frac{2\varphi(t)}{t} \right) \right) \, dt. \]
Recall that \( \varphi \) is a monotone increasing function of class \( C^{(1)}(\mathbb{R}_+) \), so that \( \varphi'(t) \geq 0 \), \( t \in \mathbb{R}_+ \), and \( \ln \frac{|z|}{t} \geq 0, 1 \leq t \leq |z| \). We continue the last-written estimate:

\[
I_2^2 \leq n(|z|) \cdot |z| + 2|z| \cdot \int_1^{1/\ln |z|} \frac{n(t) \cdot \varphi(t)}{t} \cdot \exp\left(2\varphi(t) \cdot \ln \frac{|z|}{t}\right) dt
\]

\[
\leq 3 \cdot n(|z|) \cdot |z| \cdot \varphi(|z|) \cdot \int_1^{1/|z|} \exp\left(2\varphi(t) \cdot \ln \frac{|z|}{t}\right) dt.
\]

(13)

We denote the last integral in (13) by \( \tilde{I}_2^2 \). To estimate it, we put \( \ln \frac{|z|}{t} = u \), obtaining

\[
\tilde{I}_2^2 \leq |z| \cdot \int_0^{\ln |z|} \exp(2\varphi(|z|e^{-u})u) \cdot e^{-u} du \leq |z| \cdot \int_0^{\ln |z|} \exp(2\varphi(|z|e^{-u})u) du.
\]

(14)

Since \( \varphi \) is convex, we can bound the expression \( \varphi(|z|e^{-u})u \) for \( u \geq 1 \) as follows:

\[
\varphi(|z|e^{-u})u = u \int_0^{1/\ln |z|} \varphi'(t) dt = \int_0^{u|z|e^{-u}} \varphi'(\frac{x}{u}) dx
\]

\[
\leq \int_0^{1/\ln |z|} \varphi'(\frac{x}{u}) dx \leq \int_0^{1/\ln |z|} \varphi'(x) dx = \varphi(|z|).
\]

Thus,

\[
\varphi(|z|e^{-u})u \leq \varphi(|z|)
\]

(15)

for all \( u \geq 1 \).

Obviously, if \( 0 \leq u \leq 1 \), then

\[
\varphi(|z|e^{-u})u \leq \varphi(|z|).
\]

(16)

Consequently, \( \varphi(|z|e^{-u})u \leq \varphi(|z|) \) for all \( u \in \mathbb{R}_+ \). Therefore, estimate (14) implies that

\[
\tilde{I}_2^2 \leq |z| \cdot \exp(2\varphi(|z|)) \cdot \ln |z|.
\]

(17)

A combination of (14) and (15) yields

\[
I_2^2 \leq 3 \cdot n(|z|) \cdot \varphi(|z|) \cdot |z|^2 \cdot \ln |z| \cdot \exp(2\varphi(|z|)).
\]

Since \( \varphi \) is convex and \( \varphi'(x)x \to +\infty \) as \( x \to +\infty \), we can use Lemma 6 to obtain the final estimate:

\[
I_2^2 \leq C_1 \cdot \exp(\varphi(C_2 \cdot |z|)).
\]

(18)

The quantity \( I_2^2 \) can be estimated as follows:

\[
I_2^2 = \sum_{\frac{1}{2} < m_n(z) < 1} \exp\left((p_n + 1) \cdot \ln \frac{z}{|z_n|}\right) \leq \sum_{m_n(z) > \frac{1}{2}} \sum_{|z_n| < 2|z|} 1 \leq n(2|z|).
\]

Using (12), we conclude that

\[
n(2|z|) \leq C \cdot \exp(\varphi(2|z|)).
\]

(19)

To complete the proof of Theorem 2, now it suffices to combine estimates (18) and (19).
ON ZEROS OF ENTIRE FUNCTIONS WITH MAJORANT

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