LITTLEWOOD–PALEY INEQUALITY
FOR ARBITRARY RECTANGLES IN $\mathbb{R}^2$ FOR $0 < p \leq 2$

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Abstract. The one-sided Littlewood–Paley inequality for pairwise disjoint rectangles in $\mathbb{R}^2$ is proved for the $L^p$-metric, $0 < p \leq 2$. This result can be treated as an extension of Kislyakov and Parilov’s result (they considered the one-dimensional situation) or as an extension of Journé’s result (he considered disjoint parallelepipeds in $\mathbb{R}^n$ but his approach is only suitable for $p \in (1, 2]$). We combine Kislyakov and Parilov’s methods with methods “dual” to Journé’s arguments.

§1. Statement of the theorem and the history of the problem

Let $\Delta_m$ be pairwise disjoint intervals on $\mathbb{R}$. In 1983, Rubio de Francia (see [1]) proved that

$$\| \left( \sum_m |M_{\Delta_m} f|^2 \right)^{1/2} \|_{L^p(\mathbb{R})} \leq C_p \| f \|_{L^p(\mathbb{R})}$$

for $2 \leq p < \infty$, where $M_{\Delta} f = (\hat{f} \chi_{\Delta})^\vee$ is the Fourier multiplier corresponding to a set $\Delta$, and $C_p$ depends on $p$ only. Shortly after that, Journé (see [2]) extended this result to $\mathbb{R}^n$, proving an estimate similar to (1) in the case where $f$ is defined on $\mathbb{R}^n$ and $\Delta_m$ are pairwise disjoint parallelepipeds with sides parallel to coordinate axes. Note that the $n$-dimensional version of (1) cannot be proved by $n$-fold application of the one-dimensional estimate. For this purpose Journé in [2] described and then used the theory of singular integral operators on product domains. Later, Fernando Soria offered (see [3]) a simpler proof in the case of $\mathbb{R}^2$.

Now we note that the dual version of estimate (1) can be written as

$$\left\| \sum_m f_m \right\|_{L^p(\mathbb{R})} \leq C_p \| \{ f_m \} \|_{L^p(\mathbb{R}^2)}, \quad 1 < p \leq 2,$$

where $\{ f_m \}$ is a collection of functions with $\text{supp} \hat{f}_m \subset \Delta_m$. In 1984, Bourgain (see [4]) proved that (2) remains true when $p = 1$. He used a more complicated method than that employed by Rubio de Francia. In 2005, Kislyakov and Parilov, acting in the spirit of Rubio de Francia’s methods, established (see [5]) that (2) is true for all $0 < p \leq 2$ (they considered the unit circle $\mathbb{T}$ rather than $\mathbb{R}$, but this does not play a significant role). In this paper we extend their result (namely, inequality (2) for $0 < p \leq 2$) to $\mathbb{R}^2$, proving the following theorem.

Theorem. Let $\{ f_m \}$ be a sequence of functions such that $f_m \in L^1(\mathbb{R}^2)$ and $\text{supp} \hat{f}_m \subset \Delta_m$, where the $\Delta_m$ are disjoint rectangles in $\mathbb{R}^2$ with sides parallel to coordinate axes.

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Then for $0 < p \leq 2$ the following estimate is fulfilled:

\begin{equation}
\left\| \sum_{m} f_m \right\|_{L^p(\mathbb{R}^2)} \leq C_p \left\{ \left\| f_m \right\|_{L^p(\mathbb{R}^2, \ell^2)} \right\},
\end{equation}

where $C_p$ does not depend on the functions or the rectangles.

Here the index $m$ ranges over a countable or a finite set. If $m$ ranges over a finite set, we do not show intentionally in our notation that the norm on the right-hand side is calculated in $L^p(\mathbb{R}^2, \ell^2_N)$, $N \in \mathbb{N}$, rather than in $L^p(\mathbb{R}^2, \ell^2)$ (actually, we can assume that finite sequences are supplemented by zeros to become infinite). We shall do similarly in what follows without a special mention. It should also be noted that throughout this paper we use the following definition of a Fourier transform:

\[ \hat{f}(t) = \int_{\mathbb{R}^n} f(\xi) e^{-2\pi i \langle t, \xi \rangle} \, d\xi, \quad f \in L^1(\mathbb{R}^n). \]

Our proof is based on the ideas of [5], but it will be more convenient to refer to the more recent paper [6], because in [6] the case of $\mathbb{R}$ was considered and some arguments were presented with more care. Also, we shall use implicitly the ideas of [8]. In [8], Fernando Soria employed the space $\text{BMO}(\mathbb{R}\times\mathbb{R})$ and Carleson measures on product spaces. But since we work in the dual situation, the Hardy spaces on $\mathbb{R}\times\mathbb{R}$ and the atomic decomposition for them will be used instead. The required theory (i.e., the atomic decomposition for $H^p(\mathbb{R}\times\mathbb{R})$ and singular integral operators acting on these spaces) can be found in Fefferman’s paper [7].

Note that Fefferman’s theory, as it was presented in [7], is suitable only for the product of two Euclidean spaces $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. However, there is a more complicated theory (see [8]) suitable for the situation where the number of Euclidean factors is arbitrary. That theory allows us to obtain an analog of [9] for disjoint parallelepipeds in $\mathbb{R}^n$. In this case the proof becomes more complex and will be published separately.

§2. Preliminaries: The analytic and the “real” Hardy spaces on $\mathbb{R}\times\mathbb{R}$

All statements presented in this section (except Lemma [1]) can be found in [9]. Though in [9] all arguments are for the bidisk and scalar-valued functions, reformulations for $\mathbb{R}\times\mathbb{R}$ can be found at the end of that paper. At the same time, it is clear that all definitions and results from [9] remain valid for $\ell^2$-valued functions. Throughout this section, if the range of a function is not indicated explicitly, we assume that it can be either scalar-valued or $\ell^2$-valued. We also assume that $0 < p \leq 2$.

**Definition 1.** Let $D = D_1 \times D_2$, where $D_j = \{ z_j = x_j + iy_j \mid x_j, y_j \in \mathbb{R}; y_j > 0 \}$. A function $f$ defined and analytic on the domain $D$ belongs to $H^p_A(\mathbb{R}^2)$ if

\[ \| f \|_{H^p_A} = \sup_{y_1, y_2 > 0} \int_{\mathbb{R}^2} |f(x_1 + iy_1, x_2 + iy_2)|^p \, dx_1 \, dx_2 < \infty. \]

We write $H^p_A(\mathbb{R}^2)$ instead of $H^p_A(D)$, identifying the function $f$ with its limit values as $y = (y_1, y_2) \to 0$, where we pass to the limit in the sense of tempered distributions.

**Definition 2.** Consider a function $\varphi$ in the Schwartz class on $\mathbb{R}^2$ such that $\int \varphi = 1$. Let $f$ be a tempered distribution on $\mathbb{R}^2$. Then $f$ belongs to $H^p(\mathbb{R}^2)$ if

\[ M_{\varphi} f(x_1, x_2) = \sup_{\varepsilon_1, \varepsilon_2 > 0} |f * \varphi_{\varepsilon_1, \varepsilon_2}(x_1, x_2)| \in L^p(\mathbb{R}^2), \]

where $\varphi_{\varepsilon_1, \varepsilon_2}(x_1, x_2) = \varepsilon_1^{-1} \varepsilon_2^{-1} \varphi(x_1/\varepsilon_1, x_2/\varepsilon_2)$. Put also $\| f \|_{H^p} = \| M_{\varphi} f \|_{L^p}$.

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As in the one-dimensional case, the following relations are fulfilled: $H^p_0(\mathbb{R}^2) \subset H^p(\mathbb{R}^2)$ and $\|f\|_{H^p_0} \preccurlyeq \|f\|_{H^p}$ for $f \in H^p_0(\mathbb{R}^2)$. By analogy with $H^p_0$, we can consider the space of functions analytic in one variable and anti-analytic in the other, or the space of functions anti-analytic in both variables. In fact, $H^p$ is the direct sum of all such spaces. Details can be found in [9].

The following statement will be required later.

**Lemma 1.** Let $\Delta_m$ be rectangles (maybe, overlapping) in the first quadrant of the plane $\mathbb{R}^2$, and let $g_m$ be functions in $L^1(\mathbb{R}^2)$ with $\text{supp} \hat{g}_m \subset \Delta_m$. Suppose also that only finitely many $g_m$’s are not equal to zero. Then the function $G = \{g_m\}$ belongs to $H^p_0(\mathbb{R}^2, \ell^2)$ and $\|G\|_{H^p_0(\ell^2)} \leq\|G\|_{L^p(\ell^2)}$.

This statement is trivial from the classical Hardy spaces point of view, but we present its proof here for the sake of completeness.

**Proof.** For simplicity, we assume that $G$ consists of a single component whose Fourier transform is supported on the rectangle $\Delta = I \times J$. Without loss of generality, we may also assume that $G$ is continuous and bounded. Indeed, the functions $G$ and $(\hat{G})^\vee$ are equal as distributions, and $(\hat{G})^\vee$ is a bounded and continuous function (since the function $\hat{G}$ has compact support). Thus, $G$ is equal a.e. to some continuous and bounded function.

Now we extend the function $G$ to the domain $D$:

$$G(z_1, z_2) = G(x_1 + iy_1, x_2 + iy_2) = \int_{\mathbb{R}^2} P_{y_1}(x_1 - t_1) P_{y_2}(x_2 - t_2) G(t_1, t_2) dt_1 dt_2,$$

where $(z_1, z_2) \in D$ and $P_y(x) = e^y/(x^2 + y^2)$ is the Poisson kernel ($c$ is a normalizing constant providing the identity $\int_{\mathbb{R}} P_y(x) dx = 1$). We must verify that $G$ is analytic in $D$, that it tends to its boundary values in the sense of tempered distributions as $y = (y_1, y_2) \to 0$, and that the required estimate for the norm $\|G\|_{H^p_0}$ is fulfilled. Since the Poisson kernel is the Fourier transform of the function $e^{-2\pi y|\xi|}$, it follows that

$$G(z_1, z_2) = \int_{\mathbb{R}^2} e^{2\pi i x_1 \xi_1} e^{2\pi i x_2 \xi_2} dt_1 dt_2,$$

where the Fourier transformation is applied with respect to $\xi = (\xi_1, \xi_2)$. Now, using the formula $\int f(x)g(x) = \int f(x)\hat{g}(x)$, we obtain

$$G(z_1, z_2) = \int_{\Delta} \hat{G}(\xi_1, \xi_2) e^{2\pi i x_1 \xi_1} e^{2\pi i x_2 \xi_2} d\xi_1 d\xi_2.$$

Since $\hat{G}$ is continuous and supported on a compact set and the function $e^{2\pi i(|\xi|, z)}$ is analytic in $z$ on $\mathbb{C}^2$, we see that $G$ is analytic in $D$ and tends to its boundary values. It remains to verify the estimate for the norm.

First, we prove that $G(t_1, \cdot)^\wedge(\xi_2)$ (here we fix the first variable and apply the Fourier transformation with respect to the second) is supported on $J$ for almost all $t_1 \in \mathbb{R}$. Let $\varphi$ be a function in the Schwartz class that is equal to zero on $J$. Then

$$0 = \int_{\mathbb{R}} \hat{G}(\xi_1, \xi_2) \varphi(\xi_2) d\xi_2 = \int_{\mathbb{R}} e^{-2\pi i t_1 \xi_1} \int_{\mathbb{R}} G(t_1, \cdot)^\wedge(\xi_2) \varphi(\xi_2) d\xi_2 dt_1$$

for any $\xi_1 \in \mathbb{R}$. Since the Fourier transformation is a one-to-one operator, it follows that the inner integral in the right expression is equal to zero for almost all $t_1$. But

$$\int_{\mathbb{R}} G(t_1, \cdot)^\wedge(\xi_2) \varphi(\xi_2) d\xi_2 = \int_{\mathbb{R}} G(t_1, t_2) \hat{\varphi}(t_2) dt_2.$$

Thus, since $G$ is continuous and bounded, we see that the integral in question is a continuous function of $t_1$ and, therefore, is equal to zero for all $t_1$. Hence we obtain our statement about the support of $G(t_1, \cdot)^\wedge$. 
Next, consider two functions \( G_1 = G(\cdot, x_2 + iy_2) \) and \( G_2 = G(x_1, \cdot) \) defined on \( \mathbb{R} \). The first function is the Poisson extension of the boundary function \( G \) with respect to the second variable, and the second variable (now complex) is fixed. The second function is the boundary function \( G \) with the first variable fixed. We have proved that \( \text{supp} \hat{G}_2 \subset J \).

By the same argument, we obtain \( \text{supp} \hat{G}_1 \subset I \) (the convolution with the Poisson kernel in the second variable does not interfere). It follows that \( G_k \in H^1, k = 1, 2 \) (where \( H^1 \) is the usual Hardy space on \( \mathbb{R} \)). Indeed, we can prove the analyticity and the convergence to the boundary values much as we did for \( G \) (but the simpler one-dimensional case is considered now), and since \( \int R P_y(x) \, dx = 1 \), we get the estimate for the \( H^1 \)-norm. Now, using the subharmonicity of \( |G_k(x + iy)|^p \), we easily obtain

\[
|G_k(x + iy)|^p \leq (P_y * |G_k|^p)(x), \quad k = 1, 2.
\]

To get this, we must move away from the boundary (i.e., consider \( G_k(\cdot + i\varepsilon) \) instead of \( G_k \)), use the maximum principle for harmonic and subharmonic functions, and then pass to the limit in \( \varepsilon \). Details can be found in many standard manuals in harmonic analysis, e.g., in [10] III.4.2. From (4) and the identity \( \int R P_y(x) \, dx = 1 \) it follows that \( \|G_k\|_{H^p} \leq \|G_k\|_{L^p}, k = 1, 2 \). Applying these two inequalities consecutively to the initial function \( G \), we obtain the desired estimate. \( \square \)

Note that we can formulate this lemma for any quadrants of the plane \( \mathbb{R}^2 \), replacing \( H^p \) with the space of functions analytic or anti-analytic in the corresponding variables.

§3. Preliminaries: The atomic decomposition and bounded operators on \( H^p(\mathbb{R}^2) \)

In this section, we present the main results of [7] but with a small difference: the atomic decomposition theorem is stated here so that the number of vanishing moments in it (i.e., the number \( N \) in Definition 3 below) may be increased arbitrarily (see details below). We assume that \( 0 < p \leq 1 \), and that all functions may be either scalar-valued or \( \ell^2 \)-valued.

Definition 3. Consider a function \( \alpha \) in \( L^2(\mathbb{R}^2) \) (scalar-valued or \( \ell^2 \)-valued) and a non-negative integer \( N \). We say that \( \alpha \) is an \((N,p)\) rectangle atom if the following conditions are satisfied:

(i) \( \text{supp} \alpha \subset \Delta \), where \( \Delta = I \times J \) is a rectangle with sides parallel to coordinate axes;
(ii) \( \|\alpha\|_{L^2} \leq |\Delta|^{1/2 - 1/p} \);
(iii) \( \int_J \alpha(x_1, x_2^0) x_1^s \, dx_1 = \int_J \alpha(x_1^0, x_2) x_2^s \, dx_2 = 0 \) for \( 0 \leq s \leq N \) and for all \( x_2^0 \in I \) and \( x_1^0 \in J \).

Definition 4. Suppose \( \Omega \) is an open set in \( \mathbb{R}^2 \) of finite measure, and \( \mathcal{M}(\Omega) \) is the set of the maximal dyadic rectangles contained in \( \Omega \). A function \( a \in L^2(\mathbb{R}^2) \) supported on \( \Omega \) is called an \((N,p)\) atom if the following conditions are satisfied:

(i) \( a = \sum_{\Delta \in \mathcal{M}(\Omega)} \alpha_\Delta \), where each \( \alpha_\Delta \) is a function supported on a rectangle \( \Delta \) and satisfying condition (iii) in Definition 3 (i.e., \( \alpha_\Delta \) is an \((N,p)\) rectangle atom multiplied by a coefficient);
(ii) \( \|a\|_{L^2} \leq |\Omega|^{1/2 - 1/p} \);
(iii) \( \sum_{\Delta \in \mathcal{M}(\Omega)} \|\alpha_\Delta\|_{L^2}^2 \leq |\Omega|^{1-2/p} \).
Theorem (Atomic decomposition). Fix $0 < p \leq 1$. Then we can choose a number $\nu_p$ such that for $N \geq \nu_p$ the following is true: if a function $f$ belongs to $H^p(\mathbb{R}^2)$, then $f = \sum \lambda_k a_k$, where the $a_k$ are $(N,p)$ atoms and the $\lambda_k$ are scalars such that $\sum |\lambda_k|^p \leq C_{p,N} \|f\|_{H^p}^p$.

In [7], this theorem was formulated and proved for $N = \nu_p$ (more precisely, only the case where $p = 1$ and $N = \nu_p = 0$ was considered in detail there). But to lift this restriction, it suffices to choose an appropriate function $\psi$ that occurs in the definition of the quadratic functional $S_\psi$: this function must have an appropriate number of vanishing moments (see the corresponding proof in [7]). Using atomic decomposition and Journé’s covering lemma, R. Fefferman proved the following theorem (see [7]).

Theorem (R. Fefferman). Consider an operator $T$ bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ (recall that each of the spaces $L^2(\mathbb{R}^2)$ may consist either of scalar-valued or $\ell^2$-valued functions). Let $N \geq \nu_p$. Suppose that for any $(N,p)$ rectangle atom $a$ supported on a rectangle $\Delta$ we have

$$\int_{(\gamma,\Delta)\in S} |(T\alpha)(x_1,x_2)|^p \, dx_1 \, dx_2 \leq C_{\gamma}^{-\delta}$$

for some fixed $\delta > 0$ and for every $\gamma \geq 2$. Then $T$ is a bounded operator from $H^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$.

Here and below, $\gamma \cdot \Delta$ denotes the concentric dilation of $\Delta$ with coefficient $\gamma$.

§4. Proof of the main theorem: The auxiliary operators $S$ and $R$

Now we prove the main theorem. Since all our estimates will not depend on the number of the rectangles $\Delta_m$, we can assume that there are finitely many of them.

First, consider the special case where all rectangles $\Delta_m$ can be obtained from dyadic ones by 8-fold dilation of the sides with preservation of the lower left vertex. This means that, to each $\Delta_m$, we can assign a multi-index $(k,j) = (k_1, k_2, j_1, j_2) \in \mathbb{Z}^4$ such that

$$\Delta_m = [j_1 2^{k_1}, (j_1 + 8) 2^{k_1}] \times [j_2 2^{k_2}, (j_2 + 8) 2^{k_2}].$$

Also suppose that sup $\hat{\Delta}_m \subset \frac{3}{4} \cdot \Delta_m$. We make a technical remark.

Technical Remark. Any rectangle $R = [a,b] \times [c,d]$ can be placed in a rectangle $\Delta = I \times J$ of the type described. Moreover, this can be done so that $|I| \approx |[a,b]|$, $|J| \approx |[c,d]|$, and $R \subset \frac{3}{4} \cdot \Delta$.

Indeed, let $I = [j 2^k, (j + 8) 2^k]$, where $k$ satisfies $2^k \leq |[a,b]| < 2^{k+1}$ and $j = \sup \{ j \in \mathbb{Z} \mid (j + 1) 2^k < a \}$. It is clear that $|I| \approx |[a,b]|$ and $[a,b] \subset \frac{3}{4} \cdot I$. Choosing $J$ in the same way, we get the required rectangle.

Now we prove the theorem in the special case described above. Let $A$ denote the set of multi-indices corresponding to the rectangles $\Delta_m$. Let $h = \{ h_{k,j} \}_{(k,j) \in \mathbb{Z}^4} \subset L^2(\mathbb{R}^2, \ell^2)$. Consider the operator $S$ defined by the formula

$$S(h)(x_1, x_2) = \sum_{(k,j) \in A} e^{2\pi i j_1 2^{k_1} x_1} e^{2\pi i j_2 2^{k_2} x_2} (\Phi_k * h_{k,j})(x_1, x_2),$$

where $\Phi_k(x_1, x_2) = \varphi_{k_1}(x_1) \varphi_{k_2}(x_2)$, and in their turn, the functions $\varphi_n$ are defined as follows: take a function $\varphi$ in the Schwartz class on $\mathbb{R}$ such that its Fourier transform is nonnegative, supported on $[0,8]$, and equal to one on $[1,7]$; then put $\varphi_n(t) = 2^n \varphi(2^n t)$.

Lemma 2. The operator $S$ is bounded from $H^p(\mathbb{R}^2, \ell^2)$ to $L^p(\mathbb{R}^2)$ for $0 < p \leq 2$. Moreover, the estimation constant depends on $p$ only.
We postpone the proof of this lemma. Now we take it for granted and prove the theorem in the special case indicated above. Let \( g_{k,j} \equiv 0 \) for \((k,j) \notin \mathcal{A}\), and let
\[
g_{k,j}(x_1, x_2) = e^{−2\pi i j_1 2^{k_1} x_1} e^{−2\pi i j_2 2^{k_2} x_2} f_m(x_1, x_2)
\]
for the multi-index \((k,j) \in \mathcal{A}\) corresponding to the rectangle \(\Delta_m\). Note that the Fourier transforms of the nonzero functions \(g_{k,j}\) are supported on the rectangles
\[
\frac{3}{4} \cdot \Delta_m - (j_1 2^{k_1}, j_2 2^{k_2}) = [2^{k_1}, 7 \cdot 2^{k_2}] \times [2^{k_2}, 7 \cdot 2^{k_2}].
\]
By the Plancherel theorem, this implies that \(g_{k,j} = \Phi_k \ast g_{k,j}\). Denote \(g = \{g_{k,j}\}_{(k,j) \in \mathbb{Z}^2}\).

Taking the above into consideration and using Lemmas 1 and 2 we have
\[
\left\| \sum_m f_m \right\|_{L^p} = \left\| S(g) \right\|_{L^p} \leq C_p \|g\|_{H^p(\ell^2)} \leq C'_p \|g\|_{L^p(\ell^2)} = C'_p \left\{ \{f_m\} \right\}_{L^p(\ell^2)}.
\]
This is the required estimate.

Now we reduce the theorem to the special case considered. The arguments will be similar to those in [6], but, since we consider the two-dimensional case, the computations will be more complicated. We need another auxiliary operator. Let \(\psi\) be a function in the Schwartz class on \(\mathbb{R}\) such that \(\psi \geq 0\) and supp \(\psi \subset [A^{-1}, A]\), where \(A > 1\). Put \(\psi_n(t) = A^n \psi(A^n t), n \geq 0\). Note that supp \(\hat{\psi_n} \subset [A^{n-1}, A^n]\).

Let \(\Psi_{k_1,k_2}(x_1, x_2) = \psi_k(x_1) \psi_{k_2}(x_2)\), and let \(\{h_m\} \subset L^2(\mathbb{R}^2, \ell^2)\), the required operator \(R\) is defined by the formula
\[
R(h_1, h_2, \ldots) = (r(h_1), r(h_2), \ldots),
\]
where
\[
r(h_m) = \{h_m \ast \Psi_{k_1,k_2}\}_{k_1,k_2=0}^L.
\]

**Lemma 3.** The operator \(R\) is bounded from \(H^p(\mathbb{R}^2, \ell^2)\) to \(L^p(\mathbb{R}^2, \ell^2)\) for \(0 < p \leq 2\). Moreover, the estimation constant does not depend on \(L\).

The proof of this lemma will also be given later. Now we show the reduction to the special case considered above. First, using dilation, we can assume without loss of generality that the side lengths of the rectangles \(\Delta_m\) are at least 1. Fix \(A\) sufficiently close to 1 and choose a function \(\psi\) such that \(\sum_{n \geq 0} \psi_n(t) = 1\) for \(t \geq 1\). Put \(\theta = (1,1)\) and consider the functions \(g_m(x) = e^{−2\pi i (a_m - \theta) x} f_m(x)\), where the \(a_m\) are the lower left vertices of the rectangles \(\Delta_m\). Splitting the supports of \(\hat{g_m}\) by application of \(R\) and moving them back, we get the functions \(f_{m,k_1,k_2}(x) = e^{2\pi i (a_m - \theta) x} (g_m \ast \Psi_{k_1,k_2})(x)\) such that \(f_m = \sum_{k_1,k_2} f_{m,k_1,k_2}\) (the number of summands is finite: for each \(m\), summation stops when the support of \(\Psi_{k_1,k_2}\) leaves \(\Delta_m\)). Considering the triple sequence \(\{f_{m,k_1,k_2}\}\) as a function in \(L^p(\mathbb{R}^2, \ell^2)\) and using Lemmas 1 and 2 we have
\[
\left\| \{f_{m,k_1,k_2}\} \right\|_{L^p(\ell^2)} = \left\| R(\{g_m\}) \right\|_{L^p(\ell^2)} \leq C_{p,A} \left\| \{g_m\} \right\|_{H^p(\ell^2)} \leq C'_{p,A} \left\{ \{f_m\} \right\}_{L^p(\ell^2)} = C''_{p,A} \left\{ \{f_m\} \right\}_{L^p(\ell^2)}.
\]

Now, let \(N\) be a sufficiently large natural number (e.g., \(N = 100\)). We split the sequence \(\{f_{m,k_1,k_2}\}\) into \(N^2\) subsequences in accordance with the residues of \(k_1\) and \(k_2\) modulo \(N\) in each group (i.e., for each \(m\)). Taking the above into account, we see that it suffices to prove the theorem for each subsequence separately. In such a subsequence, from each group we delete all the functions such that one of their indexes \(k_1\) or \(k_2\) is maximal. Choosing \(A\) sufficiently close to 1 and applying the Technical Remark about rectangles, we see that the supports of the remaining functions can be placed into rectangles of the same sort as in the special case described above. Now consider the deleted functions. Their Fourier transforms are associated with certain rectangles. We shift the upper right vertices of these rectangles to the point \((-1, -1)\) and split each rectangle once again,
using the method described above (but working with functions anti-analytic in both variables).

§5. Proof of the main theorem: The boundedness of $S$ and $R$

Now we verify Lemmas 2 and 3. In the paper [11] about the Calderón–Zygmund decomposition on the Hardy spaces $H^p(\mathbb{R} \times \mathbb{R})$, Chang and Fefferman proved the following: if a linear operator $T$ is bounded from $H^1(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$ and is bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, then it is bounded from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for all $1 < p < 2$. Therefore, it suffices to prove Lemmas 2 and 3 for $p = 2$ and for $0 < p < 1$. From the Plancherel theorem it follows that $S$ and $R$ are bounded on $L^2$. To prove boundedness for $0 < p \leq 1$, it suffices (by Fefferman’s theorem) to verify that $S$ and $R$ satisfy estimate (5) for any $(N, p)$ rectangle atom.

Consider $S$ first. We need the following statement.

**Lemma 4.** Consider the sequence of functions $\kappa(x, y) = \{\kappa_{k,j}(x, y)\}_{(k,j) \in \mathbb{Z}^2}$, where $\kappa_{k,j}(x, y) = e^{2\pi i x^k y^j} \varphi_k(x - y)$ (recall that the functions $\varphi_k$ were defined in (4) when we introduced the operator $S$). Let $N \geq 0$ be an integer. Then for every interval $I \subset \mathbb{R}$, there exists an $L^p$-valued function $p_I(x, y)$ that is a polynomial of degree not greater than $N$ in $y$ such that for all $\xi = \{\xi_{k,j}\}_{(k,j) \in \mathbb{Z}^2} \in l^2$, all $y \in I$, all $\gamma \geq 2$, and all integers $s \geq 1$ we have

$$\left( \int_{I_x} \left| (\kappa(x, y) - p_I(x, y), \xi)_{L^2} \right|^2 \, dx \right)^{1/2} \leq C_N \gamma^{-s B_N} |I|^{-1/2} \|\xi\|_{l^2},$$

where $B_N > 1$ and $B_N \rightarrow \infty$ as $N \rightarrow \infty$. Here $I^* = \gamma^{s+1} I \setminus \gamma^s I$.

Similar estimates can be found in the treatment of the one-dimensional case (see, e.g., [1]). However, in those estimates the pairs $(k, j)$ did not range over the entire set $\mathbb{Z}^2$, but only over a subset $B \subset \mathbb{Z}^2$ such that $\sum_{(k,j) \in B} |\varphi_k(x - 2^k y)|^2 \leq C$. In that situation the sequence $\{\kappa_{k,j}(x, y)\}_{(k,j) \in B}$ was a kernel of some Calderón–Zygmund operator (the one-dimensional analog of $S$). The $L^2$-boundedness of such an operator follows (by the Plancherel theorem) from that very restriction on the set of the pairs $(k, j)$. But Lemma 4 is true without such restrictions, and this will allow us to estimate the norm of $S$ (i.e., to consider the two-dimensional case).

The proof of Lemma 4 will be given in the next section. Now, using this lemma, we prove the boundedness of $S$. Let $\alpha = \{\alpha_{k,j}\}_{(k,j) \in \mathbb{Z}^2}$ be an arbitrary $L^2$-valued function on a rectangle atom (the rectangle is sufficiently large) supported on a rectangle $\Delta = J \times J$ (as above, $(k,j) = (k_1, k_2, j_1, j_2)$). We split the set $(\gamma \cdot \Delta)^c$ occurring in (4) into three parts:

$$A_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \gamma \cdot J; x_2 \notin \gamma \cdot J\},$$

$$A_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \notin \gamma \cdot J; x_2 \in \gamma \cdot J\},$$

$$A_3 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \notin \gamma \cdot J; x_2 \notin \gamma \cdot J\}.$$  

We integrate over each part separately. For $A_1$ and $A_2$ we use Lemma 2 with respect to one variable first and then the $L^2$-boundedness with respect to the other. For $A_3$, we use Lemma 2 twice, with respect to each variable. The idea is simple but the precise arguments will be fairly bulky.

Consider $A_1$ first. We introduce some notation. The kernel of $S$ is the following sequence of functions:

$$K(x_1, x_2, y_1, y_2) = \{e^{2\pi i x_1 x_1} e^{2\pi i x_2 x_2} \Phi_k(x_1 - y_1, x_2 - y_2)\}_{(k,j) \in \mathbb{A}}.$$
Let \( h = \{ h_{k_1,j_1} \}_{(k_1,j_1) \in \mathbb{Z}^2} \in L^2(\mathbb{R}, \ell^2) \). We define the auxiliary operators \( S_{k_2,j_2} \) by the formula

\[
S_{k_2,j_2}(h)(x_1) = \sum_{(k_1,j_1) \in \mathbb{Z}^2} \chi_A(k_1,k_2,j_1,j_2) e^{2\pi i 2^j_1 \cdot x_1} (\varphi_{k_1} * h_{k_1,j_1})(x_1).
\]

Fixing the indices \( k_2 \) and \( j_2 \), we consider the rectangles \( \Delta_m = I_m \times J_m \) corresponding to the multi-indices \( (k_1,k_2,j_1,j_2) \in \mathcal{A} \). Since the sides \( J_m = [j_2 2^{k_2}, (j_2 + 8) 2^{k_2}] \) of \( \Delta_m \) coincide and the \( \Delta_m \) are pairwise disjoint, it follows that the intervals \( I_m = [j_1 2^{k_1}, (j_1 + 8) 2^{k_1}] \) are pairwise disjoint. Thus, by the Plancherel theorem, we see that the \( S_{k_2,j_2} \) are bounded on \( L^2 \). Also denote

\[
\xi(y_2, x_1) = \left\{ S_{k_2,j_2} \left( \{ \alpha_{k_1,k_2,j_1,j_2} \}^{(k_1,j_1)} \right) \right\}^{(k_2,j_2)}(x_1).
\]

Here, for each index \((k_2, j_2)\), we take the components of \( \alpha \) corresponding to this index, and then, fixing the second variable, we apply the operator \( S_{k_2,j_2} \) to the resulting collection. In this notation we have

\[
S(\alpha)(x_1, x_2) = \int_J \langle K(x_1, x_2, y_1, y_2), \alpha(y_1, y_2) \rangle dy_1 dy_2
\]

\[
= \int_J \sum_{(k_2,j_2) \in \mathbb{Z}^2} e^{2\pi i 2^{j_2} x_2} \varphi_{k_2}(x_2 - y_2) S_{k_2,j_2} \left( \{ \alpha_{k_1,k_2,j_1,j_2} \}^{(k_1,j_1)} \right)(x_1) dy_2
\]

\[
= \int_J \langle \kappa(x_2, y_2) - p_J(x_2, y_2), \xi(y_2, x_1) \rangle dy_2,
\]

where the functions \( \kappa \) and \( p_J \) are the same as in Lemma 4, and their components are indexed by the pairs \((k_2, j_2) \in \mathbb{Z}^2 \). The last identity results from the fact that for the rectangle atom \( \alpha \) the first \( N \) moments in each variable vanish. Now, using the Hölder inequality, we have

\[
\int_{A_1} |(S\alpha)(x_1, x_2)|^p dx_1 dx_2 = \sum_{s \geq 1} \int_{J_s} \int_{J_s} |(S\alpha)(x_1, x_2)|^p dx_2 dx_1
\]

\[
\leq \sum_{s \geq 1} \int_{J_s} |J_s|^{1-p/2} \left( \int_{J_s} |(S\alpha)(x_1, x_2)|^2 dx_2 \right)^{p/2} dx_1.
\]

Consider the inner integral raised to the power \( 1/2 \), i.e., the \( L^2(J_s) \)-norm of \((S\alpha)(x_1, \cdot)\). Using formula 6 for \( S(\alpha) \), interchanging this \( L^2(J_s) \)-norm and the integral over \( J \), and applying Lemma 4, we obtain

\[
\left( \int_{J_s} |(S\alpha)(x_1, x_2)|^2 dx_2 \right)^{1/2} \leq \left( \int_{J_s} \langle \kappa(x_2, y_2) - p_J(x_2, y_2), \xi(y_2, x_1) \rangle dy_2 \right)^{1/2}
\]

\[
\leq \int_{J_s} \left( \int_{J_s} \langle \kappa(x_2, y_2) - p_J(x_2, y_2), \xi(y_2, x_1) \rangle \right)^2 dy_2
\]

\[
\leq C_N \gamma^{-s \beta_N} |J|^{-1/2} \int_{J_s} \|\xi(y_2, x_1)\|_{L^2} dy_2.
\]
Combining this estimate with the preceding one, using the Hölder inequality again, and interchanging the $L^2$-norm and the integral, we have

\[
\iint_{A_1} (|S\alpha|(x_1, x_2)|^p \, dx_2 \, dx_1 \leq \sum_{s \geq 1} C_{N,p} \gamma^{-spB_N} |J_s|^{1-p/2} |J|^{-p/2} \int_{\gamma^{-1}} \left( \int_{\gamma^{-1}} |\xi(y_2, x_1)|^\alpha \, dy_2 \right)^p \, dx_1
\]

\[
\leq \sum_{s \geq 1} C_{N,p} \gamma^{-spB_N} |J_s|^{1-p/2} |J|^{-p/2} |\gamma \cdot I|^{1-p/2} \left( \int_{\gamma^{-1}} \left( \int_{\gamma^{-1}} |\xi(y_2, x_1)|^\frac{2}{\ell_2} \, dx_1 \right)^{1/2} \, dy_2 \right)^p.
\]

Recalling the definition of $\xi(y_2, x_1)$ and using the $L^2$-boundedness of the operators $S_{k_2,j_2}$, we see that

\[
\int_{\gamma^{-1}} \left( \int_{\gamma^{-1}} |\xi(y_2, x_1)|^\frac{2}{\ell_2} \, dx_1 \right)^{1/2} \, dy_2 \leq \int_{J} \left( \sum_{(k_2,j_2) \in \mathbb{Z}^2} \int_{\mathbb{R}} \left( \{\alpha_{k_1,k_2,j_1,j_2}(x_1, y_2)\} \right)_{(k_1,j_1) \in \mathbb{Z}^2}^2 \, dx_1 \right)^{1/2} \, dy_2 \leq |J|^{1/2} \left( \iint_{\mathbb{R}^2} |\alpha(x_1, y_2)|^2_\ell_2 \, dx_1 \, dy_2 \right)^{1/2} \leq |J|^{1/2} (|I| |J|)^{1/2-1/p}.
\]

Putting it all together, we obtain

\[
\iint_{A_1} (|S\alpha|(x_1, x_2)|^p \, dx_2 \, dx_1 \leq C_{N,p} \sum_{s \geq 1} \gamma^{-spB_N} |J_s|^{1-p/2} |J|^{-p/2} |\gamma \cdot I|^{1-p/2} |J|^{p-1} |I|^{p/2-1}
\]

\[
\leq C_{N,p} \sum_{s \geq 1} \gamma^{-spB_N} \left( \gamma^s(\gamma - 1) \right)^{1-p/2} |J|^{1-p} \gamma^{1-p/2} |J|^{p-1} |I|^{p/2-1} \leq C_{N,p} \gamma^{2-p} \sum_{s \geq 1} \gamma^{s(1-p(1/2+B_N))} \leq C_p \gamma^{-\delta}.
\]

We have obtained the last inequality by choosing $N$ such that $B_N$ is sufficiently large (recall that $B_N \to \infty$ as $N \to \infty$). Our choice depends on $p$ only.

We can treat $A_2$ in the same way. It remains to consider $A_3$. Let $\tilde{\alpha}$ be an $\ell^2$-valued $(N,p)$ rectangle atom such that its components with the indices in $\mathcal{A}$ are equal to the corresponding components of $\alpha$ and the other components are equal to zero. We define a sequence $\xi$ of functions by the formula

\[
\xi(x_1, y_1, y_2) = \left\{ \langle \kappa(x_1, y_1) - p_I(x_1, y_1), \{\tilde{\alpha}_{k_1,k_2,j_1,j_2}(y_1,y_2)\} \right\}_{(k_1,j_1) \in \mathbb{Z}^2} \right\}_{(k_2,j_2) \in \mathbb{Z}^2}.
\]

Here, for each index $(k_2,j_2)$, we take the components of $\tilde{\alpha}$ corresponding to this index and calculate the inner product of the resulting collection and $\kappa - p_I$, where the functions $\kappa$ and $p_I$ are the same as in Lemma 4 and their components are indexed by the pairs $(k_1,j_1) \in \mathbb{Z}^2$. Since the first $N$ moments of $\tilde{\alpha}$ vanish in each variable, it follows that

\[
S(\alpha)(x_1, x_2) = \int_{J} \int_{J} \langle \kappa(x_2, y_2) - p_J(x_2, y_2), \xi(x_1, y_1, y_2) \rangle \, dy_1 \, dy_2.
\]
The components of \( \kappa \) and \( p_1 \) are indexed by the pairs \( (k_2, j_2) \in \mathbb{Z}^2 \) this time. Now, using the Hölder inequality and interchanging the \( L^2 \)-norm and the integral, we obtain
\[
\int \int_{A_3} |(S\alpha)(x_1, x_2)|^p \, dx_2 \, dx_1
\]
\[
= \sum_{s_1, s_2 \geq 1} \int_{J_{s_2}} \int_{J_{s_1}} \left( \int \int J \left( \kappa(x_2, y_2) - p_J(x_2, y_2), \xi(x_1, y_1, y_2) \right) \, dy_1 \, dy_2 \right)^p \, dx_1 \, dx_2
\]
\[
\leq \sum_{s_1, s_2 \geq 1} |I_{s_1}|^{1-p/2} |J_{s_2}|^{1-p/2} \left( \int \int \left( \int \int_J |\cdots|^2 \, dx_2 \, dx_1 \right)^{1/2} \, dy_1 \, dy_2 \right)^p.
\]
Using Lemma 4 twice, we have
\[
\left( \int \int_J \left( \kappa(x_2, y_2) - p_J(x_2, y_2), \xi(x_1, y_1, y_2) \right) \, dx_2 \, dx_1 \right)^{1/2}
\]
\[
\leq C_N \gamma^{-s_2 B} |J|^{-1/2} \left( \int \int |\xi(x_1, y_1, y_2)|^2 \, dx_1 \right)^{1/2}
\]
\[
\leq C'_N \gamma^{-s_1 B} |I|^{-1/2} \gamma^{-s_2 B} |J|^{-1/2} |\alpha(y_1, y_2)|^2.
\]
Using the Hölder inequality once again, we obtain
\[
\int \int J \left| \alpha(y_1, y_2) \right|^2 \, dy_1 \, dy_2 \leq |I|^{1/2} |J|^{1/2} \| \alpha \|_{L^2} \leq |I|^{1-1/p} |J|^{1-1/p}.
\]
Putting it all together, we have
\[
\int \int_{A_3} |(S\alpha)(x_1, x_2)|^p \, dx_2 \, dx_1
\]
\[
\leq C_{N, p} \sum_{s_1, s_2 \geq 1} |I_{s_1}|^{1-p/2} |J_{s_2}|^{1-p/2} \gamma^{-s_1 B} \gamma^{-s_2 B} |I|^{p/2-1} |J|^{p/2-1}
\]
\[
\leq C_{N, p} \gamma^{2-p} \sum_{s_1, s_2 \geq 1} \gamma^{(s_1 + s_2)(1-p(1/2+B))} \leq C_p \gamma^{-\delta}.
\]
The last inequality is true for the same reason as in the argument for \( A_1 \). This concludes the proof of Lemma 4 (more precisely, we have reduced it to Lemma 4).

Now, consider the operator \( R \). Its kernel \( \Lambda \) is defined by the formula
\[
\Lambda(x_1, x_2; y_1, y_2) = \left\{ \Psi_{k_1, k_2}(x_1 - y_1, x_2 - y_2) \right\}_{k_1, k_2 = 0}^L.
\]
This kernel can be treated as an \( L(\ell^2, \ell^2_{\mathbb{N}}) \)-valued function, where the components of vectors in \( \ell^2 \) are indexed by \( m \) and the components of vectors in \( \ell^2_{\mathbb{N}} \) are indexed by \( (m, k_1, k_2) \) (see the definition of \( R \)). The operator \( R \) is simpler than \( S \) and can be treated as the tensor square of its one-dimensional analog, i.e., \( R = \rho \otimes \rho \), where \( \rho \) is the operator from \( L^2(\mathbb{R}, \ell^2) \) to \( L^2(\mathbb{R}, \ell^2_{\mathbb{N}}) \) defined by the formula
\[
\rho(h_m) = \{ h_m \} \{ \psi_k \}_{m \in \mathbb{N}, k = 0, \ldots, L}.
\]
The kernel of \( \rho \) is defined by
\[
\lambda(x, y) = \{ \psi_k(x - y) \}_{k=0}^L.
\]
For the kernel \( \lambda \), the usual Calderón–Zygmund smoothness condition is fulfilled:
\[
|\lambda(x, y) - p_I(x, y)|^2 \leq C_N |I|^{N+1} \frac{1}{|x - y_0|^{N+2}},
\]
where \( N \geq 0 \), \( I \) is an arbitrary finite interval in \( \mathbb{R} \), \( y_0 \) is the center of \( I \), \( x \in I \), \( x \notin 2 \cdot I \), and \( p_I(x, y) \) is an \( \ell^2 \)-valued function that is a polynomial of degree not greater than \( N \).
in y. The proof can be found in [6] (though this had been known before). We sketch it at the beginning of the next section.

It is well known that a tensor product of singular integral operators is bounded from $H^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for $0 < p < 1$. But since the proof of this fact is hard to find in the literature, we shall give the corresponding argument for $R$ for completeness. The calculation will be similar to the above argument for $S$ but will be much simpler at the same time.

Let $\alpha = \{\alpha_m\}$ be an $\ell^2$-valued $(N, p)$ rectangle atom ($N$ is sufficiently large) supported on a rectangle $\Delta = I \times J$. We split the set $(\gamma \cdot \Delta)^c$ from Fefferman’s theorem into three parts $A_1$, $A_2$, and $A_3$ (in the same way as above) and integrate $|R(\alpha)(x_1, x_2)|^p$ over each part separately.

Consider $A_1$ first. Since the first $N$ moments of the rectangle atom $\alpha$ vanish in each variable, we have

$$R(\alpha)(x_1, x_2) = \int_I (\lambda(x_2, y_2) - p_J(x_2, y_2)) \rho(\alpha(\cdot, y_2))(x_1) \, dy_2,$$

where $p_J$ is the polynomial in $y$ as in [7]. Next, using [7], we have

$$\begin{align*}
\int_{A_1} \int_I \int_{\gamma \cdot J} |R(\alpha)(x_1, x_2)|^p \, dx_1 \, dx_2 \\
\leq \int_{\gamma \cdot I} \int_{\gamma \cdot J} \left( \int_I |\lambda(x_2, y_2) - p_J(x_2, y_2)| \rho(\alpha(\cdot, y_2))(x_1) \, dy_2 \right)^p \, dx_2 \, dx_1 \\
\leq \int_{\gamma \cdot J} \int_{\gamma \cdot I} \left( \int_I \left( \int_I |\rho(\alpha(\cdot, y_2))(x_1)| \, dx_1 \right)^p \, dy_2 \right) \, dx_1;
\end{align*}$$

here $y_2^0$ is the center of $J$. By a simple calculation, we show that the first factor in the last expression is equal to $C_{N, p} |I|^{1-p} \gamma^{1-p(N+2)}$. Using the Hölder inequality and interchanging the $L^2$-norm and the integral, we see that the second factor is majorized by the quantity

$$|\gamma \cdot I|^{1-p/2} \left( \int_I \left( \int_{\gamma \cdot I} |\rho(\alpha(\cdot, y_2))(x_1)| \, dx_1 \right)^2 \, dy_2 \right)^{1/2},$$

which, in its turn, does not exceed $|\gamma \cdot I|^{1-p/2} |J|^{p/2} \|\alpha\|_{L^p}^p$ (this follows from the $L^2$-boundness of $\rho$ and the Hölder inequality). Putting it all together, we have

$$\begin{align*}
\int_{A_1} |R(\alpha)(x_1, x_2)|^p \, dx_1 \, dx_2 &\leq C_{N, p} |I|^{1-p} \gamma^{1-p(N+2)} |\gamma \cdot I|^{1-p/2} |J|^{p/2} (|I||J|)^{p/2-1} \\
&= C_{N, p} \gamma^{2-p(N+5/2)} \leq C_p \gamma^{-\delta}.
\end{align*}$$

The set $A_2$ can be considered in the same way.

We pass to $A_3$. Since the first $N$ moments of the rectangle atom $\alpha$ vanish, we obtain

$$(R\alpha)(x_1, x_2) = \int_I \int_J (\lambda(x_2, y_2) - p_J(x_2, y_2)) (\lambda(x_1, y_1) - p_J(x_1, y_1)) \alpha(y_1, y_2) \, dy_1 \, dy_2.$$ 

Next, using [7] twice, we have

$$\begin{align*}
\int_{A_3} |R(\alpha)(x_1, x_2)|^p \, dx_1 \, dx_2 \\
\leq \int_{\gamma \cdot J} \int_{\gamma \cdot I} \left( \int_I \left( |\lambda(x_2, y_2) - p_J(x_2, y_2)| \, dx_2 \right)^p \, dy_2 \right) \, dx_1;
\end{align*}$$
here $y_0^0$ and $y_0^2$ are the centers of $I$ and $J$, respectively. Calculating the first two factors and applying the Hölder inequality to the third, we obtain

$$
\int_{A_3} \left| (R\alpha)(x_1, x_2) \right|^p dx_1 \, dx_2 \leq C_{N,p} |J|^{1-p} \gamma^{1-p(N+2)}
\times |I|^{1-p} \gamma^{1-p(N+2)} \cdot |J|^{p/2} |L|^{p/2} \|\alpha\|_{L^2}^p
\leq C_{N,p} \gamma^{2-2p(N+2)} \leq C_p \gamma^{-\delta}.
$$

Thus, by Fefferman’s theorem, the boundedness of $R$ is proved.

§6. THE CONCLUSION OF THE PROOF

The proof of (7) can be found in [6]. We give its sketch here, because we shall need some intermediate inequalities. Suppose $I$ is an interval, $y_0$ its center, $y \in I$, and $x \notin 2 \cdot I$. Let $P_{u,k}(t)$ denote the Taylor polynomial of $\psi_k$ at the point $u$ and of degree $N$. Put $p_k(x, y) = P_{x-y_0,k}(x-y)$ and $p_I(x, y) = \{p_k(x, y)\}_{k=0}^L$. Since the function $\psi$ belongs to the Schwartz class, recalling the definition of the functions $\psi_k$, we easily obtain

$$
|\psi_k(x-y) - p_k(x, y)| \leq C_{N,I} A^{k(N+2)} |y-y_0|^{N+1} (1 + A^k |x-y_0|)^{-l}, \quad l = 1, 2, \ldots.
$$

Letting $l = 0$ and $l = N + 3$, we obtain

(8)  
$$
|\psi_k(x-y) - p_k(x, y)| \leq C_N A^{k(N+2)} |I|^{N+1},
$$

(9)  
$$
|\psi_k(x-y) - p_k(x, y)| \leq C_N A^{-k} \frac{|I|^{N+1}}{|x-y_0|^{N+3}}.
$$

Estimate (8) is better than (8) exactly when $A^k \geq \frac{1}{|x-y_0|}$. We split the square of the $\ell^2$-norm in (7) into two sums: the first over all $k$ such that $A^k \geq \frac{1}{|x-y_0|}$ and the second over all remaining $k$. Using (8) and (9), we obtain the required estimate.

In the proofs of (8) and (9), we have used only the fact that $\psi$ belongs to the Schwartz class. Therefore, the same estimates are true for the functions $\varphi_k$ that occur in the definition of the operator $S$ ($A = 2$).

To conclude the proof, it remains to verify Lemma 4. For $N = 0$ and $\gamma = 2$, a similar statement was established in [1], but there the pairs $(k, j)$ did not range over the entire set $\mathbb{Z}^2$ (though the proof in [1] can be adjusted to our situation). Another statement similar to Lemma 4 was proved in [6] (for all $N \geq 0$ and $\gamma = 2$), but a slightly different setting was considered there. Despite those differences, our proof will be almost the same as the corresponding argument in [6].

Using the Taylor polynomials for the functions $\varphi_k$, we define the polynomials $p_k(x, y)$ in the same way as we did for the functions $\psi_k$. As mentioned above, estimates (8) and (9) are true for the functions $\varphi_k$ for $A = 2$. We introduce a function $p_I(x, y)$ by the formula

$$
p_I(x, y) = \{e^{2\pi i k_j} x p_k(x, y)\}_{(k,j) \in \mathbb{Z}^2}.
$$

Also denote

$$
r_{k,s} = \sup_{x \in I, y \in I} |\varphi_k(x-y) - p_k(x, y)|.
$$
In this notation, we have
\[
\left( \int_{I_s} \left| \langle \kappa(x,y) - p_I(x,y), \xi \rangle \right|^2 \, dx \right)^{1/2} \\
\leq \left( \int_{I_s} \left( \sum_{k \in \mathbb{Z}} |\varphi_k(x-y) - p_k(x,y)| \sum_{j \in \mathbb{Z}} \xi_{k,j} e^{2\pi i 2^k j x} \right)^2 \, dx \right)^{1/2} \\
\leq \sum_k r_{k,s} \left( \int_{I_s} \left| \sum_j \xi_{k,j} e^{2\pi i 2^k j x} \right|^2 \, dx \right)^{1/2} 
\]
(10)
\[
\leq \left( \sum_k r_{k,s} \right)^{1/2} \left( \sum_k \int_{I_s} \left| \sum_j \xi_{k,j} e^{2\pi i 2^k j x} \right|^2 \, dx \right)^{1/2} ; 
\]
here we have used the triangle inequality in $L^2$ and the Cauchy inequality for sums.

Next, by (3) and (11), we obtain
\[
(11)\quad r_{k,s} \leq C_N 2^{k(N+2)} |I|^{N+1}, \quad r_{k,s} \leq C_N 2^{-k} |I|^{-2} \gamma^{-(N+3)s}. 
\]
The second estimate is stronger than the first exactly when $2^k \geq |I|^{-1} \gamma^{-s}$.
Thus,
\[
\sum_k r_{k,s} \leq C_N \sum_{k: 2^k < |I|^{-1} \gamma^{-s}} 2^{k(N+2)} |I|^{N+1} + C_N \sum_{k: 2^k \geq |I|^{-1} \gamma^{-s}} 2^{-k} |I|^{-2} \gamma^{-(N+3)s} \\
\leq C'_N \gamma^{-s}(N+2) |I|^{-1}. 
\]
It remains to estimate the second factor in the last expression in (10). Substituting $t = 2^k x$ and using the Riesz–Fischer theorem, we have
\[
\int_{\gamma^{s+1},j} \left| \sum_j \xi_{k,j} e^{2\pi i 2^k j x} \right|^2 \, dx \leq \begin{cases} 
C \gamma^{s+1} |I| \sum_j |\xi_{k,j}|^2 & \text{if } \gamma^{s+1} |I| \geq 2^{-k}, \\
2^{-k} \sum_j |\xi_{k,j}|^2 & \text{if } \gamma^{s+1} |I| < 2^{-k}.
\end{cases} 
\]
By these estimates and inequalities (11), we obtain
\[
\sum_k r_{k,s} \int_{I_s} \left| \sum_j \xi_{k,j} e^{2\pi i 2^k j x} \right|^2 \, dx \\
\leq C |\xi|^2 \left( \sum_{k: 2^k < |I|^{-1} \gamma^{-s-1}} r_{k,s} 2^{-k} + \sum_{k: 2^k \geq |I|^{-1} \gamma^{-s-1}} r_{k,s} \gamma^{s+1} |I| \right) \\
\leq C_N |\xi|^2 \left( \sum_{k: 2^k < |I|^{-1} \gamma^{-s-1}} 2^{k(N+1)} |I|^{N+1} + \sum_{k: 2^k \geq |I|^{-1} \gamma^{-s-1}} 2^{-k} \gamma^{-(N+2)s+1} |I|^{-1} \right) \\
\leq C'_N |\xi|^2 \gamma^{-(N+1)s+2}. 
\]
Putting $B_N = N + 1/2$, we conclude the proof.

REFERENCES


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