FAMILIES OF FRACTIONAL CAUCHY TRANSFORMS
IN THE BALL

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Dedicated to Victor Petrovich Havin on the occasion of his 75th birthday

ABSTRACT. Let $B_n$ denote the unit ball in $\mathbb{C}^n$, $n \geq 1$. Given $\alpha > 0$, let $K_\alpha(n)$ denote the class of functions defined for $z \in B_n$ by integrating the kernel $(1 - \langle z, \zeta \rangle)^{-\alpha}$ against a complex-valued Borel measure on the sphere $\{\zeta \in \mathbb{C}^n : |\zeta| = 1\}$. The families $K_\alpha(1)$ of fractional Cauchy transforms have been investigated intensively by several authors. In the paper, various properties of $K_\alpha(n)$, $n \geq 2$, are studied. In particular, relations between $K_\alpha(n)$ and other spaces of holomorphic functions in the ball are obtained. Also, pointwise multipliers for the spaces $K_\alpha(n)$ are investigated.

§1. INTRODUCTION

For $n \geq 1$, put $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. Let $M(n)$ denote the space of complex-valued Borel measures on the sphere $\partial B_n$.

1.1. Fractional Cauchy transforms. Let $\alpha > 0$. Given a measure $\mu \in M(n)$, its fractional Cauchy transform of order $\alpha$ is defined by the identity

$$K_\alpha[\mu](z) = \frac{1}{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^{\alpha}} d\mu(\zeta), \quad z \in B_n.$$ 

Here and in what follows we use the principal branch of the logarithm. Put $K_\alpha(n) = \{K_\alpha[\mu] : \mu \in M(n)\}$.

1.2. Multipliers. Let $\mathcal{Hol}(B_n)$ denote the space of holomorphic functions in the ball $B_n$. A function $g \in \mathcal{Hol}(B_n)$ is called a multiplier for the family $K_\alpha(n)$, $\alpha > 0$, if $fg \in K_\alpha(n)$ for all $f \in K_\alpha(n)$. Let $\mathcal{M}_\alpha(n)$ denote the set of all multipliers for $K_\alpha(n)$.

1.3. Families $K_\alpha(n)$ and $\mathcal{M}_\alpha(n)$ as Banach spaces. For $f \in K_\alpha(n)$, put

$$\|f\|_{K_\alpha(n)} = \inf \{\|\mu\|_{M(n)} : f = K_\alpha[\mu]\}.$$ 

Standard arguments show that the above infimum is attained, and that $K_\alpha(n)$ with the norm $\|\cdot\|_{K_\alpha(n)}$ is a Banach space. Next, assume that $f = K_\alpha[\rho]$, where $\rho$ is a positive measure. For $\mu \in M(n)$, let $f = K_\alpha[\mu]$; then $\|\rho\| = K_\alpha[\rho](0) = K_\alpha[\mu](0) \leq \|\mu\|$. Therefore, $\|f\|_{K_\alpha(n)} = \|\rho\|_{M(n)}$.

For $g \in \mathcal{M}_\alpha(n)$, put

$$\|g\|_{\mathcal{M}_\alpha(n)} = \sup \{\|fg\|_{K_\alpha(n)} : \|f\|_{K_\alpha(n)} \leq 1\}.$$ 

The closed graph theorem guarantees that $\|g\|_{\mathcal{M}_\alpha(n)} < \infty$. The space $\mathcal{M}_\alpha(n)$ with the norm $\|\cdot\|_{\mathcal{M}_\alpha(n)}$ is a Banach space. Also, note that $\mathcal{M}_\alpha(n)$ is a Banach algebra.

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1.4. Historical remarks. The families $K_\alpha(n)$ generalize the classical family $K_\alpha(n)$ of Cauchy integrals. Investigations of the family $K_1(1)$ as a Banach space were initiated by Havin in [14, 15]. Further results and references are presented in the monograph [9]. The spaces $K_\alpha(1)$, $\alpha > 0$, were introduced by MacGregor [20]. Various properties of the families $K_\alpha(1)$ are collected in [17]. In the present paper we focus on the study of the families $K_\alpha(n)$ with $n \geq 2$. Certain properties of the spaces $K_n(n)$, $n \in \mathbb{N}$, were proved in the survey [3]. To the best knowledge of the author, for $n \geq 2$ the families $K_\alpha(n)$ have not been investigated systematically.

1.5. Related problems. Let $\sigma_n$ denote the normalized Lebesgue measure on the sphere $\partial B_n$. For $1 \leq p \leq \infty$ and $0 < \alpha < n$, put

$$K_\alpha^p(n) = \{ K_\alpha[f \sigma_n] : f \in L^p(\sigma_n) \}.$$ 

The boundary behavior of functions belonging to $K_\alpha^p(1)$ was investigated in [21], where the families $K_\alpha^p(1)$ were called Dirichlet type spaces. The families $K_\alpha^p(n)$, $n \in \mathbb{N}$, were investigated in [20]; see also [1, 19, 8].

1.6. Organization of the paper. Embedding properties for $K_\alpha(n)$ and $\mathfrak{M}_\alpha(n)$ are studied in [22]. Relations between $K_\alpha(n)$ and the classical spaces of holomorphic functions are obtained in §3 and §5. Also, in §4 and §5 we study differentiation operators on the spaces $K_\alpha(n)$. The boundary behavior of the fractional Cauchy transforms is investigated in §6. Finally, some results about the multiplier spaces $\mathfrak{M}_\alpha(n)$ are obtained in §7.

§2. Embedding properties

2.1. Embedding properties for families of fractional Cauchy transforms. Below we shall use the following lemma.

Lemma 2.1 (see [7] Lemma 1). Assume that $u, v \in B_1$, $\alpha > 0$, $\beta > 0$. Then

$$(1 - u)^{-\alpha}(1 - v)^{-\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha - 1}(1 - t)^{\beta - 1}[(1 - (tu + (1 - t)v)]^{-\alpha - \beta} dt.$$ 

Let $\mathcal{P}(n)$ denote the set of all probability measures on the sphere $\partial B_n$, $n \in \mathbb{N}$. For $n = 1$, the following result was obtained in [7].

Proposition 2.2. Put $\mathcal{F}_\alpha(n) = \{ K_\alpha[\mu] : \mu \in \mathcal{P}(n) \}$, $n \in \mathbb{N}$.

(i) If $\alpha > 0$ and $\beta > 0$, then $\mathcal{F}_\alpha(n) \cdot \mathcal{F}_\beta(n) \subset \mathcal{F}_{\alpha + \beta}(n)$.

(ii) If $0 < \alpha < \beta$, then $\mathcal{F}_\alpha(n) \subset \mathcal{F}_\beta(n)$.

Proof. First, we show how to deduce statement (i) from the following property:

$$(1 - \langle z, \zeta \rangle)^{-\alpha}(1 - \langle z, \xi \rangle)^{-\beta} \in \mathcal{F}_{\alpha + \beta}(n), \quad z \in B_n,$$

for any fixed points $\zeta, \xi \in \partial B_n$. So, let $f \in \mathcal{F}_\alpha(n)$, and let $g \in \mathcal{F}_\beta(n)$. Then, by the definition,

$$f(z)g(z) = \int_{\partial B_n \times \partial B_n} (1 - \langle z, \zeta \rangle)^{-\alpha}(1 - \langle z, \xi \rangle)^{-\beta} d\rho(\zeta, \xi), \quad z \in B_n,$$

where $\rho$ is a probability measure on the product $\partial B_n \times \partial B_n$. We approximate the measure $\rho$ in the weak* topology by probability measures $\rho_j$, where every $\rho_j$ is a finite sum of atomic charges. The set $\mathcal{F}_{\alpha + \beta}(n)$ is convex; therefore, on the one hand, we have

$$\int_{\partial B_n \times \partial B_n} (1 - \langle z, \zeta \rangle)^{-\alpha}(1 - \langle z, \xi \rangle)^{-\beta} d\rho_j(\zeta, \xi) \in \mathcal{F}_{\alpha + \beta}(n).$$


On the other hand, without loss of generality we may assume that the convergence
\[
\int_{\partial B_n} (1 - \langle z, \zeta \rangle)^{-\alpha} (1 - \langle z, \zeta \rangle)^{-\beta} \, dp_j(\zeta, \xi) \to f(z)g(z)
\]
as \(j \to \infty\) is uniform on the compact subsets of the ball \(B_n\). It remains to observe that the set \(F_{\alpha+\beta}(n)\) is closed in the topology of uniform convergence on compact subsets of the ball.

Now, we prove property (2.1). Lemma 2.4 guarantees that
\[
(1 - \langle z, \zeta \rangle)^{-\alpha} (1 - \langle z, \zeta \rangle)^{-\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1 - \langle z, w(t) \rangle]^{-\alpha-\beta} \, dt,
\]
where \(z \in B_n\) and \(w(t) = t\zeta + (1-t)\xi\). Since
\[
\Gamma(\alpha + \beta) \Gamma(\alpha)^{-1} \Gamma(\beta)^{-1} t^{\alpha-1}(1-t)^{\beta-1} \, dt
\]
is a probability measure on the interval \([0, 1]\), it suffices to show that
\[
(1 - \langle z, w(t) \rangle)^{-\alpha-\beta} \in F_{\alpha+\beta}(n), \quad z \in B_n,
\]
for every fixed parameter \(t \in [0, 1]\). If \(t = 0\) or \(t = 1\), then property (2.3) holds true by definition. So, assume that \(0 < t < 1\). Then \(w(t) \in B_n\). Next, note that \([1 - \langle z, \cdot \rangle]^{-\alpha-\beta}\) is a harmonic function in the closed ball \(B_n\). Hence,
\[
\frac{1}{1 - \langle z, w \rangle} = \frac{1}{|1 - \langle z, \xi \rangle|^{\alpha+\beta}} = \int_{\partial B_n} \frac{1 - |w|}{|w - \eta|^{2n}} \frac{1}{|1 - \langle z, \eta \rangle|^{\alpha+\beta}} \, d\sigma_n(\eta).
\]
If a point \(w \in B_n\) is fixed, then \((1 - |w|^2)|w - \eta|^{-2n} \, d\sigma_n(\eta)\) is a probability measure. Thus, to prove (2.3), it suffices to put \(w = w(t), \ 0 < t < 1\). The proof of property (i) is finished.

Let \(\beta > \alpha > 0\). Recall that the ball algebra \(A(B_n)\) is defined by
\[
A(B_n) = \text{Hol}(B_n) \cap C(B_n).
\]
If \(z \in B_n\), then \((1 - \langle z, \cdot \rangle)^{-\beta+\alpha} \in \tilde{A}(B_n)\). Hence,
\[
1 - (1 - \langle z, \zeta \rangle)^{-\beta+\alpha} = \int_{\partial B_n} \frac{1}{|1 - \langle z, \xi \rangle|^\beta} \, d\sigma_n(\zeta), \quad z \in B_n.
\]
In other words, \(1 \in F_{\beta-\alpha}(n)\); thus, (i) implies (ii).

\[\square\]

**Corollary 2.3.** Let \(n \in \mathbb{N}\).

(i) If \(\alpha > 0\) and \(\beta > 0\), then \(K_\alpha(n) \cdot K_\beta(n) \subset K_{\alpha+\beta}(n)\).

(ii) If \(0 < \alpha < \beta\), then \(K_\alpha(n) \subset K_\beta(n)\).

**Proof.** Let \(f = K_\alpha[\mu]\) and let \(g = K_\beta[\rho]\), where \(\mu, \rho \in M(n)\) and \(\alpha, \beta > 0\). Considering the Jordan decompositions of the measures \(\mu, \rho\) and applying part (i) of Proposition 2.2 we obtain properties (i) and (ii). \[\square\]

**2.2. Embedding properties for the multiplier families.** The proof of the following lemma will be omitted, because it practically coincides with that given in [29] for the case where \(n = \alpha = 1\) (see also [10], where the case of \(n = 1, \alpha > 0\) was considered).

**Lemma 2.4.** Assume that \(n \in \mathbb{N}\) and \(\alpha > 0\). Then the following properties are equivalent:

(i) \(g \in \mathfrak{M}_\alpha(n)\);

(ii) \(g(z)(1 - \langle z, \zeta \rangle)^{-\alpha} \in K_\alpha(n)\) for all \(\zeta \in \partial B_n\) and

\[
(2.5) \quad \sup \left\{ \left\| \frac{g(z)}{(1 - \langle z, \zeta \rangle)^\alpha} \right\|_{K_\alpha(n)} : \zeta \in \partial B_n \right\} < \infty.
\]
For $n = 1$, the following proposition was proved in [10].

**Proposition 2.5.** If $n \in \mathbb{N}$ and $0 < \alpha < \beta$, then $M_\alpha(n) \subset M_\beta(n)$.

**Proof.** Let $g \in M_\alpha(n)$. By Lemma 2.4, there exists a constant $C > 0$ and there exist measures $\mu_\zeta \in M(n)$, $\zeta \in \partial B_n$, such that $\|\mu_\zeta\| \leq C$ and

$$
g(z) = \frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} = \int_{\partial B_n} \frac{1}{(1 - \langle z, \xi \rangle)^\alpha} d\mu_\zeta(\xi), \quad z \in B_n.
$$

Hence,

$$
g(z) = \frac{1}{(1 - \langle z, \zeta \rangle)^\beta} = \int_{\partial B_n} \frac{1}{(1 - \langle z, \xi \rangle)^\beta} d\mu_\zeta(\xi).
$$

By (2.2) and (2.4), for every pair of points $\zeta, \xi \in \partial B_n$, there exists a probability measure $\rho_{\zeta, \xi} \in M(n)$ such that

$$
\frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} = \int_{\partial B_n} \frac{1}{(1 - \langle z, \eta \rangle)^\beta} d\rho_{\zeta, \xi}(\eta).
$$

Therefore,

$$
g(z) = \frac{1}{(1 - \langle z, \zeta \rangle)^\beta} = \int_{\partial B_n} \int_{\partial B_n} \frac{1}{(1 - \langle z, \eta \rangle)^\beta} d\rho_{\zeta, \xi}(\eta) d\mu_\zeta(\xi).
$$

Fix a point $\zeta \in \partial B_n$. We approximate the measure $\mu = \mu_\zeta$ in the weak* topology by measures $\mu_k$ such that $\|\mu_k\| \leq C$ and $\mu_k = \sum_{j=1}^{J(k)} a_{j,k}\delta_{\xi_j,k}$, $a_{j,k} \in \mathbb{C}$. Let $\lambda = \lambda_\zeta$ denote an accumulation point of the sequence $\lambda_k = \sum_{j=1}^{J(k)} a_{j,k}\mu_\zeta(\xi_j,k)$ in the weak* topology. Then

$$
g(z) = \frac{1}{(1 - \langle z, \zeta \rangle)^\beta} = \int_{\partial B_n} \frac{1}{(1 - \langle z, \eta \rangle)^\beta} d\lambda_\zeta(\eta),
$$

where $\|\lambda_\zeta\| \leq C$. Thus, Lemma 2.4 guarantees that $g \in M_\beta(n)$. □

§3. **Fractional Cauchy Transforms and Hardy Spaces**

For $f \in \text{Hol}(B_n)$, $0 < p < +\infty$ and $0 < r < 1$, put

$$M_p(f, r) = \left( \int_{\partial B_n} |f(r\zeta)|^p \, d\sigma_n(\zeta) \right)^{\frac{1}{p}}.
$$

Therefore, the classical Hardy space $H^p(B_n)$ is defined by

$$H^p(B_n) = \{ f \in \text{Hol}(B_n) : M_p(f, r) \leq C \}.
$$

Recall that if $f$ and $p$ are fixed, then the integral mean $M_p(f, r)$ is a monotone increasing function of the variable $r \in (0, 1)$.

For $n \in \mathbb{N}$, let $\mathcal{K}_0(n)$ denote the family of all functions $f$ such that

$$f(z) - f(0) = \int_{\partial B_n} \log \frac{1}{1 - \langle z, \zeta \rangle} \, d\mu(\zeta), \quad z \in B_n,
$$

for a measure $\mu \in M(n)$.

Let $f \in \mathcal{K}_0(n)$, $\alpha \geq 0$. In the present section we study the growth of the integral means $M_p(f, r)$ as $r \to 1-$. For $n = 1$, the corresponding results were obtained in [11].

The following lemma will be used below in numerous situations. The notation $a(z) \approx b(z)$ means that the quotient $a(z)/b(z)$ has a finite positive limit as $|z| \to 1-$.  

**Lemma 3.1** (see [23] Proposition 1.4.10]). *Suppose $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Put

$$I_c(z) = \int_{\partial B_n} \frac{d\sigma_n(\zeta)}{|1 - \langle z, \zeta \rangle|^{n+c}}, \quad z \in \overline{B_n}.
$$

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(i) If \( c < 0 \), then the function \( I_c(z) \) is bounded in the closed ball \( \bar{B}_n \).
(ii) If \( c > 0 \), then \( I_c(z) \approx (1 - |z|^2)^{-c} \).
(iii) Finally,
\[
I_0(z) \approx \log \frac{1}{(1 - |z|^2)}.
\]

**Proposition 3.2.** For \( \alpha \geq 0 \), let \( f \in \mathcal{K}_\alpha(n) \). The following estimates hold true as \( r \to 1^- \).

If \( \alpha = 0 \) and \( 0 < p < +\infty \), then
\[
M_p^r(f, r) = O(1).
\]
If \( 0 < \alpha \leq n \) and \( 0 < p < n/\alpha \), then
\[
M_p^r(f, r) = O(1).
\]
If \( 0 < \alpha \leq n \) and \( p = n/\alpha \), then
\[
M_p^r(f, r) = O\left( \log \frac{1}{1 - r} \right).
\]
If \( p > n/\alpha \) and \( p \geq 1 \), then
\[
M_p^r(f, r) = O\left( \frac{1}{(1 - r)^{\alpha p - n}} \right).
\]
If \( 0 < p < 1 \) and \( \alpha \geq 2n \), then
\[
M_p^r(f, r) = O\left( \frac{1}{(1 - r)^{\alpha n p}} \right).
\]

**Proof.** It is well known that \( \mathcal{K}_\alpha(n) \subset H^p(B_n) \) for all \( 0 < p < 1 \) \cite[Theorem 6.2.3]{23}. In other words, estimate (3.3) is fulfilled for \( \alpha = n \). Next, let \( f = K_{\alpha} \).

Assume that \( p \geq 1 \). Then the integral Minkowski inequality, Fubini’s theorem, and Lemma 5.1 guarantee that
\[
M_p^r(f, r) = \int_{\partial B_n} |f(r\xi)|^p \, d\sigma_n(\xi) \leq \int_{\partial B_n} \int_{\partial B_n} \frac{1}{|1 - \langle r\xi, \zeta \rangle|^\alpha} \, d\sigma_n(\xi) \, d\mu(\zeta)
\]
\[
= \begin{cases} 
O(1) & \text{if } \alpha p < n, \\
O\left( \log \frac{1}{1 - r} \right) & \text{if } \alpha p = n, \\
O\left( \frac{1}{(1 - r)^{\alpha p - n}} \right) & \text{if } \alpha p > n.
\end{cases}
\]

This proves estimates (3.4) and (3.5). Property (3.3) is proved for \( p \geq 1 \). Recall that \( H^q(B_n) \subset H^t(B_n) \) for \( q > t > 0 \). Hence, estimate (3.3) is valid for all \( 0 < p < n/\alpha \).

If \( \gamma > 0 \), then
\[
-\log 2 \leq \log \frac{1}{|1 - \langle r\xi, \zeta \rangle|} \leq C(\gamma) \frac{1}{|1 - \langle r\xi, \zeta \rangle|^\gamma},
\]
where \( 0 < r < 1 \) and \( \xi, \zeta \in \partial B_n \). Hence, estimate (3.2) is proved for all \( p \in (0, +\infty) \).

Now, assume that \( 0 < p < 1 \) and \( \alpha \geq 2n \). Note that \( (a + b)^p \leq C(p)(a^p + b^p) \) for \( a \geq 0, b \geq 0 \). Hence, by the Jordan decomposition theorem, we may assume that \( \mu \) is a probability measure. Therefore, we obtain
\[
|f(z)| \leq \int_{\partial B_n} \frac{1}{|1 - \langle z, \zeta \rangle|^\alpha} \, d\mu(\zeta)
\]
\[
\leq \frac{1}{(1 - |z|)^{\alpha - 2n}} \int_{\partial B_n} \frac{1}{|1 - \langle z, \zeta \rangle|^{2n}} \, d\mu(\zeta)
\]
\[
= \frac{1}{(1 - |z|)^{\alpha - n}} \int_{\partial B_n} \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}} \, d\mu(\zeta).
\]
Note that
\[ u(z) = \int_{\partial B_n} \frac{(1-|z|^2)^n}{|1-\langle z, \zeta\rangle|^{2n}} \, d\mu(\zeta) \]
is a positive $M$-harmonic function in the ball. Thus,
\[ \int_{\partial B_n} u(r\zeta) \, d\sigma_n(\zeta) = u(0) \]
for $0 < r < 1$; see [23, Theorem 4.2.4]. Next, since $1/p > 1$, Minkowski’s inequality yields
\[ (\int_{\partial B_n} u^p(r\zeta) \, d\sigma_n(\zeta))^{\frac{1}{p}} \leq \int_{\partial B_n} u^{\frac{p}{n}}(r\zeta) \, d\sigma_n(\zeta) = u(0) \]
for $0 < r < 1$.

Estimates (3.7) and (3.8) imply that
\[ M^p_p(f,r) = \int_{\partial B_n} |f(r\zeta)|^p \, d\sigma_n(\zeta) \leq \frac{1}{(1-r)(\alpha-n)p} u^p(0) = \frac{(\mu(\partial B_n))^p}{(1-r)(\alpha-n)p}. \]

This proves (3.6).

\[ \text{Corollary 3.3.} \quad \text{Let } n \in \mathbb{N}. \]

(i) If $0 \leq \alpha \leq n$, then $K_\alpha(n) \subset H^p(B_n)$ for $0 < p < n/\alpha$.

(ii) If $f \in H^1(B_n)$, then $f \in K_n(n)$; moreover, $\|f\|_{K_n(n)} \leq \|f\|_{H^1(B_n)}$.

(iii) $K_0(n) \subset K_n(n)$.

\[ \text{Proof.} \quad \text{Part (i) is equivalent to estimates (3.2) and (3.3). Next, if } f \in H^1(B_n), \text{ then } f \in K_n(n); \text{ moreover, } \|f\|_{K_n(n)} \leq \|f\|_{H^1(B_n)}. \]

Finally, part (iii) follows from (i) and (ii). \[ \square \]

Part (iii) of Corollary 3.3 will be improved in Corollary 5.5. Also, estimate (3.3) can be refined in terms of the weak Hardy spaces $H^{p,\infty}(B_n)$. Let $p > 0$. By definition, $f \in H^{p,\infty}(B_n)$ if $f \in H^p(B_n)$ for some $q > 0$ and
\[ \|f\|_{p,\infty} = \sup_{t>0} \int_{\partial B_n} |f^*(\zeta)|^t \, d\sigma_n(\zeta) \]
for all $f \in H^{p,\infty}(B_n)$, for some positive constants $C_1(p)$ and $C_2(p)$ (see, e.g., [10]).

\[ \text{Proposition 3.4.} \quad \text{If } 0 < \alpha \leq n, \text{ then } K_\alpha(n) \subset H^{\frac{n}{\alpha}}(B_n). \]

\[ \text{Proof.} \quad \text{The argument below is due to A. B. Aleksandrov. The case of } \alpha = n \text{ is well known. So, assume that } p = n/\alpha > 1; \text{ there exists a norm } \|\cdot\|_p \text{ on the space } H^{p,\infty}(B_n) \text{ such that} \]
\[ \|f\|_{p,\infty} \leq C_1(p) \|f\|_p \leq C_2(p) \|f\|_{p,\infty} \]
for all $f \in H^{p,\infty}(B_n)$, for some positive constants $C_1(p)$ and $C_2(p)$ (see, e.g., [10]).

Now, assume that $f \in K_\alpha(n)$. We must prove that $f \in H^{p,\infty}(B_n)$. Suppose, without loss of generality, that $\mu$ is a probability measure. We approximate the measure $\mu$ in the weak* topology by probability measures $\mu_k = \sum_{j=1}^{J(k)} a_{k,j} \delta_{\xi_{k,j}}$, where $a_{k,j} > 0$ and $\delta_{\xi_{k,j}}$ denotes the $\delta$-measure at the point $\xi_{k,j} \in \partial B_n$. Applying estimates (3.3), the triangle inequality, and Proposition 5.1.4 in [23], we obtain
\[ \|K_\alpha \mu_k\|_{p,\infty} \leq C_1(p) \|K_\alpha \mu_k\|_p \leq C_1(p) \|1-z_1\|^{-\alpha} \|f\|_p \]
\[ \leq C_2(p) \|1-z_1\|^{-\alpha} \|f\|_{p,\infty} \leq C(p,n). \]

Finally, since $\mu_k \to \mu$ in the weak* topology, we see that $f \in H^{p,\infty}(B_n)$. The proof is finished. \[ \square \]
§4. Families $K_\alpha(n)$ and Differentiation

Let $f \in \mathcal{H}o\ell(B_1)$, and let $\alpha \geq 0$. It is well known that $f \in K_\alpha(1)$ if and only if $f' \in K_{\alpha+1}(1)$. In the present section, we prove similar assertions for $f \in \mathcal{H}o\ell(B_n)$, $n \in \mathbb{N}$.

4.1. Radial derivatives. Let $f \in \mathcal{H}o\ell(B_n)$. The radial derivative $Rf$ is defined by the identity

$$ Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z). $$

Recall that

(4.1) $f(z) - f(0) = \int_{0}^{1} \frac{Rf(tz)}{t} dt, \quad z \in B_n.$

Proposition 4.1. Let $f \in \mathcal{H}o\ell(B_n)$, $n \in \mathbb{N}$. Then $f \in K_\alpha(n)$ if and only if $Rf \in K_{\alpha+1}(n)$.

Proof. Suppose that $f \in K_\alpha(n)$, that is, (4.1) is true. Then

$$ Rf(z) = \int_{\partial B_n} \frac{\langle z, \zeta \rangle}{\langle z, \zeta \rangle} d\mu(\zeta) = K_1[\rho](z), $$

where $\rho = \mu - \mu(\partial B_n)\sigma_1$.

To prove the reverse implication, assume that

$$ Rf(z) = \int_{\partial B_n} \frac{1}{1 - \langle z, \zeta \rangle} d\rho(\zeta), $$

where $\rho \in M(n)$. Note that $Rf(0) = 0$; thus, $\rho(\partial B_n) = 0$. Applying identity (4.1) and the property $\rho(\partial B_n) = 0$, we obtain

$$ f(z) - f(0) = \int_{0}^{1} \int_{\partial B_n} \frac{1}{t(1 - t\langle z, \zeta \rangle)} dt \frac{1}{1 - t\langle z, \zeta \rangle} d\rho(\zeta) dt $$
$$ = \int_{0}^{1} \int_{\partial B_n} \frac{1}{t} \frac{\langle z, \zeta \rangle}{1 - t\langle z, \zeta \rangle} d\rho(\zeta) dt $$
$$ = \int_{\partial B_n} \int_{0}^{1} \frac{\langle z, \zeta \rangle}{1 - t\langle z, \zeta \rangle} dt d\rho(\zeta) $$
$$ = \int_{\partial B_n} \log \frac{1}{1 - \langle z, \zeta \rangle} d\rho(\zeta), $$

as required. $\square$

Proposition 4.2. Assume that $\alpha > 0$ and $f \in \mathcal{H}o\ell(B_n)$, $n \in \mathbb{N}$. Then $f \in K_\alpha(n) + K_\alpha(n)$ if and only if $Rf \in K_{\alpha+1}(n)$.

Proof. Let $g \in K_\alpha(n)$ and $h \in K_\alpha(n)$, that is,

$$ h(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} d\mu(\zeta), \quad z \in B_n. $$

Straightforward calculations show that

$$ Rh(z) = \int_{\partial B_n} \frac{\alpha}{(1 - \langle z, \zeta \rangle)^{\alpha+1}} d\mu(\zeta) - \alpha h(z) \in K_{\alpha+1}(n) $$

because $\alpha h \in K_\alpha(n) \subset K_{\alpha+1}(n)$ by part (ii) of Corollary 2.3. On the other hand, Proposition 4.1 guarantees that $Rg \in K_1(n) \subset K_{\alpha+1}(n)$.

It is convenient to split the proof of the reverse implication into two steps.
Step 1. Let $\alpha = m \in \mathbb{N}$. By the hypothesis, we have
\[ \mathcal{R}f(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^{m+1}} d\mu(\zeta), \quad z \in B_n. \]
Hence,
\[ f(z) - f(0) = \int_0^1 \frac{\mathcal{R}f(tz)}{t} dt = \int_0^1 \int_{\partial B_n} \frac{1}{t(1 - t\langle z, \zeta \rangle)^{m+1}} d\mu(\zeta) dt, \quad z \in B_n. \]
If $w \in \mathbb{C}$ and $|w| < 1$, then
\[ \frac{1}{t(1 - tw)^{m+1}} = \frac{1}{t} + \sum_{j=1}^{m+1} \frac{w}{(1 - tw)^j}. \]
Since $\mathcal{R}f(0) = 0$, we have $\mu(\partial B_n) = 0$. Putting $w = \langle z, \zeta \rangle$, we obtain
\[ f(z) - f(0) = \sum_{j=1}^{m+1} \int_{\partial B_n} \int_0^1 \frac{\langle z, \zeta \rangle}{(1 - t\langle z, \zeta \rangle)^j} dt d\mu(\zeta) \]
\[ = - \int_{\partial B_n} \log(1 - \langle z, \zeta \rangle) d\mu(\zeta) + \sum_{j=2}^{m+1} \int_{\partial B_n} \frac{1}{(j-1)(1 - \langle z, \zeta \rangle)^{j-1}} d\mu(\zeta). \]
The embeddings $K_{j-1}(n) \subset K_m(n)$, $j = 2, \ldots, m$, guarantee that $f \in K_0(n) + K_m(n)$.

Step 2. Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Repeating the arguments of Step 1 and reversing the order of integration, we have
\[ f(z) - f(0) = \int_0^1 \int_{\partial B_n} \frac{1}{t(1 - t\langle z, \zeta \rangle)^{\alpha+1}} d\mu(\zeta) dt \]
\[ = \int_{\partial B_n} \int_0^1 \frac{\langle z, \zeta \rangle}{(1 - t\langle z, \zeta \rangle)^\alpha} dt d\mu(\zeta) + \int_0^1 \int_{\partial B_n} \frac{1}{t(1 - t\langle z, \zeta \rangle)^\alpha} d\mu(\zeta) dt. \]
The inner integral in the first summand can be calculated explicitly. So, consider the second summand. Put $[\alpha] = m \in \mathbb{N} \cup \{0\}$. Note that $m+1 > \alpha$ and $\mu(\partial B_n) = 0$. Hence, by part (ii) of Corollary 5.6, there exists a measure $\rho \in M(n)$ such that $K_{m+1}[\rho] = K_m[\mu]$. Also, we have $\rho(\partial B_n) = K_{m+1}[\rho](0) = K_m[\mu](0) = \mu(\partial B_n) = 0$. Therefore, reversing the order of integration once again, we obtain
\[ f(z) - f(0) = \frac{1}{\alpha} \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} d\mu(\zeta) + \int_{\partial B_n} \int_0^1 \frac{1}{t(1 - t\langle z, \zeta \rangle)^{m+1}} dt d\rho(\zeta). \]
By the definition, the first summand in the above sum belongs to $K_{\alpha}(n)$. If $m = 0$, then the proof of Proposition 5.1 shows that the second summand is in the family $K_0(n)$. Finally, for $m \in \mathbb{N}$, at Step 1 it was proved that the second summand belongs to $K_0(n) + K_m(n) \subset K_0(n) + K_{\alpha}(n)$. □

Note that Proposition 5.2 admits refinement (see Corollary 5.6).

4.2. Fractional differential operators of order 1. By definition, we put $R^1 f = f + \mathcal{R}f$ for $f \in \mathcal{H}ol(B_n)$. The operator $R^1$ is called the fractional differential operator of order 1. Observe that
\[ f(z) = \int_0^1 R^1 f(tz) dt, \quad z \in B_n. \]
The following proposition can be proved with the help of elementary methods.
Proposition 4.3. Let $f \in \mathcal{H}oI(B_n)$, $n \in \mathbb{N}$.

(i) If $\alpha > 0$, then the property $f \in K_\alpha(n)$ implies that $R^1 f \in K_{\alpha+1}(n)$.

(ii) If $\alpha \geq 1$, then the property $R^1 f \in K_{\alpha+1}(n)$ implies that $f \in K_{\alpha}(n)$.

Proof. Let $f \in K_\alpha(n)$, that is, $f = K_\alpha[\mu]$. Then

$$f + \mathcal{R}f = \alpha K_{\alpha+1}[\mu] + (1 - \alpha) f \in K_{\alpha+1}(n)$$

because $(1 - \alpha) f \in K_\alpha(n) \subset K_{\alpha+1}(n)$. The proof of part (i) is complete.

We split the proof of part (ii) into two steps.

Step 1. $\alpha = m \in \mathbb{N}$. By assumption, we have

$$R^1 f(tz) = \int_{\partial B_n} \frac{1}{(1 - t(z, \zeta))^{m+1}} d\mu(\zeta), \quad z \in B_n, \ t \in [0, 1].$$

Hence, using (4.2), we obtain

$$f(z) = \int_{\partial B_n} \int_0^1 \frac{1}{(1 - t(z, \zeta))^{m+1}} dt \ d\mu(\zeta), \quad z \in B_n.$$ 

Let $I(z, \zeta)$ denote the inner integral. If $\langle z, \zeta \rangle = 0$, then $I(z, \zeta) = 1$. If $\langle z, \zeta \rangle \neq 0$, then

$$I(z, \zeta) = \frac{1}{m} \frac{1 - (1 - \langle z, \zeta \rangle)^m}{\langle z, \zeta \rangle^{m+1}} = \frac{1}{m} \sum_{j=1}^{m} (1 - \langle z, \zeta \rangle)^{-j}.$$ 

Therefore, in both cases we have

$$I(z, \zeta) = \frac{1}{m} \sum_{j=1}^{m} (1 - \langle z, \zeta \rangle)^{-j}.$$ 

Consequently,

$$f(z) = \frac{1}{m} \sum_{j=1}^{m} \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^{m+1}} d\mu(\zeta).$$ 

Finally, we see that $f \in K_m(n)$, because $K_j(n) \subset K_m(n)$ for $j = 1, \ldots, m - 1$.

Step 2. $\alpha > 1$, $\alpha \notin \mathbb{N}$. Repeating the arguments of Step 1, we get

$$f(z) = \int_{\partial B_n} \int_0^1 \frac{1}{(1 - t(z, \zeta))^{\alpha+1}} dt \ d\mu(\zeta), \quad z \in B_n.$$ 

We represent the inner integral as a sum of two summands and integrate by parts in the second summand, obtaining

$$\int_0^1 \frac{1}{(1 - t(z, \zeta))^{\alpha+1}} dt = \int_0^1 \frac{1}{(1 - t(z, \zeta))^\alpha} dt + \int_0^1 \frac{t(z, \zeta)}{(1 - t(z, \zeta))^{\alpha+1}} dt$$

$$= \frac{1}{\alpha} \frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} + \frac{\alpha - 1}{\alpha} \int_0^1 \frac{1}{(1 - t(z, \zeta))^{\alpha}} dt.$$ 

Hence, Fubini’s theorem yields

$$f(z) = \frac{1}{\alpha} \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^\alpha} d\mu(\zeta) + \frac{\alpha - 1}{\alpha} \int_0^1 \int_{\partial B_n} \frac{1}{(1 - (t z, \zeta))^\alpha} d\mu(\zeta) dt.$$ 

By definition, the first summand in the above sum belongs to $K_\alpha(n)$. 

Consider the second summand. Let \([\alpha] = m \in \mathbb{N}\). Note that \(m + 1 > \alpha\). Hence, by part (ii) of Corollary 2.3, there exists a measure \(\rho \in M(n)\) such that \(K_{m+1}[\rho] = K_\alpha[\mu]\). Therefore, the second summand has the following form:

\[
\frac{\alpha - 1}{\alpha} \int_0^1 \int_{\partial B_n} \frac{1}{(1 - (tz, \zeta))^{m+1}} \, d\rho(\zeta) \, dt.
\]

At Step 1 it was proved that the above function belongs to \(K_m(n) \subset K_\alpha(n)\). \(\square\)

To refine Proposition 4.3, we need the following lemma.

**Lemma 4.4.** Assume that \(\alpha > 0\) and \(g(w) = 1 - (1 - w)^\alpha\), \(w \in B_1\). Then there exists a measure \(\rho \in M(1)\) such that \((1 - w)^{-\alpha} g(w) / w = K_\alpha[\rho](w), w \in B_1\).

**Proof.** The definitions imply that \((g(w)/w)'w^2 = g'(w)w - g(w) \in H^1(B_1)\), where \(H^1(B_1)\) denotes the Hardy space. Note that \(h \in H^1(B_1)\) if \(h \in \mathcal{H}(B_n)\) and \(h(w)w^2 \in H^1(B_1)\). Putting \(h(w) = (g(w)/w)'\), we obtain \((g(w)/w)' \in H^1(B_1)\). Hence, \((g(w)/w) \in M_\alpha(1)\) by [16, Theorem 3.5]. Since \((1 - w)^{-\alpha} \in K_\alpha(1)\), the required measure \(\rho \in M(1)\) exists by the definition of the multiplier space \(M_\alpha(1)\). \(\square\)

**Theorem 4.5.** Assume that \(\alpha > 0\) and \(f \in \mathcal{H}(B_n), n \in \mathbb{N}\). Then \(f \in K_\alpha(n)\) if and only if \(R^1 f \in K_{\alpha+1}(n)\).

**Proof.** Suppose that \(R^1 f = K_{\alpha+1}[\mu]\), where \(\mu \in M(n)\). Applying formula (4.2) and reversing the order of integration, we obtain

\[
\alpha f(z) = \int_{\partial B_n} \int_0^1 \frac{\alpha \, dt}{(1 - t(z, \zeta))^{\alpha+1}} \, d\mu(\zeta)
= \int_{\partial B_n} \frac{1 - (1 - (z, \zeta))^{\alpha}}{(z, \zeta)\left(1 - (z, \zeta)\right)^{\alpha}} \, d\mu(\zeta), \quad z \in B_n,
\]

where the integrand is assumed to be equal to \(\alpha\) if \((z, \zeta) = 0\). We put \(w = (z, \zeta)\) and apply Lemma 4.4. Reversing the order of integration, we have

\[
\alpha f(z) = \int_{\partial B_1} \int_{\partial B_n} \frac{1 - (z, \lambda \zeta)}{(z, \lambda \zeta)^{-\alpha}} \, d\mu(\zeta) \, d\rho(\lambda)
= \int_{\partial B_1} K_\alpha[\mu_{\lambda}](z) \, d\rho(\lambda), \quad z \in B_n,
\]

where \(\mu_{\lambda}(E) = \mu(\lambda E)\) for the Borel sets \(E \subset \partial B_n\). Note that \(K_\alpha[\mu_{\lambda}](z)\) is a continuous function of the variable \(\lambda \in \partial B_1\). Below we show that the last integral is in the family \(K_\alpha(n)\). Without loss of generality, we may assume that \(\rho\) is a probability measure. We approximate the measure \(\rho\) in the weak* topology by the probability measures \(\rho_k = \sum_{j=1}^{\#k} a_k,j \delta_{\lambda_k,j}\), where \(\delta_{\lambda_k,j}\) denotes the \(\delta\)-measure at \(\lambda_k,j \in \partial B_1\). By the definition of weak* convergence, we have

\[
\sum_{j=1}^{\#k} a_k,j K_\alpha[\mu_{\lambda_k,j}](z) \xrightarrow{\rho_k} \int_{\partial B_1} K_\alpha[\mu_{\lambda}](z) \, d\rho(\lambda), \quad z \in B_n.
\]

Next, observe that the sequence \(\left\{\sum_{j=1}^{\#k} a_k,j \mu_{\lambda_k,j}\right\}\) is norm bounded. Hence, there exists a subsequence which converges in the weak* topology to a measure \(\nu \in M(n)\). Therefore,

\[
\sum_{j=1}^{\#k} a_k,j K_\alpha[\mu_{\lambda_k,j}](z) \xrightarrow{\rho_k} \int K_\alpha[\nu](z), \quad z \in B_n.
\]

Finally, we obtain \(f = K_\alpha[\nu/\alpha]\). Now, it suffices to refer to Proposition 4.3. \(\square\)
Note that Theorem 4.5 extends to the case of $\alpha = 0$ (see Corollary 5.7).

§5. Families $\mathcal{K}_\alpha(n)$ and Bergman–Sobolev spaces

5.1. Modified operators of fractional differentiation. Consider a pair of parameters $(\beta, t) \in \mathbb{R}^2$ with the property that neither $n - 1 - \beta$ nor $n - 1 - \beta + t$ is a strictly negative integer. We define an operator

$$R^{\beta, t} : \mathcal{H}ol(B_n) \rightarrow \mathcal{H}ol(B_n)$$

as follows. If

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

is the homogeneous expansion of $f \in \mathcal{H}ol(B_n)$, then

$$R^{\beta, t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n-\beta)\Gamma(n+k-\beta+t)}{\Gamma(n-\beta+t)\Gamma(n+k-\beta)} f_k(z).$$

The inverse of $R^{\beta, t}$, denoted by $R_{\beta, t}$, is given by the formula

$$R_{\beta, t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n-\beta+t)\Gamma(n+k-\beta)}{\Gamma(n-\beta)\Gamma(n+k-\beta+t)} f_k(z).$$

Assume that $\beta < n$, $t > 0$, and identity (5.1) is true. Also, suppose that $r \in [0, 1]$ and $z \in B_n$. By the definition of the operator $R^{\beta, t}$, we have

$$\Gamma(n-\beta+t) \int_0^1 R^{\beta, t} f(rz)r^{\alpha-1}(1-r)^{t-1} dr = \sum_{k=0}^{\infty} f_k(z) = f(z).$$

Straightforward calculations imply the following lemma.

Lemma 5.1 (cf. [30] Proposition 1.14). Let $n \in \mathbb{N}$. Suppose that neither $n - 1 - \beta$ nor $n - 1 - \beta + t$ is a strictly negative integer. Then

$$R^{\beta, t} \left( \frac{1}{(1-(z,\zeta))^{n-\beta}} \right) = \frac{1}{(1-(z,\zeta))^{n-\beta+t}}$$

for all points $z \in B_n$ and $\zeta \in \partial B_n$.

5.2. A condition sufficient for membership in $\mathcal{K}_\alpha(n)$ with $\alpha > n$. For $n = 1$, the following fact was proved in [13].

Proposition 5.2. Suppose $n \in \mathbb{N}$ and $\alpha > n$. If $g \in \mathcal{H}ol(B_n)$ and

$$\int_0^1 \int_{\partial B_n} |g(r\zeta)|(1-r)^{\alpha-n-1} d\sigma_n(\zeta) dr = V < +\infty,$$

then $g \in \mathcal{K}_\alpha(n)$ and $\|g\|_{\mathcal{K}_\alpha(n)} \leq C(\alpha, n, V) = (\alpha - n) \cdots (\alpha - 1)V/(n-1)!$.

Proof. Put $t = \alpha - n$ and $f = R_{0, t} g$. Then $g = R_{0, t} f$ and

$$f(z) = \frac{\Gamma(n+t)}{\Gamma(n)\Gamma(t)} \int_0^1 g(rz)r^{\alpha-1}(1-r)^{t-1} dr$$
by \((\ref{5.3})\). If \(0 < s < 1\), then
\[
\frac{\Gamma(n)\Gamma(t)}{\Gamma(n+t)} \int_{\partial B_n} |f(s\zeta)| \, d\sigma_n(\zeta) \leq \int_0^1 \int_{\partial B_n} |g(r\zeta)| \, d\sigma_n(\zeta)(1-r)^{t-1} \, dr \\
\leq \int_0^1 \int_{\partial B_n} |g(r\zeta)| \, d\sigma_n(\zeta)(1-r)^{t-1} \, dr \\
= V < +\infty.
\]

Therefore, \(f \in H^1(B_n) \subset \mathcal{K}_n(n)\) and \(\|f\|_{\mathcal{K}_n(n)} \leq \|f\|_{H^1(B_n)} \leq C(\alpha, n, V)\). In other words,
\[
f(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^n} \, d\mu(\zeta), \quad z \in B_n,
\]
where \(\|\mu\|_{M(n)} \leq C(\alpha, n, V)\). Thus, with the help of Lemma \((\ref{5.1})\) we obtain
\[
g(z) = R^{\alpha+t}f(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^{n+t}} \, d\mu(\zeta), \quad z \in B_n.
\]
Hence, \(\|g\|_{\mathcal{K}_n(n)} \leq C(\alpha, n, V)\). \hfill \qedsymbol

### 5.3. Bergman–Sobolev spaces.

Suppose \(j \in \mathbb{N}\) and \(q > 0\). By definition, the Bergman–Sobolev space \(A^1_{\alpha,j}(B_n)\) consists of all functions \(f \in \mathcal{H}ol(B_n)\) for which
\[
\int_0^1 \int_{\partial B_n} |R^j f(r\zeta)| (1-r)^q \, d\sigma_n(\zeta) \, dr < +\infty,
\]
where \(R^j = (R^1)^j\). For \(n = 1\), the following assertion coincides with \((\ref{13})\) Lemma 2.

**Proposition 5.3.** Suppose \(n \in \mathbb{N}\), \(j \in \{1, \ldots, n\}\), and \(\alpha > n - j\).

(i) If \(f \in A^1_{\alpha-n+j,j}(B_n)\), then \(f \in \mathcal{K}_n(n)\).

(ii) Let \(\beta > \alpha\). Then \(\mathcal{K}_n(n) \subset A^1_{\beta-n+j,j}(B_n)\).

**Proof.** We prove part (i). Since \(\alpha + j > n\), Proposition \((\ref{5.2})\) shows that \(R^j f \in \mathcal{K}_{\alpha+j}(n)\). Since \(\alpha > 0\), repeated application of Theorem \((\ref{4.5})\) guarantees that \(f \in \mathcal{K}_n(n)\).

Now, we turn to part (ii). Let \(f \in \mathcal{K}_n(n)\), \(\alpha > n - j\). By Theorem \((\ref{4.0})\) we have \(R^j f \in \mathcal{K}_{\alpha+j}(n)\); that is,
\[
R^j f(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \zeta \rangle)^{\alpha+j}} \, d\mu(\zeta), \quad z \in B_n,
\]
for some measure \(\mu \in M(n)\). Therefore, if \(0 \leq r < 1\), then
\[
\int_{\partial B_n} |R^j f(r\zeta)| \, d\sigma_n(\zeta) \leq \int_{\partial B_n} \int_{\partial B_n} \frac{1}{|1 - r \langle \zeta, \zeta' \rangle|^{\alpha+j}} \, d\sigma_n(\zeta) \, d|\mu|(\zeta).
\]

Since \(\alpha + j > n\), Lemma \((\ref{3.1})\) yields
\[
\int_{\partial B_n} |R^j f(r\zeta)| \, d\sigma_n(\zeta) \leq C|\mu|_{M(n)} (1-r)^{n-\alpha-j}.
\]

Now, assume that \(\beta > \alpha\). Then the above inequality guarantees that
\[
\int_0^1 \int_{\partial B_n} |R^j f(r\zeta)|(1-r)^{\beta-n+j-1} \, d\sigma_n(\zeta) \, dr \leq C|\mu| \int_0^1 (1-r)^{\beta-\alpha-1} < +\infty.
\]
In other words, \(f \in A^1_{\beta-n+j,j}(B_n)\). \hfill \qedsymbol
5.4. Bergman–Sobolev spaces and $K_0(n)$.

**Lemma 5.4.** Let $n \in \mathbb{N}$. Then $K_0(n) \subset A^1_{\alpha,n}(B_n)$ for any $\varepsilon > 0$.

**Proof.** Let $f \in K_0(n)$. Recall that $K_0(n) \subset K_n(n)$ by part (iii) of Corollary 5.3. Next, $R^j f \in K_j(n) \subset K_n(n)$ for $j = 1, 2, \ldots, n$ by Propositions 4.1, 4.2 and Corollary 5.3. Hence, $R^n f = (I + R)^n f \in K_n(n)$; that is,

$$R^n f(z) = \frac{1}{(1 - \langle z, \zeta \rangle)^n} d\mu(\zeta), \quad z \in B_n,$$

for some measure $\mu \in M(n)$. Hence, if $0 \leq r < 1$, then

$$\int_{\partial B_n} |R^n f(r\xi)| \, d\sigma_n(\xi) \leq \int_{\partial B_n} \int_{\partial B_n} \frac{1}{|1 - r(\xi, \zeta)|^n} d\sigma_n(\xi) \, d\mu(\zeta).$$

By Lemma 3.1 we have

$$\int_{\partial B_n} |R^n f(r\xi)| \, d\sigma_n(\xi) \leq C \log \frac{e}{1 - r}.$$

The above estimate implies that

$$\int_0^1 \int_{\partial B_n} |R^n f(r\xi)|(1 - r)^{\varepsilon - 1} \, d\sigma_n(\xi) \, dr \leq C \int_0^1 (1 - r)^{\varepsilon - 1} \log \frac{e}{1 - r} < +\infty,$$

as required. \hfill \Box

5.5. An embedding property for $K_0(n)$. It is well known that $K_0(1) \subset K_\alpha(1)$ for all $\alpha > 0$.

**Corollary 5.5.** Let $n \in \mathbb{N}$. Then $K_0(n) \subset K_\alpha(n)$ for all $\alpha > 0$.

**Proof.** Lemma 5.4 guarantees that $K_0(n) \subset A^1_{\alpha,n}(B_n), \alpha > 0$. It remains to note that $A^1_{\alpha,n}(B_n) \subset K_\alpha(n)$ by part (i) of Proposition 5.3. As B. Aleksandrov observed that Corollary 5.5 can also be deduced from the embedding $K_0(1) \subset K_\alpha(1)$ for all $\alpha > 0$.

5.6. The families $K_\alpha(n), \alpha \geq 0,$ and differentiation. We have the following refinement of Proposition 4.2.

**Corollary 5.6.** Suppose $n \in \mathbb{N}$ and $\alpha \geq 0$. Then $f \in K_\alpha(n)$ if and only if $R f \in K_{\alpha + 1}(n)$.

**Proof.** We apply Propositions 4.1, 4.2 and Corollary 5.5. \hfill \Box

The next corollary refines Theorem 4.3.

**Corollary 5.7.** Suppose $n \in \mathbb{N}$ and $\alpha \geq 0$. Then $f \in K_\alpha(n)$ if and only if $R^j f \in K_{\alpha + j}(n)$.

**Proof.** The case of $\alpha > 0$ is covered by Theorem 4.3. So, assume that $f \in K_0(n)$. Corollary 5.5 guarantees that $f \in K_1(n)$. On the other hand, $R f \in K_1(n)$ by Corollary 5.6. Therefore, $R^j f = f + R f \in K_1(n)$.

To prove the reverse implication, assume that $R^j f \in K_j(n)$. Since $K_1(n) \subset K_2(n)$, we have $R^j f \in K_0(n)$. Hence, $f \in K_1(n)$ by Theorem 4.5. Therefore, $R f = R^j f - f \in K_1(n)$. So, $f \in K_0(n)$ by Corollary 5.6. \hfill \Box
5.7. Modified Bergman–Sobolev spaces and $\mathcal{K}_\alpha(n)$. Suppose $n \in \mathbb{N}$, $j \in \mathbb{N}$, and $q > 0$. In some applications, it is convenient to use the modified Bergman–Sobolev space $\tilde{A}_q^1(B_n)$ in place of $A_q^1(B_n)$. By definition, the space $\tilde{A}_q^1(B_n)$ consists of functions $f \in \mathcal{H}(B_n)$ such that

$$
\| f \|_{\tilde{A}_q^1(B_n)} = \sum_{|m| \leq 1} \left| \frac{\partial^m f}{\partial z^m}(0) \right| + \int_0^1 \int_{\partial B_n} |R^j f(r\zeta)|(1-r)^{q-1} d\sigma_n(\zeta) dr
$$

where $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ and $|m| = m_1 + \cdots + m_n$. The space $\tilde{A}_q^1(B_n)$ with the norm $\| \cdot \|_{\tilde{A}_q^1(B_n)}$ is a Banach space (cf. [30, Proposition 6.2]).

**Proposition 5.8.** Suppose $n \in \mathbb{N}$, $j \in \{1, \ldots, n\}$, and $\alpha > n - j$.

(i) The embedding $\tilde{A}_q^1(B_n) \subset \mathcal{K}_\alpha(n)$ holds true. If $f \in \tilde{A}_q^1(B_n)$, then $\| f \|_{\mathcal{K}_\alpha(n)} \leq C \| f \|_{\tilde{A}_q^1(B_n)}$, where the constant $C > 0$ does not depend on $f$.

(ii) Let $\beta > \alpha$. Then $\mathcal{K}_\alpha(n) \subset \tilde{A}_q^1(B_n)$.

**Proof.** Assume that $f \in \tilde{A}_q^1(B_n)$. We have

$$
\int_0^1 \int_{\partial B_n} |R^j f(r\zeta)|(1-r)^{\alpha+j-1} d\sigma_n(\zeta) dr < +\infty.
$$

Since $\alpha + j > n$, Proposition 5.2 yields $R^j f \in \mathcal{K}_\alpha(n)$. Repeated application of Corollary 5.6 shows that $f \in \mathcal{K}_\alpha(n)$.

Now, consider the operator $I : \tilde{A}_q^1(B_n) \to \mathcal{K}_\beta(n)$ defined by $If = f$. If a functional sequence converges in the space $\tilde{A}_q^1(B_n)$ or in the space $\mathcal{K}_\alpha(n)$, then this sequence converges uniformly on the compact subsets of the ball $B_n$. Hence, the graph of the operator $I$ is closed. So, by the closed graph theorem,

$$
\| f \|_{\mathcal{K}_\alpha(n)} \leq C \| f \|_{\tilde{A}_q^1(B_n)}
$$

for all $f \in \tilde{A}_q^1(B_n)$. Part (i) is proved.

Part (ii). Let $f \in \mathcal{K}_\alpha(n)$, $\alpha > n - j$. We have $R^j f \in \mathcal{K}_\alpha(n)$ by Proposition 4.2. To finish the argument, it suffices to repeat the proof of part (ii) of Proposition 5.3. \(\square\)

5.8. Fractional Cauchy transforms and inner functions. Assume that $n \in \mathbb{N}$, $j \in \mathbb{N}$, and $q > 0$. The space $\tilde{A}_q^2(B_n)$ consists of all functions $f \in \mathcal{H}(B_n)$ such that

$$
\int_0^1 \int_{\partial B_n} |R^j f(r\zeta)|^2(1-r)^{-q} d\sigma_n(\zeta) dr < +\infty.
$$

**Lemma 5.9.** Suppose $n \in \mathbb{N}$, $j \in \{1, \ldots, n\}$, and $\beta > \alpha > n - j$. Then

$$
H^\infty(B_n) \cap \mathcal{K}_\alpha(n) \subset \tilde{A}_q^2(B_n).
$$

**Proof.** Let $f \in H^\infty(B_n) \cap \mathcal{K}_\alpha(n)$. Since $f \in H^\infty(B_n)$, we have

$$
f(z) = \int_{\partial B_n} \frac{f^*(\zeta)}{(1-z, \zeta)^n} d\sigma_n(\zeta), \quad z \in B_n.
$$

Therefore,

$$
R f(z) = \int_{\partial B_n} \frac{n(z, \zeta)f^*(\zeta)}{(1-z, \zeta)^{n+1}} d\sigma_n(\zeta), \quad z \in B_n.
$$
Hence, using Lemma 3.1, we obtain

\[ |Rf(z)| \leq \int_{\partial B_n} \frac{C}{|1 - (z, \zeta)|^{n+1}} \, d\sigma_n(\zeta) \leq \frac{C}{(1 - |z|)^j}, \quad z \in B_n. \]

Arguing similarly, we verify that

\[ (5.4) \quad |R^j f(z)| \leq \frac{C}{(1 - |z|)^j}, \quad z \in B_n. \]

Since \( f \in \mathcal{K}_\alpha(n) \), Proposition 4.2 guarantees that \( R^j f \in \mathcal{K}_{\alpha+j}(n) \). Since \( \alpha + j > n \) and \( \beta > \alpha \), part (ii) of Proposition 5.8 yields

\[ \int_0^1 \int_{\partial B_n} |R^j f(r\zeta)|(1 - r)^{\beta-n+j-1} \, d\sigma_n(\zeta) \, dr < \infty. \]

Hence, applying inequality (5.4), we get

\[ \int_0^1 \int_{\partial B_n} |R^j f(r\zeta)|(1 - r)^j \cdot |R^j f(r\zeta)|(1 - r)^{\beta-n+j-1} \, d\sigma_n(\zeta) \, dr < +\infty, \]

as required. \( \square \)

Recall that a nonconstant function \( f \in H^\infty(B_n) \) is said to be inner if \( |f^*| = 1 \) \( \sigma_n \)-a.e. On the one hand, the families \( \mathcal{K}_\alpha(n) \) with \( \alpha \geq n \) contain all inner functions because \( H^\infty(B_n) \subset H^1(B_n) \subset \mathcal{K}_n(n) \subset \mathcal{K}_\alpha(n) \). On the other hand, the following assertion is true for \( n \geq 2 \).

**Proposition 5.10.** Let \( n \geq 2 \). If \( 0 \leq \alpha < n - 1/2 \), then the family \( \mathcal{K}_\alpha(n) \) contains no inner functions.

**Proof.** Let \( f \in \mathcal{K}_\alpha(n) \) be an inner function. By Corollary 5.5 and part (ii) of Corollary 2.3, there is no loss of generality in assuming that \( \alpha \in (n - 1, n - 1/2) \).

Suppose \( \rho \in [1/2, 1) \) and \( f_\rho(\zeta) = f(\rho\zeta) \), \( \zeta \in \partial B_n \). For \( \zeta \in \partial B_n \) and \( \lambda \in B_1 \), put \( f_\lambda(\zeta) = f(\lambda\zeta) \). Observe that \( R f(\lambda\zeta) = \lambda f'_\lambda(\zeta) \); hence,

\[ (5.5) \quad |f^*(\zeta) - f_\rho(\zeta)| \leq 2 \int_0^1 |R f(r\zeta)| \, dr \]

provided that \( f^*(\zeta) \) is well defined. Applying estimate (5.5), Hölder’s inequality, and Fubini’s theorem, we obtain

\[ \|f_\rho - f^*\|_{L^2(\partial B_n)}^2 \leq C \int_{\partial B_n} \left( \int_0^1 |R f(r\zeta)|(1 - r)^{\frac{j}{2}}(1 - r)^{-\frac{n}{2}} \, dr \right)^2 \, d\sigma_n(\zeta) \]

\[ \leq C(1 - \rho)^{\frac{j}{2}} \int_{\partial B_n} \int_0^1 |R f(r\zeta)|^2(1 - r)^{\frac{j}{2}} \, dr \, d\sigma_n(\zeta) \]

\[ = o(1 - \rho)^{\frac{j}{2}} \text{ as } \rho \to 1 \]

by Lemma 5.9. Tamm [27] proved that no inner function can satisfy the estimate \( \|f_\rho - f^*\|_{L^2(\partial B_n)}^2 = o(1 - \rho)^{\frac{j}{2}} \) as \( \rho \to 1 \). This contradiction finishes the proof of the proposition. \( \square \)
§6. Boundary behavior

6.1. Directions of the maximal radial growth. The definition of the norm in the space $K_\alpha(n)$, $\alpha > 0$, implies that

$$|f(z)| \leq \frac{|f|_{K_\alpha(n)}}{(1 - |z|)^\alpha}, \quad z \in B_n.$$ 

Proposition 6.1 shows that this maximal growth is possible for an at most countable set of radial directions. For $n = 1$, this was proved in [28]. Suppose $\xi \in \partial B_n$ and $C > 1$. Recall that the corresponding Korányi domain $D_C(\xi)$ is defined by

$$D_C(\xi) = \{z \in B_n : |1 - \langle z, \xi \rangle| < C(1 - |z|)\}.$$ 

Proposition 6.1. Suppose $n \in \mathbb{N}$, $\mu \in M(n)$, $\alpha > 0$, and $f = K_\alpha[\mu]$. If $\xi \in \partial B_n$, then

$$\lim_{z \to \xi} \frac{(1 - \langle z, \xi \rangle)^\alpha}{(1 - |z|)^\alpha} f(z) = \mu(\{\xi\})$$

for all $C > 1$.

Proof. Let $\zeta, \xi \in \partial B_n$. We have

$$\lim_{z \to \xi} \frac{(1 - \langle z, \zeta \rangle)^\alpha}{(1 - |z|)^\alpha} = \delta_{\zeta, \xi},$$

where $\delta_{\zeta, \zeta} = 0$ for $\zeta \neq \xi$, and $\delta_{\zeta, \zeta} = 1$. If $z \in D_C(\xi)$, then

$$\left|\frac{1 - \langle z, \xi \rangle}{1 - \langle z, \zeta \rangle}\right| \leq \frac{|1 - \langle z, \xi \rangle|^\alpha}{|1 - \langle z, \zeta \rangle|^\alpha} \leq C^\alpha.$$ 

Hence, if $f = K_\alpha[\mu]$, then

$$\lim_{z \to \xi} \frac{(1 - \langle z, \xi \rangle)^\alpha}{(1 - |z|)^\alpha} f(z) = \lim_{z \to \xi} \int_{\partial B_n} \frac{(1 - \langle z, \xi \rangle)^\alpha}{(1 - |z|)^\alpha} d\mu(\zeta) = \mu(\{\xi\})$$

by the dominated convergence theorem. □

6.2. Radial behavior of functions in $K_\alpha(n)$ with $\alpha > n$. Recall that the Bloch space $\mathcal{B}(B_n)$ consists of the functions $f \in \mathcal{H}(B_n)$ such that

$$|R f(z)| \leq \frac{C}{1 - |z|}, \quad z \in B_n,$$

for some constant $C > 0$ (see [30, Chapter 3] for equivalent definitions).

Proposition 6.2. Suppose $n \in \mathbb{N}$ and $\alpha > n$. Then $\mathcal{B}(B_n) \subset K_\alpha(n)$.

Proof. Let $f \in \mathcal{B}(B_n)$, that is, $|R f(r\zeta)| \leq C(1 - r)^{-1}$ for all $\zeta \in \partial B_n$ and $r \in [0, 1)$. The assumption $\alpha > n$ guarantees that

$$\int_0^1 \int_{\partial B_n} |R f(r\zeta)| (1 - r)^{\alpha - n - 1} d\sigma_n(\zeta) dr < +\infty.$$ 

Since $\alpha + 1 > n$, we have $R f \in K_{\alpha + 1}(n)$ by Proposition 5.2. Finally, applying Corollary 5.6, we obtain $f \in K_\alpha(n)$. □

Corollary 6.3. Let $n \in \mathbb{N}$. Then there exists a function $f \in \bigcap_{\alpha > n} K_\alpha(n)$ such that the finite limit $\lim_{r \to 1^-} f(r\zeta)$ fails to exist for all $\zeta \in \partial B_n$.

Proof. Ullrich [28] constructed a function $f \in \mathcal{B}(B_n)$ with no finite radial limits. It remains to apply Proposition 6.2. □
6.3. Exceptional sets and Hausdorff contents: definitions. Let $0 < \tau \leq 1$, and let $\zeta \in \partial B_n$. Put

$$D_{\tau,C}(\zeta) = \{z \in B_n : |1 - (z, \zeta)| < C(1 - |z|)^\tau\},$$

where $C > 1$ for $\tau = 1$, and $C > 0$ for $\tau \in (0, 1)$. Note that the sets $D_{1,C}(\zeta)$ coincide with the Korányi domains $D_C(\zeta)$, which were considered in Subsection 6.1. Also, note that the order of contact between the set $D_{\tau,C}(\zeta)$ and the sphere $\partial B_n$ increases as the parameter $\tau$ decreases.

Consider a function $f : B_n \to \mathbb{C}$. By definition, the set $E_{\tau,C}(f)$ consists of all $\zeta \in \partial B_n$ for which the function $f(z)$ fails to have a limit at the point $\zeta$ as $z$ approaches $\zeta$ through $D_{\tau,C}(\zeta)$. The sets $E_{1,C}(f)$ are said to be exceptional; the sets $E_{\tau,C}(f)$, $0 < \tau < 1$, are tangentially exceptional.

To estimate the size of the sets $E = E_{\tau,C}(f)$ for $f \in K_\alpha(n)$, it is natural to apply the nonisotropic Hausdorff contents, which are defined as follows:

$$(6.1) \quad H_m(E) = H_m(E, \partial B_n) = \inf \left\{ \sum_k \delta_k^m : E \subset \bigcup_k Q(\zeta_k, \delta_k) \right\},$$

where $\zeta_k \in \partial B_n$ and $Q(\zeta, \delta) = \{\xi \in \partial B_n : |1 - (\zeta, \xi)| < \delta\}$.

6.4. Hardy–Sobolev spaces. Assume that a function $f \in \mathcal{H}ol(B_n)$ has homogeneous expansion $f = \sum_k f_k$. Then the fractional derivative of order $\beta > 0$ is defined by the formula

$$R^\beta f(z) = \sum_k (k + 1)^\beta f_k(z), \quad z \in B_n.$$ 

For $\beta > 0$ and $0 < p < \infty$, the Hardy–Sobolev spaces are defined by

$$H_p^\beta(B_n) = \left\{ f \in \mathcal{H}ol(B_n) : \sup_{0 < r < 1} \int_{\partial B_n} |R^\beta f(r\zeta)|^p \, d\sigma_n(\zeta) < \infty \right\}.$$ 

Theorem 6.4 (Ahern and Cohn [1]). Let $n - \beta p > 0$. If $f \in H_p^\beta(B_n)$, then

$$H_{n-\beta p+\varepsilon}(E(f)) = 0$$

for all $\varepsilon > 0$.

Theorem 6.5 (Cascante and Ortega [8, Corollary 2.1]; see also [26]). Assume that $n \in \mathbb{N}$, $0 < p < \infty$, $0 < \beta < n/p$, $0 < \tau < 1$, and $m = (n - \beta p)/\tau$. If $f \in H_p^\beta(B_n)$, then $H_m(E_{\tau,C}(f)) = 0$ for all $C > 0$.

6.5. Exceptional sets. By Corollary 6.3, there exists a function $f \in \bigcap_{\alpha > n} K_\alpha(n)$ such that $E_{1,C}(f) = \partial B_n$ for all $C > 1$. On the other hand, if $0 \leq \alpha \leq n$, then $K_\alpha(n) \subset H_p^\beta(B_n)$ for all $0 < p < 1$. Hence, if $f \in K_\alpha(n)$, $0 \leq \alpha \leq n$, then $\sigma_n(E_{1,C}(f)) = 0$ for all $C > 1$. This observation can be refined in the case where $0 \leq \alpha < n$. For the spaces $K_\alpha(1)$, $0 \leq \alpha < 1$, similar refinements in terms of the Bessel capacities were obtained in [12].

Proposition 6.6. Suppose $n \in \mathbb{N}$ and $\alpha \in (0,n)$. If $f \in K_\alpha(n)$, then $H_{n+\varepsilon}(E_{1,C}(f)) = 0$ for all $\varepsilon > 0$ and $C > 1$.

Proof. Let $n - \alpha > \varepsilon > 0$. Then $K_\alpha(n) \subset A^1_{n+\varepsilon,n}(B_n) \subset H^1_{n-\alpha-\varepsilon}(B_n)$ by Proposition 5.3 and by [4, Theorem 5.12], respectively. It remains to apply Theorem 6.3.

The following example corresponds to Proposition 6.6.

Proposition 6.7. Suppose $n \in \mathbb{N}$, $\alpha \in (0,n)$, $0 \leq m < \alpha$, $E \subset \partial B_n$ is a compact set, $H_m(E) = 0$, and $C > 1$. Then $E = E_{1,C}(f)$ for some function $f \in K_\alpha(n)$.
Proof. Fix $p_0 > 1$ such that $m = n - (n - \alpha)p_0$. We have $H_m(E) = 0$; hence, Theorem 1.2 in [11] guarantees that $E = E_{I,C}(f)$ for some function $f \in H_{n-\alpha}^p(B_n)$. Next, recall that 
\[ K_{n-\alpha}^p(n) = \{ K_\alpha[g\sigma_n] : g \in L^p_\alpha(\sigma_n) \}. \]
By [8, Theorem 2.1], we have $H_{n-\alpha}^p(B_n) = K_{n-\alpha}^p(n)$. It remains to note that $K_{n-\alpha}^p(n) \subset K_\alpha(n)$. □

### 6.6. Tangentially exceptional sets.
For the spaces $K_\alpha(1) \leq \alpha < 1$, tangentially exceptional sets were investigated in [10]. In the case of an arbitrary dimension $n \in \mathbb{N}$, the following assertion holds true.

**Proposition 6.8.** Suppose $n \in \mathbb{N}$, $\alpha \in (0, n)$, $\tau \in (\alpha/n, 1)$, and $m > \alpha/\tau$. If $f \in K_\alpha(n)$, then $H_m(E_{\tau,C}(f)) = 0$ for all $C > 0$.

**Proof.** We have $K_\alpha(n) \subset H_{n-\alpha-\epsilon}(B_n)$ for $n - \alpha > \epsilon > 0$. It remains to apply Theorem 6.5. □

The following example corresponds to Proposition 6.8.

**Proposition 6.9.** Suppose $n \in \mathbb{N}$, $\alpha \in (0, n)$, and $\tau = \alpha/n, 1)$, $m < \alpha/\tau$, and $C > 0$. If $E \subset \partial B_n$ is a compact set and $H_m(E) = 0$, then $E = E_{\tau,C}(f)$ for some function $f \in K_\alpha(n)$.

**Proof.** Since $m < \alpha/\tau$, there exists $p_0 > 1$ such that 
\[ m = (n - (n - \alpha)p_0)/\tau. \]
Since $p_0 > 1$ and $H_m(E) = 0$, we have $E = E_{\tau,C}(f)$ for some function $f \in K_{n-\alpha}^p(n)$; see [26, Remark after Corollary 3.11]. It remains to note that $K_{n-\alpha}^p(n) \subset K_\alpha(n)$. □

### §7. Multipliers

In this section, we study the spaces $\mathfrak{M}_\alpha(n)$ that consist of multipliers for the families $K_\alpha(n)$. Note that multipliers for the Hardy–Sobolev spaces were investigated in the recent papers [22, 5, and 26].

### 7.1. Necessary conditions.
For $n = 1$, the results of this subsection were obtained in [10].

**Lemma 7.1.** Suppose $n \in \mathbb{N}$, $\alpha > 0$, and $f \in \mathfrak{M}_\alpha(n)$. Then $f \in H^\infty(B_n)$; moreover, 
\[ \|f\|_{H^\infty(B_n)} \leq \|f\|_{\mathfrak{M}_\alpha(n)}. \]

**Proof.** Fix a point $\zeta \in \partial B_n$ and fix a constant $K$ for which $\|f\|_{\mathfrak{M}_\alpha(n)} < K$. Since $\|(1 - \langle \cdot, \zeta \rangle)^{-\alpha}\|_{K_\alpha(n)} = 1$, there exists a measure $\mu_\zeta \in \mathcal{M}(n)$ such that $\|\mu_\zeta\|_{\mathcal{M}(n)} < K$ and 
\[ f(z) = \int_{\partial B_n} \frac{1}{(1 - \langle z, \xi \rangle)^\alpha} d\mu_\zeta(\xi). \]
In other words, 
\[ f(z) = \int_{\partial B_n} \left( \frac{1 - \langle z, \xi \rangle}{1 - \langle \zeta, \xi \rangle} \right)^\alpha d\mu_\zeta(\xi). \]
Put $z = r\zeta$, $0 \leq r < 1$. Then 
\[ |f(r\zeta)| \leq \int_{\partial B_n} \left( \frac{1 - r}{1 - r|\zeta, \xi|} \right)^\alpha d\mu_\zeta(\xi) \leq \|\mu_\zeta\|_{\mathcal{M}(n)} < K. \]
Since the point $\zeta \in \partial B_n$ and the constant $K > \|f\|_{\mathfrak{M}_\alpha(n)}$ are arbitrary, we have $\|f\|_{H^\infty(B_n)} \leq \|f\|_{\mathfrak{M}_\alpha(n)}$. □
Lemma 7.2. Suppose \( n \in \mathbb{N}, \alpha > 0, \) and \( f \in \mathcal{M}_\alpha(n). \) Then \( f \in \mathcal{K}_\alpha(n); \) moreover, \( \|f\|_{\mathcal{K}_\alpha(n)} \leq \|f\|_{\mathcal{M}_\alpha(n)}. \)

Proof. Put \( I(z) = 1 \) for \( z \in B_n. \) Recall that \( I = K_\alpha[\sigma_n]. \) Since \( \sigma_n \) is a positive measure, we have \( \|I\|_{\mathcal{K}_\alpha(n)} = \|\sigma_n\| = 1. \) The assumption \( f \in \mathcal{M}_\alpha(n) \) implies that \( f = fI \in \mathcal{K}_\alpha(n). \) Moreover, \( \|f\|_{\mathcal{K}_\alpha(n)} = \|fI\|_{\mathcal{K}_\alpha(n)} \leq \|f\|_{\mathcal{M}_\alpha(n)} \|I\|_{\mathcal{K}_\alpha(n)} = \|f\|_{\mathcal{M}_\alpha(n)}. \) \( \square \)

Proposition 7.3. Let \( \alpha > 0. \) Then there exists a constant \( C_\alpha \) such that, for any \( f \in \mathcal{M}_\alpha(n) \) and any \( n \in \mathbb{N}, \) the radial variation in the direction \( \xi \in \partial B_n \) satisfies the estimate \( V(f, \xi) \leq C_\alpha\|f\|_{\mathcal{M}_\alpha(n)} \) for all \( \xi \in \partial B_n. \)

Proof. Suppose \( f \in \mathcal{M}_\alpha(n), \) \( \xi \in \partial B_n, \) and \( \varepsilon > 0. \) By the definition of the space \( \mathcal{M}_\alpha(n), \) there exists a measure \( \mu_\xi \) such that \( \|\mu_\xi\|_{M(n)} \leq \|f\|_{\mathcal{M}_\alpha(n)} + \varepsilon \) and

\[
f(z) = \int_{\partial B_n} \frac{(1 - (z, \xi))^\alpha}{(1 - (z, \zeta))^\alpha} d\mu_\xi(\zeta), \quad z \in B_n.
\]

Let \( \lambda \in B_1. \) We put \( z = \lambda \xi \) and \( f_\xi(\lambda) = f(\lambda \xi). \) Then

\[
f_\xi(\lambda) = \int_{\partial B_n} \frac{(1 - \lambda)^\alpha}{(1 - \lambda(\xi, \zeta))^\alpha} d\mu_\xi(\zeta), \quad f_\xi'(\lambda) = \alpha \int_{\partial B_n} \frac{(1 - \lambda)^{\alpha - 1}(\xi, \zeta) - 1}{(1 - \lambda(\xi, \zeta))^{\alpha + 1}} d\mu_\xi(\zeta).
\]

Now, set \( \lambda = r, 0 \leq r < 1. \) Fubini’s theorem guarantees that

\[
(7.1) \quad \int_0^1 |f'_\xi(r)| dr \leq \alpha \int_{\partial B_n} \int_0^1 \frac{(1 - r)^{\alpha - 1}|1 - (\xi, \zeta)|}{|1 - r(\xi, \zeta)|^{\alpha + 1}} dr d\mu_\xi(\zeta).
\]

Put \( w = (\xi, \zeta) \) and \( b = |1 - (\xi, \zeta)|. \) Observe that

\[|1 - rw|^2 = (1 - r)(1 - r|w|^2) + r|1 - w|^2 \geq (1 - r)^2 + r^2b^2.\]

Hence,

\[
\int_0^1 \frac{(1 - r)^{\alpha - 1}b}{|1 - r(\xi, \zeta)|^{\alpha + 1}} dr \leq \int_0^1 \frac{(1 - r)^{\alpha - 1}b}{((1 - r)^2 + r^2b^2)^{\alpha + 1/2}} dr := I(\alpha, b).
\]

If \( b = 0, \) then the proof is finished. Otherwise, as was shown in the proof of Theorem 2.6 in [10], we have \( I(\alpha, b) \leq C_\alpha < \infty. \) Therefore, estimate (7.1) implies that \( V(f, \xi) \leq C_\alpha\|f\|_{\mathcal{M}_\alpha(n)}. \) The proof of the proposition is finished. \( \square \)

Recall that a bounded nonconstant function \( f \in \mathcal{H}ol(B_n) \) is said to be inner if \( |f^*| = 1 \) \( \sigma_n \)-a.e. A complete description of the inner functions belonging to \( \mathcal{M}_1(1) \) was obtained in [18]. For arbitrary \( \alpha > 0, \) the inner functions in the family \( \mathcal{M}_\alpha(1) \) were investigated in [13]. For \( n \geq 2, \) the answer to the corresponding question follows from Proposition 7.3.

Corollary 7.4. Assume that \( n \geq 2 \) and \( \alpha > 0. \) If \( f \) is an inner function in the ball \( B_n, \) then \( f \notin \mathcal{M}_\alpha(n). \)

Proof. For \( n \geq 2, \) it is well known that any inner function has no radial limits on a dense subset of the sphere \( \partial B_n; \) see [24, §1]. \( \square \)
7.2. Sufficient conditions. Recall that $A(B_n)$ denotes the ball algebra. If $f \in A(B_n)$, then put $f^* = f|_{\partial B_n}$. The complex modulus of continuity of the function $f^*$ is defined by

$$\omega_C(f^*, \delta) = \sup_{|\zeta - \xi| \leq \delta} \left\{ |f^*(\zeta) - f^*(\xi)| : d(\zeta, \xi) \leq \delta \right\},$$

where $\delta \in (0, 2]$ and $d(\zeta, \xi) = |1 - \langle \zeta, \xi \rangle|$. Note that $d(\zeta, \xi)$ is a quasimetric on the sphere.

Let $\nu_n$ denote the Lebesgue measure on $B_n$ normalized by the condition $\nu_n(B_n) = 1$.

Suppose that $n \geq 2$. For a Borel function $f : B_1 \to \mathbb{C}$, it is well known that the identity

$$(7.2) \quad \int_{\partial B_n} f(\langle \zeta, \xi \rangle) \, d\nu_n(\xi) = (n - 1) \int_{B_1} f(z)(1 - |z|^2)^{n-2} \, dv_1(z)$$

is fulfilled if its left-hand side or right-hand side is well defined.

An analog of the following statement for $n = 1$ was proved in [29].

**Proposition 7.5.** Let $\alpha \geq n \geq 2$. Suppose that a function $g \in A(B_n)$ satisfies the following condition:

$$(7.3) \quad (n - 1) \int_{B_1} \frac{(1 - |z|^2)^{n-2}\omega_C(g^*, |1 - z|)}{|1 - z|^n} \, dv_1(z) = C_\omega < +\infty.$$  

Then $g \in \mathcal{M}_\alpha(n)$.

**Proof.** By Lemma 2.4 and Proposition 2.5, it suffices to verify that the function $g$ satisfies condition (2.5) with $\alpha = n$.

Fix a point $\zeta \in \partial B_n$. Then

$$\frac{g(w)}{(1 - \langle w, \zeta \rangle)^n} = \frac{g(w) - g(\zeta)}{(1 - \langle w, \zeta \rangle)^n} + \frac{g(\zeta)}{(1 - \langle w, \zeta \rangle)^n} = h_1(w) + h_2(w), \quad w \in B_n.$$  

Observe that $\|h_2\|_{K_\alpha(n)} \leq \|g\|_{A(B_n)}$.

On the other hand, we have $g(\cdot) - g(\zeta) \in H^\infty(B_n)$ and $(1 - \langle \cdot, \zeta \rangle)^{-n} \in H^p(B_n)$ for all $0 < p < 1$. Hence, $h_1 \in H^p(B_n)$ for all $0 < p < 1$. Let $h_1^*(\xi) = \lim_{r \to 1^-} h_1(r\xi), \xi \in \partial B_n, \xi \neq \zeta$. Formula (7.2) and condition (7.3) show that $\|h^*\|_{L^1(\partial B_n)} \leq C_\omega$. Therefore, $h_1 \in H^1(B_1)$; also, we obtain $\|h_1\|_{K_\alpha(n)} \leq \|h_1\|_{H^1(B_n)} \leq C_\omega$. This proves (2.5). \hfill \Box

Assume that $0 < \beta < 1$. By definition, the standard Lipschitz space $\Lambda^\beta(\partial B_n)$ consists of all functions $f : \partial B_n \to \mathbb{C}$ such that

$$|f(\zeta) - f(\xi)| \leq C_f |\zeta - \xi|^\beta$$

for all $\zeta, \xi \in \partial B_n$. In other words, $\Lambda^\beta(\partial B_n)$ are the Lipschitz spaces with respect to the Euclidean metric on the sphere. Elements of these spaces can be used to construct examples of functions belonging to families $\mathcal{M}_\alpha(n)$ with $\alpha \geq n$.

**Corollary 7.6.** Suppose $n \geq 2$, $0 < \beta < 1/2$, and $f \in \Lambda^\beta(\partial B_n)$. Then $K_n[f] \in \mathcal{M}_\alpha(n)$ for $\alpha \geq n$.

**Proof.** Let $f \in \Lambda^\beta(\partial B_n)$. It is well known that $\omega_C(K_n[f]^*, \delta) = \mathcal{O}(\delta^\beta)$ (see [2]). Hence, condition (7.3) is satisfied. \hfill \Box

Explicit examples of functions belonging to $\mathcal{M}_\alpha(n)$, $\alpha \geq 0$, can be obtained with the help of the following assertion.

**Proposition 7.7.** Suppose $n \in \mathbb{N}$, $j \in \{0, 1, \ldots, n\}$, and $\alpha \geq n - j$. Also, suppose that the derivative $R^j g \in A(B_n)$ satisfies condition (7.3). Then $g \in \mathcal{M}_\alpha(n)$.

**Proof.** We argue by induction. As the base of induction, we take the case of $j = 0$. Then the claim is true by Proposition 7.5.
Now, let \( k \in \{0, 1, \ldots, n - 1\} \). Assume that the required implication is proved for \( j = k \).

Suppose \( j = k + 1 \in \{1, \ldots, n\} \) and \( f \in K_\alpha(n), \alpha \geq n - j \). We must prove that \( f g \in K_\alpha(n) \). Since \( \alpha \geq n - j \geq 0 \), Corollary 5.6 guarantees that \( f g \in K_\alpha(n) \) if and only if \( R(f g) \in K_{\alpha+1}(n) \). Note that

\[ R(f g) = f Rg + g Rf. \]

The function \( R^{-1}(Rg) \in A(B_n) \) satisfies condition \((7.3)\). We have \( \alpha + 1 \geq n - (j - 1) \); hence, \( Rg \in M_{\alpha+1}(n) \) by the inductive hypothesis. Since \( f \in K_\alpha(n) \subset K_{\alpha+1}(n) \), we obtain \( f Rg \in K_{\alpha+1}(n) \) by the definition of the multiplier space.

On the other hand, \( f \in K_\alpha(n) \) if and only if \( Rf \in K_{\alpha+1}(n) \). Next, the function \( R^{-1}g \in A(B_n) \) satisfies condition \((7.3)\). By the inductive hypothesis, we have \( g \in M_{\alpha+1}(n) \); thus, \( g Rf \in K_{\alpha+1}(n) \).

Therefore, \( R(f g) \in K_{\alpha+1}(n) \) and \( f g \in K_\alpha(n) \). The induction step is finished, and the proposition is proved.

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**REFERENCES**


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