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L^p-BOUNDED POINT EVALUATIONS FOR POLYNOMIALS AND UNIFORM RATIONAL APPROXIMATION

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To Victor Havin on his 75th birthday

Abstract. A connection is established between uniform rational approximation and approximation in the mean by polynomials on compact nowhere dense subsets of the complex plane $\mathbb{C}$. Peak points for $R(X)$ and bounded point evaluations for $H^p(X, dA)$, $1 \le p < \infty$, play a fundamental role.

§1. Introduction

Let μ be a finite positive compactly supported Borel measure in the complex plane $\mathbb{C}$ having no point masses. For each p, $1 \le p < \infty$, let $H^p(\mu)$ be the closed subspace of $L^p(\mu)$ that is spanned by the complex analytic polynomials. Over the years considerable attention has been directed to understanding the conditions under which $H^p(\mu) = L^p(\mu)$, due in part to its connection with the invariant subspace problem for subnormal operators on a Hilbert space when $p = 2$. On the other hand, equality evidently fails whenever there exists a point $z_0 \in \mathbb{C}$ such that the map $P \to P(z_0)$ can be extended from the polynomials to a bounded linear functional on $H^p(\mu)$, that is, if

$$|P(z_0)| \le C \, \|P\|_{L^p(\mu)} \tag{1.1}$$

for every polynomial P and some absolute constant $C > 0$. Such a point z_0 is said to be a *bounded point evaluation* or *bpe* for $H^p(\mu)$, and the question arises: Is it true that either

(1) $H^p(\mu)$ has a bpe, or
(2) $H^p(\mu) = L^p(\mu)$?

The initial step in dealing with the proposed alternative was taken by Wermer [26] in 1955. At that time he was able to show that if μ is carried on a compact set X having planar measure zero (i.e., if $|X| = 0$), then the alternative is indeed valid. His argument is roughly this. Let $R(X)$ be the class of functions that can be uniformly approximated on X by rational functions whose poles lie outside of X, and let $C(X)$ be the space of all continuous functions on X. If $H^p(\mu)$ has no bpe's, then it must contain every rational function analytic on X, and so also $R(X)$ (cf. [4, p. 218]). On the other hand, since $|X| = 0$ it follows from a theorem of Hartogs and Rosenthal [11] that $R(X) = C(X)$, and the latter is dense in $L^p(\mu)$. Therefore, if (1) fails, then (2) holds.

Although the alternative embodied in (1) and (2) above is now known to be valid for all measures μ (cf. Thomson [21]), we are nevertheless led to consider the relationship between polynomial and rational approximation. Following Wermer's early success, the natural question was this: Does the suggested alternative persist if $d\mu \ll dA$? In

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particular, does it remain in force if $d\mu = dA$ restricted to a compact set X having positive area, but no interior? In an attempt to answer these and similar questions one of the authors was led in 1973 to ask: Does there exist a compact set X such that $H^p(X, dA) = L^p(X, dA)$ for all p, $1 \le p < \infty$, but $R(X) \ne C(X)$ (cf. [3, p. 174]). Several years later in a major survey article Mel′nikov and Sinanyan [16] made further reference to the problem just stated, and it has remained open throughout the intervening years. Our goal here in §4 is to settle the matter in the negative.

Almost a decade prior to the publication of [16], Sinanyan [20] had considered, and answered, the corresponding question for $R^p(X, dA)$, the closed subspace of $L^p(X, dA)$ that is spanned by the rational functions having no poles on X. He showed that there exists a compact set X such that $R(X) \ne C(X)$, but nevertheless $R^p(X, dA) = L^p(X, dA)$ for all p, $1 \le p < \infty$. In §5 we present another example of this kind, motivated by an as yet unsolved problem concerning the possible underlying structure of a compact set X where $R(X) \ne C(X)$.

Whenever possible the page numbers on individual citations to Russian articles will refer to the English translation.

§2. Bounded point evaluations and the Swiss cheese

In order to develop a greater appreciation for what might be valid in the most general situation let us consider initially a special class of compact nowhere dense sets, a typical member of which is often referred to as the Swiss cheese. Such sets were first studied in connection with rational approximation by Alice Roth [18] in 1938, rediscovered in a similar context by Mergelyan [17] in 1952, and are constructed as follows. Remove from the closed unit disk $\overline{D}$ countably many disjoint open disks D_j, $j = 1, 2, 3, \ldots$, having radii r_j in such a way that

(1) $\overline{D_j} \subset \operatorname{int}(\overline{D})$ for each $j = 1, 2, 3, \ldots$;
(2) $\overline{D_j} \cap \overline{D_k} = \varnothing$ whenever $j \ne k$;
(3) $\overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ has no interior;
(4) $\sum_j r_j < \infty$.

The resulting set $E = \overline{D} \setminus \bigcup_{j=1}^{\infty} D_j$ is compact and nowhere dense. Letting $d\mu$ be dz on ∂D and $-dz$ on the remaining circles $\Gamma_j = \partial D_j$, $j = 1, 2, 3, \ldots$, we obtain a nonzero measure of finite total variation on E such that

$$\int_E f\,d\mu = \int_{\partial D} f\,dz - \sum_j \int_{\Gamma_j} f\,dz = 0$$

for all $f \in R(E)$. Thus, $R(E) \ne C(E)$ and so, by the Hartogs–Rosenthal theorem, E has positive area. The space $H^p(E, dA)$ is therefore nonempty, and we can ask whether $H^p(E, dA) = L^p(E, dA)$.

Although the Hartogs–Rosenthal theorem allows us to conclude indirectly that $|E| > 0$, it fails to provide any additional information on the specific geometric structure of the Swiss cheese. For this reason it is important here to recall an argument due to W. K. Allard (cf. [5, p. 163]) which is considerably more informative on that point. For each $x \in [-1, 1]$, let $E_x = \{z \in E : \operatorname{Re} z = x\}$, and for each $n = 1, 2, 3, \ldots$, let $I_n(x)$ be the number of points in $E_x \cap \Gamma_n$. Evidently, $I_n(x) = 0, 1$ or 2. By the monotone convergence theorem we have

$$\int_{-1}^{1} \sum_{n=1}^{\infty} I_n(x)\,dx = \sum_{n=1}^{\infty} \int_{-1}^{1} I_n(x)\,dx = 4 \sum_{n=1}^{\infty} r_n < \infty.$$

Hence, $\sum I_n(x) < \infty$ for almost every $x \in [-1,1]$. For any such x all but finitely many $I_n(x)$ must be zero, and the corresponding set E_x consists of a finite number of nondegenerate intervals. Consequently $|E| > 0$ by Fubini's Theorem.

An equally important implication for the question raised in the introduction is the following.

Lemma 2.1. *Let E be a Swiss cheese and for each $z \in E$ let $\mathcal{F}_z$ denote the union of all circles centered at z and lying entirely in E. Then, there exists at least one point $z \in E$ where $|\mathcal{F}_z| > 0$.*

Proof. Let E_x be as above, denote by $l(E_x)$ its total length or linear measure, and set $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$. Since there are uncountably many $x \in [-1,1]$ where E_x consists of a finite number of nondegenerate intervals we can choose a sequence of such points x_n in such a manner that $l(E_{x_n}) \geq C > 0$ for some constant C and all $n = 1, 2, 3, \ldots$. Moreover, we can assume with no loss of generality that $x_n < x_{n+1}$. Now select a finite collection of disjoint open disks $\Delta_{j_1} = \Delta(z_{j_1}, r_{j_1})$ with centers $z_{j_1} \in E_{x_1}$ so that $r_{j_1} < \frac{1}{2}(x_2 - x_1)$ and $\sum r_{j_1} > \frac{C}{2}$. Next, choose another finite collection of disjoint open disks $\Delta_{j_2} = \Delta(z_{j_2}, r_{j_2})$ with centers $z_{j_2} \in E_{x_2}$ so that $r_{j_2} < \frac{1}{2}\min(x_3 - x_2, x_2 - x_1)$ and $\sum r_{j_2} > \frac{C}{2}$. The disks Δ_{j_2} are clearly disjoint from any of the Δ_{j_1}. Continuing in this way we obtain a collection of disjoint open disks which we redesignate as Δ_j with centers $z_j \in E$ and radii r_j such that $\sum r_j = \infty$.

If we assume that $|\mathcal{F}_{z_j}| = 0$ for all $j = 1, 2, 3, \ldots$, then almost every circle in each Δ_j with center at z_j will meet Γ and we will be forced to conclude that

$$l(\Gamma) \geq \sum_{j=1}^{\infty} l(\Gamma \cap \Delta_j) \geq \sum_{j=1}^{\infty} r_j = \infty,$$

contradicting our construction ensuring that $l(\Gamma) < \infty$. Thus $|\mathcal{F}_{z_j}| > 0$ for, not only one, but many z_j. □

Theorem 2.2. *If E is an arbitrary Swiss cheese there exists a point $x_0 \in E$ which is a bpe for $H^p(E, dA)$, for $1 \leq p < \infty$. In particular, $H^p(E, dA) \neq L^p(E, dA)$ for any $p \geq 1$.*

Proof. Choose a point $x_0 \in E$ for which $|\mathcal{F}_{x_0}| > 0$. Assume for convenience that $x_0 = 0$ and let $X = \mathcal{F}_{x_0} \cap [0,1]$. For any polynomial P,

$$\int_{\mathcal{F}_{x_0}} P\, dA = \int_0^{2\pi} \int_X P(re^{i\theta}) r\, dr\, d\theta = \int_X \left(\int_0^{2\pi} P(re^{i\theta})\, d\theta \right) r\, dr = 2\pi C P(0),$$

where $C = \int_X r\, dr$. It follows that

$$|P(0)| \leq \frac{1}{2\pi C} \|P\|_{L^1(E,dA)},$$

and so by Hölder's inequality, $x_0 = 0$ is a bpe for $H^p(E, dA)$ whenever $1 \leq p < \infty$. □

In addition to what has been established it can be shown that whenever $|\mathcal{F}_{x_0}| > 0$ every function $f \in H^p(E, dA)$ actually admits an analytic continuation to a fixed neighborhood of x_0. To that end, choose $\epsilon > 0$ and small enough to ensure that the portion of $\mathcal{F}_{x_0}$ lying outside of $D_\epsilon = \{z : |z - x_0| < \epsilon\}$ has positive dA-measure. By the argument in the proof of the preceding theorem, there exists a function $h \in L^\infty$ with support in $E \setminus D_\epsilon$ such that

$$P(x_0) = \int P\, h\, dA$$

for all polynomials P. Thus $k\,dA = (z-x_0)h\,dA$ is an annihilating measure and so for any polynomial P and $\zeta \in \mathbb{C}$, it follows from an argument due essentially to Cauchy that

$$P(\zeta) = \frac{1}{\widehat{k}(\zeta)} \int \frac{P(z)}{z-\zeta}\, k\, dA$$

at any point where $\widehat{k}(\zeta) = \int \frac{k(z)}{z-\zeta}\, dA$ is defined and not equal to zero. Since $\widehat{k}$ is analytic in D_ϵ and $\widehat{k}(x_0) = 1$ it follows that $|\widehat{k}(\zeta)| \geq C > 0$ in a neighborhood U of x_0. Since the kernel $(z-\zeta)^{-1}$ is bounded on the support of the measure dA we conclude that

$$P(\zeta) \leq K\|P\|_{L^1(E,dA)}$$

for some suitable constant K, and therefore ζ is a bpe for $H^1(E, dA)$.

§3. Analytic capacity and construction of barriers

The notion of analytic capacity was introduced by Ahlfors in 1947 in connection with the problem of characterizing sets of removable singularities for bounded analytic functions. In subsequent years others, and Vitushkin in particular, further developed the concept and used it to settle a number of questions concerning uniform approximation by rational functions on compact subsets of the plane.

As initially conceived, the *analytic capacity* of a compact set X, denoted $\gamma(X)$, is defined as follows:

$$\gamma(X) = \sup |f'(\infty)|,$$

where the supremum is extended over all functions f analytic in $\widehat{\mathbb{C}} \setminus X$ and normalized so that

(a) $\|f\|_\infty = \sup_{\widehat{\mathbb{C}}\setminus X} |f| \leq 1$;
(b) $f(\infty) = 0$.

Here $\widehat{\mathbb{C}}$ is the extended complex plane or Riemann sphere. For an arbitrary planar set E we let $\gamma(E) = \sup \gamma(X)$, the supremum now being taken over all compact sets $X \subseteq E$. For a more thorough discussion of analytic capacity and its properties, see [8] and [27], where it is shown that

(1) $\gamma(B_r) = r$ for every disk B_r of radius r;
(2) $\gamma(X) \approx \operatorname{diam}(X)$ whenever X is compact and connected; in particular, $\gamma(X) \leq \operatorname{diam}(X) \leq 4\gamma(X)$.

From the outset it was not known and is still not known whether γ is subadditive, and so possibly not a capacity in the usual sense. We now know, however, that γ is at least semiadditive in that

$$\gamma(E \cup F) \leq C\,(\gamma(E) + \gamma(F))$$

for all compact (and even Borel) sets $E, F \subseteq \mathbb{C}$ and some absolute constant C. The key point is that γ is equivalent to a second auxiliary capacity γ^+ defined as follows. For a compact set X and positive measure ν supported on X we form the Cauchy integral

$$\widehat{\nu}(z) = \int \frac{d\nu(\zeta)}{\zeta - z} \tag{3.1}$$

and we define

$$\gamma^+(X) = \sup_\nu \nu(X)$$

to be the supremum over all positive measures ν such that $\widehat{\nu} \in L^\infty(\mathbb{C})$ and $\|\widehat{\nu}\|_\infty \leq 1$. Since $\widehat{\nu}$ is analytic in $\widehat{\mathbb{C}} \setminus X$ and $|\widehat{\nu}'(\infty)| = \nu(X)$, the function $\widehat{\nu}$ is admissible for γ and so

$$\gamma^+(X) \leq \gamma(X).$$

As before, if E is an arbitrary planar set we let $\gamma^+(E) = \sup \gamma^+(X)$, where X is compact and $X \subset E$.

The essential equivalence of γ and γ^+ was established by Tolsa [22]. Here is what he proved: There exists an absolute constant $C > 0$ so that

(3) $\gamma^+(E) \le \gamma(E) \le C\gamma^+(E)$ for all sets $E \subseteq \mathbb{C}$;

(4) if E_n, $n = 1, 2, 3, \ldots$, are Borel sets, then $\gamma(\bigcup_n E_n) \le C \sum_n \gamma(E_n)$.

Since Tolsa had previously shown that γ^+ is itself countably semiadditive, (3) implies (4).

Because γ^+ is defined directly in terms of the Cauchy integral, it can be used to establish a certain lower semi-continuity enjoyed by such integrals. If μ is a finite, complex, compactly supported measure, then $\widehat{\mu}$ is defined as in (3.1) and

$$U^{|\mu|}(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|}$$

is the corresponding Newtonian potential.

Lemma 3.1. *Let μ be a finite complex, compactly supported, Borel measure in $\mathbb{C}$, and let x_0 be any point where $U^{|\mu|}(x_0) < \infty$. For each $r > 0$ let $B_r = B(x_0, r)$ be the disk with center at x_0 and radius r, and let E be a set with the property that for every $r > 0$ there is a relatively large subset $E_r \subseteq (E \cap B_r)$ on which $U^{|\mu|}$ is bounded; that is,*

(1) *$U^{|\mu|} \le M_r < \infty$ on E_r;*

(2) *$\gamma(E_r) \ge \epsilon\gamma(E \cap B_r)$ for some absolute constant ϵ. If, moreover, E is thick at x_0 in the sense that*

$$\limsup_{r \to 0} \frac{\gamma(E \cap B_r)}{r} > 0, \tag{3.2}$$

then

$$|\widehat{\mu}(x_0)| \le \limsup_{z \to x_0,\, z \in E} |\widehat{\mu}(z)|.$$

A proof can be found in [4, p. 224].

In order to extend Theorem 2.2 to the most general setting where $R(X) \ne C(X)$ it is essential that we have a way to decide whether a point $x_0 \in X$ is a bpe for $H^p(X, dA)$ or not. With this in mind we shall adopt a scheme due, in broad outline, to Thomson [21] and having its roots in the work of Mel′nikov [15]. Throughout the discussion ν will be a measure on X, not necessarily absolutely continuous with respect to area. For each $\lambda > 0$ the sets $E_\lambda = \{z : |\widehat{\nu}(z)| < \lambda\}$ will play a critical role.

Let $x_0 \in X$ be a point where $U^{|\mu|}(x_0) < \infty$ and fix $\lambda > 0$. For each positive integer n form a grid in the plane consisting of lines parallel to the coordinate axes and intersecting at those points whose coordinates are both integral multiples of 2^{-n}. The resulting collection of squares $\mathcal{G}_n = \{S_n\}_{j=1}^{\infty}$ of side length 2^{-n} is an edge-to-edge tiling of the plane; its members will be referred to as squares of the *n-th generation.*

Beginning with a fixed generation, the n-th say, choose a square $S^* \in \mathcal{G}_n$ with $x_0 \in S^*$. Denote by $\mathcal{G}_n^\lambda$ the collection of all squares in $\mathcal{G}_n$ for which

$$|E_\lambda \cap S| > \frac{1}{100}\,|S|. \tag{3.3}$$

K_n will denote the union of those squares in $\mathcal{G}_n^\lambda$ that can be joined to S^* by a finite chain of squares lying in $\mathcal{G}_n^\lambda$. If K_n is bounded, or empty, there exists a closed *corridor* or *barrier*, $Q_n = \bigcup_j S_{nj}$, composed of squares S_{nj} from $\mathcal{G}_n$ abutting $S^* \cup K_n$, separating the latter from ∞, adjacent to one another along their sides, and such that

$$|E_\lambda \cap S_{nj}| \le \frac{1}{100}|S_{nj}| \tag{3.4}$$

for each j. The polynomial convex hull of Q_n is a polygon Π_n with its boundary Γ_n lying along sides of squares for which (3.4) is satisfied. Thus $|\widehat{\nu}| \geq \lambda$ on a large portion of every square from S_{nj} meeting Γ_n. By adjoining to Π_n additional squares from $\mathcal{G}_n$ we obtain another polygon Π_n^* with boundary Γ_n^* in such a way that

(i) $\Gamma_n^* \supseteq \Gamma_n$;
(ii) $n^2 2^{-n} \leq \operatorname{dist}(\Gamma_n^*, \Gamma_n) \leq 3n^2 2^{-n}$.

This can be done by simply adjoining to Π_n additional squares from $\mathcal{G}_n$.

At this point let K_{n+1} denote the union of all squares in $\mathcal{G}_{n+1}^\lambda$ that can be joined to Π_n^* by a chain of squares in $\mathcal{G}_{n+1}^\lambda$. Again, if K_{n+1} is bounded, or empty, there is a second barrier Q_{n+1} abutting $\Pi_n^* \cup K_{n+1}$ and

$$|E_\lambda \cap S| \leq \frac{1}{100}|S|$$

for every square S in Q_{n+1}. The polygon Π_{n+1} is defined to be the polynomial convex hull of Q_{n+1}, and the process continues. In this way we obtain a nested sequence of polygons

$$\Pi_n \subseteq \Pi_{n+1} \subseteq \cdots \subseteq \Pi_{n+l} \subseteq \cdots$$

and compact sets $K_j \subseteq \Pi_j \setminus \Pi_{j-1}$, some of which may be empty, such that if $K_j \neq \varnothing$, then

(a) K_j is the union of squares in $\mathcal{G}_j$ and connects Γ_{j-1}^* to Q_j;
(b) $|E_\lambda \cap S| > \frac{1}{100}|S|$ for each $S \subseteq K_j$;
(c) $\operatorname{dist}(K_j, \Gamma_j^*) \leq \operatorname{dist}(K_j, \Gamma_j) + \operatorname{dist}(\Gamma_j, \Gamma_j^*) < 4j^2 2^{-j}$.

Given an arbitrary disk $B_r = B(x_0, r)$ with center at x_0 there are two mutually exclusive possibilities:

(A) the sets K_j eventually exit B_r;
(B) there exists an infinite sequence of barriers Q_j, $j = n, n+1, n+2, \ldots$ extending outward from x_0 and lying entirely in B_r.

Should the first alternative (A) be the case for all $r > 0$ it can be shown (cf. [4, p. 233]) that the set E_λ satisfies all the hypotheses of Lemma 3.1, and therefore

$$\limsup_{r \to 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0.$$

The second alternative (B) implies that E_λ surrounds x_0 and contains sufficient mass to ensure that $H^1(X, dA)$ admits a bpe at x_0. This phenomenon replaces the collection of circles lying inside a Swiss cheese as discussed in §2. It is the subject of our next lemma.

Lemma 3.2. *If there exists an infinite sequence of barriers Q_j, $j = n, n+1, n+2, \ldots$, surrounding a point x_0, then $H^1(X, dA)$ admits a bpe at x_0.*

Proof. Because Q_n is a barrier, Γ_n is the union of certain specified sides of n-th generation squares S such that $|E_\lambda \cap S| \leq \frac{1}{100}|S|$, or, setting $F_\lambda = \{z : |\widehat{\nu}(z)| \geq \lambda\}$, squares S for which

$$|F_\lambda \cap S| \geq \frac{99}{100}|S|.$$

We can assume for the purpose of argument that $\lambda = 1$, and we set $F = F_1$ and $E = E_1$.

The map $L : P \to P(x_0)$ can be viewed as a bounded linear functional on the space of polynomials when the latter is endowed with the norm $\|P\|_{L^\infty(\Gamma_n)} = \sup_{\Gamma_n} |P|$. As such, L can be extended in a norm-preserving way to $C(\Gamma_n)$, the full space of continuous

functions on Γ_n likewise endowed with the uniform norm. Hence, there exists a measure ω of finite total variation on Γ_n such that $\|\omega\| = \|L\|$ and

$$P(x_0) = \int_{\Gamma_n} P\, d\omega$$

for all polynomials P. The first step in the proof of the lemma is to replace $\int_{\Gamma_n} P\, d\omega$ by an area integral over $F \cap Q_n$, committing only a small error.

Assume for the moment that P is a *fixed* polynomial. Take $\epsilon > 0$ and let $\Gamma_n = \bigcup I_j$ be the union of finitely many closed intervals I_j with mutually disjoint interiors chosen so that

$$\left| \int_{\Gamma_n} P\, d\omega - \sum_j P(\xi_j)\omega_j \right| < \epsilon \tag{3.5}$$

whenever $\xi_j \in I_j$ and $\omega_j = \omega(I_j)$. We can arrange that each I_j is contained entirely in the side of a single square S in the barrier Q_n. Moreover, ω can have no point masses and so there is no ambiguity associated with the approximating sums for $\int_{\Gamma_n} P\, d\omega$ in (3.5).

For a fixed barrier square $S \subseteq Q_n$ with one or more of its sides in Γ_n, let x_S denote its center, and let ξ_j be one of the points in (3.5) situated on ∂S. Since by construction $\operatorname{dist}(\Gamma_n, \Gamma_{n+1}) \geq n^2 2^{-n}$, a form of Schwartz's lemma implies that

$$|P(\xi_j) - P(x_S)| \leq \frac{2^{n+1}}{n^2}\, |\xi_j - x_S|\, \|P\|_{L^\infty(\Gamma_{n+1})} \leq \frac{\sqrt{2}}{n^2}\, \|P\|_{L^\infty(\Gamma_{n+1})}.$$

If $S_1, S_2, \ldots, S_k$ represent the totality of squares in Q_n with sides along Γ_n and $B_1, B_2, \ldots, B_k$ are the corresponding inscribed disks, it follows by summing on j and the fact that Γ_n lies entirely inside the region bounded by Γ_{n+1} that

$$\left| P(x_0) - \int_{F \cap Q_n} P\, h_n\, dA \right| \leq \epsilon + \Big(\frac{\sqrt{2}}{n^2} + \frac{2}{100} \Big) \|\omega\|\, \|P\|_{L^\infty(\Gamma_{n+1})},$$

where $h_n = \sum_{j=1}^k \frac{4}{\pi} 2^{2n} \omega(S_j) \chi_{F \cap B_j}$ and $\chi_{F \cap B_j}$ is the characteristic function of $F \cap B_j$. Since $\epsilon > 0$ is arbitrary it can now be dropped from the inequality, and by choosing n sufficiently large we can arrange that

$$\left| P(x_0) - \int_{F \cap Q_n} P\, h_n\, dA \right| \leq \frac{3}{100} \|\omega\|\, \|P\|_{L^\infty(\Gamma_{n+1})}$$

for *all* polynomials P, and $\|h_n\|_\infty \leq \frac{2^{2n+2}}{\pi} \|\omega\|$. Now repeat the process noting that the map

$$L_{n+1} \,:\, P \longrightarrow P(x_0) - \int_{F \cap Q_n} P h_n\, dA$$

can be extended from the space of polynomials and viewed as a bounded linear functional on $C(\Gamma_{n+1})$ with $\|L_{n+1}\| \leq \frac{3}{100} \|L\|$, where $L = L_n$ and $\|L\| = \|\omega\|$. Since all squares adjacent to Γ_{n+1} are barrier squares from Q_{n+1}, the argument above gives a function h_{n+1} with support in $F \cap Q_{n+1}$ so that

$$\left| P(x_0) - \int P\, h_n\, dA - \int P\, h_{n+1}\, dA \right| \leq \Big(\frac{3}{100} \Big)^2 \|L\|\, \|P\|_{L^\infty(\Gamma_{n+2})},$$

for all polynomials P, and $\|h_{n+1}\|_\infty \leq \frac{4}{\pi} 2^{2(n+1)} \frac{3}{100} \|L\|$. Continuing in this way we obtain an infinite sequence of functions $h_n, h_{n+1}, \ldots$ such that for any $k > 0$,

$$\left| P(x_0) - \int P(h_n + \cdots + h_{n+k})\, dA \right| \leq \Big(\frac{3}{100} \Big)^{k+1} \|L\|\, \|P\|_{L^\infty(\Gamma_{n+k+1})}.$$

For a given polynomial P the right side tends to zero as $k \to \infty$, since the curves Γ_{n+k} all lie in a bounded portion of the plane. Setting $h = \sum_{k=0}^{\infty} h_{n+k}$ it follows that

$$P(x_0) = \int P\, h\, dA$$

for all polynomials P. Moreover, $h \in L^\infty$ because the individual h_j's have disjoint supports and $\|h_{n+k}\|_\infty \le \|h_n\|_\infty$ for all $k > 0$; hence, $\|h\| = \|h_n\|_\infty$. □

§4. Bounded point evaluations for $H^p(dA)$

We have now reached the point where we can address the question raised in the introduction concerning the relation between uniform rational approximation and the existence of L^p-bounded point evaluations for the polynomials. Even in the most general situation, however, bpe's arise for essentially the same reason as in the case of the Swiss cheese. That is, a point $x_0 \in X$ is a bpe if it is surrounded by a portion of X having sufficient mass to ensure that the inequality (1.1) is satisfied at x_0. From our reasoning it would appear that in some cases the local geometry of X in a neighborhood of a bpe x_0 must be quite complicated, but we have not been able to rule out the possibility that there is a collection of concentric circles about x_0, lying entirely in X and having positive dA measure.

Theorem 4.1. *Let X be a compact subset of $\mathbb{C}$ with empty interior. If $R(X) \neq C(X)$, then there exists at least one point x_0 that yields a bpe for every $H^p(X, dA)$, $1 \le p < \infty$. Moreover, every function $f \in H^p(X, dA)$ admits an analytic extension to a fixed neighborhood of x_0.*

Proof. By assumption there exists a nonzero measure ν such that $\nu \perp R(X)$. Since $\nu \neq 0$ as a measure there is at least one point x_0 such that

(a) $U^{|\nu|}(x_0) < \infty$;
(b) $\widehat{\nu}(x_0) \neq 0$.

We can conclude, therefore, that there exists an infinite sequence of barriers relative to the set where $|\widehat{\nu}|$ is bounded away from zero, and surrounding the point x_0 as described in §3. Such a collection of barriers must, of course, lie entirely in X since $\widehat{\nu} \equiv 0$ in $\mathbb{C} \setminus X$.

Suppose for the moment that no such sequence of barriers exists. For an arbitrary, but fixed, $\lambda > 0$ consider the set $E_\lambda = \{z : |\widehat{\nu}(z)| < \lambda\}$. By assumption, E_λ must in a sense escape from x_0 to ∞. More precisely, we can find a connected set X linking x_0 to ∞ such that X is the union of squares from some generation, the n-th say, and higher, and certain narrow rectangles R_j, $j > n$, where

(1) $|E_\lambda \cap S| > \frac{1}{100}|S|$ for each square $S \subseteq X$;
(2) $\operatorname{diam}(R_j) \approx j^2 2^{-j}$.

Given $r > 0$, let $B_r = B(x_0, r)$. By discarding certain superfluous pieces we can assume that $X \cap B_r$ is connected and joins x_0 to ∂B_r. Thus,

$$\gamma(X \cap B_r) \ge \frac{1}{4} \operatorname{diam}(X \cap B_r) \ge \frac{r}{8}.$$

On the other hand, it follows from the countable semiadditivity of analytic capacity that

$$\frac{r}{16} \le \gamma(X \cap B_{r/2}) \le C\Big(\gamma(K) + \sum_{j=n}^{\infty} j^2 2^{-j}\Big),$$

where K is the union of squares in X for which (1) is satisfied, and C is an absolute constant. Since we are free to begin with an arbitrary generation, we can let $n \to \infty$ and conclude that

$$\gamma(E_\lambda \cap B_r) \ge Cr$$

(cf. [4, p. 233] for details). In particular,

$$\limsup_{r\to 0} \frac{\gamma(E_\lambda \cap B_r)}{r} > 0,$$

and so Lemma 3.1 implies that

$$|\widehat{\nu}(x_0)| \le \limsup_{z\to x_0,\, z\in E_\lambda} |\widehat{\nu}(z)| \le \lambda.$$

Since this is valid for all $\lambda > 0$, we are led to infer, contrary to assumption, that $\widehat{\nu}(x_0) = 0$. Consequently, for some $\lambda > 0$ there exists an infinite sequence of barriers surrounding x_0 that correspond to a set where $|\widehat{\nu}| > \lambda$.

Lemma 3.2 now implies that the point x_0 yields a bpe for $H^1(X, dA)$ and so a bpe for all $H^p(X, dA)$, $1 \le p < \infty$. From the discussion following Theorem 2.2, it follows that there is a fixed neighborhood U of x_0 such that every function $f \in H^p(X, dA)$ extends analytically to U. □

Let us recall by way of a scholium the key role played by peak points in the theory of rational approximation. By definition, a point $x \in X$ is a *peak point* for $R(X)$ if there exists an $f \in R(X)$ such that $f(x) = 1$, but $|f(y)| < 1$ whenever $y \ne x$. Bishop's peak point criterion for rational density is this (cf. [5, p. 172] or [8, p. 54]).

Theorem 4.2 (Bishop)**.** *$R(X) = C(X)$ if and only if dA-almost every point of X is a peak point for $R(X)$.*

That fact in itself suggests that the conclusion of Theorem 4.1 is linked in a fundamental way to the presence of nonpeak points, and that is indeed the case.

Corollary 4.3. *If $x_0 \in X$ is not a peak point for $R(X)$, then x_0 yields a bpe for $H^p(X, dA)$, $1 \le p < \infty$; that is,*

$$|P(x_0)| \le C_p \, \|P\|_{L^p(X,\, dA)}$$

for every polynomial P and some constant C_p depending only on p.

The proof of the corollary is a direct consequence of the following well-known fact: x_0 is a peak point for $R(X)$ if and only if $\widehat{\nu}(x) = 0$ whenever $\nu \perp R(X)$ and $U^{|\nu|}(x) < \infty$. The argument in the proof of Theorem 4.1 can therefore proceed exactly as before. It would be remiss, however, if we did not mention that Aleman, Richter and Sundberg [2] have also shown that almost every point where (a) and (b) are both satisfied corresponds to a bpe for $H^p(X, dA)$. That is sufficient for the proof of Theorem 4.1, but not for the corollary. The underlying feature here is that Lemma 3.1 applies at *every point* x_0 where (a) and (b) are satisfied simultaneously.

§5. A counterexample for $R^p(dA)$

In 1966, Sinanyan [20] announced the following result, which now stands in striking contrast to Theorem 4.1.

Theorem 5.1 (Sinanyan)**.** *There exists a compact nowhere dense set X such that $R(X) \ne C(X)$, but nevertheless $R^p(X, dA) = L^p(X, dA)$ for all p, $1 \le p < \infty$.*

His proof depends on a construction of Mergelyan [17, p. 315] and actually produces a Swiss cheese X with the desired properties. The reader, however, is referred to an earlier paper [19] for many of the computational details. Here we shall describe an entire family of compact nowhere dense sets X having a locally nonrectifiable perimeter such that $R(X) \ne C(X)$, and still $R^p(X, dA) = L^p(X, dA)$ for all $p < \infty$.

In order to verify that $R^p(X, dA) = L^p(X, dA)$ in any given instance we shall argue by duality in the standard way. Suppose k is any function in $L^q(X, dA)$, $q = \frac{p}{p-1}$, with the property that $\int fk\, dA = 0$ for every rational function f that is analytic on X, and form the Cauchy integral

$$\widehat{k}(\zeta) = \int \frac{k(z)}{z-\zeta}\, dA_z.$$

By assumption $\widehat{k} \equiv 0$ in $\mathbb{C} \setminus X$ and we need only show that $\widehat{k} = 0$ dA-a.e. on X, from which it follows that $k\, dA = 0$ as a measure. Since we are assuming that X has no interior, and $\widehat{k}$ is continuous when $q > 2$, we can infer that $R^p(X, dA) = L^p(X, dA)$ for $1 \le p < 2$ with no additional requirements. The import of Theorem 5.1 consists in the assertion of rational density when $p \ge 2$.

In the less transparent case where $1 < q \le 2$ the Cauchy integral $\widehat{k}$ belongs to the Sobolev space W_1^q and as such enjoys a certain residual continuity which is best described in terms of an associated nonlinear capacity C_q. Nothing in the way of generality is lost if we further assume that $q < 2$, and we will do so. By definition, for any Borel set E,

$$C_q(E)^{1/q} = \sup_{\nu} \nu(E),$$

the supremum being taken over all positive measures ν concentrated on E for which $\|U^\nu\|_{L^p(dA)} \le 1$. For additional information and background material on these nonlinear capacities the reader is referred to the books [1, 13], and articles [7, 14] where proofs of the following can be found:

(i) if Φ is a contraction, then $C_q(\Phi E) \le kC_q(E)$ for some constant k depending only on q;
(ii) $C_q(B_r) \approx C_q(\operatorname{diam} B_r) \approx r^{2-q}$ for any disk B_r of radius r.

A property is said to hold *q-quasieverywhere* if the set where it fails has q-capacity zero.

As an element of W_1^q, the transform $\widehat{k}$ is *q-quasicontinuous* in the sense that: Given any $\epsilon > 0$ there exists an open set U such that $C_q(U) < \epsilon$ and $\widehat{k}$ is continuous in the complement of U. In addition there is a much subtler pointwise notion of continuity associated with functions in W_1^q, called *fine continuity*. A function $h \in W_1^q$, which we can assume to be defined q-quasieverywhere, is said to be *q-finely continuous* at a point x_0 if there exists a set E that is *thin*, or *sparse*, in a potential-theoretic sense at x_0 and

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{C} \setminus E}} h(x) = h(x_0).$$

If E is not thin at x_0, then it is said to be *thick* at that point. In our case it is sufficient to know that E is thick at x_0 if

$$\liminf_{r \to 0} \frac{C_q(E \cap B_r)}{C_q(B_r)} > 0,$$

where $B_r = B_r(x_0)$ is the disk with center at x_0 and radius r (cf. [4, p. 221]).

In our treatment of Theorem 5.1 we are able to avoid many of the more extended computations found in Sinanyan's construction (cf. [19, 20]) by taking advantage of the fact that the capacities C_q decrease modulo a constant under a contraction, whereas analytic capacity may not. Equally important to us, of course, is Vitushkin's criterion for rational approximation (cf. [25] and [8, p. 207]).

Theorem 5.2 (Vitushkin). *$R(X) = C(X)$ if and only if for dA-almost all points $x \in X$,*

$$\limsup_{r \to 0} \frac{\gamma(B_r(x) \setminus X)}{r} > 0.$$

Proof of Theorem 5.1. We begin with the construction of a planar Cantor set as follows. Let $Q^0 = [0,1] \times [0,1]$ be the closed unit square. Choose 4 closed squares inside Q^0 with side length $1/4$, having sides parallel to the coordinate axes, and so that each square contains a vertex of Q^0. Next, apply the same procedure to each of the four squares obtained in the first step. In this way we obtain 16 squares, each having side length $1/16$. Continuing in this way, at the n-th stage we obtain 4^n closed squares Q_j^n, $j = 1$, 2, 3, ..., 4^n, each having side length $1/4^n$. For each n let

$$E_n = \bigcup_{j=1}^{4^n} Q_j^n$$

and define

$$K = \bigcap_{n=1}^{\infty} E_n.$$

The set K is most commonly referred to as the *corner quarters Cantor set*. It can easily be checked that the orthogonal projection of K onto the line $2y = x$ covers an interval of length $3/\sqrt{5}$, in particular, a line segment of length greater than $\frac{1}{2}\operatorname{diam}(Q^0)$. Not quite as obvious, however, is the fact that $\gamma(K) = 0$. Cantor sets with these properties were first produced by Vitushkin [24], and his construction was later simplified by Garnett [9] and Ivanov [12, pp. 346–348].

Now iterate the procedure outlined above. Decompose Q^0 into 4 congruent squares S_j^1, $j = 1$, 2, 3, 4 by lines through midpoints of the opposite sides. In each square S_j^1 construct a Cantor set K_j^1 similar to K and differing only by a scaling factor of $1/4$. Let $K_1 = \bigcup_j K_j^1$ be the union of the four scaled-down Cantor sets, and continue the bisection process in the same manner, thereby obtaining a sequence of Cantor sets K_1, K_2, K_3, ... having these properties:

(1) $\gamma(K_n) = 0$ for all $n = 1, 2, 3, \ldots$;
(2) $E = \bigcup_n K_n$ is dense in Q^0;
(3) $\Lambda(\operatorname{proj}(K_j^n)) > \frac{1}{2}\operatorname{diam}(S_j^n)$.

Here $K_n = \bigcup_j K_j^n$, $\operatorname{proj}(K_j^n)$ denotes the orthogonal projection of K_j^n onto the line $2y = x$, and $\Lambda(\operatorname{proj}(K_j^n))$ denotes the 1-dimensional Hausdorff measure or length of this projection. It follows from Tolsa's theorem on the countable semiadditivity of analytic capacity that $\gamma(E) = 0$. In this case, however, where E is the countable union of compact sets of capacity zero, the full force of Tolsa's theorem is not needed (cf. [8, p. 237] or [10, p. 12]).

Because $\gamma(E) = 0$, and therefore $|E| = 0$, we are able to select a compact set X_0 lying in the interior of $Q = Q^0$, and so that $|X_0| > 0$ and $E \cap X_0 = \varnothing$. Pick $r_1 > 0$, but small enough to ensure that $\{z : \operatorname{dist}(z, X_0) < r_1\}$ lies entirely inside Q. Since K_1 is a compact totally disconnected set with $\gamma(K_1) = 0$, we can cover K_1 by finitely many open rectangles, having mutually disjoint closures with sides parallel to the coordinate axes, and so that the union Ω_1 of the open pieces satisfies $\gamma(\Omega_1) < \frac{1}{2}r_1$. Next, choose $r_2 < r_1$, but small enough that $\{z : \operatorname{dist}(z, X_0) < r_2\}$ does not meet $\overline{\Omega}_1$. Proceed as above to cover $K_2 \setminus \overline{\Omega}_1$ by finitely many open rectangles, with mutually disjoint closures, in such a way that

(i) $\gamma(\Omega_2) < \frac{1}{2^2}r_2$,
(ii) $\gamma(\Omega_1 \cup \Omega_2) < C\left(\frac{r_1}{2} + \frac{r_2}{2^2}\right) < Cr_1$,

where C is the absolute constant guaranteed by Tolsa's theorem. Continuing in this way,

we obtain a sequence of numbers $r_j \downarrow 0$ and a decreasing sequence of open sets Ω_1, Ω_2, Ω_3, ... with these properties:

(a) $E \subset \bigcup_j \Omega_j$;
(b) $X_0 \subseteq Q \setminus \bigcup_j \Omega_j$;
(c) $\gamma(\Omega_j) < \frac{1}{2^j} r_j$;
(d) $\gamma(\Omega_j \cup \cdots \cup \Omega_k) < \frac{C}{2^{j-1}} r_j$ for all $j = 1, 2, 3, \ldots$.

Setting $X = Q \setminus \bigcup_j \Omega_j$ we obtain a compact nowhere dense set with $X_0 \subseteq X$, and we must prove that $R(X) \neq C(X)$, but $R^p(X, dA) = L^p(X, dA)$ for all $p \geq 2$.

By construction, for each point $x \in X_0$, the inequality

$$\frac{\gamma(B(x, r_j) \setminus X)}{r_j} \leq \frac{C}{2^{j-1}}$$

is satisfied for all $j = 1, 2, 3, \ldots$, where C is an absolute constant throughout. Hence, at each point of X_0 the lower capacitary density of the complement $\mathbb{C} \setminus X$ is zero. It follows from the instability of analytic capacity as established by Vitushkin [25, p. 190] that

$$\lim_{r \to 0} \frac{\gamma(B(x, r) \setminus X)}{r} = 0$$

at dA-a.e. point $x \in X_0$ (cf. also [8, p. 207]). By Vitushkin's Theorem 5.2, it follows that $R(X) \neq C(X)$.

Again by construction, for dA-a.e. point $x \in X$, and r sufficiently small depending on x, we have $\Lambda(\mathrm{proj}(B(x, r) \setminus X)) \geq Cr$, where C is an absolute constant independent of r. For a fixed $q < 2$ this implies that $C_q(B(x, r) \setminus X) \geq Cr^{2-q}$, since q-capacity decreases modulo a constant under a contraction. Hence, at dA-a.e. point $x \in X$ the complement $\mathbb{C} \setminus X$ is thick in the sense of q-capacity. Suppose then that $k \in L^q(X, dA)$ and that $k \perp R^p(X, dA)$. Then $\widehat{k} \equiv 0$ in $\mathbb{C} \setminus X$ and by fine continuity $\widehat{k} = 0$ dA-a.e. on X. Therefore, $k = 0$ a.e. and it follows that $R^p(X, dA) = L^p(X, dA)$, and this holds for all $p > 2$. □

As indicated at the beginning of §4 we do not know of a single example of a compact nowhere dense set X such that $R(X) \neq C(X)$, and for which no point $x_0 \in X$ admits a collection of concentric circles $\mathcal{F}_{x_0}$ having positive dA measure and lying entirely inside X. The construction in the proof of Theorem 5.1, however, precludes the corresponding phenomenon for rectangles oriented so as to have two of their sides orthogonal to the line $2y = x$. The projection properties of irregular sets such as those that underlie the entire construction here are studied extensively in [6, Chapter 7].

If X happens to be a set of finite perimeter *in the sense of DeGiorgi*, it can be shown that there exist sufficiently many rectangles contained entirely in X to ensure that $H^p(X, dA)$ has a bounded point evaluation for all $p < \infty$, thereby extending Theorem 2.2 to this more general setting (cf. Trent [23]).

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