Lp-BOUNDED POINT EVALUATIONS FOR POLYNOMIALS
AND UNIFORM RATIONAL APPROXIMATION

J. E. BRENNAN AND E. R. MILITZER

To Victor Havin on his 75th birthday

ABSTRACT. A connection is established between uniform rational approximation and approximation in the mean by polynomials on compact nowhere dense subsets of the complex plane \( \mathbb{C} \). Peak points for \( R(X) \) and bounded point evaluations for \( H^p(X, dA) \), \( 1 \leq p < \infty \), play a fundamental role.

§1. INTRODUCTION

Let \( \mu \) be a finite positive compactly supported Borel measure in the complex plane \( \mathbb{C} \) having no point masses. For each \( p, 1 \leq p < \infty \), let \( H^p(\mu) \) be the closed subspace of \( L^p(\mu) \) that is spanned by the complex analytic polynomials. Over the years considerable attention has been directed to understanding the conditions under which \( H^p(\mu) = L^p(\mu) \), due in part to its connection with the invariant subspace problem for subnormal operators on a Hilbert space when \( p = 2 \). On the other hand, equality evidently fails whenever there exists a point \( z_0 \in \mathbb{C} \) such that the map \( P \to P(z_0) \) can be extended from the polynomials to a bounded linear functional on \( H^p(\mu) \), that is, if

\[
|P(z_0)| \leq C \|P\|_{L^p(\mu)}
\]

for every polynomial \( P \) and some absolute constant \( C > 0 \). Such a point \( z_0 \) is said to be a bounded point evaluation or bpe for \( H^p(\mu) \), and the question arises: Is it true that either

1. \( H^p(\mu) \) has a bpe, or
2. \( H^p(\mu) = L^p(\mu) \)?

The initial step in dealing with the proposed alternative was taken by Wermer [26] in 1955. At that time he was able to show that if \( \mu \) is carried on a compact set \( X \) having planar measure zero (i.e., if \( |X| = 0 \)), then the alternative is indeed valid. His argument is roughly this. Let \( R(X) \) be the class of functions that can be uniformly approximated on \( X \) by rational functions whose poles lie outside of \( X \), and let \( C(X) \) be the space of all continuous functions on \( X \). If \( H^p(\mu) \) has no bpe’s, then it must contain every rational function analytic on \( X \), and so also \( R(X) \) (cf. [4, p. 218]). On the other hand, since \( |X| = 0 \) it follows from a theorem of Hartogs and Rosenthal [11] that \( R(X) = C(X) \), and the latter is dense in \( L^p(\mu) \). Therefore, if (1) fails, then (2) holds.

Although the alternative embodied in (1) and (2) above is now known to be valid for all measures \( \mu \) (cf. Thomson [21]), we are nevertheless led to consider the relationship between polynomial and rational approximation. Following Wermer’s early success, the natural question was this: Does the suggested alternative persist if \( d\mu \ll dA \)? In

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particular, does it remain in force if \( d\mu = dA \) restricted to a compact set \( X \) having positive area, but no interior? In an attempt to answer these and similar questions one of the authors was led in 1973 to ask: Does there exist a compact set \( X \) such that \( H^p(X, dA) = L^p(X, dA) \) for all \( p, 1 \leq p < \infty \), but \( R(X) \neq C(X) \) (cf. [3, p. 174]). Several years later in a major survey article Mel’nikov and Sinanyan [16] made further reference to the problem just stated, and it has remained open throughout the intervening years. Our goal here in §4 is to settle the matter in the negative.

Almost a decade prior to the publication of [16], Sinanyan [20] had considered, and answered, the corresponding question for \( R^p(X, dA) \), the closed subspace of \( L^p(X, dA) \) that is spanned by the rational functions having no poles on \( X \). He showed that there exists a compact set \( X \) such that \( R(X) \neq C(X) \), but nevertheless \( R^p(X, dA) = L^p(X, dA) \) for all \( p, 1 \leq p < \infty \). In §5 we present another example of this kind, motivated by an as yet unsolved problem concerning the possible underlying structure of a compact set \( X \) where \( R(X) \neq C(X) \).

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§2. Bounded point evaluations and the Swiss cheese

In order to develop a greater appreciation for what might be valid in the most general situation let us consider initially a special class of compact nowhere dense sets, a typical member of which is often referred to as the Swiss cheese. Such sets were first studied in connection with rational approximation by Alice Roth [18] in 1938, rediscovered in a similar context by Mergelyan [17] in 1952, and are constructed as follows. Remove from the closed unit disk \( \mathcal{D} \) countably many disjoint open disks \( D_j, j = 1, 2, 3, \ldots \), having radii \( r_j \) in such a way that

1. \( \overline{D}_j \subset \text{int}(\mathcal{D}) \) for each \( j = 1, 2, 3, \ldots \);
2. \( D_j \cap D_k = \emptyset \) whenever \( j \neq k \);
3. \( \mathcal{D} \setminus \bigcup_{j=1}^\infty D_j \) has no interior;
4. \( \sum r_j < \infty \).

The resulting set \( E = \mathcal{D} \setminus \bigcup_{j=1}^\infty D_j \) is compact and nowhere dense. Letting \( d\mu = dz \) on \( \partial D \) and \( -dz \) on the remaining circles \( \Gamma_j = \partial D_j, j = 1, 2, 3, \ldots \), we obtain a nonzero measure of finite total variation on \( E \) such that

\[
\int_E f d\mu = \int_{\partial D} f dz - \sum_j \int_{\Gamma_j} f dz = 0
\]

for all \( f \in R(E) \). Thus, \( R(E) \neq C(E) \) and so, by the Hartogs–Rosenthal theorem, \( E \) has positive area. The space \( H^p(E, dA) \) is therefore nonempty, and we can ask whether \( H^p(E, dA) = L^p(E, dA) \).

Although the Hartogs–Rosenthal theorem allows us to conclude indirectly that \( |E| > 0 \), it fails to provide any additional information on the specific geometric structure of the Swiss cheese. For this reason it is important here to recall an argument due to W. K. Allard (cf. [5, p. 163]) which is considerably more informative on that point. For each \( x \in [-1, 1] \), let \( E_x = \{ z \in E : \Re z = x \} \), and for each \( n = 1, 2, 3, \ldots \), let \( I_n(x) \) be the number of points in \( E_x \cap \Gamma_n \). Evidently, \( I_n(x) = 0, 1 \) or 2. By the monotone convergence theorem we have

\[
\int_{-1}^1 \sum_{n=1}^\infty I_n(x) \, dx = \sum_{n=1}^\infty \int_{-1}^1 I_n(x) \, dx = 4 \sum_{n=1}^\infty r_n < \infty.
\]
Hence, $\sum I_n(x) < \infty$ for almost every $x \in [-1,1]$. For any such $x$ all but finitely many $I_n(x)$ must be zero, and the corresponding set $E_x$ consists of a finite number of nondegenerate intervals. Consequently $|E| > 0$ by Fubini’s Theorem.

An equally important implication for the question raised in the introduction is the following.

**Lemma 2.1.** Let $E$ be a Swiss cheese and for each $z \in E$ let $\mathcal{F}_z$ denote the union of all circles centered at $z$ and lying entirely in $E$. Then, there exists at least one point $z \in E$ where $|\mathcal{F}_z| > 0$.

**Proof.** Let $E_x$ be as above, denote by $l(E_x)$ its total length or linear measure, and set $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j$. Since there are uncountably many $x \in [-1,1]$ where $E_x$ consists of a finite number of nondegenerate intervals we can choose a sequence of such points $x_n$ in such a manner that $l(E_{x_n}) \geq C > 0$ for some constant $C$ and all $n = 1, 2, 3, \ldots$. Moreover, we can assume with no loss of generality that $x_n < x_{n+1}$. Now select a finite collection of disjoint open disks $\Delta_{j_1} = \Delta(z_{j_1}, r_{j_1})$ with centers $z_{j_1} \in E_{x_1}$ so that $r_{j_1} < \frac{1}{3}(x_2 - x_1)$ and $\sum r_{j_1} > \frac{C}{3}$. Next, choose another finite collection of disjoint open disks $\Delta_{j_2} = \Delta(z_{j_2}, r_{j_2})$ with centers $z_{j_2} \in E_{x_2}$ so that $r_{j_2} < \frac{1}{3} \min(x_3 - x_2, x_2 - x_1)$ and $\sum r_{j_2} > \frac{C}{2}$. The disks $\Delta_{j_2}$ are clearly disjoint from any of the $\Delta_{j_1}$. Continuing in this way we obtain a collection of disjoint open disks which we redesignate as $\Delta_j$ with centers $z_j \in E$ and radii $r_j$ such that $\sum r_j = \infty$.

If we assume that $|\mathcal{F}_{z_j}| = 0$ for all $j = 1, 2, 3, \ldots$, then almost every circle in each $\Delta_j$ with center at $z_j$ will meet $\Gamma$ and we will be forced to conclude that

$$l(\Gamma) \geq \sum_{j=1}^{\infty} l(\Gamma \cap \Delta_j) \geq \sum_{j=1}^{\infty} r_j = \infty,$$

contradicting our construction ensuring that $l(\Gamma) < \infty$. Thus $|\mathcal{F}_{z_j}| > 0$ for, not only one, but many $z_j$. \hfill $\square$

**Theorem 2.2.** If $E$ is an arbitrary Swiss cheese there exists a point $x_0 \in E$ which is a bpe for $H^p(E,dA)$, for $1 \leq p < \infty$. In particular, $H^p(E,dA) \neq L^p(E,dA)$ for any $p \geq 1$.

**Proof.** Choose a point $x_0 \in E$ for which $|\mathcal{F}_{x_0}| > 0$. Assume for convenience that $x_0 = 0$ and let $X = \mathcal{F}_{x_0} \cap [0,1]$. For any polynomial $P$,

$$\int_{\mathcal{F}_{x_0}} P \, dA = \int_{0}^{2\pi} \int_{X} P(r e^{i\theta}) r \, dr \, d\theta = \int_{X} \left( \int_{0}^{2\pi} P(r e^{i\theta}) \, d\theta \right) r \, dr = 2\pi C P(0),$$

where $C = \int_{X} r \, dr$. It follows that

$$|P(0)| \leq \frac{1}{2\pi C} \|P\|_{L^1(E,dA)},$$

and so by Hölder’s inequality, $x_0 = 0$ is a bpe for $H^p(E,dA)$ whenever $1 \leq p < \infty$. \hfill $\square$

In addition to what has been established it can be shown that whenever $|\mathcal{F}_{x_0}| > 0$ every function $f \in H^p(E,dA)$ actually admits an analytic continuation to a fixed neighborhood of $x_0$. To that end, choose $\epsilon > 0$ and small enough to ensure that the portion of $\mathcal{F}_{x_0}$ lying outside of $D_\epsilon = \{z : |z - x_0| < \epsilon\}$ has positive $dA$-measure. By the argument in the proof of the preceding theorem, there exists a function $h \in L^\infty$ with support in $E \setminus D_\epsilon$ such that

$$P(x_0) = \int P h \, dA$$
for all polynomials $P$. Thus $kdA = (z - x_0)h dA$ is an annihilating measure and so for any polynomial $P$ and $\zeta \in \mathbb{C}$, it follows from an argument due essentially to Cauchy that

$$P(\zeta) = \frac{1}{k(\zeta)} \int \frac{P(z)}{z - \zeta} k dA$$

at any point where $k(\zeta) = \int \frac{k(z)}{z - \zeta} dA$ is defined and not equal to zero. Since $\hat{k}$ is analytic in $D_\epsilon$ and $\hat{k}(x_0) = 1$ it follows that $|\hat{k}(\zeta)| \geq C > 0$ in a neighborhood $U$ of $x_0$. Since the kernel $(z - \zeta)^{-1}$ is bounded on the support of the measure $dA$ we conclude that

$$P(\zeta) \leq K \|P\|_{L^1(E, dA)}$$

for some suitable constant $K$, and therefore $\zeta$ is a bpe for $H^1(E, dA)$.

§3. Analytic capacity and construction of barriers

The notion of analytic capacity was introduced by Ahlfors in 1947 in connection with the problem of characterizing sets of removable singularities for bounded analytic functions. In subsequent years others, and Vitushkin in particular, further developed the concept and used it to settle a number of questions concerning uniform approximation by rational functions on compact subsets of the plane.

As initially conceived, the analytic capacity of a compact set $X$, denoted $\gamma(X)$, is defined as follows:

$$\gamma(X) = \sup \|f'(\infty)\|,$$

where the supremum is extended over all functions $f$ analytic in $\hat{\mathbb{C}} \setminus X$ and normalized so that

(a) $\|f\|_\infty = \sup_{\hat{\mathbb{C}} \setminus X} |f| \leq 1$;

(b) $f(\infty) = 0$.

Here $\hat{\mathbb{C}}$ is the extended complex plane or Riemann sphere. For an arbitrary planar set $E$ we let $\gamma(E) = \sup \gamma(X)$, the supremum now being taken over all compact sets $X \subseteq E$. For a more thorough discussion of analytic capacity and its properties, see [8] and [27], where it is shown that

1. $\gamma(B_r) = r$ for every disk $B_r$ of radius $r$;
2. $\gamma(X) \approx \text{diam}(X)$ whenever $X$ is compact and connected; in particular, $\gamma(X) \leq \text{diam}(X) \leq 4\gamma(X)$.

From the outset it was not known and is still not known whether $\gamma$ is subadditive, and so possibly not a capacity in the usual sense. We now know, however, that $\gamma$ is at least semiadditive in that

$$\gamma(E \cup F) \leq C (\gamma(E) + \gamma(F))$$

for all compact (and even Borel) sets $E, F \subseteq \mathbb{C}$ and some absolute constant $C$. The key point is that $\gamma$ is equivalent to a second auxiliary capacity $\gamma^+$ defined as follows. For a compact set $X$ and positive measure $\nu$ supported on $X$ we form the Cauchy integral

$$(3.1) \quad \hat{\nu}(z) = \int \frac{d\nu(\zeta)}{\zeta - z}$$

and we define

$$\gamma^+(X) = \sup_{\nu} \nu(X)$$

to be the supremum over all positive measures $\nu$ such that $\hat{\nu} \in L^\infty(\mathbb{C})$ and $\|\hat{\nu}\|_\infty \leq 1$. Since $\hat{\nu}$ is analytic in $\hat{\mathbb{C}} \setminus X$ and $|\hat{\nu}(\infty)| = \nu(X)$, the function $\hat{\nu}$ is admissible for $\gamma$ and so

$$\gamma^+(X) \leq \gamma(X).$$
As before, if $E$ is an arbitrary planar set we let $\gamma^+(E) = \sup \gamma^+(X)$, where $X$ is compact and $X \subset E$.

The essential equivalence of $\gamma$ and $\gamma^+$ was established by Tolsa [22]. Here is what he proved: There exists an absolute constant $C > 0$ so that

1. $\gamma^+(E) \leq \gamma(E) \leq C \gamma^+(E)$ for all sets $E \subseteq \mathbb{C}$;
2. if $E_n$, $n = 1, 2, 3, \ldots$, are Borel sets, then $\gamma(\bigcup E_n) \leq C \sum \gamma(E_n)$.

Since Tolsa had previously shown that $\gamma^+$ is itself countably semiadditive, (3) implies (4).

Because $\gamma^+$ is defined directly in terms of the Cauchy integral, it can be used to establish a certain lower semi-continuity enjoyed by such integrals. If $\mu$ is a finite, complex, compactly supported measure, then $\hat{\mu}$ is defined as in (3.1) and

$$U[\mu](z) = \int \frac{d\mu(\xi)}{|\xi - z|}$$

is the corresponding Newtonian potential.

**Lemma 3.1.** Let $\mu$ be a finite complex, compactly supported, Borel measure in $\mathbb{C}$, and let $x_0$ be any point where $U[\mu](x_0) < \infty$. For each $r > 0$ let $B_r = B(x_0, r)$ be the disk with center at $x_0$ and radius $r$, and let $E$ be a set with the property that for every $r > 0$ there is a relatively large subset $E_r \subseteq (E \cap B_r)$ on which $U[\mu]$ is bounded; that is,

1. $U[\mu] \leq M_r < \infty$ on $E_r$;
2. $\gamma(E_r) \geq \epsilon \gamma(E \cap B_r)$ for some absolute constant $\epsilon$. If, moreover, $E$ is thick at $x_0$ in the sense that

$$\limsup_{r \to 0} \frac{\gamma(E \cap B_r)}{r} > 0,$$

then

$$|\hat{\mu}(x_0)| \leq \limsup_{z \to x_0, z \in E} |\hat{\mu}(z)|.$$

A proof can be found in [3] p. 224.

In order to extend Theorem 2.2 to the most general setting where $R(X) \neq C(X)$ it is essential that we have a way to decide whether a point $x_0 \in X$ is a bpe for $H^p(X, dA)$ or not. With this in mind we shall adopt a scheme due, in broad outline, to Thomson [21] and having its roots in the work of Mel’nikov [15]. Throughout the discussion $\nu$ will be a measure on $X$, not necessarily absolutely continuous with respect to area. For each $\lambda > 0$ the sets $E_\lambda = \{z : |\hat{\nu}(z)| < \lambda\}$ will play a critical role.

Let $x_0 \in X$ be a point where $U[\mu](x_0) < \infty$ and fix $\lambda > 0$. For each positive integer $n$ form a grid in the plane consisting of lines parallel to the coordinate axes and intersecting at those points whose coordinates are both integral multiples of $2^{-n}$. The resulting collection of squares $\mathcal{G}_n = \{S_n\}_{n=1}^\infty$ of side length $2^{-n}$ is an edge-to-edge tiling of the plane; its members will be referred to as squares of the $n$-th generation.

Beginning with a fixed generation, the $n$-th say, choose a square $S^* \in \mathcal{G}_n$ with $x_0 \in S^*$. Denote by $\mathcal{G}_n^\lambda$ the collection of all squares in $\mathcal{G}_n$ for which

$$|E_\lambda \cap S| > \frac{1}{100} |S|.$$ 

$K_n$ will denote the union of those squares in $\mathcal{G}_n^\lambda$ that can be joined to $S^*$ by a finite chain of squares lying in $\mathcal{G}_n^\lambda$. If $K_n$ is bounded, or empty, there exists a closed corridor or barrier, $Q_n = \bigcup S_{nj}$, composed of squares $S_{nj}$ from $\mathcal{G}_n$ abutting $S^* \cup K_n$, separating the latter from $\infty$, adjacent to one another along their sides, and such that

$$|E_\lambda \cap S_{nj}| \leq \frac{1}{100} |S_{nj}|.$$
for each \( j \). The polynomial convex hull of \( Q_n \) is a polygon \( \Pi_n \) with its boundary \( \Gamma_n \) lying along sides of squares for which (3.4) is satisfied. Thus \( |\hat{\nu}| \geq \lambda \) on a large portion of every square from \( S_n \), meeting \( \Gamma_n \). By adjoining to \( \Pi_n \) additional squares from \( G_n \), we obtain another polygon \( \Pi_n^* \) with boundary \( \Gamma_n^* \) in such a way that

1. \( \Gamma_n^* \supseteq \Gamma_n \);
2. \( n^22^{-n} \leq \text{dist}(\Gamma_n^*, \Gamma_n) \leq 3n^22^{-n} \).

This can be done by simply adjoining to \( \Pi_n \) additional squares from \( G_n \).

At this point let \( K_{n+1} \) denote the union of all squares in \( G_{n+1} \), which can be joined to \( \Pi_n^* \) by a chain of squares in \( G_{n+1} \). Again, if \( K_{n+1} \) is bounded, or empty, there is a second barrier \( Q_{n+1} \) abutting \( \Pi_n^* \) and

\[ |E_{\lambda} \cap S| \leq \frac{1}{100} |S| \]

for every square \( S \) in \( Q_{n+1} \). The polygon \( \Pi_{n+1} \) is defined to be the polynomial convex hull of \( Q_{n+1} \), and the process continues. In this way we obtain a nested sequence of polygons

\[ \Pi_n \subseteq \Pi_{n+1} \subseteq \cdots \subseteq \Pi_{n+t} \subseteq \cdots \]

and compact sets \( K_j \subseteq \Pi_j \setminus \Pi_{j-1} \), some of which may be empty, such that if \( K_j \neq \emptyset \), then

1. \( K_j \) is the union of squares in \( G_j \) and connects \( \Gamma_j^{*} \) to \( Q_j \);
2. \( |E_{\lambda} \cap S| > \frac{1}{100} |S| \) for each \( S \subseteq K_j \);
3. \( \text{dist}(K_j, \Gamma_j^{*}) \leq \text{dist}(K_j, \Gamma_j) + \text{dist}(\Gamma_j, \Gamma_j^{*}) < 4j2^{-j} \).

Given an arbitrary disk \( B_r = B(x_0, r) \) with center at \( x_0 \) there are two mutually exclusive possibilities:

(A) the sets \( K_j \) eventually exit \( B_r \);
(B) there exists an infinite sequence of barriers \( Q_j \), \( j = n, n+1, n+2, \ldots \) extending outward from \( x_0 \) and lying entirely in \( B_r \).

Should the first alternative (A) be the case for all \( r > 0 \) it can be shown (cf. [4, p. 233]) that the set \( E_{\lambda} \) satisfies all the hypotheses of Lemma 3.1, and therefore

\[ \limsup_{r \to 0} \frac{\gamma(E_{\lambda} \cap B_r)}{r} > 0. \]

The second alternative (B) implies that \( E_{\lambda} \) surrounds \( x_0 \) and contains sufficient mass to ensure that \( H^1(X, dA) \) admits a bpe at \( x_0 \). This phenomenon replaces the collection of circles lying inside a Swiss cheese as discussed in §2. It is the subject of our next lemma.

**Lemma 3.2.** If there exists an infinite sequence of barriers \( Q_j \), \( j = n, n+1, n+2, \ldots \), surrounding a point \( x_0 \), then \( H^1(X, dA) \) admits a bpe at \( x_0 \).

**Proof.** Because \( Q_n \) is a barrier, \( \Gamma_n \) is the union of certain specified sides of \( n \)-th generation squares \( S \) such that \( |E_{\lambda} \cap S| \leq \frac{1}{100} |S| \), or, setting \( F_{\lambda} = \{ z : |\hat{\nu}(z)| \geq \lambda \} \), squares \( S \) for which

\[ |F_{\lambda} \cap S| \geq \frac{99}{100} |S|. \]

We can assume for the purpose of argument that \( \lambda = 1 \), and we set \( F = F_1 \) and \( E = E_1 \).

The map \( L : P \to P(x_0) \) can be viewed as a bounded linear functional on the space of polynomials when the latter is endowed with the norm \( \| \| P \|_L^{\infty}(\Gamma_n) = \sup_{\Gamma_n} |P| \). As such, \( L \) can be extended in a norm-preserving way to \( C(\Gamma_n) \), the full space of continuous
functions on $\Gamma_n$ likewise endowed with the uniform norm. Hence, there exists a measure $\omega$ of finite total variation on $\Gamma_n$ such that $\|\omega\| = \|L\|$ and

$$P(x_0) = \int_{\Gamma_n} P \, d\omega$$

for all polynomials $P$. The first step in the proof of the lemma is to replace $\int_{\Gamma_n} P \, d\omega$ by an area integral over $F \cap Q_n$, committing only a small error.

Assume for the moment that $P$ is a fixed polynomial. Take $\epsilon > 0$ and let $\Gamma_n = \bigcup I_j$ be the union of finitely many closed intervals $I_j$ with mutually disjoint interiors chosen so that

$$\left| \int_{\Gamma_n} P \, d\omega - \sum_j P(\xi_j)\omega_j \right| < \epsilon$$

whenever $\xi_j \in I_j$ and $\omega_j = \omega(I_j)$. We can arrange that each $I_j$ is contained entirely in the side of a single square $S$ in the barrier $Q_n$. Moreover, $\omega$ can have no point masses and so there is no ambiguity associated with the approximating sums for $\int_{\Gamma_n} P \, d\omega$ in (3.5).

For a fixed barrier square $S \subseteq Q_n$ with one or more of its sides in $\Gamma_n$, let $x_S$ denote its center, and let $\xi_j$ be one of the points in (3.5) situated on $\partial S$. Since by construction $\text{dist}(\Gamma_n, \Gamma_{n+1}) \geq n^{\frac{1}{2}}2^{-n}$, a form of Schwartz’s lemma implies that

$$|P(\xi_j) - P(x_S)| \leq \frac{2n+1}{n^2} |\xi_j - x_S| \|P\|_{L^\infty(\Gamma_{n+1})} \leq \frac{\sqrt{2}}{n^2} \|P\|_{L^\infty(\Gamma_{n+1})}.$$ 

If $S_1, S_2, \ldots, S_k$ represent the totality of squares in $Q_n$ with sides along $\Gamma_n$ and $B_1, B_2, \ldots, B_k$ are the corresponding inscribed disks, it follows by summing on $j$ and the fact that $\Gamma_n$ lies entirely inside the region bounded by $\Gamma_{n+1}$ that

$$\left| P(x_0) - \int_{F \cap Q_n} P \, h_n \, dA \right| \leq \epsilon + \left(\frac{\sqrt{2}}{n^2} + \frac{2}{100}\right) \|\omega\| \|P\|_{L^\infty(\Gamma_{n+1})},$$

where $h_n = \sum_{j=1}^k \frac{4}{2n\omega(S_j)\chi_{F \cap B_j}}$ and $\chi_{F \cap B_j}$ is the characteristic function of $F \cap B_j$. Since $\epsilon > 0$ is arbitrary it can now be dropped from the inequality, and by choosing $n$ sufficiently large we can arrange that

$$\left| P(x_0) - \int_{F \cap Q_n} P \, h_n \, dA \right| \leq \frac{3}{100} \|\omega\| \|P\|_{L^\infty(\Gamma_{n+1})}$$

for all polynomials $P$, and $\|h_n\|_\infty \leq \frac{2^{n+2}}{\pi} \|\omega\|$. Now repeat the process noting that the map

$$L_{n+1} : P \mapsto P(x_0) - \int_{F \cap Q_n} P \, h_n \, dA$$

can be extended from the space of polynomials and viewed as a bounded linear functional on $C(\Gamma_{n+1})$ with $\|L_{n+1}\| \leq \frac{3}{100} \|L\|$, where $L = L_n$ and $\|L\| = \|\omega\|$. Since all squares adjacent to $\Gamma_{n+1}$ are barrier squares from $Q_{n+1}$, the argument above gives a function $h_{n+1}$ with support in $F \cap Q_{n+1}$ so that

$$\left| P(x_0) - \int P \, h_n \, dA - \int P \, h_{n+1} \, dA \right| \leq \left(\frac{3}{100}\right)^2 \|L\| \|P\|_{L^\infty(\Gamma_{n+2})},$$

for all polynomials $P$, and $\|h_{n+1}\|_\infty \leq \frac{4}{\pi} 2^{2(n+1)} \frac{3}{100} \|L\|$. Continuing in this way we obtain an infinite sequence of functions $h_n, h_{n+1}, \ldots$ such that for any $k > 0$, 

$$\left| P(x_0) - \int P(h_n + \cdots + h_{n+k}) \, dA \right| \leq \left(\frac{3}{100}\right)^{k+1} \|L\| \|P\|_{L^\infty(\Gamma_{n+k+1})}.$$
For a given polynomial $P$ the right side tends to zero as $k \to \infty$, since the curves $\Gamma_{n+k}$ all lie in a bounded portion of the plane. Setting $h = \sum_{k=0}^{\infty} h_{n+k}$ it follows that

$$P(x_0) = \int P \, h \, dA$$

for all polynomials $P$. Moreover, $h \in L^\infty$ because the individual $h_j$’s have disjoint supports and $\|h_{n+k}\|_\infty \leq \|h_n\|_\infty$ for all $k > 0$; hence, $\|h\| = \|h_n\|_\infty$. □

§4. **Bounded point evaluations for $H^p(dA)$**

We have now reached the point where we can address the question raised in the introduction concerning the relation between uniform rational approximation and the existence of $L^p$-bounded point evaluations for the polynomials. Even in the most general situation, however, bpe’s arise for essentially the same reason as in the case of the Swiss cheese. That is, a point $x_0 \in X$ is a bpe if it is surrounded by a portion of $X$ having sufficient mass to ensure that the inequality (1.1) is satisfied at $x_0$. From our reasoning it would appear that in some cases the local geometry of $X$ in a neighborhood of a bpe $x_0$ must be quite complicated, but we have not been able to rule out the possibility that there is a collection of concentric circles about $x_0$, lying entirely in $X$ and having positive $dA$ measure.

**Theorem 4.1.** Let $X$ be a compact subset of $\mathbb{C}$ with empty interior. If $R(X) \neq C(X)$, then there exists at least one point $x_0$ that yields a bpe for every $H^p(X,dA)$, $1 \leq p < \infty$. Moreover, every function $f \in H^p(X,dA)$ admits an analytic extension to a fixed neighborhood of $x_0$.

**Proof.** By assumption there exists a nonzero measure $\nu$ such that $\nu \perp R(X)$. Since $\nu \neq 0$ as a measure there is at least one point $x_0$ such that

(a) $U^{[\nu]}(x_0) < \infty$;

(b) $\nu(x_0) \neq 0$.

We can conclude, therefore, that there exists an infinite sequence of barriers relative to the set where $|\nu|$ is bounded away from zero, and surrounding the point $x_0$ as described in §3. Such a collection of barriers must, of course, lie entirely in $X$ since $\nu = 0$ in $C \setminus X$.

Suppose for the moment that no such sequence of barriers exists. For an arbitrary, but fixed, $\lambda > 0$ consider the set $E_\lambda = \{z : |\nu(z)| < \lambda\}$. By assumption, $E_\lambda$ must in some sense escape from $x_0$ to $\infty$. More precisely, we can find a connected set $X$ linking $x_0$ to $\infty$ such that $X$ is the union of squares from some generation, the $n$-th say, and higher, and certain narrow rectangles $R_j$, $j > n$, where

1. $|E_\lambda \cap S| > \frac{1}{100} |S|$ for each square $S \subseteq X$;
2. $\text{diam}(R_j) \approx j^2 2^{-j}$.

Given $r > 0$, let $B_r = B(x_0, r)$. By discarding certain superfluous pieces we can assume that $X \cap B_r$ is connected and joins $x_0$ to $\partial B_r$. Thus,

$$\gamma(X \cap B_r) \geq \frac{1}{4} \text{diam}(X \cap B_r) \geq \frac{r}{8}.$$

On the other hand, it follows from the countable semiadditivity of analytic capacity that

$$\frac{r}{16} \leq \gamma(X \cap B_{r/2}) \leq C \left( \gamma(K) + \sum_{j=n}^{\infty} j^2 2^{-j} \right),$$

where $K$ is the union of squares in $X$ for which (1) is satisfied, and $C$ is an absolute constant. Since we are free to begin with an arbitrary generation, we can let $n \to \infty$ and conclude that

$$\gamma(E_\lambda \cap B_r) \geq Cr.$$
that {\(R\)}earlier paper \([19]\) for many of the computational details. Here we shall describe an entire a Swiss cheese \(X\)uropean way to the presence of nonpeak points, and that is indeed the case.

Let us recall by way of a scholium the key role played by peak points in the theory of rational approximation. By definition, a point \(x \in X\) is a peak point for \(R(X)\) if there exists an \(f \in R(X)\) such that \(f(x) = 1\), but \(|f(y)| < 1\) whenever \(y \neq x\). Bishop’s peak point criterion for rational density is this (cf. \([5, p. 172]\) or \([8, p. 54]\)).

**Theorem 4.2** (Bishop). \(R(X) = C(X)\) if and only if \(dA\)-almost every point of \(X\) is a peak point for \(R(X)\).

That fact in itself suggests that the conclusion of Theorem 4.1 is linked in a fundamental way to the presence of nonpeak points, and that is indeed the case.

**Corollary 4.3.** If \(x_0 \in X\) is not a peak point for \(R(X)\), then \(x_0\) yields a bpe for \(H^p(X, dA)\), \(1 \leq p < \infty\); that is,

\[
|P(x_0)| \leq C_p \|P\|_{L^p(X, dA)}
\]

for every polynomial \(P\) and some constant \(C_p\) depending only on \(p\).

The proof of the corollary is a direct consequence of the following well-known fact: \(x_0\) is a peak point for \(R(X)\) if and only if \(\tilde{\nu}(x) = 0\) whenever \(\nu \perp R(X)\) and \(U^{\nu,1}(x) < \infty\). The argument in the proof of Theorem 4.1 can therefore proceed exactly as before. It would be remiss, however, if we did not mention that Aleman, Richter and Sundberg \([2]\) have also shown that almost every point where (a) and (b) are both satisfied corresponds to a bpe for \(H^p(X, dA)\). That is sufficient for the proof of Theorem 4.1, but not for the corollary. The underlying feature here is that Lemma 3.1 applies at every point \(x_0\) where (a) and (b) are satisfied simultaneously.

§5. A COUNTEREXAMPLE FOR \(R^p(dA)\)

In 1966, Sinanyan \([20]\) announced the following result, which now stands in striking contrast to Theorem 4.1.

**Theorem 5.1** (Sinanyan). There exists a compact nowhere dense set \(X\) such that \(R(X) \neq C(X)\), but nevertheless \(R^p(X, dA) = L^p(X, dA)\) for all \(p, 1 \leq p < \infty\).

His proof depends on a construction of Mergelyan \([17, p. 315]\) and actually produces a Swiss cheese \(X\) with the desired properties. The reader, however, is referred to an earlier paper \([19]\) for many of the computational details. Here we shall describe an entire family of compact nowhere dense sets \(X\) having a locally nonrectifiable perimeter such that \(R(X) \neq C(X)\), and still \(R^p(X, dA) = L^p(X, dA)\) for all \(p < \infty\).
In order to verify that \( R^p(X, dA) = L^p(X, dA) \) in any given instance we shall argue by duality in the standard way. Suppose \( k \) is any function in \( L^q(X, dA) \), \( q = \frac{p}{p-1} \), with the property that \( \int f k \, dA = 0 \) for every rational function \( f \) that is analytic on \( X \), and form the Cauchy integral

\[
\hat{k}(\zeta) = \int \frac{k(z)}{z - \zeta} \, dA_z.
\]

By assumption \( \hat{k} \equiv 0 \) in \( \mathbb{C} \setminus X \) and we need only show that \( \hat{k} = 0 \) dA.e. on \( X \), from which it follows that \( k \, dA = 0 \) as a measure. Since we are assuming that \( X \) has no interior, and \( \hat{k} \) is continuous when \( q > 2 \), we can infer that \( R^p(X, dA) = L^p(X, dA) \) for \( 1 \leq p < 2 \) with no additional requirements. The import of Theorem 5.1 consists in the assertion of rational density when \( p \geq 2 \).

In the less transparent case where \( 1 < q \leq 2 \) the Cauchy integral \( \hat{k} \) belongs to the Sobolev space \( W^q_1 \) and as such enjoys a certain residual continuity which is best described in terms of an associated nonlinear capacity \( C_q \). Nothing in the way of generality is lost if we further assume that \( q < 2 \), and we will do so. By definition, for any Borel set \( E \),

\[
C_q(E)^{1/q} = \sup_{\nu} \nu(E),
\]

the supremum being taken over all positive measures \( \nu \) concentrated on \( E \) for which \( \|U^\nu\|_{L^p(dA)} \leq 1 \). For additional information and background material on these nonlinear capacities the reader is referred to the books [1, 13], and articles [7, 14] where proofs of the following can be found:

(i) if \( \Phi \) is a contraction, then \( C_q(\Phi E) \leq kC_q(E) \) for some constant \( k \) depending only on \( q \);
(ii) \( C_q(B_r) \approx C_q(\text{diam } B_r) \approx r^{2-q} \) for any disk \( B_r \) of radius \( r \).

A property is said to hold \( q \)-quasieverywhere if the set where it fails has \( q \)-capacity zero.

As an element of \( W^q_1 \), the transform \( \hat{k} \) is \( q \)-quasicontinuous in the sense that: Given any \( \epsilon > 0 \) there exists an open set \( U \) such that \( C_q(U) < \epsilon \) and \( \hat{k} \) is continuous in the complement of \( U \). In addition there is a much subtler pointwise notion of continuity associated with functions in \( W^q_1 \), called fine continuity. A function \( h \in W^q_1 \), which we can assume to be defined \( q \)-quasieverywhere, is said to be \( q \)-finely continuous at a point \( x_0 \) if there exists a set \( E \) that is thin, or sparse, in a potential-theoretic sense at \( x_0 \) and

\[
\lim_{x \to x_0, x \notin E} h(x) = h(x_0).
\]

If \( E \) is not thin at \( x_0 \), then it is said to be thick at that point. In our case it is sufficient to know that \( E \) is thick at \( x_0 \) if

\[
\liminf_{r \to 0} \frac{C_q(E \cap B_r)}{C_q(B_r)} > 0,
\]

where \( B_r = B_r(x_0) \) is the disk with center at \( x_0 \) and radius \( r \) (cf. [4] p. 221).

In our treatment of Theorem 5.1 we are able to avoid many of the more extended computations found in Sinanyan’s construction (cf. [13, 20]) by taking advantage of the fact that the capacities \( C_q \) decrease modulo a constant under a contraction, whereas analytic capacity may not. Equally important to us, of course, is Vitushkin’s criterion for rational approximation (cf. [25] and [8] p. 207).

Theorem 5.2 (Vitushkin). \( R(X) = C(X) \) if and only if for dA-almost all points \( x \in X \),

\[
\limsup_{r \to 0} \frac{\gamma(B_r(x) \setminus X)}{r} > 0.
\]
Proof of Theorem 5.1. We begin with the construction of a planar Cantor set as follows. Let \( Q^0 = [0, 1] \times [0, 1] \) be the closed unit square. Choose 4 closed squares inside \( Q^0 \) with side length 1/4, having sides parallel to the coordinate axes, and so that each square contains a vertex of \( Q^0 \). Next, apply the same procedure to each of the four squares obtained in the first step. In this way we obtain 16 squares, each having side length 1/16. Continuing in this way, at the \( n \)-th stage we obtain \( 4^n \) closed squares \( Q^n_j, j = 1, 2, 3, \ldots, 4^n \), each having side length \( 1/4^n \). For each \( n \) let

\[
E_n = \bigcup_{j=1}^{4^n} Q^n_j
\]

and define

\[
K = \bigcap_{n=1}^{\infty} E_n.
\]

The set \( K \) is most commonly referred to as the corner quarters Cantor set. It can easily be checked that the orthogonal projection of \( K \) onto the line \( 2y = x \) covers an interval of length \( 3/\sqrt{5} \), in particular, a line segment of length greater than \( \frac{1}{2} \text{diam}(Q^0) \). Not quite as obvious, however, is the fact that \( \gamma(K) = 0 \). Cantor sets with these properties were first produced by Vitushkin [24], and his construction was later simplified by Garnett [9] and Ivanov [12] pp. 346–348.

Now iterate the procedure outlined above. Decompose \( Q^0 \) into 4 congruent squares \( S^1_j, j = 1, 2, 3, 4 \) by lines through midpoints of the opposite sides. In each square \( S^1_j \) construct a Cantor set \( K^1_j \) similar to \( K \) and differing only by a scaling factor of 1/4. Let \( K_1 = \bigcup_j K^1_j \) be the union of the four scaled-down Cantor sets, and continue the bisection process in the same manner, thereby obtaining a sequence of Cantor sets \( K_1, K_2, K_3, \ldots \) having these properties:

1. \( \gamma(K_n) = 0 \) for all \( n = 1, 2, 3, \ldots \);
2. \( E = \bigcup_n K_n \) is dense in \( Q^0 \);
3. \( \Lambda(\text{proj}(K^n_j)) > \frac{1}{2} \text{diam}(S^n_j) \).

Here \( K_n = \bigcup_j K^n_j \), \( \text{proj}(K^n_j) \) denotes the orthogonal projection of \( K^n_j \) onto the line \( 2y = x \), and \( \Lambda(\text{proj}(K^n_j)) \) denotes the 1-dimensional Hausdorff measure or length of this projection. It follows from Tolsa’s theorem on the countable semiadditivity of analytic capacity that \( \gamma(E) = 0 \). In this case, however, where \( E \) is the countable union of compact sets of capacity zero, the full force of Tolsa’s theorem is not needed (cf. [8] p. 237 or [10] p. 12).

Because \( \gamma(E) = 0 \), and therefore \( |E| = 0 \), we are able to select a compact set \( X_0 \) lying in the interior of \( Q = Q^0 \), and so that \( |X_0| > 0 \) and \( E \cap X_0 = \emptyset \). Pick \( r_1 > 0 \), but small enough to ensure that \( \{ z : \text{dist}(z, X_0) < r_1 \} \) lies entirely inside \( Q \). Since \( K_1 \) is a compact totally disconnected set with \( \gamma(K_1) = 0 \), we can cover \( K_1 \) by finitely many open rectangles, having mutually disjoint closures with sides parallel to the coordinate axes, and so that the union \( \Omega_1 \) of the open pieces satisfies \( \gamma(\Omega_1) < \frac{1}{2} r_1 \). Next, choose \( r_2 < r_1 \), but small enough that \( \{ z : \text{dist}(z, X_0) < r_2 \} \) does not meet \( \Omega_1 \). Proceed as above to cover \( K_2 \setminus \Omega_1 \) by finitely many open rectangles, with mutually disjoint closures, in such a way that

1. \( \gamma(\Omega_2) < \frac{1}{2^2} r_2 \),
2. \( \gamma(\Omega_1 \cup \Omega_2) < C \left( \frac{r_1}{2} + \frac{r_2}{2^2} \right) < Cr_1 \),

where \( C \) is the absolute constant guaranteed by Tolsa’s theorem. Continuing in this way,
we obtain a sequence of numbers \( r_j \downarrow 0 \) and a decreasing sequence of open sets \( \Omega_1, \Omega_2, \Omega_3, \ldots \) with these properties:

(a) \( E \subset \bigcup_j \Omega_j \);
(b) \( X_0 \subseteq Q \setminus \bigcup_j \Omega_j \);
(c) \( \gamma(\Omega_j) < \frac{1}{2^j} r_j \);
(d) \( \gamma(\Omega_j \cup \cdots \cup \Omega_k) < \frac{C}{2^j} r_j \) for all \( j = 1, 2, 3, \ldots \).

Setting \( X = Q \setminus \bigcup_j \Omega_j \) we obtain a compact nowhere dense set with \( X_0 \subseteq X \), and we must prove that \( R(X) \neq C(X) \), but \( R^p(X, dA) = L^p(X, dA) \) for all \( p \geq 2 \).

By construction, for each point \( x \in X_0 \), the inequality

\[
\frac{\gamma(B(x, r_j) \setminus X)}{r_j} \leq \frac{C}{2^j - 1}
\]

is satisfied for all \( j = 1, 2, 3, \ldots \), where \( C \) is an absolute constant throughout. Hence, at each point of \( X_0 \) the lower capacity density of the complement \( C \setminus X \) is zero. It follows from the instability of analytic capacity as established by Vitushkin [25] p. 190] that

\[
\lim_{r \to 0} \frac{\gamma(B(x, r) \setminus X)}{r} = 0
\]

at \( dA \)-a.e. point \( x \in X_0 \) (cf. also [8, p. 207]). By Vitushkin’s Theorem 5.2, it follows that \( R(X) \neq C(X) \).

Again by construction, for \( dA \)-a.e. point \( x \in X \), and \( r \) sufficiently small depending on \( x \), we have \( \Lambda(\text{proj}(B(x, r) \setminus X)) \geq Cr \), where \( C \) is an absolute constant independent of \( r \). For a fixed \( q < 2 \) this implies that \( C_q(B(x, r) \setminus X) \geq Cr^{2-q} \), since \( q \)-capacity decreases modulo a constant under a contraction. Hence, at \( dA \)-a.e. point \( x \in X \) the complement \( C \setminus X \) is thick in the sense of \( q \)-capacity. Suppose then that \( k \in L^q(X, dA) \) and that \( k \perp R^p(X, dA) \). Then \( \hat{k} \equiv 0 \) in \( C \setminus X \) and by fine continuity \( \hat{k} = 0 \) \( dA \)-a.e. on \( X \). Therefore, \( k = 0 \) \( a.e. \) and it follows that \( R^p(X, dA) = L^p(X, dA) \), and this holds for all \( p > 2 \).

As indicated at the beginning of §4 we do not know of a single example of a compact nowhere dense set \( X \) such that \( R(X) \neq C(X) \), and for which no point \( x_0 \in X \) admits a collection of concentric circles \( F_{x_0} \) having positive \( dA \) measure and lying entirely inside \( X \). The construction in the proof of Theorem 5.1, however, precludes the corresponding phenomenon for rectangles oriented so as to have two of their sides orthogonal to the line \( 2y = x \). The projection properties of irregular sets such as those that underlie the entire construction here are studied extensively in [6, Chapter 7].

If \( X \) happens to be a set of finite perimeter in the sense of DeGiorgi, it can be shown that there exist sufficiently many rectangles contained entirely in \( X \) to ensure that \( H^p(X, dA) \) has a bounded point evaluation for all \( p < \infty \), thereby extending Theorem 2.2 to this more general setting (cf. Trent [23]).

References


Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
E-mail address: brennan@ms.uky.edu

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
E-mail address: militzer@ms.uky.edu

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