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ON RATIONAL SYMPLECTIC PARAMETRIZATION OF THE COADJOINT ORBIT OF GL(N). DIAGONALIZABLE CASE

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Dedicated to L. D. Faddeev on the occasion of his 75th birthday

ABSTRACT. A method for constructing birational Darboux coordinates on a coadjoint orbit of the general linear group is presented. This method is based on the Gauss decomposition of a matrix in the product of an upper-triangular and a lower-triangular matrix. The method works uniformly for the orbits formed by the diagonalizable matrices of any size and for arbitrary dimensions of the eigenspaces.

§0. INTRODUCTION

Our aim in this paper is to present a method for canonical parametrization of an important algebraic symplectic manifold, namely, a coadjoint orbit of the complex general linear group; see [4, 5, 10].

The problem of description of any manifold consists of two steps. First, we should construct *charts*. They are sets keeping the information about the local structure of the manifold. The charts should have a global structure as simple as possible. The second step is creation of the atlas. We should glue the charts in a proper way. The law of gluing is stated by the *transition functions*, which identify the overlapping parts of the charts. The transition functions set the global structure of the manifold.

Largely, this article is devoted to the first step. We construct one chart, a Zariski open subset of the orbit. Such a domain covers the entire orbit except for several submanifolds of dimension smaller than the dimension of the orbit. The parametrization of the charts is given analytically in Theorems 2 and 3.

To describe the covering, we point out what subspaces must be in general position with the coordinate subspaces. Different charts are parametrized by renumberings of the coordinates. The transition functions can be obtained by reparametrization of the domain already parametrized in the renumerated basis. We do not present these formulas: they are bulky and useless.

It should be noted that the relative arrangement of the coordinate domains of the orbit should be well understood for the following reason. There are problems where we need to glue *different orbits* (i.e., orbits that differ by the spectral structure of matrices they involve) to one algebraic symplectic manifold. The organization of the maps in the atlases of these manifolds is similar to the organization of the maps in one orbit. As important examples, we mention the phase spaces of the systems of equations of the isomonodromic deformations [6]–[8].

We identify $\mathrm{gl}(N,\mathbb{C})$ with its dual $\mathrm{gl}^*(N,\mathbb{C})$ by using the nondegenerate form $\langle A, B\rangle = \operatorname{tr} AB$. Then the coadjoint orbits are identified with the adjoint orbits. Let $\mathcal{O}_J$ be the

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orbit that contains a Jordan matrix[1] J. It is formed by the matrices

$$A = g\,J\,g^{-1},$$

where $g \in \mathrm{GL}(N, \mathbb{C})$. Thus, the orbit is parametrized by the matrix entries of A. They form the set of coordinate functions $A_{i,j} : \mathcal{O}_J \to \mathbb{C}$.

It is our aim to find a set of functions p_k, q_k birational with respect to A_{ij} and canonically conjugated with respect to the symplectic structure of the orbit. We use the following formula [1] for the symplectic form on the orbit:

$$\omega_{\mathcal{O}}(\xi, \eta) = \operatorname{tr} U_\xi \dot{A}_\eta = -\operatorname{tr} U_\eta \dot{A}_\xi, \tag{1}$$

where $\xi, \eta \in T_A\mathcal{O}$ are two vectors tangent to the orbit $\mathcal{O}$ at the point $A \in \mathcal{O}$,

$$\xi = [U_\xi, A] = \frac{d}{dt}\Big|_{t=0} A_\xi(t) =: \dot{A}_\xi, \quad A_\xi(0) = A,$$
$$\eta = [U_\eta, A] = \frac{d}{dt}\Big|_{t=0} A_\eta(t) =: \dot{A}_\eta, \quad A_\eta(0) = A.$$

This means that

$$\omega_{\mathcal{O}}(\partial_{p_i}, \partial_{q_j}) = \delta_{ij}, \quad 0 = \omega_{\mathcal{O}}(\partial_{p_i}, \partial_{p_j}) = \omega_{\mathcal{O}}(\partial_{q_i}, \partial_{q_j}),$$

where the $\partial_{p_k}, \partial_{q_k}$ are dual to the dp_k, dq_k.

We treat A as the matrix of a linear transformation $\mathcal{A} : V \to V$ in some basis $(e_1, e_2, \dots, e_N)$ that is fixed initially.

So, there is a basis in which $\mathcal{A}$ has matrix J. Of course, there is a nontrivial family of such bases parametrized by matrices commuting with J. In this paper we consider a diagonal J only.

Restriction 1. The matrix A has a complete set of eigenvectors, and the matrix J is diagonal.

We order the eigenvalues of A in some way:

$$\lambda_0, \lambda_1, \dots, \lambda_N, \quad \lambda_i = \lambda_j \iff i = j.$$

The method we propose is iterative. Each iteration reduces the parametrization of an orbit to that of an orbit of smaller dimension. The number of different eigenvalues on the parametrizing orbit decreases by one; we remove λ_k's from the set of eigenvalues sequentially. The difference of the dimensions of the matrices after an iteration is equal to the multiplicity of the removed eigenvalue $\lambda_k : \dim\ker(A - \lambda_k I) =: n_k$.

We get the sequence

$$A = \widetilde{A}_0, A_0 \in \mathrm{End}(V_0) \to \widetilde{A}_1, A_1 \in \mathrm{End}(V_1) \to \cdots$$
$$\to \widetilde{A}_{N-1}, A_{N-1} \in \mathrm{End}(V_{N-1}) \to \widetilde{A}_N, A_N = 0 \in \mathrm{End}(V_N).$$

The dimension of the space V_k where the transformations $\widetilde{A}_k$ and A_k are defined is equal to $n_k + n_{k+1} + \cdots + n_N = \dim V_k$. The matrix A_k differs from $\widetilde{A}_k$ by a matrix proportional to the unit matrix:

$$A_k = \widetilde{A}_k - (\lambda_k - \lambda_{k-1})I, \quad k = 0, 1, \dots, N; \quad \lambda_{-1} := 0.$$

We have $\dim\ker A_k = n_k$, so that the initial A should be denoted by $\widetilde{A}_0$. After that, we consider $A_0 := \widetilde{A}_0 - \lambda_0 I$ and the first iteration gives an $\widetilde{A}_1$ that acts on the space V_1. The second iteration starts with a transformation of the eigenspace corresponding to $\lambda_1 - \lambda_0$ to the root space: $A_1 := \widetilde{A}_1 - (\lambda_1 - \lambda_0)I$.

[1] That is, a matrix in the normal Jordan form; see [2, 3].

The eigenvalues of the matrix $\widetilde{A}_k$ and its normal Jordan form $J_{\widetilde{A}_k}$ are $\lambda_k - \lambda_{k-1}$, $\lambda_{k+1} - \lambda_{k-1}, \dots, \lambda_N - \lambda_{k-1}$. The multiplicity of $\lambda_j - \lambda_{k-1}$ is equal to n_j, $j = k, \dots, N$.

The eigenvalues of the matrix $A_k := \widetilde{A}_k - (\lambda_k - \lambda_{k-1})I$ and the eigenvalues of its normal Jordan form $J_{A_k} = J_{\widetilde{A}_k} - (\lambda_k - \lambda_{k-1})I$ are 0, $\lambda_{k+1} - \lambda_k, \dots, \lambda_N - \lambda_k$. The multiplicity of the kernel is equal to n_k; the multiplicity of $\lambda_j - \lambda_k$ is equal to n_j, $j = k+1, \dots, N$.

The eigenvalues of the matrix $\widetilde{A}_{k+1}$ obtained by iteration from A_k are $\lambda_{k+1} - \lambda_k$, $\lambda_{k+2} - \lambda_k, \dots, \lambda_N - \lambda_k$. The multiplicity of the eigenvalue $\lambda_j - \lambda_k$ is still equal to n_j. The normal Jordan form $J_{\widetilde{A}_{k+1}}$ is obtained by crossing out the zero lines and zero rows from J_{A_k}.

Finally, we get a zero-dimensional orbit.

Each iteration $k \to k+1$ gives a couple of sets of $n_k \times (n_{k+1} + n_{k+2} + \dots + n_N)$ functions, which determine the positions of two dual subspaces, namely, the kernel and the image of A_k.

The coordinate functions that give the position of the kernel are the coordinates of a basis of the kernel. The basis is normalized in such a way that the basis vector with number "s" has a unit coordinate with the number "s", and all its first n_k coordinates different from "s" vanish.

The coordinate functions that give the position of the image are the coordinates of the projection along V_k of the A_k-image of the fixed basis of V_k to the coordinate subspace enveloping the first n_k basis vectors. This projection is the projection to the coordinate subspace enveloping the first n_k basis vectors parallel to the subspace enveloping the last $n_{k+1} + n_{k+2} + \dots + n_N$ basis vectors.

As has already been mentioned, we consider diagonalizable matrices only. In the case of the arbitrary Jordan structure of A, it is not convenient to split off the entire invariant subspace corresponding to the eigenvalue λ_k of A_k at one iteration. The subspace has nontrivial internal structure; it is "too big" in a sense.

It is possible to split off a subspace of $\ker A_k$, i.e., a subspace of the invariant space in the Jordan case. Each eigenvalue will be exhausted in several steps. The number of the steps is equal to the size of the maximal Jordan block corresponding to the eigenvalue. The coordinate functions will be obtained by projections in the same way as in the diagonalizable case.

However, we shall not consider the case of general Jordan matrices in this paper. This requires a detailed analysis of the structure defined in the space by the general linear transformation, but the idea of the method remains the same. The corresponding projections that describe the positions of the kernel and the image are canonically conjugated. Their common level set is isomorphic to the orbit of smaller dimension, and the restriction of $\omega_{\mathcal{O}}$ to this level set coincides with the canonical symplectic form of this smaller orbit.

§1. Main construction

Now we describe one iteration $k \to k+1$ in detail. We start from the transformation $\widetilde{\mathcal{A}}_k$ that is a result of the preceding iteration. We subtract $(\lambda_k - \lambda_{k-1})$I in order to get A_k such that $\dim \ker A_k = n_k$. To reduce the number of indices, we omit the indices "k" and "$k+1$". This can be done without ambiguity, because all objects supplied by the tilde should have index "$k+1$" and no one object with index "k" has a tilde.

Consider $\mathcal{A} : V \to V$, and let J be the corresponding normal Jordan form. Let $\ker \mathcal{A}$ be the kernel, and let n be its dimension: $\dim \ker \mathcal{A} =: n \in \mathbb{N}$. By Restriction 1, the space V is the direct sum

$$V = \ker \mathcal{A} \oplus \operatorname{im} \mathcal{A} =: K \oplus M,$$

where M is the image of $\mathcal{A}$. Obviously, in this case, a linear transformation can be given by the position of the image, the position of the kernel, and the action of the transformation on the image.

We emphasize that we deal with the case of the absence of generalized eigenvectors, which are the vectors from the intersection of the image and the kernel.

Proposition 1. *The normal Jordan form of the restriction of a diagonalizable transformation to its image is the diagonal matrix obtained by crossing out the zero rows and the zero columns from the normal Jordan form of the initial transformation.*

The image and the kernel of $\mathcal{A}$ are subspaces of V. Their dimensions are m and n, respectively, so that they are points of the Grassmanians $G(n,V)$ and $G(m,V)$. Since $m+n=\dim V$, the Grassmanians are isomorphic. The image and the kernel of $\mathcal{A}$ are an *arbitrary* pair of spaces of the given dimensions and transversal to each other. The map $\mathcal{O}_J \to G(n,V)\times G(m,V)$ is well defined. Its image is a Zariski open submanifold

$$(G(n,V)\times G(m,V))_\Delta := (G(n,V)\times G(m,V))\setminus \Delta,$$

where Δ stands for a pair $(K,\, M)$ such that $K\cap M\neq 0$.

We map the orbit $\mathcal{O}_J$ to $(G(n,V)\times G(m,V))_\Delta \times \mathcal{O}_{\widetilde{J}}$, where $\widetilde{J}$ is the diagonal matrix obtained by crossing out the zero rows and columns from J. It is the orbit that contains the restriction of A to its image:

$$\mathcal{A}\to (\ker\mathcal{A}, \operatorname{im}\mathcal{A}, \widetilde{\mathcal{A}}), \quad \text{where} \ \ \widetilde{\mathcal{A}} = \mathcal{A}\big|_{\operatorname{im}\mathcal{A}}. \tag{2}$$

Proposition 2. *This map is a bijection.*

The map itself and the inverse map are well defined because the kernel and the image are arbitrary. The action on the image is an arbitrary $\widetilde{\mathcal{A}}$ belonging to the orbit $\mathcal{O}_{\widetilde{J}}$. □

In order to introduce coordinates, we split the basis vectors $(e_1, e_2, \dots, e_N)$ of V into two families. The first family is $(\mathbf{e}) = (e_{n+1}, e_{n+2}, \dots, e_{n+m})$ and the second is $(\mathbf{f}) = (e_1, e_2, \dots, e_n)$:

$$(e_1, e_2, \dots, e_N) = ((\mathbf{f}), (\mathbf{e})).$$

Let $\widetilde{V}$ denote the linear envelope of $(\mathbf{e})$ and F the linear envelope of $(\mathbf{f})$. Consider the open set ${}_e\mathcal{O}_J$ that consists of all maps with the property that no nonzero vector in $\ker\mathcal{A}$ belongs to $\widetilde{V}$. Let ${}_eG(n,V)$ consist of all n-dimensional subspaces transversal to $\widetilde{V}$. The set ${}_eG(n,V)$ is an open subset of $G(n,V)$. Let ${}_e(G(n,V)\times G(m,V))_\Delta$ consist of the couples

$$(K,M)\in (G(n,V)\times G(m,V))_\Delta$$

such that $K\cap E = 0$.

Proposition 3. *The restriction of the map* (2) *to* ${}_e\mathcal{O}_J$ *is a bijection onto*

$${}_e(G(n,V)\times G(m,V))_\Delta \times \mathcal{O}_{\widetilde{J}}.$$

We have reduced the domain and the target of the bijection (2) in a consistent way. □

Since the spaces $\ker\mathcal{A}$ and $\widetilde{V}$ have complementary dimensions and are transversal to each other, the projection to $\widetilde{V}$ along $\ker\mathcal{A}$ is well defined. We denote this projection by $\rho^{\|\ker}$:

$$\rho^{\|\ker} : V\to \widetilde{V}.$$

The subspace $\operatorname{im}\mathcal{A}$ is also transversal to $\ker\mathcal{A}$ and also has complementary dimension, so that the projection along the kernel sets an isomorphism

$$(\rho^{\|\ker})\big|_{\operatorname{im}\mathcal{A}} : \operatorname{im}\mathcal{A} \xrightarrow{\sim} \widetilde{V}.$$

This isomorphism induces an isomorphism between the sets of linear automorphisms of the spaces $\operatorname{im}\mathcal{A}$ and $\widetilde{V}$. Since these automorphisms have the same matrices in the corresponding bases, an isomorphism between subspaces preserves the Jordan forms of the automorphisms.

Remark 1. The normal Jordan form of the restriction of the diagonalizable transformation $\widetilde{\mathcal{A}}$ to its image is obtained from the normal Jordan form of $\mathcal{A}$ by crossing out the zero rows and columns. So we have set a diagonalizable transformation of the space $\widetilde{V}$ of dimension smaller than that of V. The Jordan form of the transformation is fixed; in other words, it belongs to the orbit of a smaller dimension. If we reduce the problem of parametrization of the given orbit to parametrization of this smaller orbit, we shall solve the problem announced in the title.

Now we describe the *coordinate submanifolds* (the level surfaces) of the system of coordinate functions to be constructed.

The splitting of V into the direct sum of the subspaces F and $\widetilde{V}$ determines two projections. The first is $\pi^{\|F} : V \to \widetilde{V}$ parallel to F. The second is $\pi^{\|\widetilde{V}} : V \to F$ parallel to $\widetilde{V}$. A linear isomorphism $(\pi^{\|\widetilde{V}}\big|_{\ker\mathcal{A}})^{-1} : \ker\mathcal{A} \to F$ that is the projection of the kernel of $\mathcal{A}$ to the coordinate subspace F parallel to $\widetilde{V}$ is well defined because $\ker\mathcal{A}$ is transversal to $\widetilde{V}$.

We define the mappings

$$\begin{aligned} &\mathcal{Q} : F \to \widetilde{V}, \qquad \mathcal{Q} := \pi^{\|F}(\pi^{\|\widetilde{V}}\big|_{\ker\mathcal{A}})^{-1}, \\ &\mathcal{P} : \widetilde{V} \to F, \qquad \mathcal{P} := \pi^{\|\widetilde{V}}\mathcal{A}\big|_{\widetilde{V}}. \end{aligned}$$

Denote by $\widetilde{\mathcal{A}}$ the transformation of $\widetilde{V}$ in question, i.e., the restriction of $\mathcal{A}$ to its image, transported by $(\rho^{\|\ker})\big|_{\operatorname{im}\mathcal{A}}$:

$$\widetilde{\mathcal{A}} : \widetilde{V} \to \widetilde{V}, \quad \widetilde{\mathcal{A}} := \rho^{\|\ker}\mathcal{A}\big|_{\widetilde{V}}.$$

The coordinate surfaces are the level sets of the mappings $\mathcal{P} \in \operatorname{Hom}(\widetilde{V}, F)$, $\widetilde{\mathcal{A}} \in \mathcal{O}_{\widetilde{J}} \subset \operatorname{End}(\widetilde{V}, \widetilde{V})$.

We denote the resulting map by π:

$$\pi : {}_e\mathcal{O}_J \to \operatorname{Hom}(F, \widetilde{V}) \times \operatorname{Hom}(\widetilde{V}, F) \times \mathcal{O}_{\widetilde{J}}. \tag{3}$$

Theorem 1. *The map π is birational and bijective.*

We have used rational operations only: the calculation of a root space and the image, restriction to a subspace, projection, and inversion of linear transformations.

The transformation π is bijective. This follows from Propositions 2, 3 and the fact that $\mathcal{P}$ and $\mathcal{Q}$ determine the kernel and the image of $\mathcal{A}$ uniquely. □

It will follow from Theorem 3 that the tangent space to the orbit at the point $A \in {}_e\mathcal{O}_J$ is the direct sum of the three spaces tangent to the coordinate surfaces in question. We denote these surfaces by $\mathcal{Q}\big|_{\substack{\mathcal{P}(A)=\text{const}\\ \widetilde{A}=\text{const}}}$, $\mathcal{P}\big|_{\substack{\mathcal{Q}(A)=\text{const}\\ \widetilde{A}=\text{const}}}$, and $\widetilde{\mathcal{O}}\big|_{\substack{\mathcal{P}(A)=\text{const}\\ \mathcal{Q}(A)=\text{const}}}$:

$$\mathcal{T}_A\mathcal{O}_J = \mathcal{T}_A\mathcal{Q}\big|_{\substack{\mathcal{P}(A)=\text{const}\\ \widetilde{A}=\text{const}}} \oplus \mathcal{T}_A\,\mathcal{P}\big|_{\substack{\mathcal{Q}(A)=\text{const}\\ \widetilde{A}=\text{const}}} \oplus \mathcal{T}_A\,\widetilde{\mathcal{O}}\big|_{\substack{\mathcal{P}(A)=\text{const}\\ \mathcal{Q}(A)=\text{const}}}.$$

Theorem 2. *The spaces $\mathcal{T}_A\,\mathcal{Q}\big|_{\substack{\mathcal{P}(A)=\text{const}\\ \widetilde{A}=\text{const}}}$ and $\mathcal{T}_A\,\mathcal{P}\big|_{\substack{\mathcal{Q}(A)=\text{const}\\ \widetilde{A}=\text{const}}}$ are isotropic and orthogonal to $\mathcal{T}_A\widetilde{\mathcal{O}}\big|_{\substack{\mathcal{P}(A)=\text{const}\\ \mathcal{Q}(A)=\text{const}}}$ with respect to the symplectic structure $\omega_{\mathcal{O}}$ on the orbit* (*see formula* (1)).

In other words, if the tangent vectors $\xi_i \in \mathcal{T}_A\mathcal{O}$, $i = 1, 2$, are written as $\xi_i = {}_i\partial_Q + {}_i\partial_P + {}_i\partial_{\widetilde{\mathcal{O}}}$, where

$$ {}_i\partial_Q \in \mathcal{T}_A\mathcal{Q}\big|_{\substack{\mathcal{P}(A)=\text{const},\\ \widetilde{A}=\text{const}}}, \quad {}_i\partial_P \in \mathcal{T}_A\ \mathcal{P}\big|_{\substack{\mathcal{Q}(A)=\text{const},\\ \widetilde{A}=\text{const}}}, \quad {}_i\partial_{\widetilde{\mathcal{O}}} \in \mathcal{T}_A\ \widetilde{\mathcal{O}}\big|_{\substack{\mathcal{P}(A)=\text{const},\\ \mathcal{Q}(A)=\text{const}}} $$

then

$$ 0 = \omega_{\mathcal{O}}({}_1\partial_Q, {}_2\partial_Q) = \omega_{\mathcal{O}}({}_1\partial_P, {}_2\partial_P) = \omega_{\mathcal{O}}({}_i\partial_P, {}_j\partial_{\widetilde{\mathcal{O}}}) = \omega_{\mathcal{O}}({}_i\partial_Q, {}_j\partial_{\widetilde{\mathcal{O}}}), \quad i, j \in \{1, 2\}. $$

We shall write our $N \times N$ matrices as the block 2×2 matrices in accordance with the splitting of the basis into two groups $((\mathbf{f}), (\mathbf{e}))$. The matrix A can be uniquely represented in the form

$$ A = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix} $$

provided $\ker A$ does not intersect the envelope of $(\mathbf{e})$. In the above formula, Q is an $m \times n$ matrix, P is of size $n \times m$, $\widetilde{A}$ is of size $m \times m$, and 1, 0 are the unit and the zero matrix of the due dimensions.

Restriction 1 implies that the entire root space of A is the envelope of the first n columns of $\begin{pmatrix} 1 & 0 \\ Q & 1. \end{pmatrix}$ Consequently, $\widetilde{A}$ is nonsingular. Consider the $n \times m$ matrix $\check{P} := P\widetilde{A}^{-1}$; since

$$ \begin{pmatrix} 1 & \check{P} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & -\check{P} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & P \\ 0 & \widetilde{A} \end{pmatrix}, $$

we have

$$ A = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 1 & \check{P} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & -\check{P} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix}. $$

- We calculate $\omega_{\mathcal{O}}({}_1\partial_Q, {}_2\partial_Q)$. For such vectors, $P = \text{const}$, and $\widetilde{A} = \text{const}$, so that

$$ {}_i\partial_Q = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & 0 \\ Q_i(t) & 1 \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q_i(t) & 1 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ \dot{Q}_i & 0 \end{pmatrix}, A \right], $$

whence

$$ \omega_{\mathcal{O}}({}_1\partial_Q, {}_2\partial_Q) = \operatorname{tr} \begin{pmatrix} 0 & 0 \\ \dot{Q}_1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 0 \\ \dot{Q}_2 & 0 \end{pmatrix}, A \right] = 0. $$

- We calculate $\omega_{\mathcal{O}}({}_1\partial_P, {}_2\partial_P)$. For such vectors, $\widetilde{A} = \text{const}$ and $Q = \text{const}$.

The tangent vectors ${}_1\partial_P$ and ${}_2\partial_P$ are defined as the velocity vectors $\begin{pmatrix} 0 & P_i(t) \\ 0 & \widetilde{A} \end{pmatrix}$ transformed by the similarity transformation with the matrix $\begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix}$. After taking the trace, the transformation $\begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix}$ disappears, and we can put $Q = 0$ and $\widetilde{A} = \text{const}$:

$$ \begin{aligned} {}_i\partial_P &= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 0 & P_i(t) \\ 0 & \widetilde{A} \end{pmatrix} = \begin{pmatrix} 0 & \dot{P}_i \\ 0 & 0 \end{pmatrix} \\ &= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & \check{P}_i(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & -\check{P}_i(t) \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 0 & \dot{\check{P}}_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A} \end{pmatrix} \right]. \end{aligned} $$

Consequently,

$$ \omega_{\mathcal{O}}({}_1\partial_P, {}_2\partial_P) = \operatorname{tr} \begin{pmatrix} 0 & \dot{\check{P}}_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \dot{P}_2 \\ 0 & 0 \end{pmatrix} = 0. $$

- We calculate $\omega_{\mathcal{O}}(\partial_Q, \partial_{\widetilde{\mathcal{O}}})$:

$$\partial_Q = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & 0 \\ Q(t) & 1 \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q(t) & 1 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ \dot{Q} & 0 \end{pmatrix}, A \right],$$

$$\partial_{\widetilde{\mathcal{O}}} = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \dot{\widetilde{A}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix},$$

$$\omega_{\mathcal{O}}(\partial_Q, \partial_{\widetilde{\mathcal{O}}}) = \operatorname{tr} \begin{pmatrix} 0 & 0 \\ \dot{Q} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \dot{\widetilde{A}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix}$$

$$= \operatorname{tr} \begin{pmatrix} 0 & 0 \\ \dot{Q} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \dot{\widetilde{A}} \end{pmatrix} = 0.$$

- Finally, we calculate $\omega_{\mathcal{O}}(\partial_P, \partial_{\widetilde{\mathcal{O}}})$. As above, we can put $Q = 0$:

$$\partial_P = \frac{d}{dt}\Big|_{\substack{t=0 \\ \widetilde{A}=\text{const}}} \begin{pmatrix} 1 & P(t)\widetilde{A}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & -P(t)\widetilde{A}^{-1} \\ 0 & 1 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & \dot{P}\widetilde{A}^{-1} \\ 0 & 0 \end{pmatrix}, A \right], \quad \partial_{\widetilde{\mathcal{O}}} = \begin{pmatrix} 0 & 0 \\ 0 & \dot{\widetilde{A}} \end{pmatrix},$$

$$\omega_{\mathcal{O}}(\partial_P, \partial_{\widetilde{\mathcal{O}}}) = \operatorname{tr} \begin{pmatrix} 0 & \dot{P}\widetilde{A}^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \dot{\widetilde{A}} \end{pmatrix} = 0.$$

$\square$

Theorem 3. *The restriction of $\omega_{\mathcal{O}}$ from the orbit $\mathcal{O}_J$ to the submanifold $\widetilde{\mathcal{O}}\big|_{\substack{\mathcal{P}(A)=\text{const} \\ \mathcal{Q}(A)=\text{const}}} \simeq \mathcal{O}_{\widetilde{J}}$ coincides with the form $\omega_{\widetilde{\mathcal{O}}}$ defined on the orbit $\mathcal{O}_{\widetilde{J}}$.*

Let

$$\partial_i = \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & \widetilde{A}_i(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix},$$

and let $\widetilde{A}_i(t) = g_i(t)\, \widetilde{J}\, g_i^{-1}(t)$. Then $\dot{\widetilde{A}_i} = \left[\widetilde{U}_i, \widetilde{A}\right]$, where $\widetilde{U}_i := \dot{g}_i g^{-1}$. Note that we can put $Q = 0$ again. We have

$$\frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & P\widetilde{A}_i^{-1}(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A}_i(t) \end{pmatrix} \begin{pmatrix} 1 & -P\widetilde{A}_i^{-1}(t) \\ 0 & 1 \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & P\widetilde{A}_i^{-1}(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g_i(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{A}(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g_i^{-1}(t) \end{pmatrix} \begin{pmatrix} 1 & -P\widetilde{A}^{-1}(t) \\ 0 & 1 \end{pmatrix}$$

$$= \left[\left(\frac{d}{dt}\Big|_{t=0} \left(\begin{pmatrix} 1 & P\widetilde{A}_i^{-1}(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g_i(t) \end{pmatrix} \right) \right) \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & -P\widetilde{A}^{-1} \\ 0 & 1 \end{pmatrix}, A \right].$$

The matrices $U_i \,:\, \dot{A}_i = [U_i, A]$ are represented as follows:

$$U_i = \begin{pmatrix} 1 & P\widetilde{A}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{U}_i \end{pmatrix} \begin{pmatrix} 1 & -P\widetilde{A}^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & P\dot{\widetilde{A}_i^{-1}} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & P\widetilde{A}^{-1}\widetilde{U}_i \\ 0 & \widetilde{U}_i \end{pmatrix} + \begin{pmatrix} 0 & P\dot{\widetilde{A}_i^{-1}} \\ 0 & 0 \end{pmatrix}.$$

Finally,

$$\omega_{\mathcal{O}}(\partial_1,\partial_2) = \operatorname{tr}\left(\begin{pmatrix}0 & P\widetilde{A}^{-1}\widetilde{U}_1\\ 0 & \widetilde{U}_1\end{pmatrix} + \begin{pmatrix}0 & P\dot{\widetilde{A}}{}_1^{-1}\\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}0 & 0\\ 0 & \dot{\widetilde{A}}_2\end{pmatrix}$$
$$= \operatorname{tr}\widetilde{U}_1\dot{\widetilde{A}}_2 = \omega_{\widetilde{\mathcal{O}}}(\partial_1,\partial_2).$$

§2. Lie–Poisson bracket

So far we have avoided using explicit coordinates in calculations, trying to explain our general point in the language of symplectic geometry. To give a complete picture, here we adduce the calculation of the Lie–Poisson bracket by using explicit coordinates.

We recall the general construction of the Lie–Poisson bracket. A canonical Lie–Poisson bracket exists on the space $\mathbf{g}^*$ dual to the Lie algebra $\mathbf{g}$. The linear functions X_1, X_2 on $\mathbf{g}^*$ are elements of the Lie algebra itself: $X_k(A) = \langle A, X_k\rangle$. The bracket is defined by the formula

$$\{X_1, X_2\}(A) = \langle A, [X_1, X_2]\rangle\,; \quad X_1, X_2 \in \mathbf{g}, \quad A \in \mathbf{g}^*.$$

The Leibniz identity shows that the bracket extends from the set of linear functions to the set of arbitrary smooth functions $\varphi_k \in C^\infty(\mathbf{g}^*)$ in such a way that the general formula reads

$$\{\varphi_1, \varphi_2\}(A) = \langle A, [\mathrm{d}\varphi_1, \mathrm{d}\varphi_2]\rangle.$$

We rewrite the general formula in coordinates. For this, we select a basis $\mathrm{e}_{\mathbf{k}}$ in the Lie algebra $\mathbf{g}$ and a dual basis $\mathrm{e}^{\mathbf{k}}$ in $\mathbf{g}^*$:

$$[\mathrm{e}_{\mathbf{j}}, \mathrm{e}_{\mathbf{k}}] = \sum_{\mathbf{i}} \mathrm{C}^{\mathbf{i}}_{\mathbf{jk}}\mathrm{e}_{\mathbf{i}}\,; \quad \langle \mathrm{e}^{\mathbf{k}}, \mathrm{e}_{\mathbf{i}}\rangle = \delta^{\mathbf{k}}_{\mathbf{i}},$$

where $\mathbf{k}$ enumerates the elements of the basis and the $\mathrm{C}^{\mathbf{i}}_{\mathbf{jk}}$ are structure constants of the Lie algebra $\mathbf{g}$. An arbitrary element $A \in \mathbf{g}^*$ can be written as $A = \sum_{\mathbf{k}} A_{\mathbf{k}}\mathrm{e}^{\mathbf{k}}$, so that the coordinates of A are the $A_{\mathbf{k}}$, and "the decoding" of the general formula in coordinates looks like this:

$$\{\varphi_1, \varphi_2\}(A) = \sum_{\mathbf{ijk}} \mathrm{C}^{\mathbf{i}}_{\mathbf{jk}} A_{\mathbf{i}} \cdot \frac{\partial\varphi_1}{\partial A_{\mathbf{j}}}\frac{\partial\varphi_2}{\partial A_{\mathbf{k}}}.$$

We identify the Lie algebra $\mathrm{gl}(N,\mathbb{C})$ and its dual $\mathrm{gl}^*(N,\mathbb{C})$ again. In the Lie algebra $\mathrm{gl}(N,\mathbb{C})$, a standard basis consists of the matrix units $\mathbf{e}_{ij}$: $A = \sum_{ij} A_{ij}\mathbf{e}_{ij}$, so that the pair ij plays the role of $\mathbf{k}$. The only nonzero entry equal to unity in the matrix $\mathbf{e}_{ij}$ is located at the intersection of the ith line and the jth column. Thus, the coordinate functions A_{ij} are entries of the matrix A. The structure constants of the Lie algebra $\mathrm{gl}(N,\mathbb{C})$ can easily be found from the defining commutation relations

$$[\mathbf{e}_{ik}, \mathbf{e}_{nm}] = \delta_{kn}\mathbf{e}_{im} - \delta_{im}\mathbf{e}_{nk}.$$

We get a formula for the Lie–Poisson bracket of smooth functions $\varphi_k(A)$:

$$\{\varphi_1, \varphi_2\}(A) = A_{im} \cdot \frac{\partial\varphi_1}{\partial A_{ip}}\frac{\partial\varphi_2}{\partial A_{pm}} - A_{nk} \cdot \frac{\partial\varphi_1}{\partial A_{pk}}\frac{\partial\varphi_2}{\partial A_{np}},$$

where the summation is meant over all repeated indices. The bracket of coordinate functions reproduces the commutation relations of the initial Lie algebra,

$$\{A_{ik}, A_{nm}\} = \delta_{kn}A_{im} - \delta_{im}A_{nk}. \tag{4}$$

The points of the orbit $\mathcal{O}_J \ni J$ are matrices of the form

$$A = g\,J\,g^{-1}, \quad g \in \mathrm{GL}(N,\mathbb{C}).$$

We need to find coordinate functions p_k, q_k on the orbit that are birational with respect to A_{ij} and canonically conjugated with respect to the Lie–Poisson bracket

$$\{q_i, q_k\} = \{p_i, p_k\} = 0, \quad \{p_i, q_k\} = \delta_{ik}.$$

For simplicity, we shall confine ourselves to the case of general position, where all the eigenvalues of $J = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ are different. Consider the first step of the iterative process of selection of canonical coordinates on the orbit. For almost any matrix $g \in \mathrm{GL}(N, \mathbb{C})$, the matrix A belonging to the orbit can be presented as a product of the following triangular matrices:

$$A = \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & P \\ 0 & \widetilde{A} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ q_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_N & 0 & \ldots & 1 \end{pmatrix}, \qquad \begin{pmatrix} \lambda_1 & P \\ 0 & \widetilde{A} \end{pmatrix} = \begin{pmatrix} \lambda_1 & p_2 & \cdots & p_N \\ 0 & \widetilde{A}_{22} & \ldots & \widetilde{A}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widetilde{A}_{N2} & \ldots & \widetilde{A}_{NN} \end{pmatrix},$$

and the matrix $\widetilde{A}$ lies on the orbit of smaller dimension:

$$\widetilde{A} = \widetilde{g}\,\widetilde{J}\,\widetilde{g}^{-1}, \quad \widetilde{g} \in \mathrm{SL}(N-1, \mathbb{C}), \quad \widetilde{J} = \mathrm{diag}(\lambda_2, \ldots, \lambda_N).$$

Thus, the matrix entries of A can be expressed via those of $\widetilde{A}$ and the set of coordinates q_k, p_k:

$$A_{11} = \lambda_1 - q_k p_k\,, \qquad A_{1k} = p_k\,,$$
$$A_{k1} = q_k\,(\lambda_1 - q_i p_i) - \widetilde{A}_{ki} q_i\,, \qquad A_{ik} = \widetilde{A}_{ik} + q_i p_k\,,$$

where $i, k = 2, \ldots, N$ and repeated indices imply summation. If we require that the entries of $\widetilde{A}$ commute with the variables q_k, p_k, $\{\widetilde{A}_{ik}, q_j\} = \{\widetilde{A}_{ik}, p_j\} = 0$, then the entries of A will satisfy relations (4) if

$$\{q_i, q_k\} = \{p_i, p_k\} = 0, \quad \{p_i, q_k\} = \delta_{ik},$$
$$\{\widetilde{A}_{ik}, \widetilde{A}_{nm}\} = \delta_{kn}\widetilde{A}_{im} - \delta_{im}\widetilde{A}_{nk}.$$

Thus, after the first step of our iterative procedure, we naturally obtain the set of $N-1$ pairs of canonical coordinates q_k, p_k, and the problem reduces to the construction of canonical coordinates on the orbit of smaller dimension. We can continue the iterative procedure, getting finally $(N-1) + (N-2) + \cdots + 1 = \frac{N(N-1)}{2}$ pairs of canonical coordinates on the orbit in general position. As an example, consider the formulas for $N = 2, 3$:

$$A = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & p \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix},$$
$$A = \begin{pmatrix} 1 & 0 & 0 \\ q_2 & 1 & 0 \\ q_3 & q_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & p_2 + p_3\, q_1 & p_3 \\ 0 & \lambda_2 & p_1 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ q_2 & 1 & 0 \\ q_3 & q_1 & 1 \end{pmatrix}^{-1}.$$

General formulas can be derived easily; they are similar to those for the generators of the representation of the basic series for the group $\mathrm{GL}(N, \mathbb{C})$, which were obtained in [11] on the basis of the method of [12]. Note that everything can easily be generalized to the case of the multiple eigenvalues λ_k [12]. In all formulas, the matrix entries can be rectangular matrices; it is only necessary to keep their correct order in products.

For example, in the first formula, let λ_1 have multiplicity n_1 and λ_2 multiplicity n_2. Then A is a matrix of size $(n_1+n_2)\times(n_1+n_2)$, q is a block of size $n_2\times n_1$, and p is a block of size $n_1\times n_2$. In the second formula, let λ_k have multiplicity n_k. Then A is a matrix of size $(n_1+n_2+n_3)\times(n_1+n_2+n_3)$, the block q_1 is of size $n_3\times n_2$, the block p_1 is of size $n_2\times n_3$, the block q_2 has size $n_2\times n_1$, the block p_2 has size $n_1\times n_2$, the block q_3 has size $n_3\times n_1$, and block p_3 has size $n_1\times n_3$.

The inverse mapping, i.e., the formulas $p_i=p_i(A), q_i=q_i(A)$ can be obtained by the successive finding of eigenvectors corresponding to the initially specified eigenvalues for smaller and smaller matrices. Obviously, this is a rational operation.

The canonical parametrization of the coadjoint orbit is a well-known problem. Among many publications, the papers [13, 14], and [15] should be mentioned. The method introduced here is closer to the methods of [16] and [17].

The papers [14] and [15] also present explicit formulas for the parametrization, but they employ other coordinates. An important property of our coordinates is *rationality*. To find p_k, q_k we need to solve systems of linear equations only. The coefficients of the equations are rational functions of the matrix entries $A_{i,j}$. The methods of [14] and [15] give rational formulas for some polynomials of degree $N-1$. The canonical coordinates are the roots of these polynomials. Thus, starting with the case of 3×3-matrices, the coordinates are different. Finally, we notice that in our method the dimensions of the eigenspaces of A are arbitrary, but the methods of [14] and [15] are applicable in the case of one-dimensional eigenspaces only.

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References

[1] N. Hitchin, *Geometrical aspects of Schlesinger's equation*, J. Geom. Phys. **23** (1997), 287–300. MR1484592 (99a:32023)

[2] I. M. Gel′fand, *Lectures on linear algebra*, 4th ed., Nauka, Moscow, 1971; English transl. of 2nd Russian ed., Dover Publ., Inc., New York, 1989. MR0352111 (50:4598); MR1034245 (90j:15001)

[3] F. Gantmacher, *Theory of matrices*, 2nd ed., Nauka, Moscow, 1966; English transl. of 1st ed., Chelsea Publ. Co., New York, 1959. MR0202725 (34:2585); MR0107649 (21:6372c)

[4] V. I. Arnol′d, *Mathematical methods of classical mechanics*, 3rd ed., Nauka, Moscow, 1989; English transl., Grad. Texts in Math., vol. 60, Springer-Verlag, New York, 1989. MR1037020 (93c:70001); MR0997295 (90c:58046)

[5] A. T. Fomenko, *Symplectic geometry. Methods and applications*, Moskov. Gos. Univ., Moscow, 1988; English transl., Adv. Stud. in Contemp. Math., vol. 5, Gordon and Breach Sci. Publ., New York, 1988. MR0964470 (90k:58082); MR0994805 (90k:58065)

[6] K. Okamoto, *Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé*, Japan. J. Math. (N.S.) **5** (1979), no. 1, 1–79. MR0614694 (83m:58005)

[7] M. V. Babich, *Coordinates on the phase spaces of the system of Schlesinger equations and the system of Garnier–Painlevé* 6 equations, Dokl. Akad. Nauk **412** (2007), no. 4, 439–443. (Russian) MR2451331 (2009e:34011)

[8] ———, *Rational symplectic coordinates on the space of Fuchs equations, $m\times m$-case*, Lett. Math. Phys. **86** (2008), no. 1, 63–77. MR2460728 (2010f:53151)

[9] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974), no. 1, 121–130. MR0402819 (53:6633)

[10] A. G. Reĭman and M. A. Semenov-Tyan-Shanskiĭ, *Integrable systems. Group-theoretical approach*, Inst. Kompyuter. Issled., Moscow–Izhevsk, 2003; *Group-theoretical methods in the theory of integrable systems*, Dynamical Systems–7, Itogi Nauki i Tekhniki Sovrem. Probl. Mat. Fundam. Naprav., vol. 16, VINITI, Moscow, 1987, pp. 119–194; English transl., Encyclopaedia Math. Sci., vol. 16, Springer-Verlag, Berlin, 1994, pp. 116–225. MR1256257 (94h:58069)

[11] S. E. Derkachov and A. N. Manashov, *R-matrix and Baxter Q-operators for the noncompact $SL(N,\mathbb{C})$ invariant spin chain*, SIGMA Symmetry Integrability Geom. Methods Appl. **2** (2006), 084, 20 pp.; `arXiv:nlin.SI/0612003` MR2264900 (2008g:81119)

[12] I. M. Gel′fand and M. A. Naĭmark, *Unitary representations of the classical groups*, Trudy Mat. Inst. Steklov. **36** (1950), 288 pp. (Russian) MR0046370 (13:722f)

[13] A. Alekseev, L. Faddeev, and S. Shatashvili, *Quantization of symplectic orbits of compact Lie groups by means of the functional integral*, J. Geom. Phys. **5** (1988), 391–406. MR1048508 (91d:58087) MR1048508 (91d:58087)

[14] A. Gerasimov, S. Kharchev, and D. Lebedev, *Representation theory and quantum inverse scattering method: the open Toda chain and the hyperbolic Sutherland model*, Int. Math. Res. Not. **2004**, no. 17, 823–854. MR2040074 (2006m:81134)

[15] B. Dubrovin and M. Mazzocco, *Canonical structure and symmetries of the Schlesinger equations*, Comm. Math. Phys. **271** (2007), no. 2, 289–373. MR2287909 (2008a:32015)

[16] I. M. Krichever, *An analogue of the d'Alembert formula for the equations of a principal chiral field and the sine–Gordon equation*, Dokl. Akad. Nauk SSSR **253** (1980), no. 2, 288–292; English transl., Soviet Math. Dokl. **32** (1980), no. 1, 79–84. MR0581396 (82k:35095)

[17] A. P. Veselov and S. P. Novikov, *Poisson brackets and complex tori*, Algebraic Geometry and its Applications, Trudy Mat. Inst. Steklov. **165** (1984), 49–61; English transl. in Proc. Steklov Inst. Math. **165** (1985), no. 3, 53–65. MR0752932 (86c:58076)

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