GRÖBNER–SHIRSHOV BASES OF THE LIE ALGEBRA $D_n^+$

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Abstract. Over a field of characteristic 0, the reduced Gröbner–Shirshov bases (RGShB) are computed in the positive part $D_n^+$ of the simple finite-dimensional Lie algebra $D_n$ for the canonical generators corresponding to simple roots, under an arbitrary ordering of these generators (i.e., an arbitrary basis among the $n!$ bases is fixed and analyzed). In this setting, the RGShBs were previously computed by the author for the Lie algebras $A_n^+$, $B_n^+$, and $C_n^+$. For one ordering of the generators, the RGShBs of these algebras were calculated by Bokut and Klein (1996).

Introduction

In 1962 Shirshov [1], and in 1965 Buchberger [2] suggested essentially the same ideas for Lie polynomials (Shirshov) and commutative polynomials (Buchberger). Actually, they developed a universal way of computing a basis of an arbitrary algebra $L$ of a certain variety. For this, the structure of the algebra $L$ should be enriched somewhat: one needs to choose a set $X$ of generators in $L$, to specify a basis of the free algebra of this variety that is determined by this set of generators, and to give a linear ordering of this basis.

After that, everything reduces to the properties of relations in the algebra $L$. Shirshov introduced the notion of a composition of (Lie) polynomials and the notion of a set of polynomials closed under composition. Later on, the sets closed under composition were called the Gröbner bases (GB) in the commutative case and the Gröbner–Shirshov bases (GShB) in the case of Lie algebras. Among other sets of Lie polynomials, the GsHBs are distinguished by the fact that, knowing a GShB, one can easily find a basis of the algebra $L$ associated with it.

With the knowledge of a basis of the algebra, it is rather easy to solve some problems (for example, the word problem). Moreover, a GShB in fact contains much more information than a basis of the algebra. For this reason, the GShBs have gained recognition rather quickly, and a new area of study has been opened.

These new ideas found application immediately after the paper [1]. The first applications were given in the paper [1] itself; namely, the word problem was solved and a theorem was proved concerning the property of being free for Lie algebras with one defining relation. At the same time, in the paper [3], Shirshov used GShBs to disprove a conjecture on the structure of subalgebras of the free product of Lie algebras.

In [4], with the help of GShBs, Bokut’ proved that an arbitrary Lie algebra can be embedded in an algebraically closed algebra.

At present, the GBs and GShBs are actively used both in theoretic investigations and in computational techniques. One of the advantages of the new method is that it brings an algorithmic flavor in a search for answers to a large number of questions (for example, see the papers of Buchberger [5] and Gerdt [6] and also many other papers of

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As a consequence, it became possible to apply computational techniques to many problems. For example, if a GShB of an arbitrary Lie algebra is recursive, then this algebra contains a linear recursive basis, and the question as to whether an arbitrary Lie polynomial belongs to an ideal of relations is solved algorithmically.

Thus, a GShB is an important object of study; an active investigation of this object began in the 1990s. Bokut’ and Klein found GShBs of the simple finite-dimensional Lie algebras $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$; see [7]–[10]. Moreover, together with Malcolmson, they described the GShBs for quantum universal enveloping algebras of type $A_n$ [11] and proved that a set of Lie polynomials is a GShB of the corresponding Lie algebra if and only if it is a GShB in the associative sense [12].

The idea of a GShB, suggested in the cases of commutative algebras and Lie algebras, has received further development. The recognition of some properties of algebras given with the help of GShBs was the subject of the paper [13] by Gateva-Ivanova and Latyshev. In the 1980s, Mikhalev extended the techniques of compositions to the case of superalgebras by proving the composition lemma for color superalgebras; see [14]–[16]. In [17], Bokut’, Kang, Lee, and Malcolmson found GShBs for the simple finite-dimensional Lie superalgebras of type $A_n$, $B_n$, $C_n$, or $D_n$. In [18, 19], Kang and Lee constructed the theory of GShBs for modules over Lie algebras and found GShBs (which were called Gröbner–Shirshov pairs in this case) for the irreducible $\mathfrak{sl}_n$-modules. Chibrikov [20] simplified significantly the notion of a GShB for modules: it became not a pair but a single set in a free module over a free algebra.

Moreover, the theory of GShBs was employed in the cases of conformal and vertex algebras. In the papers [21, 22], Roitman proved the existence of free conformal algebras (this fact is not a consequence of theorems in universal algebra, because conformal algebras do not form a variety). In [23], Bokut’, Fong, and Ke obtained the results of Roitman in a different way. Those authors applied ideas and techniques related to GShBs to associative conformal algebras [24].

Vasil’iev and Pavlov [25] listed all monomial orders (in the commutative case), and they plan to review all GBs of a finitely generated commutative algebra. The history of the theory of GShBs was presented in [26, 27].

In the papers [28, 29], for a certain ordering of generators, Bokut’ and Klein computed the reduced GShBs (BGShBs) for series of simple Lie algebras (the RGShBs are distinguished by the fact that they are uniquely determined by generators and the order on generators). For affine nontwisted Kac–Moody algebras, Poroshenko [28, 29] computed first a certain basis of a Lie algebra as a linear space (this basis was uniquely determined by the order on the Lie monomials that correspond to regular words, a so-called reduced basis), and then he found the RGShB uniquely determined by that basis. A conjecture was stated that, for the classical simple Lie algebras, the reduced basis, which is uniquely determined by the root system and the order on the set of monomials, can be described in terms of the root system. Later, this conjecture was confirmed for fields of characteristic 0.

So far, the RGShBs have been computed only for one ordering of the generators. Bokut’ proposed to compute RGShBs (for the order deg-lex) under an arbitrary ordering of generators. Thus, an entirely new problem arose: for classical simple Lie algebras, under an arbitrary ordering of generators, to describe the reduced bases and then the RGShBs uniquely determined by them in terms of the root system. In this wording, the generators of the Lie algebra are fixed, but the order on them is arbitrary, and the $n!$ RGShBs are analyzed simultaneously ($n$ is the number of generators). This entirely new problem was solved by the author for the Lie algebras $A_n^+$, $B_n^+$, $C_n^+$ over a field of characteristic 0 in the papers [30–32], and for the Lie algebra $D_n^+$ in the present...
This is a new step in the development of the theory of GShBs. For this paper, the reduced bases of the Lie algebra $D_n^+$ were computed earlier in [33]. Before that, the reduced bases for the algebra $D_n^+$ had been found in [34], but for only one ordering of generators.

§1. Reduced Gröbner bases (the commutative case)

First, we give a definition of a GB in the commutative case (which is much easier than the case of Lie algebras). Let $A$ be a finitely generated commutative algebra given by a finite set of generators $X = \{x_1, \ldots, x_n\}$ and a collection of relations. It can be regarded as the quotient algebra of polynomials $k[x_1, \ldots, x_n]$ by a certain ideal $I$. The relations of the algebra $A$ can be identified with the elements of $I$.

The set $\mathbb{Z}_{\geq 0}^n$ and, what is equivalent, the set of monomials $\{x^\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^n\}$ are linearly ordered so that if $\alpha < \beta$ and $\gamma \in \mathbb{Z}_{\geq 0}^n$, then $\alpha + \gamma < \beta + \gamma$ (here the order relation is denoted by $\leq$).

For an arbitrary polynomial $f$, the greatest monomial occurring in the representation of $f$ as a linear combination of monomials is called the highest monomial for $f$, and its coefficient is the highest coefficient of $f$ [35, p. 56, Definition 7].

A finite subset $G = \{g_1, \ldots, g_s\}$ of an ideal $I$ of the algebra of polynomials $k[x_1, \ldots, x_n]$ is called a Gröbner basis or a standard basis of $I$ if the ideal generated by the highest monomials of the polynomials in $G$ coincides with the ideal generated by the highest monomials of the polynomials in $I$ [35, p. 74, Definition 5].

In other words, a Gröbner basis (GB) of $I$ is a finite set of relations (polynomials in $I$) such that the highest monomial of any relation is divisible by the highest monomial of a certain element in the GB. A GB of the ideal $I$ may be also called a GB of the quotient algebra $k[x_1, \ldots, x_n]/I$.

A GB depends on the ideal $I$ and the monomial order. But, given an ideal $I$ and a monomial order, many GBs may exist. This stimulates a search for conditions that determine a GB uniquely.

A minimal BG ([35, p. 89, Definition 4]) is a Gröbner basis $G$ such that 1) the highest coefficient of any polynomial of $G$ is equal to 1; 2) for all $p \in G$, the highest monomial of $p$ does not belong to the ideal generated by the highest monomials of the polynomials in $G - \{p\}$, or, for two distinct polynomials in $G$, the highest monomial of any of them does not divide the highest monomial of the other.

In this definition, there is a restriction only on the highest words of the polynomials in a GB, and a minimal GB is not determined uniquely by the ideal $I$ and the monomial order.

A reduced GB ([35, p. 90, Definition 5]) is a Gröbner basis $G$ such that 1) the highest coefficient of any polynomial in $G$ is equal to 1; 2) for all $p \in G$, no monomial of $p$ is divisible by any highest monomial of the polynomials in $G - \{p\}$.

A reduced GB is a minimal GB. For any nonzero ideal $I$ and any monomial order, a reduced GB exists and is determined uniquely ([35, p. 90, Proposition 6]). In the present paper, we need some analysis of the notion of a reduced GB.

It is clear that the set of highest monomials of the elements in $I$ is closed under multiplication by an arbitrary monomial. For this reason, all monomials can be split into two parts relative to the ideal $I$.

1) The highest monomials of the relations (elements of $I$). These monomials can be represented modulo $I$ by linear combinations of lesser monomials.

2) The monomials that are not the highest monomials of relations. These monomials cannot be represented modulo $I$ by linear combinations of lesser monomials. They form a basis of the quotient algebra $k[x_1, \ldots, x_n]/I$. 

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The monomials that are not representable modulo $I$ by a linear combination of lesser monomials are said to be \textit{reduced}, and the other monomials are \textit{nonreduced}. The set of reduced monomials forms a basis of the quotient algebra $k[x_1, \ldots, x_n]/I$. This basis is said to be \textit{reduced}. Any relation (an element of the ideal $I$) is a linear combination of relations of the form $e - \pi(e)$, where $e$ is a nonreduced monomial and $\pi(e)$ is a unique linear combination of reduced monomials smaller than $e$.

The relations $e - \pi(e) = 0$ are said to be \textit{canonical}. Associating the polynomial $e - \pi(e)$ with each nonreduced monomial $e$, we obtain a one-to-one correspondence between the set of nonreduced monomials and the set of canonical relations.

Now we can define a reduced GB of the ideal $I$ as a set of canonical relations that generates the ideal $I$ and is such that the highest monomials of the elements in this GB are not divisors of each other (we do not mention that a reduced GB consists of finitely many elements, because in a finitely generated commutative algebra this is true automatically). In other words, we can define a reduced Gröbner basis $G$ as the set of all canonical relations $e - \pi(e)$ in which the highest monomial $e$ is minimal among the nonreduced monomials under the order “the monomial $e_1$ is a divisor of the monomial $e_2$”.

From this definition, the property that the set of relations $G$ generates the ideal $I$ is obtained immediately.

A reduced Gröbner basis can also be defined as the smallest set of canonical relations that generates the quotient algebra $k[x_1, \ldots, x_n]/I$ ($I$ is an ideal). Indeed, first, if $e$ is a nonreduced monomial such that any proper divisor of it (a monomial in the algebra of polynomials) is a reduced monomial, then the canonical relation $e - \pi(e)$ is an element of any Gröbner basis of canonical relations. Second, the set of all canonical relations $e - \pi(e)$, where $e$ is a nonreduced monomial such that any proper divisor of it is a reduced monomial, forms a Gröbner basis of canonical relations and even a reduced Gröbner basis.

In the case of Lie algebras,

1) instead of an arbitrary monomial ordering, only the ordering deg-lex is used;

2) instead of (commutative) monomials, the Lie monomials corresponding to the Lyndon–Shirshov words (nonassociative regular words) are considered;

3) the ordering $\preceq$, “a monomial $e_1$ divides a monomial $e_2$”, converts into the following ordering: “removing brackets in a nonassociative regular word $e_1$ results in a subword of the associative regular word obtained by removing the brackets in a nonassociative regular word $e_2$”;

4) a reduced Gröbner basis converts into a reduced Gröbner–Shirshov basis, namely, the smallest set of canonical relations that generates the quotient Lie algebra, or the set of canonical relations $e - \pi(e)$ that correspond to minimal words $e$ (under the ordering $\preceq$) among the nonreduced words.

\section*{§2. Reduced Gröbner–Shirshov Bases of Lie Algebras}

In the sequel, $X$ is an arbitrary linearly ordered finite set (of letters); $\text{Lie}(X)$ is the Lie algebra over an arbitrary field $k$ of characteristic 0 freely generated by the set $X$. The words are tacitly assumed to be associative and composed of elements of $X$. On the set of words, we consider the linear ordering lex, which differs from the lexicographic order only in the condition that a proper beginning of any word is greater than the word itself.

As in \cite{36, 37}, we say that a nonempty word $u$ is \textit{regular} if, for any two nonempty words $v$ and $w$, from $u = vw$ it follows that $vw > wv$. The regular words were introduced independently by Lyndon in \cite{38}. In \cite{38, 39, 40}, the authors also considered \textit{regular nonassociative words}, which satisfy the following conditions: the letters are regular nonassociative words; removing the brackets in a regular nonassociative word results
in a regular word; if a nonassociative word \( u \) can be represented in the form \( u = u_1 z \), then the nonassociative words \( u_1 \) and \( z \) are regular; if \( u_1 = xy \), then \( y \leq z \).

At present, the regular words (both associative and nonassociative) are commonly named the Lyndon–Shirshov words. Western authors call them the Lyndon words (see the survey [40]).

In [36], Shirshov showed that, in a regular associative word \( u \), the brackets can be arranged uniquely so as to yield a regular nonassociative word, which will be denoted by \((u)\). This arrangement of brackets is said to be regular.

We denote by \( \mathsf{kReg}(X) \) the space with the basis consisting of all regular words in letters from \( X \). We extend the regular arrangement of brackets to a linear mapping \((\cdot)\) of the space \( \mathsf{kReg}(X) \) to the space with a basis of regular nonassociative words in letters from \( X \). This mapping is bijective.

For an arbitrary regular word \( u \), the image of the word \((u)\) under the canonical mapping to the free algebra \( \mathsf{Lie}(X) \) is denoted by \([u]\). Consider the linear mapping \([\cdot]\) of the space \( \mathsf{kReg}(X) \) to the free algebra \( \mathsf{Lie}(X) \) that takes each regular word \( u \) to the Lie monomial \([u]\). An element \( w \) of the space \( \mathsf{kReg}(X) \) is called the support for \((w)\) and \([w]\).

In [36, 39], it was shown that the mapping \([\cdot]\) is bijective and the set of Lie monomials \([u]\), where \( u \) is a regular associative word, is a basis of the algebra \( \mathsf{Lie}(X) \).

Now we turn to bases and relations of an arbitrary finitely generated Lie algebra \( L = \mathsf{Lie}(X)/J \) given by a finite set of generators \( X \) and an ideal of relations \( J \). Relations that are different from zero will be called nontrivial.

On the set of words, we consider the linear ordering \( \text{deg-lex} \): a word \( x \) is less than a word \( y \) if the length of \( x \) is less than that of \( y \), or these words are of equal length but \( x \) is less than \( y \) in the lexicographic order.

A word \( \bar{w} \) will be called the highest word of a nonzero element \( w \) of \( \mathsf{kReg}(X) \) if for a nonzero element \( \alpha \) of the field \( k \), the word \( w - \alpha \bar{w} \) is a linear combination of words that are less than \( u \) in the deg-lex order. For \( S \subseteq \mathsf{kReg}(X) \) the set of highest words of the elements of \( S \) different from zero will be denoted by \( \bar{S} \).

A regular word \( u \) is said to be reduced if \([u] \) modulo the ideal \( J \) is not a linear combination of elements \([v]\), where \( v \) is a regular word that is less than \( u \). The elements of the form \([u] + J \), where \( u \) is a reduced word, constitute a basis of the algebra \( \mathsf{Lie}(X)/J \). This basis is said to be reduced.

Remark 2.1. Shirshov showed that if \( 0 \neq s \in \mathsf{kReg}(X) \) and \( u \) is a regular word that contains the subword \( \bar{s} \), then in the Lie ideal of the free Lie algebra generated by the element \([s]\), there is a nonzero element for which the highest word of its support coincides with \( u \) (see [36, Lemma 4]).

Remark 2.2. Remark 2.1 implies that if a regular word has a subword that is the highest word of the support of a nontrivial relation, then this word itself is the highest word of the support of a nontrivial relation.

Thus, if \( S \subseteq \mathsf{kReg}(X) - \{0\} \) and \([S]\) is a set of nontrivial relations of the algebra \( L \), then the supports of the elements of a reduced basis are \( \bar{S} \)-reduced, i.e., they contain no subwords from \( S \).

Question 2.3. What are the sets \([S]\) of nontrivial relations for which the converse is true, i.e., when are the \( \bar{S} \)-reduced words reduced?

A set of relations as in Question 2.3 enables one to find a reduced basis easily. Shirshov found partial operations (he called them compositions) with respect to which the sets as in Question 2.3 are closed; he showed in [36] that such sets are precisely the sets of nontrivial relations that are closed with respect to compositions. Subsequently, such sets of relations
were called the Gröbner–Shirshov bases (GShB) (see [7]). This definition suggests taking arbitrary sets of relations and closing them with respect to compositions, which was done in many papers. But if we consider an arbitrary ordering of the generators, then calculation of compositions seems to be impossible. For this reason, in the present paper we give the following definition of a GShB.

**Definition 2.4.** A GShB of a Lie algebra $L$ is a set $[S]$ of nontrivial relations of $L$ for which all $S$-reduced words are reduced.

Now there is no reason for considering compositions, and our nearest goal is to further analyze the notion of a GShB. It is easy to show that any GShB $[S]$ is a set of generating relations.

In papers concerning GShBs, the Gröbner–Shirshov bases under consideration are composed exclusively of relations of the form $[u - \pi(u)]$, where $u$ is a regular word and $\pi(u)$ is a linear combination of reduced words that are less than $u$ in the deg-lex order. Such relations will be called canonical relations. It is clear that $\pi(u)$ is uniquely determined by $u$, and that any relation can be represented uniquely as a linear combination of canonical ones. We extend $\pi$ to a linear mapping of the space $k\text{Reg}(X)$.

As was shown in [30, §2], the intersection of all GsHBs of canonical relations of the algebra $L = \text{Lie}(X)/J$ is a GShB of this algebra. It is called the reduced GShB (RGShB); see [30, §2].

**Remark 2.5.** In [30, §2], it was also shown that the RGShB of an arbitrary finitely generated Lie algebra $L$ is precisely the set of canonical relations $[u - \pi(u)]$ in which the highest word $u$ is not reduced, but any proper regular subword of it is reduced.

This will be our working definition of the RGShB of an arbitrary finitely generated algebra. Remark 2.5 shows that to calculate the RGShB it remains to learn how to calculate regular words that are not reduced but any proper regular subword of them is reduced, and also to construct canonical relations with such highest words.

Now we consider the case where the ideal $J$ is homogeneous with respect to the number of occurrences of each letter. Let $ZX$ denote the free Abelian group generated by the set $X$. The operation in this group will be denoted additively. The algebra $\text{Lie}(X)$ is graded by the group $ZX$: as an element of the algebra $\text{Lie}(X)$, any generator $x \in X$ is homogeneous and lies in the homogeneous component that corresponds to $x$ regarded as an element of $ZX$. This graduation induces a graduation on the algebra $L = \text{Lie}(X)/J$.

The image of an arbitrary word $u$ under the natural homomorphism of the semigroup of words to the group $ZX$ will be denoted by $|u|$ and will be called the composition of the word $u$. Moreover, if the homogeneous component of the algebra $L$ that corresponds to the element $|u|$ is not zero, we shall say that the composition of the word $u$ is rooted. Now we recall some useful properties of reduced words.

**Lemma 2.6.** Let $L = \text{Lie}(X)/J$ be a finitely generated algebra such that the ideal of its relations is homogeneous with respect to the number of occurrences of each letter. Then

1) the composition of any reduced word is rooted;
2) each regular subword of a reduced word is reduced;
3) any regular subword of a reduced word (including the word itself) has rooted composition.

Indeed, 1) follows immediately from the definitions; 2) is another form of Remark 2.2; 3) is a consequence of 1) and 2).

As in [30, 33], a regular word is called an RR-word if any regular subword of it (including the word itself) has rooted composition.
Remark 2.7. Under the assumptions of Lemma 2.6, the least word among the RR-words with one and the same composition is reduced.

In [33], it was shown that, for the algebra \(D_n^+\), for any ordering of generators the RR-words are uniquely determined by their compositions, and the set of these words was calculated.

Now we consider the special case where for the algebra \(L = \text{Lie}(X)/J\) every RR-word is determined by its composition. A regular word is called an SR-word if its composition is not rooted but the compositions of all proper regular subwords of it are rooted.

Lemma 2.8. Let \(L = \text{Lie}(X)/J\) be a finitely generated Lie algebra such that (i) its ideal of relations is homogeneous with respect to the number of occurrences of each letter; (ii) every RR-word is determined by its composition. Then:
1) the set of reduced words coincides with the set of RR-words;
2) the set of regular nonreduced words such that all proper regular subwords of them are reduced coincides with the set of SR-words;
3) the RGShB of \(L\) coincides with the set of elements of the free Lie algebra \(\text{Lie}(X)\) of the form \([u]\), where \(u\) is an arbitrary SR-word.

Proof. 1) follows from 3) of Lemma 2.6 and from Remark 2.7. 2) is a consequence of Remark 2.5 and statement 1).

From Remark 2.5 and statement 1) we deduce that the RGShB of the algebra \(L\) is the set of relations of the form \([u - \pi(u)]\), where \(u\) is an SR-word and \(\pi(u)\) is a uniquely determined linear combination of reduced words. Since \(u\) is an SR-word, its composition is not rooted, i.e., the homogeneous component of the algebra \(L\) that corresponds to the element \([u]\) consists only of 0. Therefore, \(\pi(u) = 0\) and \([u - \pi(u)] = [u]\), as required. \(\square\)

Our aim in the present paper is to calculate the RGShB of the Lie algebra \(D_n^+\) for the generators corresponding to simple roots, under any ordering of these generators. For this, we construct the set of SR-words for the algebra \(D_n^+\) for any ordering of generators.

§3. The algebra \(D_n^+\), the root system, graphs of positive roots, and systems of occurrences

To describe the algebra \(D_n^+\) and its generators that correspond to simple roots, we recall some well-known facts from the classical theory of finite-dimensional simple Lie algebras over a field of characteristic 0 (see [11, 12]).

The algebra \(D_n\) is a finite-dimensional simple decomposable Lie algebra. Let \(L\) be any finite-dimensional simple decomposable Lie algebra.

The term decomposable means that \(L\) contains a nilpotent subalgebra \(H\) (Cartan subalgebra) such that the Lie \(H\)-module \(L\) is the direct sum of \(H\) and the submodules \(L_\alpha = \{x \in L \mid \exists n \geq 1 \forall h \in H \ (\text{ad}_h - \alpha(h) \cdot 1)^n \cdot x = 0\}\), where \(\alpha\) ranges over the set of nonzero weights of the \(H\)-semiinvariants in \(L\).

The nonzero weights of the \(H\)-semiinvariants are called roots. By definition, the roots are elements of the space \(H^*\) dual to \(H\). The Cartan subalgebra is Abelian. The spaces \(L_\alpha\) are one-dimensional; in particular, their elements are \(H\)-semiinvariants.

There exists a subset \(\Pi\) of the set of roots such that any root \(\alpha\) can be uniquely represented in the form \(\alpha = \sum a_\pi \cdot \pi\) (where the sum is taken over some elements \(\pi\) of \(\Pi\); \(a_\pi\) is an integer different from zero), and the coefficients \(a_\pi\) have one and the same sign. Such a subset \(\Pi\) is called a system of simple roots. The roots that are sums of simple roots (as elements of the space \(H^*\)) are said to be positive.

The simple roots are linearly independent elements of the space \(H^*\); in particular, the subgroup \(\mathbb{Z}\Pi\) generated by simple roots in the additive group of \(H^*\) is the free Abelian
group generated by simple roots. Any root is an element of the group \( L^+ \). Thus, the algebra \( L \) is graded by the group \( L^+ \). In this graduation, any homogeneous component corresponding to a root is one-dimensional.

We denote by \( L^+ \) the subspace of \( L \) generated by all components \( L_\alpha \), where \( \alpha \) is a positive root; \( L^+ \) is a subalgebra of \( L \), and it is natural to call it the “positive part” of \( L \). This subalgebra is generated by the \( H \)-semiinvariants the weights of which are simple roots. For each simple root \( \pi \), we choose a nonzero \( H \)-semiinvariant of weight \( \pi \). The set \( X \) of these elements generates the algebra \( L^+ \). The elements of \( X \) are precisely the generators that are of interest to us.

The algebra \( D_n^+ \) is the “positive part” of the algebra \( D_n \), and \( n \) is the number of simple roots.

**Remark 3.1.** The relations of the algebra \( L \) are homogeneous with respect to the number of occurrences of each letter. Therefore, \( L \) is graded by the group \( \mathbb{Z}X \). Moreover, the simple roots and the generators of the algebra \( L^+ \) are in one-to-one correspondence. Consequently, the gradations of the algebra \( L \) by the group \( \mathbb{Z}L_\Pi \) and by the group \( \mathbb{Z}X \) coincide. Hence, to know what elements of \( \mathbb{Z}L \) are positive roots is the same as to know what homogeneous components of the algebra \( L^+ \) are nonzero.

**Remark 3.2.** It should be noted that to calculate the RGShB of the algebra \( D_n^+ \), we only need to know the ordering of the generators and to know what elements of the group \( \mathbb{Z}L_\Pi \) are positive roots (i.e., what homogeneous components of the algebra \( L^+ \) are nonzero).

Indeed, this information suffices for the calculation of the sets of RR-words and SR-words. As was shown in [33], the RR-words are determined by their compositions. Therefore, by Lemma 2.8, we can calculate the RGShB as well.

Remarks 3.1 and 3.2 show that there is no need to distinguish between letters and the simple roots corresponding to these letters. But if it is important to emphasize that we regard a letter \( q \) as a simple root, we shall write \( |q| \) in place of \( q \).

By a (nonoriented) graph we mean a pair \( (X, R) \), where \( X \) is a finite set (the set of vertices) and \( R \) is a symmetric antireflexive binary relation on \( X \) (the adjacency relation). Usually, the vertices of a graph are represented by points, and vertices \( x \) and \( y \) are joined by an edge if \( x \) and \( y \) are adjacent, i.e., \( (x, y) \in R \). The vertices of an arbitrary graph that have at most one adjacent vertex are called end vertices.

A graph is connected if for any two vertices \( x \) and \( y \) there is a finite sequence of its vertices such that it begins with the vertex \( x \), ends with the vertex \( y \), and each subsequent vertex is adjacent to the preceding one. A connected graph is called an interval if at least one vertex of it has at most one adjacent vertex and each vertex has at most two adjacent vertices.

An arbitrary interval \( I \) either consists of only one vertex or has precisely two end vertices. If an interval \( I \) is a subgraph of a graph \( D \) without cycles, then \( I \) is determined by its end vertices. We denote \( I = [x, y] \), where \( \{x, y\} \) is the set of end vertices of the interval \( I \); \( [x, y] = [x, y] - \{y\} \); \( (x, y) = [x, y] - \{x\} \). Clearly, \( [x, y] = [y, x] \), \( [x, y] = (y, x) \), \( (x, y) = (y, x) \), and the sets \( [x, y] \) and \( (x, y) \) are intervals.

For the root system \( D_n \), we consider the graph the vertices of which are simple roots, and two vertices \( \alpha \) and \( \beta \) are adjacent if the element \( \alpha + \beta \) of the group \( \mathbb{Z}L_\Pi \) is again a root. Using the tables given at the end of the book [43], we can easily check the following:

1. this graph is connected and has no cycles;
2. any vertex of it has at most three adjacent vertices;
3. there exists precisely one vertex \( z^* \) (\( * \)-center) that has three adjacent vertices;
(4) at least two vertices adjacent to the \( * \)-center are end vertices;
(5) this graph is determined by the number \( n \) of its vertices.

We denote this graph by \( D_n \).

The elements of the group \( \mathbb{Z}X \) that can be represented as a sum of elements of \( X \) are called positive. We define a partial order in the group \( \mathbb{Z}I \): \( \alpha \leq \beta \) if the element \( \beta - \alpha \) is positive.

**Remark 3.3.** From [43] it can be seen that any positive root of the root system \( D_n \) is a positive element of the group \( \mathbb{Z}I \), obtained by summation of vertices of a connected subgraph of the graph \( D_n \), or the element \( \sum D + [z^*, x] \), where \( D \) is a connected subgraph of the graph \( D_n \) containing the \( * \)-center and all three adjacent vertices; \( x \) is an arbitrary nonend vertex of \( D \). Here \( \sum D \) is the element of \( \mathbb{Z}X \) obtained by summation of all vertices of \( D \): \( \sum D + [z^*, x] \) is the element \( \sum D + \sum [z^*, x] \) (we sometimes omit the symbol \( \sum \) to simplify the notation).

For any positive element \( \alpha \in \mathbb{Z}I \), we denote by \( \bar{D}(\alpha) \) the subgraph of the graph \( \bar{D}_n \) (the vertices of which are simple roots \( \beta \) such that \( \beta \leq \alpha \)) endowed with an additional structure: each vertex \( \beta \) is marked by the greatest positive integer \( d \) such that \( d\beta \leq \alpha \) (this \( d \) is called the multiplicity of \( \beta \)).

From Remark 3.3 it is seen that any root \( \alpha \) is a linear combination of simple roots with coefficients 0, 1, 2. We say that a vertex \( \beta \) of \( \bar{D}(\alpha) \) is single if \( 2\beta \not\leq \alpha \), and \( \beta \) is a double vertex if \( 2\beta \leq \alpha \).

Also, the definitions show that, if we know the graph \( \bar{D}(\alpha) \), we can find the root \( \alpha \) itself (as an element of the group \( \mathbb{Z}I \)).

We denote by \( 0 \) the set of end vertices of \( \bar{D}_n \) that are adjacent to the \( * \)-center. The graph \( \bar{D}_n \) contains at least four vertices. If \( \bar{D}_n \) consists precisely of four vertices, then the set \( 0 \) has three elements. If \( \bar{D}_n \) contains at least five vertices, then the set \( O \) has precisely two elements.

For any positive root \( \alpha \), we denote by \( M^+(\alpha) \) the set of simple roots \( \beta \) such that \( \alpha - \beta \) is a positive root.

For any positive root \( \alpha \) such that \( z^* \leq \alpha \), we denote by \( O(\alpha) \) the set of end vertices in the graph \( \bar{D}(\alpha) \) adjacent to the \( * \)-center.

If the graph \( \bar{D}(\alpha) \) has double vertices, then we denote by \( x^+_2(\alpha) \) a unique double vertex of \( \bar{D}(\alpha) \) that belongs to \( M^+(\alpha) \). It is a unique double vertex of \( \bar{D}(\alpha) \) that has at most one adjacent double vertex and is different from the \( * \)-center if the graph has at least two double vertices.

If the graph \( \bar{D}(\alpha) \) contains \( z^* \) and three vertices adjacent to it and has at least five vertices, then we denote by \( x^+_1(\alpha) \) a unique end vertex of \( \bar{D}(\alpha) \) that is not adjacent to \( z^* \). Moreover, if \( \bar{D}(\alpha) \) has double vertices, then we denote by \( x^-_1(\alpha) \) a unique (single) vertex of this graph that is adjacent to \( x^+_2(\alpha) \) and is such that \( x^-_1(\alpha) \in (x^+_2(\alpha), x^+_1(\alpha)) \).

If the vertices of \( \bar{D}_n \) are linearly ordered, then we shall talk about the graph \( D_n \). The graph \( \bar{D}(\alpha) \) with the linear order of the vertices induced from \( D_n \) will be called the graph \( D(\alpha) \).

For an arbitrary ordered set \( M \), we denote its greatest element by \( \max(M) \). The greatest vertices of graphs and intervals will be denoted in the same way. We assume that the maximum of the empty set is always less than the maximum of a nonempty set.

**Definition 3.4.** A \( * \)-system (of occurrences) is a quadruple \( \langle V, b, \leq, p \rangle \), where

1. \( V \) is a finite set (of occurrences of letters);
2. \( b : V \rightarrow \bar{D}_n \) is a mapping;
3. \( \leq \) is a partial order on the set \( V \) such that any elements \( x, y \) of \( V \) are comparable if and only if \( b(x) = b(y) \).
(4) \( p \in V \) is a marked occurrence (of the letter \( b(p) \)) such that from \( x \in V \) and \( b(x) = b(p) \) it follows that \( x = p \).

In an arbitrary \( * \)-system \( (V, b, \preceq, p) \), for an arbitrary set of occurrences \( M \) we denote by \( \sum M \) the element of the group \( \mathbb{Z} \mathbb{II} \) obtained by summation of the letters \( b(x) \) over all \( x \in M \). If \( p \in M \), then the structure of a \( * \)-system is induced on the set \( M \). We shall call it the subsystem of the \( * \)-system \( \Omega \) and denote it by \( \Omega|_M \). A \( * \)-system will be called the \( * \)-system (of occurrences) of an element \( \alpha \in \mathbb{Z} \mathbb{II} \) if \( \sum V = \alpha \); it will be denoted by \( V^*(\alpha) \) or \( \dot{V}(\alpha) \).

In an arbitrary \( * \)-system \( (V, b, \preceq, p) \), we can consider the occurrences of any letter \( x \); these are elements of the set \( b^{-1}(x) \). If this is a \( * \)-system of a root, then any letter \( x \) has at most two occurrences. They may be called the left and right ones. The left occurrence of the letter \( x \) is the smallest element of the set \( b^{-1}(x) \) (under the order \( \preceq \)); the right occurrence is the greatest element. If the letter \( x \) has only one occurrence (i.e., if the set \( b^{-1}(x) \) consists of one element), then the right and left occurrences coincide.

If the set of vertices has a linear order and in the \( * \)-system \( V^*(\alpha) \) the marked occurrence is the occurrence of the leading vertex in the graph \( D(\alpha) \), then we shall talk about the system \( V(\alpha) \) of occurrences of the element \( \alpha \). Any system of occurrences can be viewed as a \( * \)-system.

An arbitrary word \( a \) can be regarded as a finite linearly ordered set \( V \) of occurrences of letters with the following order: an occurrence of \( x \) is less than an occurrence of \( y \) if the occurrence of \( x \) in the word \( a \) is placed to the left of the occurrence of \( y \). In this situation, there are a finite set of letters and a finite set of occurrences of these letters. The notion of the “occurrence of a letter” can be regarded as a mapping \( b \) of the set of occurrences to the set of letters. We shall distinguish different occurrences of one and the same letter with the help of a partial order \( \preceq \) on the set of occurrences under which occurrences are comparable if and only if they are occurrences of one and the same letter.

An arbitrary RR-word with one occurrence of its leading letter will be called an RR1-word. An RR2-word is an RR-word with two occurrences of its leading letter. Let \( a \) be an arbitrary RR1-word of composition \( \alpha \). If we mark a unique occurrence \( p \) of a certain letter, then the quadruple \( (V, b, \preceq, p) \) constructed above is a \( * \)-system of the root \( \alpha \). Moreover, if the marked occurrence \( p \) is the occurrence of the leading vertex in the graph \( D(\alpha) \), then \( (V, b, \preceq, p) \) is the system of occurrences of the root \( \alpha \). We call it the system of occurrences of letters in the RR-word \( a \). We may talk about subsystems of the \( * \)-system of occurrences of letters in an arbitrary RR-word.

We have introduced the graphs \( D_n, D(\alpha) \) and \( * \)-systems, because in some cases we have no need to know the order on the set of vertices: only a certain marked occurrence is of importance.

In a \( * \)-system of an arbitrary root \( \alpha \), the set of all left occurrences will be called the lower layer, and the complement of this set is the upper layer.

An arbitrary \( * \)-system of a root \( \alpha \) may be presented in the following way. We draw the graph \( D(\alpha) \) on a horizontal plane. On each of the single vertices of the graph \( D(\alpha) \), we place a checker piece, and on each double vertex we place a column of two pieces. The pieces on the bottom form the lower layer, and the remaining ones constitute the upper layer. The mapping “the occurrence of a letter” associates each piece with the vertex on which it is laid. We choose a column that consists of a single piece, and we mark this piece. If we mark the piece that is placed on the leading vertex of the graph \( D(\alpha) \), we obtain the system of occurrences of the root \( \alpha \).

**Definition 3.5.** For an arbitrary \( * \)-system of occurrences of a root \( \alpha \), the following properties uniquely determine a binary relation \( R \) on the set of occurrences.
Lemma 3.9. Let $\Omega$ be a nonempty open set. Then:

1. if $x, y$ are occurrences from the lower layer, then $(x, y) \in R$ is equivalent to the fact that $b(x) \in \{b(p), b(y)\}$ and the vertices $b(x)$ and $b(y)$ are adjacent;
2. if $x$ is an occurrence from the lower layer and $y$ is an occurrence from the upper layer, then $(x, y) \in R$ is equivalent to the fact that $b(y) \in [z^*, b(x)]$ and the vertices $b(x)$ and $b(y)$ are adjacent;
3. if $x$ and $y$ are occurrences from the upper layer, then $(x, y) \in R$ is equivalent to the fact that $b(x) \in [z^*, b(y)]$ and the vertices $b(x)$ and $b(y)$ are adjacent.

Consider the oriented graph the vertices of which are occurrences; we join occurrences $x$ and $y$ by an arrow (oriented edge) if $(x, y) \in R$. This graph of the $*$-system $\Omega$ will be called the occurrence graph of $\Omega$.

Remark 3.6. For an arbitrary root $\alpha$, the occurrence graph $\tilde{V}(\alpha)$ can be thought of in the following way. The mapping “occurrence of a letter”, denoted by $b$, gives rise to a one-to-one correspondence between the occurrences of the lower layer and all vertices of the graph $D(\alpha)$ and between the occurrences of the upper layer and the double vertices of the graph $\tilde{D}(\alpha)$. In both layers we draw the corresponding edges and orient them; in the lower layer we choose the direction away from the marked occurrence, and in the upper layer the edges are directed away from the occurrence of the vertex $z^*$. For each occurrence $x$ in the lower layer such that the vertex $b(x)$ is different from $z^*$ and has an adjacent double vertex, we draw the oriented edge $(x, y)$, where $y$ is an occurrence from the upper layer such that $b(y) \in [z^*, b(x)]$ and the vertices $b(x)$ and $b(y)$ are adjacent.

We obtain the graph $\tilde{V}(\alpha)$.

For the occurrence graph $\tilde{V}(\alpha)$ of an arbitrary root $\alpha$, we consider the family of all subsets $U$ of the set of occurrences that satisfy the following condition.

$T$. If $x \in U$ and $(x, y)$ is an edge of the graph $\tilde{V}(\alpha)$, then $y \in U$.

Clearly, the empty set and the entire set $V$ possess property $T$; the union and the intersection of subsets satisfying $T$ also possess this property. Since the set of occurrences is finite, on the set of occurrences there exists a structure of topological space in which the closed sets are sets with property $T$. This topology is said to be $p$-root. The topological space arising on the set of occurrences is called the $p$-space of occurrences and is denoted in the same way as the occurrence graph. In the $p$-space of occurrences, we denote the closure of an arbitrary set $M$ by $\tilde{M}$ and the closure of an arbitrary occurrence $x$ by $\tilde{x}$.

Remark 3.7. In the $p$-space of occurrences of an arbitrary root, a subset $U$ is open if and only if the following conditions are satisfied:

1. if the set $U$ is nonempty, then $p \in U$ (here $p$ is the marked occurrence);
2. if $x$ and $y$ are, respectively, the left and right occurrences of one and the same letter, then $y \in U$ implies $x \in U$;
3. if the set $U$ is nonempty, then the element $\sum U$ is a root.

Remark 3.8. In the $p$-space of occurrences of a root $\alpha$, the closed sets $M$ are characterized by the following properties:

1. if $M \neq V$, then $p \notin M$, or the closure $\tilde{p}$ of the marked occurrence $p$ coincides with $V$ (here $V$ is the set of occurrences);
2. if $x$ and $y$ are, respectively, the left and the right occurrences of one and the same letter, then $x \in M$ implies $y \in M$;
3. if $M \neq V$, then the element $\sum(V - M)$ is a root.

Lemma 3.9. Let $\Omega$ be the $p$-space of occurrences of an arbitrary root $\alpha$, and let $U$ be a nonempty open set. Then:

1. if $U' \subseteq U$, then the set $U'$ is open in $\Omega$ if and only if $U'$ is open in $\Omega|_U$;
2. if $M$ is a set closed in $\Omega$, then $U \cap M$ is closed in $\Omega|_U$;
3) if $P$ is a set closed in $\Omega|_U$, then the set $P \cup (V - U)$ is closed in $\Omega$ (here $V$ is the set of occurrences);

4) for $x \in U$ the set $x \cap U$ is the closure of the element $x$ in $\Omega|_U$.

Proof. Since $U$ is a nonempty open set, the element $\sum U$ is a root. Moreover, being a nonempty open set, $U$ contains the marked occurrence. Therefore, the restriction $\Omega|_U$ is a *-system (of the root $\sum U$). If $\Omega$ is a system of occurrences of the root $\alpha$, then $\Omega|_U$ is also a system of occurrences. Assertion 1) is easily verified by definition.

2) If a set $M$ is closed, then the set $V - M$ is open. Therefore, the set $(V - M) \cap U$ is open. For this reason, by property 1), the set $(V - M) \cap U$ is open in $\Omega|_U$. Therefore, its complement $U \cap M$ is closed in $\Omega|_U$.

3) The set $U - P$ is open in $\Omega|_U$. Hence, by property 1), the set $U - P$ is open in $\Omega$. Consequently, its complement $P \cup (V - U)$ in $V$ is closed in $\Omega$.

4) Denote by $x'$ the closure of the element $x$ in $\Omega|_U$. By 2), the set $x' \cap U$ is closed in $\Omega|_U$. Moreover, $x \in x' \cap U$. Therefore, $x' \subseteq x \cap U$.

By 3), the set $x' \cup (V - U)$ is closed. Moreover, $x \in x' \cup (V - U)$. Therefore, $x \subseteq x' \cup (V - U)$. Thus, $x \cap U \subseteq (x' \cup (V - U)) \cap U = (x' \cap U) \cup ((V - U) \cap U) = x' \cap U$. The inclusions $x' \subseteq x \cap U$, $x \cap U \subseteq x'$ imply that $x' = x \cap U$, as required.

Lemma 3.10. If $a$ is an arbitrary RR1-word and $c$ is a beginning of the word $a$, then the set $U$ of the occurrences in a that are occurrences in $c$ is open.

Proof. 1) Since the word $a$ is regular, it begins with the occurrence of its leading letter. Therefore, its beginning $c$, if nonempty, also starts with the occurrence of the same letter. Consequently, property 1) of Remark 3.12 is fulfilled for the set $U$.

2) Since $c$ is a beginning of the word $a$, property 2) is also fulfilled.

3) The beginning $c$ of the word $a$, if nonempty, starts with a unique occurrence of its leading letter. Therefore, it is regular. Consequently, being a subword of the RR-word $a$, it has rooted composition, i.e., property 3) is also fulfilled.

Corollary 3.11. If $x$ and $y$ are occurrences from the system of occurrences of an arbitrary RR1-word $a$ and $y \in x$, then the occurrence $y$ in $a$ is not placed in $a$ to the left of the occurrence $x$.

Proof. Let $c$ be the ending of the word $a$ that begins with the occurrence $x$. By Lemma 3.11, the subset of occurrences $M$ in the subword $c$ is closed. Therefore, $y \in x \subseteq M$, i.e., $y$ is the occurrence in the ending $c$.

Remark 3.12. For the $p$-space of occurrences of an arbitrary root, the property “a set $\{x\}$ is closed” is equivalent to the following:

1) the occurrence $x$ is right;
2) if $V \neq \{p\}$, then the element $\sum V - |b(x)|$ is a root (here $V$ is the set of occurrences and $p$ is the marked occurrence);
3) if $x = p$, then $V = \{p\}$.

Lemma 3.13. In the space of occurrences of an arbitrary root, every nonempty closed set $M$ has a closed one-element subset.

Proof. The set $U = V - M$ is open (here $V$ is the set of occurrences). Therefore, the element $\sum U$ is either zero or a positive root. From the structure of roots it is easily seen that there exists a simple root $\gamma$ such that $\sum U \leq \sum V - \gamma$; the element $\sum V - \gamma$ is either zero or a positive root. Let $x$ denote a right occurrence such that $|b(x)| = \gamma$. Then $x \notin U$, i.e., $\{x\} \subseteq M$. The set $V - \{x\}$ is open, i.e., the set $\{x\}$ is closed.

Now using Lemma 3.10 and Remark 3.12, we can obtain the following inductive (with respect to the number of elements) characterization of closed sets.
Corollary 3.14. In the $p$-space of occurrences $\Omega$ of an arbitrary root,  
1) the empty set of occurrences is closed;  
2) the closed singletons are described in Remark 3.12;  
3) if $V \neq \{p\}$, $\{x\}$ is a closed singleton, and $M$ is a set closed in the system of occurrences $\Omega|_{V - \{x\}}$, then the set $M \cup \{x\}$ is closed (here $V$ is the set of occurrences).

Remark 3.16 and the definition of closed sets imply the following statement.

Corollary 3.15. Let $\Omega = \langle V, b, \preceq, p \rangle$ be the $*$-system of occurrences of an arbitrary root $\alpha$, and let $x$ be a right occurrence. Then:
1) if $x = p$, then the closure $\overline{x}$ of the occurrence $x$ coincides with $V$;  
2) if $x \neq p$, the graph $D(\alpha)$ has double vertices, and $b(x) \in O$ or $x$ belongs to the upper layer, then $\overline{x} - \{x\}$ is the intersection of the upper layer and the set $b^{-1}(b(x), x_1^+(\alpha))$;  
3) if $x \neq p$, there are double vertices in the graph $D(\alpha)$, this graph contains at least five vertices, $b(p) \in O$, and $b(x) \in [x_1^+(\alpha), x_1^-(\alpha)]$;  
4) if $b(x) \neq b(p)$, the vertex $b(x)$ and the vertex of the interval $(b(x), b(p)]$ adjacent to $b(x)$ is single in the graph $D(\alpha)$, then $\overline{x}$ coincides with the set of occurrences $y$ such that $b(y) \in [b(y), b(p)]$.

Corollary 3.16. Let $\langle V, b, \preceq, p \rangle$ be the $*$-system of occurrences of an arbitrary root $\alpha$, and let $x$ be a right occurrence. Then the element $\sum x$ is a root and the vertex $b(x)$ has precisely one occurrence in the set $\overline{x}$. Therefore, the quadruple $(\overline{x}, b(x), \preceq|_{\sum x}, x)$ is the $*$-system of occurrences of the root $\sum x$, and it has the structures of a graph and of a topological space.

Lemma 3.17. Let $\Omega = \langle V, b, \preceq, p \rangle$ be the $*$-system of occurrences of an arbitrary root $\alpha$, let $p' \in V$ be a right occurrence, and let $M \subseteq V$ be a closed set. Then the element $\sum p'$ is a root, $\Omega' = \langle p', b, \preceq|_{\sum p'}, p' \rangle$ is the $*$-system of occurrences of the root $\sum p'$, and the set $M \cap \sum p'$ is closed in the space of occurrences $\Omega'$.

Proof. By Corollary 3.16 the element $\sum p'$ is a root, the vertex $b(p')$ in the graph $D(\sum p')$ is single, and $\Omega' = \langle p', b, \preceq|_{\sum p'}, p' \rangle$ is the $*$-system of occurrences of the root $\sum p'$. Therefore, the space $\Omega'$ of occurrences is well defined.

It suffices to consider the case where $M \subseteq \overline{p'}$ and to prove that if $M$ is closed in $\Omega$, then it is also closed in $\Omega'$. The proof proceeds by induction on the number $m$ of elements of the set $\overline{p'}$. The base of induction: for $m = 1$ the statement is obvious. The inductive step: consider $m \geq 2$ and assume that the statement is valid for $m - 1$. By Lemma 3.13 the set $M$ has a closed one-element subset $\{x\}$. Then the set $U = V - \{x\}$ is open. Moreover, it is not empty $(p' - \{x\} \subseteq U$ and the set $p' - \{x\}$ is not empty because $m \geq 2$). Therefore, $p \in U$ and $\Omega|_U$ is the system of occurrences of the root $\sum U$. By statement 2) of Lemma 3.13 the set $U \cap M$ is closed in $\Omega|_U$.

Since the set $\{x\}$ is closed and the closed set $\overline{p'}$ contains at least two elements $(m \geq 2)$, it follows that $x \neq p'$. Therefore, $p' \in V - \{x\} = U$. Consequently, by statement 4) of Lemma 3.13 the set $A = \overline{p'} \cap U$ is the closure of the occurrence $p'$ in $\Omega|_U$.

By Corollary 3.16 (for the closure $A$ in $\Omega|_U$ of the right occurrence $p'$), the element $\sum A$ is a root and $\Omega'|_{\sum A}$ is the system of occurrences of the root $\sum A$.

By the inductive hypothesis, since the set $U \cap M$ closed in $\Omega|_U$, it is closed in $\Omega'|_{\sum A}$.

Moreover, the element $\sum A = \sum (p' - \{x\})$ is a root and $x \neq p'$. Therefore, by the same remark, the set $\{x\}$ is closed in $\Omega'$. Consequently, the set $A = p' - \{x\}$ is open in $\Omega'$. Since, moreover, the set $U \cap M$ is closed in $\Omega'|_{\sum A}$, we see that the set $(U \cap M) \cup (\overline{p'} - A) = (U \cap M) \cup \{x\} = M$ is closed in $\Omega'$ by statement 3) of Lemma 3.13. \[\square\]
§4. RR-words of the Lie Algebra $D_n^+$

In the sequel we tacitly assume that the set of letters is linearly ordered. We introduce the following notation:

- $\mathcal{R}_1$ is the set of roots whose graph has the property that its leading vertex $p_1$ is single;
- $\mathcal{R}_{1,1}$ is the set of roots in $\mathcal{R}_1$ whose graph involves double vertices and either the second-by-priority vertex $p_2$ is double or $p_2 \in O(\alpha)$;
- $\mathcal{R}_{1,2}$ is the set of roots $\alpha$ in $\mathcal{R}_1$ such that the graph of $\alpha$ has double vertices and contains at least five vertices, and $p_1 \in O(\alpha)$, $p_2 = x_1^- (\alpha)$;
- $\mathcal{R}_{1,3}$ is the set of nonsimple roots $\alpha$ of $\mathcal{R}_1$ whose graph has the property that the vertex $p_2$ and the vertex of the interval $[p_1, p_2)$ that is adjacent to it are single.

Lemma 4.1. The set $\{\mathcal{R}_{1,1}, \mathcal{R}_{1,2}, \mathcal{R}_{1,3}\}$ is a partition of the set of all nonsimple roots in $\mathcal{R}_1$.

Proof. Clearly, $\mathcal{R}_{1,1}$, $\mathcal{R}_{1,2}$, and $\mathcal{R}_{1,3}$ are subsets of the set of nonsimple roots in $\mathcal{R}_1$. Let $p_1$ and $p_2$ denote the leading and the second-by-priority vertices of the graph $D(\alpha)$, respectively, and let $p'$ be the vertex of the interval $(p_2, p_1]$ adjacent to $p_2$.

If $\alpha \in \mathcal{R}_{1,2}$, then the vertex $p_2$ in the graph $D(\alpha)$ is single and $p_2 \notin O(\alpha)$, so that $\alpha \notin \mathcal{R}_{1,1}$. Consequently, the sets $\mathcal{R}_{1,1}$ and $\mathcal{R}_{1,2}$ do not intersect.

If $\alpha \in \mathcal{R}_{1,3}$, then the vertex $p_2$ in the graph $D(\alpha)$ is single and has a single adjacent vertex. Then $\alpha \notin \mathcal{R}_{1,1}$. Therefore, the sets $\mathcal{R}_{1,1}$ and $\mathcal{R}_{1,3}$ do not intersect.

Assume that there exists a nonsimple root $\alpha \in \mathcal{R}_1$ such that $\alpha \notin \mathcal{R}_{1,1} \cup \mathcal{R}_{1,2} \cup \mathcal{R}_{1,3}$. Since $\alpha \notin \mathcal{R}_{1,1}$, the vertex $p_2$ in $D(\alpha)$ is single and $p_2 \notin O(\alpha)$. Since $\alpha \notin \mathcal{R}_{1,3}$, the vertex $p'_2$ is double in $D(\alpha)$. Therefore, $p_2 = x_1^- (\alpha)$ and $p_1 \in O(\alpha)$, i.e., $\alpha \in \mathcal{R}_{1,2}$, a contradiction. Consequently, any nonsimple root in $\mathcal{R}_1$ is an element of the set $\mathcal{R}_{1,1} \cup \mathcal{R}_{1,2} \cup \mathcal{R}_{1,3}$. \[\square\]

Lemma 4.2. Let $u$ be an arbitrary RR1-word of length at least 2. Let $p_2$ be its second-by-priority letter, and let $v$ be the ending of the word $u$ that begins with the right occurrence of the letter $p_2$. Then

1) if $\alpha \in \mathcal{R}_{1,1}$, then $|v| = [p_2, x_1^+(\alpha)]$ (see [33] Lemma 2.5));
2) if $\alpha \in \mathcal{R}_{1,2}$, then the graph $D(\alpha)$ contains at least five vertices, $p_2 = x_1^- (\alpha)$, and $|v| = [x_2^+(\alpha), x_1^+(\alpha)]$ (see [33] Lemma 2.6));
3) if $\alpha \in \mathcal{R}_{1,3}$, then the graph $D(|v|)$ coincides with the connected component that contains the vertex $p_2$ of the graph obtained from $D(\alpha)$ by removing the edge $[p_2, p_2']$ (here $p'_2$ is the vertex of the interval $[p_1, p_2)$ that is adjacent to $p_2$); see [33] Lemma 2.4).

Corollary 4.3. An arbitrary RR1-word is determined by its composition, i.e., it is reduced.

For an arbitrary root $\alpha$, we denote by $\text{red}(\alpha)$ a unique reduced word of composition $\alpha$. In the sequel, we shall refer to Lemma 4.2 when talking about the structure of RR1-words. The description in Corollary 4.3 of the closure of an arbitrary right occurrence enables us to state results on the structure of RR1-words without searching through all cases.

Corollary 4.4. Let $\alpha$ be a positive root such that the leading vertex $p_1$ in its graph is single. Then an RR1-word $u$ of composition $\alpha$ is unique. Moreover, if the root $\alpha$ is not simple, then the reduced word of composition $\sum p'$ is an ending of the word $u$ (here $p'$ is the right occurrence of the second-by-priority letter of $u$, and $p'$ is the closure of the occurrence $p'$ in the space of occurrences of $u$).
**Lemma 4.5.** Let $a$ be an RR1-word, let $\Omega = \langle V, f, \preceq, p \rangle$ be its system of occurrences, and let $M$ be a set of occurrences closed in $\Omega$ and such that $M \neq V$. Then the word that is obtained from $a$ by elimination of all occurrences belonging to $M$ is an RR1-word.

**Proof.** The proof proceeds by induction on the length $n$ of $a$. The assertion for $n = 1$ is obvious. Let $n \geq 2$, and suppose the lemma is true for all $k < n$. Since $n \geq 2$, the word $a$ has length at least 2. Therefore, being an RR-word, it has occurrences of at least two letters. Consequently, by Corollary 4.4 $a = \text{red}(\sum U')\text{red}(p')$, where $p'$ is the right occurrence of the second-by-priority letter of the word $a$, $U' = V - p'$. We introduce the following notation: $U = V - M; b = \text{red}(\sum U'), c = \text{red}(p'); a', b', c'$ are the words obtained from $a, b, c$, respectively, by elimination of the occurrences belonging to $M$.

Then $a = bc$ and $a' = b'c'$.

The set $U' = V - p'$ is open. Moreover, it is not empty (because $U'$ contains the occurrence of the leading letter of $a$). Therefore, the element $\sum U'$ is a root and $\Omega[U']$ is a system of occurrences of the RR1-word $b$.

Since $M$ is closed and $U'$ is open, the set $U' \cap M$ is closed in $\Omega[U']$ by statement 2) of Lemma 4.3. Moreover, the length of $b$ is less than $n$. Therefore, by the inductive assumption, $b'$ is an RR1-word.

If $p' \subseteq M$, then $c' = 1$ and $a' = b'c' = b'$. Since $b'$ is an RR1-word, it remains to consider the case where $p' \nsubseteq M$, i.e., $p' \notin M$, or $p' \in U$.

By Lemma 3.17 the element $\sum p'$ is a root, and the set $M \cap p'$ is closed in the system of occurrences $\langle p', f|p', \preceq|p', p' \rangle$ of this root. Therefore, by the inductive assumption, $c'$ is an RR1-word. Its set of occurrences is $p' - M = p' \cap U$. Thus, $c' = \text{red}(\sum (p' \cap U))$.

The set of occurrences in the word $a'$ coincides with $U$. Now we show that the word $a'$ is reduced, i.e., $a' = \text{red}(\sum U')$.

The set $U = V - M$ is open. Moreover, it is not empty ($p' \in U$). Therefore, the element $\sum U$ is a root and $\Omega[U]$ is the system of occurrences of this root. Consequently, by statement 4) of Lemma 4.3 the closure of the element $p'$ in the system $\Omega[U]$ coincides with $p' \cap U$. Now Corollary 4.4 shows that the word $\text{red}(\sum p' \cap U) = c'$ is an ending of the word $\text{red}(\sum U)$.

Thus, the words $a' = b'c'$ and $\text{red}(\sum U)$ have one and the same composition and one and the same ending $c'$, and the word $b'$ (which can be obtained by elimination of the ending $c'$) is reduced. Therefore, $a' = \text{red}(\sum U)$. \hfill $\square$

**Corollary 4.6.** Let $a$ be an RR1-word, let $p_1$ be its first letter, and let $q \neq p_1$ be a letter occurring in the word $a$ and such that $|a| - |q|$ is a root. Then the word obtained from $a$ by elimination of the right occurrence of $q$ is an RR1-word.

**Lemma 4.7.** Let $a$ be an RR1-word, let $q \in M^+([a])$, and let $q$ be different from the first and the last letter of $a$. Then $q$ is less than the letter that occurs in $a$ immediately after the right occurrence of $q$.

**Proof.** We denote by $b$ the longest beginning of the word $a$ that does not contain the right occurrence (in the word $a$) of the letter $q$. Since $q$ is different from the first letter of the word $a$, the word $b$ is not empty. Since $q$ is different from the last letter of $a$, there exists a letter that occurs in $a$ immediately after the right occurrence of $q$. We denote this letter by $m$. By the choice of $m$, we have $m \neq q$.

Suppose $q > m$. Then the word $qm$ is regular, and moreover, it is a subword of the RR-word $a$. Therefore, the word $qm$ has rooted composition, i.e., the vertices $q$ and $m$ are adjacent.

Since $a$ is an RR1-word, it begins with a unique occurrence of the leading letter $p_1$. Moreover, since its beginning $b$ is not empty, it follows that $b$ also begins with a unique occurrence of the leading letter $p_1$ (in particular, $p_1 \leq |b|$) and $q < p_1, m < p_1$. 


Therefore, the word $bq$ is regular. Consequently, being a subword of the RR-word $a$, the word $bq$ has rooted composition, i.e., $m \in M^+(|bqm|)$.

Since $m \neq q$ and, by the choice of the word $b$, the vertex $q$ has the same multiplicity in the graphs $D(|a|)$ and $D(|bqm|)$. Moreover, $q \in M^+(|a|)$. Therefore, $q \in M^+(|bqm|)$.

For an arbitrary positive root $\beta$, the set $M^+(\beta)$ has adjacent vertices only in the case where the graph $D(\beta)$ contains precisely two vertices. Therefore, the graph $D(|bqm|)$ has two vertices. But this is impossible, because $p_1 \leq |b|$ and the vertices $p_1$, $q$, and $m$ are distinct.

Thus, $q \leq m$ and $q \neq m$. Therefore, $q < m$, as required. \hfill $\Box$

With the help of Lemma 4.7 it is easy to obtain the following assertion.

**Corollary 4.8.** For arbitrary RR1-words $a$ and $b$ with one and the same leading letter, if $|a| \leq |b|$, then $b \leq a$.

**Definition 4.9.** For an arbitrary vertex $p$ of the graph $D_n$, we introduce the following notation: $O_p$ is the set of vertices $x$ such that $\max[x, p] = p$; $\leq_p$ is a linear order on $O_p$ such that $x <_p y$ and the greatest vertex of the intervals $[x, p]$ and $[y, p]$ that belongs to only one of them lies in the interval $[y, p]$.

**Definition 4.10.** Let $\alpha \in \mathcal{R}_1$, let $\Omega = \langle V, b, \leq, p \rangle$ be the system of occurrences of the root $\alpha$, and let $V_1$ be the set of all occurrences from the lower layer. Then an arbitrary set $M \subseteq V_1$ is said to be $\leq_1$-closed if for arbitrary occurrences $x$ and $y$ from the lower layer, the relations $b(x) \leq_{b(p)} b(y)$ and $y \in M$ imply $x \in M$. The intersection of $\leq_1$-closed sets is $\leq_1$-closed. For $M \subseteq V_1$, the intersection of all $\leq_1$-closed sets that contain $M$ is called the $\leq_1$-closure of $M$.

**Definition 4.11.** For any system of occurrences $(V, b, \leq, p)$ of an arbitrary RR1-word $v$, we denote by $\leq_V$ the linear order on the set $V$ under which $x \leq_V y$ (for $x, y \in V$) is equivalent to the fact that the occurrence $x$ is not placed in the word $b$ to the right of the occurrence $y$.

If we know the composition of an RR1-word, then, with the help of Lemma 4.2 we can determine what letter stands at the end of this word.

This is done in the following lemma.

**Lemma 4.12.** Let $\alpha \in \mathcal{R}_1$, and let $p_1$ and $q$ be the first and the last letter, respectively, in the word $\text{red}(\alpha)$. Then:

1. If there are no double vertices in the graph $D(\alpha)$, then $q$ is the greatest vertex under the order $\leq_{p_1}$ among the end vertices of this graph.
2. If the graph $D(\alpha)$ has double vertices and the following conditions a)–c) are fulfilled, then $q = x^+_1(\alpha)$:
   a) in the graph $D(\alpha)$ the second-by-priority vertex $p_2$ is single and has an adjacent single vertex;
   b) $p_2 \in \{p_1, x^+_1(\alpha)\}$;
   c) if $p_1 \in O(\alpha)$ and $p_2 = x^-_1(\alpha)$, then $x^+_2(\alpha) < \max(p_2, x^+_1(\alpha))$.
3. If the graph $D(\alpha)$ has double vertices and at least one of the conditions a)–c) is not satisfied, then $q = x^+_2(\alpha)$.

Now we can describe the order $\leq_V$.

**Lemma 4.13.** Let $\alpha \in \mathcal{R}_1$, let $\Omega = \langle V, b, \leq, p \rangle$ be the system of occurrences of the root $\alpha$, and let $x, y \in V$.

1. If $x$ and $y$ are occurrences from the lower layer, then $x \leq_V y \iff b(x) \leq_{b(p)} b(y)$. 

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2. Let $U_x$ and $U_y$ be the smallest sets open in the $p$-root topology that satisfy the condition $x \in U_x$, $y \in U_y$. If one of the sets $U_x$ and $U_y$ is a subset of the other, then $x \leq_V y \iff U_x \subseteq U_y$.

3. If the sets $U_x$ and $U_y$ are not comparable with respect to inclusion and one of the occurrences $x$ and $y$ belongs to the upper layer and the other to the lower layer, then $b(x) \neq b(y)$, the vertices $b(x)$ and $b(y)$ are not adjacent, the graph $D(\beta)$ has double vertices and adjacent single vertices ($\beta = \sum U_x \cup U_y$), \{b(x), b(y)\} = $M(\beta) = \{x^2_1(\beta), x^1_1(\beta)\}$, $b(p) \notin \ M(\beta)$. Moreover, the property that an occurrence $x_1^1(\beta)$ from the upper layer is less under the order $\leq_V$ than an occurrence $y_1^1(\beta)$ from the lower layer is equivalent to the fact that the second-by-priority letter $p_2$ of $b(U_x \cup U_y)$ lies in the interval $[x_1^1(\beta), b(y_1^1(\beta))] \cap (b(p), b(y_1^1(\beta)))$, and if $p_2 = x_1^1(\beta)$, then $b(x_1^1(\beta)) < \max(x_1^1(\beta), b(y_1^1(\beta)))$.

Proof. 1. The set of occurrences $V_2$ from the upper layer is closed in the $p$-root topology. Consequently, the set $V_1$ in the lower layer is open in the $p$-root topology. Therefore, by Lemma 4.13, the word $d$ obtained from $\text{red}(\alpha)$ by elimination of all occurrences belonging to $V_2$ is an RR1-word and $\Omega|_{V_1}$ is its system of occurrences. Applying statement 1 of Lemma 4.12 to the word $d$, we obtain assertion 1.

2. Assertion 2 follows from Lemma 3.10.

3. We apply statements 2, 3 of Lemma 4.12 to the shortest beginning of the word $\text{red}(\alpha)$ that contains the occurrences $x$ and $y$; this yields assertion 3. \hfill $\Box$

Now, for an arbitrary RR1-word, we describe the set of occurrences of the beginnings of that word that end with an arbitrary occurrence $q$.

**Corollary 4.14.** Let $a$ be an RR1-word, let $\Omega = \langle V, b, \leq, p \rangle$ be the system of occurrences of the word $a$, and let $p_1 = b(p)$. Let $V_1 \subseteq V$ be the set of all occurrences from the lower layer, let $q \in V$, and let $M \subseteq V$ be the set of occurrences of the beginning of the word $\text{red}(a)$ that ends with the occurrence $q$.

1. If $q \in V_1$ and condition $A$ below is not satisfied for $q$, then $M = q$, where $q$ is the $\leq_1$-closure of the set \{q\}.

   **A.** $V \neq V_1$ and the set $V_1$ has at least five elements; $q \neq p_1$, $b(q) \notin O \cup [z^*, p_1]$, and $\max(o, p_1) < p_2$ for any element $o \in O$ such that $o \neq p_1$, (here $p_2 = \max(p_1, b(q))$); $z^* \neq p_2$. If the vertex $p_2$ is adjacent to $z^*$, then $z^* > \max(p_2, q)$.

2. If $q \in V_1$ and condition $A$ is fulfilled for $q$, then the set $M$ is open in the $p$-root topology and satisfies $\sum M = \sum q + [z^*, x]$, where $x$ is a double vertex of the interval $[z^*, p_2]$ that is most distant from $z^*$ and below $p_1 \notin [z^*, x]$, and if $x$ is adjacent to $p_2$, then $x > \max(p_2, b(q))$.

3. If $q \notin V_1$ and condition $B$ below is not satisfied for $q$, then $M = U_q \cup \overline{U_q \cap V_1}$, where $U_q \subseteq V$ is the smallest set among the open sets in the $p$-root topology for which $q \in U_q$; $U_q \cap V_1$ is the $\leq_1$-closure of the set $U_q \cap V_1$.

   **B.** $p_1 \in O$, the second-by-priority vertex $p_2$ from $b(U_q \cap V_1)$ is adjacent to $b(q)$, and $p_2 \notin [z^*, b(q)] \cup O$; moreover, $p_2$ has an adjacent single vertex in the graph $D(\{a\})$ and this vertex is less than $b(q)$.

4. If $q \notin V_1$ and condition $B$ is fulfilled for $q$, then $\sum M = [z^*, b(q)] + \sum O + [z^*, x]$, where $x$ is a vertex of the graph $D(\{a\})$ most distant from $z^*$ and such that $x \notin O \cup [z^*, p_2]$, $b(q) > \max(p_2, x)$.

Now, for an arbitrary RR1-word, we describe the set of occurrences of an arbitrary beginning of that word.

**Corollary 4.15.** Let $a$ be an RR1-word, let $\Omega = \langle V, b, \leq, p \rangle$ be the system of occurrences of the word $a$, let $V_1$ be the set of all occurrences from the lower layer, and let $p_1 = b(p)$,
\[ M \subseteq V. \] Then \( M \) is the set of occurrences of a beginning of the word \( a \) if and only if the following conditions are satisfied:

1. The set \( M \) is open in the \( p \)-root topology.
2. The set \( M \cap V_1 \) is \( \leq_1 \)-closed.
3. Suppose the graph \( D(|a|) \) has double vertices and contains at least five vertices and the set \( M \cap V_1 \) contains occurrences of the vertex \( z^* \) and of the three vertices adjacent to it. Let \( M \cap V_1 \) have at least five vertices, and let \( y \) be an occurrence from the lower layer such that the vertex \( b(y) \) does not coincide with and is not adjacent to \( z^* \). Moreover, let \( b(p) \in O \cup \{z^*, b(y)\} \) and let \( b(y) \) be greater than all elements of the set \( O \cup \{z^*, b(y)\} \), except for \( p_1 \). Let \( x \) be an occurrence from the upper layer such that \( b(x) \in [z^*, b(y)] \), \( p_1 \notin [z^*, b(x)] \), and the vertices \( b(x) \) and \( b(y) \) are not adjacent. Then \( y \in M \Rightarrow x \in M \).

Moreover, let \( p_1 \in O \), let \( y' \) be an occurrence from the lower layer such that \( b(y') \in [z^*, b(y')] \) and \( \max(b(y), b(y')) < b(y) \), and let \( x' \) be an occurrence from the upper layer such that \( b(x') \) is a vertex of the interval \([z^*, b(y)]\) adjacent to \( b(y) \) and such that \( b(x') < \max(b(y), b(y')) \). Then \( y' \in M \Rightarrow x' \in M \).

4. Suppose the graph \( D(|a|) \) have double vertices and contains at least five vertices. Let \( p_1 \in O \), let \( p_2 \) be a vertex of \( D(|a|) \) such that \( p_2 \notin O \cup \{z^*\} \) and \( p_2 \) is greater than all elements of the set \( O \cup \{z^*, p_2\} \) except for \( p_1 \). Let \( y \) be an occurrence from the upper layer such that \( b(y) \in [z^*, p_2] \) and the vertex \( b(y) \) is adjacent to \( p_2 \), and let \( x \) be an occurrence from the lower layer such that \( p_2 \in [z^*, b(x)] \) and \( b(y) > \max(p_2, b(x)) \). Then \( y \in M \Rightarrow x \in M \).

For the graph \( D_n, n \geq 5 \), we denote by \( o_1 \) the lesser vertex among the two end vertices adjacent to \( z^* \).

We introduce the following notation: \( Rt_2 \) is the set of roots whose graph is such that its leading vertex is double; \( Rt_{2,1} \) is the set of roots from \( Rt_2 \) whose graph has only one double vertex; \( Rt_{2,2} \) and \( Rt_{2,3} \) are the sets of roots \( \alpha \) from \( Rt_2 \) whose graph has at least two double vertices and \( \max(o_1, p_1) < \max(p_1, x_1^{-1}(\alpha)), \max(o_1, p_1) > \max(p_1, x_1^{-1}(\alpha)) \) (respectively). Note that \( \{Rt_{2,1}, Rt_{2,2}, Rt_{2,3}\} \) is a partition of the set \( Rt_2 \).

**Lemma 4.16.** Let \( u \) be an RR2-word of composition \( \alpha \), let \( p_1 \) be the leading letter of \( u \), and let \( u_2 \) be the ending of \( u \) that begins with the right occurrence of the letter \( p_1 \). Then one of the following statements is true:

1. \( \alpha \in Rt_{2,1}, p_1 = z^* \) and \( |u_2| = [z^*, k] \), where \( k \) is an end vertex of the graph \( D(\alpha) \) such that the interval \([z^*, k]\) contains the least vertex among those adjacent to \( z^* \) (see \[83\] Lemma 3.8));
2. \( \alpha \in Rt_{2,2} \) and \( |u_2| = [o_1, x_2^+(\alpha)] \), where \( o_1 \) is the lesser element among the two elements of the set \( O(\alpha) \) (see \[83\] Lemma 3.8));
3. \( \alpha \in Rt_{2,3} \) and \( |u_2| = (m, x_1^{-1}(\alpha)) \), where \( m \) is the vertex closest to \( p_1 \) among the vertices of the interval \([p_1, p_1] \) satisfying \( m > \max(p_1, x_1^{-1}(\alpha)) \) (see \[83\] Lemma 3.9)).

**Corollary 4.17.** An arbitrary RR-word is determined by its composition, i.e., it is reduced (see Corollary \[43\] and Lemma 4.16).

Referring to Lemma 4.16 we shall speak about the structure of RR2-words. Now we state these results without searching through the cases.

**Lemma 4.18.** Let \( \alpha \) be a positive root whose graph is such that its leading vertex \( p_1 \) is double. Then the RR2-word \( u \) of composition \( \alpha \) is unique. We introduce the following notation: \( u_2 \) is the ending of \( u \) that begins with the right occurrence of the letter \( p_1 \); \( m \) is the least vertex (in the order \( \leq_{p_1} \)) among the single vertices of the graph \( D(\alpha) \) that have a double adjacent vertex; \( k \) is an end vertex of the graph \( D(\alpha) \) such that \( m \in (p_1, k] \); \( v \) is the beginning of the word \( \text{red}([z^*, x_2^+(\alpha)] + [m, k]) \) that ends with the letter \( k \). Then \( |u_2| \) is the interval generated by the vertex \( x_2^+(\alpha) \) and the interval \(|v|\).
§5. Regular words and SR-words

For an arbitrary word $u$ of length at least 2, we introduce the following notation: $l(u)$ is the longest proper regular beginning of the word $u$; $r(u)$ is the longest proper regular ending of the word $u$; $l(u)$ is the ending of the word $u$ obtained by elimination of the beginning $l(u)$; $r(u)$ is the beginning of the word $u$ obtained by elimination of the ending $r(u)$.

Lemma 5.1. A nonempty word is regular if and only if any proper ending of it is less than the word itself.

Proof. Let $v$ be a regular word, and let $v = ab$, where $a$ and $b$ are nonempty. Then $ba < ab = v$. Assume that $v \leq b$. Then $ba < ab \leq b$. Therefore, $b$ is a beginning of the word $ab$, i.e., $ab = bc$ for some word $c$. Since $ba < ab = bc$, we have $a < c$.

Since $a \neq 1$, we have $c \neq 1$. Since the word $v$ is regular, it follows that $cb < bc = ab$ and $c < a$, a contradiction. Thus, $b < v$.

Conversely, let $v$ be a nonempty word any proper ending of which is less than $v$. Let $v = ab$, $a \neq 1 \neq b$. Then $b < ab$. Moreover, $ba \leq b$ (a nonempty beginning is not less than the word itself). Therefore, $ba \leq b < ab$, $ba < ab$. Since the representation $v = ab$ was chosen arbitrarily, the word $v$ is regular. \qed

Corollary 5.2. For any two regular words $u$ and $v$, the word $uv$ is regular if and only if $v < u$.

Proof. Consider arbitrary regular words $u$ and $v$. Since the regular words are nonempty, we have $u \neq 1 \neq v$. If the word $uv$ is regular, then $v < uv$ by Lemma 5.1. Moreover, $uv < u$, because a proper beginning of an arbitrary word is greater than the word itself. Therefore, $v < uv < u$.

Conversely, let $v < u$. By Lemma 5.1 it suffices to prove that any proper ending $b$ of the word $uv$ is less than the word itself.

Assume that $uv \leq v$. Then $uv \leq v < u$. Therefore, $u$ is a beginning of the word $v$, i.e., $v = uc$ for some $c$. Since $uc = v < u$, we have $c \neq 1$. Now, the inequality $uv \leq v$ yields $u^2c \leq uc$, $uc \leq c$. But the word $uc = v$ is regular. Consequently, by Lemma 5.1 $c < uc$, a contradiction. Thus, $v < uv$.

If $b$ is an ending of the word $v$, then $b \leq v$ by Lemma 5.1. Therefore, $b \leq v < uv$.

Now we may assume that $v$ is a proper ending of the word $b$, i.e., $u = ay$, $b = yv$, and $y \neq 1$. Since $b \neq uv$, we have $y \neq u$, i.e., $a \neq 1$. Therefore, $y$ is a proper ending of the regular word $u$ and, by Lemma 5.1, $y < u$. Thus, $b = yv < uv$. \qed

Lemma 5.3. The union of two intersecting regular subwords of an arbitrary word is a regular subword.

Proof. It suffices to show that if $ab$ and $bc$ are regular words and $b \neq 1$, then the word $abc$ is regular. We may assume that $a \neq 1$ and $c \neq 1$. By Lemma 5.1 it suffices to prove that $d < abc$ for an arbitrary proper ending $d$ of the word $abc$.

Since the word $ab$ is regular and $a \neq 1 \neq b$, we have $b < ab$ by Lemma 5.1. Therefore, $bc < abc$. If $d \neq 1$ is an ending of the word $bc$, then, again by Lemma 5.1 $d \leq bc < abc$.

It remains to consider the case where $bc$ is a proper ending of the word $d$, i.e., $d = ybc$, $a = xy$, $y \neq 1$. Since $d \neq abc$, we have $y \neq a$ and $x \neq 1$. Therefore, $yb$ is a proper ending of the regular word $ab$ and, by Lemma 5.1, $yb < ab$. Thus, $d = ybc < abc$. \qed

Corollary 5.4. Any nonempty word $u$ admits a unique representation as the product $u = u_1u_2 \cdots u_m$ of a nondecreasing sequence of regular words $u_1 \leq u_2 \leq \cdots \leq u_m$. 
Corollary 5.5. Let $u$ be a regular word different from a letter, and let $m$ be a positive integer such that the word $l(u)^m$ is a beginning of $u$. Then the ending of $u$ obtained by removing the beginning $l(u)^m$ is regular. In particular, the word $l(u)$ is regular.

Proof. Suppose the contrary. Let $c$ be the ending of the word $u$ obtained by removing the beginning $l(u)^m$, and let $a = u_1 \ldots u_k$ be the decomposition as in Corollary 5.4. Then $k > 1$.

If $u_1 > l(u)$, then $l(u)u_1$ is a regular proper beginning of $u$, which is longer than $l(u)$; this contradicts the definition of $l(u)$. Therefore, $u_1 \leq l(u)$. Consequently, $u = l(u)^m u_1 \ldots u_k$, $l(u) \leq u_1 \leq \cdots \leq u_m$, $m + k > 1$, which contradicts Corollary 5.4 (the word $u$ is regular). Therefore, the word $c$ is regular.

For arbitrary words $u$ and $v$, we shall write $v \prec u$ if $u = wy_1u_1$ and $v = wy_2v_1$ for some (possibly empty) words $w$, $u_1$, $v_1$ and letters $y_1$, $y_2$ such that $y_1 > y_2$. Moreover, if the word $v_1$ is empty, we shall write $v \sqsubset u$. Now we present a property of regular words that is most important in the present paper.

Corollary 5.6. Let $a$ be a regular word different from a letter. Then:

1) if $m$ is the greatest positive integer such that $l(a)^m$ is a beginning of $a$, and $c$ is an ending of $a$ such that $a = l(a)^mc$, then the word $c$ is regular and $c \sqsubset a$ (see 31 Lemma 5.1);

2) two regular words $b$ and $c$ and a positive integer $m$ such that $a = b^mc$ and $b$ is not a beginning of $c$ are uniquely determined by $a$.

For an arbitrary regular word $u$ different from a letter, we consider the representation $u = a^i u b$ as in Corollary 5.6. Here $i(u)$ is a positive integer and $a$, $b$ are regular words such that $b \sqsubset a$. Let $j(u)$ and $k(u)$ denote the number of occurrences of the leading letter of $u$ in the word $a$ (respectively, $b$). For arbitrary nonnegative integers $i$, $j$, $k$, we denote by $S(i, j, k)$ the set of SR-words $u$ such that $i = i(u)$, $j = j(u)$, $k = k(u)$.

Lemma 5.7. The set of SR-words splits into the sets $S(i, j, k)$, where 1) $i, j \in \{1, 2\}$; 2) $0 \leq k \leq j$; 3) for $i = 2$ and $j = 2$, only $k = 0$ is possible.

Proof. 1) For an arbitrary SR-word $u$, we consider the representation $u = a^i u b$, where $a$ and $b$ are regular words such that $b \sqsubset a$. Suppose $i \geq 3$. Then, by Corollary 5.2, the word $a^2b$ is regular. Moreover, $a^2b$ is a proper subword of $u$. Therefore, the word $a^2b$ has rooted composition. It follows that $k = 0$ (otherwise the word $a^2b$ has at least three occurrences of its leading letter).

Next, $k = 0$ and $b \sqsubset a$ imply that $b$ is a letter. But then in the graph $D([a^2b])$ there is precisely one single vertex ($b$), which is impossible in the root system of $D_m$ (the word $a^2b$ has rooted composition), a contradiction. Thus, $i \in \{1, 2\}$.

Now, $a$ and $b$ are proper subwords of $u$. Moreover, since $a$ and $b$ are regular, they have rooted composition. Therefore, $a$ has at most two occurrences of $p_1$. Moreover, $a$ has at least one occurrence of the leading letter $p_1$ of $u$ (the word $a$ begins with this letter). Consequently, $j \in \{1, 2\}$. Statement 1) is proved.
The word $b$ does not end with the letter $p_1$ (because it is regular). This fact and $b \sqsupset a$ imply statement 2).

3) For $i = 2$, the word $ab$ is a proper subword of the SR-word $a' b$. Moreover, $b < a$ because $b \sqsupset a$. Therefore, the word $ab$ is regular, and, being a regular proper subword of the SR-word $a' b$, the word $ab$ is an RR-word. Consequently, $ab$ has rooted composition. Therefore, the number of occurrences $j + k$ of the letter $p_1$ in the word $ab$ does not exceed 2. Hence, for $i = 2$ and $j = 2$, we have $k = 0$. 

Now it is easy to enumerate the triples $(i, j, k)$ from Lemma 5.7: $(1, 1, 0)$, $(1, 1, 1)$, $(1, 2, 0)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 0)$, $(2, 1, 1)$, and $(2, 2, 0)$. This allows us to make the statement of Lemma 5.7 more specific.

**Lemma 5.8.** The set of SR-words splits into the sets $S(1, 1, 0)$, $S(1, 1, 1)$, $S(1, 2, 0)$, $S(1, 2, 1)$, $S(2, 1, 0)$, $S(2, 1, 1)$, and $S(2, 2, 0)$.

Our purpose in this paper is to study each of the eight sets $S(i, j, k)$ separately.

**Corollary 5.9.** For an arbitrary regular word $x$, any proper regular subword of it is a subword either of $l(x)$ or of $r(x)$ (see Lemma 5.3).

**Corollary 5.10.** An arbitrary word $u$ is an RR-word if and only if it is regular, it has rooted composition, and $l(u)$ and $r(u)$ are RR-words (see Corollary 5.9).

**Corollary 5.11.** An arbitrary word $u$ is an SR-word if and only if it is regular, its composition is not rooted, and $l(u)$ and $r(u)$ are RR-words (see Corollary 5.9).

§6. The set $S(1, 1, 0)$

From Lemma 5.8 it is easily seen that $S(1, 1, 0)$ is precisely the set of SR-words with one occurrence of the leading letter. This fact and Corollary 5.11 imply the following statement.

**Corollary 6.1.** $S(1, 1, 0)$ is precisely the set of words $u$ such that

1) upon removing the last letter, we obtain an RR1-word;
2) the last letter of $u$ is less than its first letter;
3) the element $|u|$ is not a root;
4) the word $r(u)$ is reduced; in particular, the ending of $u$ that begins with the right occurrence of the second-by-priority letter of $u$ is an RR1-word.

By property 1) of Corollary 6.1 any word in $S(1, 1, 0)$ is uniquely represented in the form $\text{red}(\alpha)q$, where $\alpha$ is a positive root and $q$ is a letter. Next we analyze properties 3) and 4) in this corollary. For an arbitrary root $\alpha$, let $M^- (\alpha)$ denote the set of simple roots $\beta$ such that $\alpha + \beta$ is a root.

**Remark 6.2.** Let $(\alpha, q)$ be a pair in which $\alpha$ is a positive root and $q$ is a letter. Then $q \in M^- (\alpha)$ (i.e., the element $\alpha + |q|$ is a root) if and only if either 1) $q$ is not a vertex of the graph $D(\alpha)$ but has an adjacent vertex in it, or 2) $q$ is a single vertex of the graph $D(\alpha)$ and the total multiplicity of the vertices adjacent to $q$ is equal to 3.

**Lemma 6.3.** Let $v$ be an RR1-word, and let $q$ be a letter that is less than the first letter $p_1$ of $v$. Moreover, suppose that the ending $v_{\geq q}$ of the word $vq$ that begins with the right occurrence of the second-by-priority letter $p_2$ of this word is reduced. Then the property “the word $r(vq)$ has two occurrences of its leading letter” is equivalent to the fact that

1) $q < p_2$;
2) the vertex $p_2$ in the graph $D(\alpha)$ is double (here $\alpha = |v|$);
3) the graph $D(\alpha)$ has at least five vertices and $p_1 \notin O(\alpha)$;
4) the least of the vertices adjacent to $p_2$ is a vertex of the interval $[p_2, p_1]$, and it is a vertex of the graph $D(|v_{\geq q}|)$.

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Proof. Since \( q < p_1 \), it follows that the word \( vq \) has occurrences of at least two letters and the second-by-priority letter \( p_2 \) of the word \( vq \) exists and is the first and the leading letter of the word \( r(vq) \).

Assume that the word \( r(vq) \) has two occurrences of its leading letter, i.e., the word \( r(vq) \) has two occurrences of the letter \( p_2 \).

1. If \( q \geq p_2 \), then \( q = p_2 \) (because \( q < p_1 \)). In this case \( r(vq) = p_2 \), which contradicts the fact that the word \( r(vq) \) has two occurrences of the letter \( p_2 \).

2. Since the word \( r(vq) \) has two occurrences of the letter \( p_2 \), property 1) implies property 2).

3. Assume that \( p_1 \in O(\alpha) \). Introduce the following notation: \( v_2 \) is the ending of \( v \) that begins with the right occurrence of the letter \( p_2 \); \( v_1 \) is the longest subword of \( u \) that begins with the left occurrence of the letter \( p_2 \). Then the structure of RR-words shows that \( |v_2| = [p_2, x_2^+] \) and \( |v_1| = I \), where \( x_2^+ = x_2^+(\alpha) \) and \( I \) is an interval; if \( p_2 = z^* \), then \( I \) coincides with the interval of all vertices of \( D(\alpha) \) except for \( p_1 \), and if \( p_2 \neq z^* \), then \( I = [x, k] \), where \( k \) is an end vertex of \( D(\alpha) \) such that \( p_2 \in [z^*, k] \); \( x \) is the vertex of the interval \( [o, p_2] \) adjacent to \( p_2 \); \( o \) is an end vertex of \( D(\alpha) \) such that \( o \notin \{p_1, k\} \).

In any case, the interval \( I \) contains all vertices of \( [p_2, x_2^+] \), as well as all vertices of \( D(\alpha) \) that have an adjacent vertex in \( [p_2, x_2^+] \) (except for the vertex \( p_1 \)).

The word \( v_2q \) is the ending of \( vq \) that begins with the right occurrence of the letter \( p_2 \). Thus, by assumption, the word \( v_2q \) is reduced. Therefore, the vertex \( q \) does not lie in the interval \( [p_2, x_2^+] \) but has an adjacent vertex in it. The proved above shows that \( |v_2q| \leq |v_1| \).

The word \( v_1 \) begins with the only occurrence of its leading letter. Therefore, it is a regular RR-word (as a regular proper subword of the RR-word \( v \)). Moreover, \( v_2q \) is also a regular word and \( |v_2q| \leq |v_1| \). Therefore, \( v_1 \leq v_2q \) by Corollary \( \text{(4.8)} \). Thus, the word \( v_1v_2q \) is not regular.

But the regular word \( r(vq) \) is the ending of \( vq \) that begins with the left occurrence of the letter \( p_2 \), i.e., \( r(vq) = v_1v_2q \). This contradiction implies that \( p_1 \notin O(\alpha) \). Moreover, the graph \( D(\alpha) \) has double vertices and its vertex \( p_1 \) is single. Therefore, the graph \( D(\alpha) \) has at least five vertices.

4. Since \( p_1 \notin O(\alpha) \), the structure of RR1-words shows that \( |v_2| = [p_2, x_2^+] \). The vertices of \( D(|v_1|) \) are the vertices \( t \) of \( D(\alpha) \) such that \( p_2 \in [t, p_1] \); moreover, the vertex \( p_2 \) in \( D(|v_1|) \) is single, but the other vertices of \( D(|v_1|) \) have the same multiplicity in it as in the graph \( D(\alpha) \). We denote by \( m' \) the vertex of the interval \( [p_2, p_1] \) that is adjacent to \( p_2 \).

If \( m' \) is not a vertex of the graph \( D(|v_2q|) \), then \( |v_2q| \leq |v_1| \). In this case, \( v_1 \leq v_2q \) by Corollary \( \text{(4.8)} \). Then the word \( v_1v_2q \) is not regular. But the word \( v_1v_2q = r(vq) \) is regular. Thus, \( m' \) is a vertex of the graph \( D(|v_2q|) \). Therefore, \( m' \) is the second letter of the word \( v_2q \).

Denote by \( m \) the second letter of the word \( v_1 \). Then the word \( p_2m \) is a beginning of \( v_1 \). Being a regular beginning of the RR-word \( v_1 \), the word \( p_2m \) has rooted composition, i.e., the vertices \( p_2 \) and \( m \) are adjacent.

From the description of the graph \( D(|v_1|) \) given above, it follows that \( p_2 \in [m, p_1] \). Therefore, \( m \notin [p_2, p_1] \). Also, \( m' \notin [p_2, p_1] \). Therefore, \( m 
eq m' \). Since the word \( r(vq) = v_1v_2q \) is regular, we have \( v_1 > v_2q \) by Corollary \( \text{(5.2)} \). Since in addition \( m \) (\( m' \)) is the second letter of the word \( v_1 \) (respectively, \( v_2q \)) and \( m 
eq m' \), we have \( m > m' \).

The graph \( D(|v_1|) \) contains any vertex adjacent to \( p_2 \) and different from \( m' \). Therefore, the second letter of the word \( v_2 \) is the least of the vertices adjacent to \( p_2 \) and different from \( m' \). This fact and \( m > m' \) imply that \( m' \) is the least of the vertices adjacent to \( p_2 \).

Conversely, suppose all properties 1)–4) are fulfilled. Then the words \( v_1 \) and \( v_2 \) are
well defined and, as in the proof of property 4), we have $v_1 > v_2q$, i.e., the word $v_1v_2q$ is regular. Thus, $r(vq) = v_1v_2q$.

**Lemma 6.4.** If under the assumptions of Lemma 6.3 properties 1)–4) are satisfied for the word $vq$, then either $vq$ has rooted composition, or the composition of the word $r(vq)$ is not rooted.

**Proof.** Indeed, since $q < p_2$, it follows that $p_2$ is the second-by-priority vertex in the graph $D(\alpha)$. Therefore, the word $v_2$ is not empty and begins with the letter $p_2$. Since the word $v_2q$ is reduced, it has rooted composition, i.e., the element $|v_2| + q$ is a root. Therefore, the graph $D(|v_2| + q)$ is connected. Consequently, either $q$ is a vertex of the graph $D(|v_2|)$, or it has an adjacent vertex in it. Therefore, either $q$ is a vertex of the graph $D(|v|)$, or it has an adjacent vertex in it.

Therefore, if $q$ is not a vertex of $D(|v|)$, then $q$ has an adjacent vertex in this graph. In this case, the word $vq$ has rooted composition.

It remains to consider the case where $q$ is a vertex of $D(|v|)$. Since $v$ is an RR1-word and the vertex $p_2$ in the graph $D(|v|)$ is double, the structure of RR1-words shows that $|v_2| = [p_2, x_q^+(|v|)]$. Moreover, $q$ is either a vertex of the graph $D(|v_2|)$ or has an adjacent vertex in it. If $q$ is vertex of $D([p_2, x_q^+(|v|)])$, then the composition of $vq$ is not rooted, which is not true. Thus, $q$ is not a vertex of the graph $D([p_2, x_q^+(|v|)])$ but has an adjacent vertex in it.

If $q \in [p_2, p_1]$, then $q = x_1^-(|v|)$ and the word $vq$ has rooted composition. If $q \notin [p_2, p_1]$, then, using the structure of RR1-words, we compute the ending of $v$ that begins with the left occurrence of the letter $p_2$. As a result, we conclude that if $p_2 \neq z^*$, then $q$ has three occurrences in the word $vq$; if $p_2 = z^*$, then $q \in O$ and the letter $q$ has two occurrences in the word $vq$. In both cases, the composition of the word $r(vq)$ is not rooted.

Now we can make the description of the words in $S(1, 1, 0)$ given in Corollary 6.1 more precise.

**Corollary 6.5.** $S(1, 1, 0)$ is precisely the set of words $vq$, where $v$ is an RR1-word and $q$ is a letter that is less than the first letter of $v$. Moreover,

1) the ending of $vq$ that begins with the right occurrence of the second-by-priority letter of this word is an RR1-word;
2) the element $|v| + q$ is not a root;
3) properties 1)–4) in Lemma 6.3 cannot be satisfied simultaneously.

**Remark 6.6.** An arbitrary word $w$ is an RR1-word if and only if either it is a letter, or

1) the word $\bar{w}$ obtained from $w$ by removing the last letter $q$ is an RR1-word;
2) $q \in M^-(|w|)$ and $q$ is less than the leading letter of $\bar{w}$;
3) a reduced word with composition $|w|$ ends with the letter $q$.

**Theorem 6.7.** The following statements I and II are valid.

I. $S(1, 1, 0)$ is precisely the set of words $vq$, where $v$ is an RR1-word and $q$ is a letter that is less than the first letter $p_1$ of $v$; moreover, one of the following alternative cases 1 and 2 occurs.

1. $|q| \leq \alpha$ (here $\alpha = |v|$). In this case, the graph $D(\alpha)$ contains no vertices adjacent to $q$, and $q$ is greater than any vertex of $D(\alpha)$ except for $p_1$.
2. $q$ is a vertex of $D(\alpha)$. In this case, the root $\alpha$ is not simple and one of the following alternative cases 2.1 or 2.2 occurs.

2.1. $q$ coincides with the second-by-priority vertex $p_2$ of $D(\alpha)$. In this case, if the vertex $q$ in $D(\alpha)$ is single, then in this graph the total multiplicity of the vertices adjacent to $q$ is not equal to 3.
2.2. $q < p_2$. In this case,
Corollary 6.5, we show that i.e., the element by removing the last letter. Then

\[ x^2 \in \text{D} \]

addition, 0  

\[ \nu \]

Therefore, property 2) of Corollary 6.5 is fulfilled.

\[ \rho \]

fails. Therefore, property 3) of Corollary 6.5 holds. By Corollary 6.5, \( v \in S(1, 1, 0) \).

II. Under the conditions of case 2.2, one of the following mutually excluding cases 2.2.1, 2.2.2, and 2.2.3 occurs.

2.2.1. \( \alpha \in \text{Rt}_{1,1} \). In this case, properties a)–c) in case 2.2 are equivalent to the following properties 1)–3):

1) \( q \notin [p_2, x^2_1(\alpha)] \), but \( q \) has an adjacent vertex in the interval \([p_2, x^2_1(\alpha)]\);

2) it is impossible that the vertex \( q \) in the graph \( \text{D(\alpha)} \) is single and simultaneously has an adjacent single vertex;

3) \( q \) is the greatest vertex (in the order \( \leq p_2 \)) among the end vertices of the graph \([p_2, x^2_1(\alpha)] \cup \{q\}\).

2.2.2. \( \alpha \in \text{Rt}_{1,2} \). In this case, the graph \( \text{D(\alpha)} \) has at least five vertices and properties a)–d) of case 2.2 are equivalent to the fact that \( (q, x^2_1(\alpha)) = [x^2_2(\alpha), x^2_1(\alpha)] \),  

\[ \max(q, x^2_1(\alpha)) > \max(x^2_2(\alpha), x^2_1(\alpha)). \]

2.2.3. \( \alpha \in \text{Rt}_{1,3} \). In this case, properties a)–d) of case 2.2 are equivalent to the following properties 1)–3):

1) the vertices \( p_1 \) and \( p_2 \) are not adjacent and \( q \) is the vertex of the interval \((p_2, p_1)\) adjacent to \( p_2 \);

2) the vertex \( q' \) of the interval \((q, p_1)\) adjacent to \( q \) is single in the graph \( \text{D(\alpha)} \). In \( \text{D(\alpha)} \), there are precisely two vertices adjacent to \( q' \); these are \( p_2 \) and \( q' \);

3) \( q \) is greater than any vertex \( x \) of \( \text{D(\alpha)} \) such that \( p_2 \in (x, q) \).

Proof. I. Consider an arbitrary word \( u \) in \( S(1, 1, 0) \). We introduce the following notation: \( p_1 \) and \( q \) are the first and the last letters, respectively, of \( u; v \) is the word obtained from \( u \) by removing the last letter. Then \( u = vq; \) moreover, by Corollary 6.5, \( v \) is an RR1-word and \( q < p_1 \). Therefore, the element \( \alpha = |v| \) is a root and \( p_1 \) is the leading vertex of the graph \( \text{D(\alpha)} \).

1. Consider the case where \( |q| \leq \alpha \). Then \( q \) has no adjacent vertices in \( \text{D(\alpha)} \); otherwise the element \( \alpha + |q| \) is a root.

Assume that in the word \( v \) there is a letter that is less than \( p_1 \) but greater than \( q \). Then \( v \) has occurrences of at least two letters and its second-by-priority letter \( p_2 \) is greater than \( q \). Denote by \( v_2 \) the ending of \( v \) that begins with the right occurrence of \( p_2 \). Then \( v_2q \) is a regular proper subword of the SR-word \( u \), and thus it has rooted composition, i.e., the element \([v_2q]\) is a root. Therefore, the graph \( \text{D([v_2q])} \) is connected. Since, in addition, \( 0 \neq [v_2q] \leq |v| \), the graph \( \text{D([vq])} \) is also connected. But since \( q \) does not have adjacent vertices in the graph \( \text{D([v])} \), this is possible only if \( v = q \). But the word \( v \) is not a letter. Thus, the vertex \( q \) is greater than any vertex of \( \text{D(\alpha)} \) except for \( p_1 \).

Conversely, consider a pair \((v, q)\) (where \( v \) is an RR1-word and \( q \) is a letter that is less than the last letter \( p_1 \) of \( v \)) for which the assertions of case 1 are true. With the help of Corollary 6.5, we show that \( vq \in S(1, 1, 0) \).

Since \( q \) is greater than any vertex of \( \text{D(\alpha)} \) except for \( p_1 \) (here \( \alpha = |v| \)), we have \( r(vq) = q \). Therefore, property 1) of Corollary 6.5 is fulfilled.

Since \( \text{D(\alpha)} \) contains no vertices adjacent to \( q \), the graph \( \text{D(\alpha + |q|)} \) is not connected. Therefore, property 2) of Corollary 6.5 is fulfilled.

Since \( q \) is greater than any vertex of \( \text{D(\alpha)} \) except for \( p_1 \), property 1) of Lemma 6.3 fails. Therefore, property 3) of Corollary 6.5 holds. By Corollary 6.5, \( vq \in S(1, 1, 0) \).
Let $q$ be a vertex of $D(\alpha)$. Then the root $\alpha$ is not simple; otherwise $v = q$ and $vq = q^2$, but the word $vq$ is regular.

Denote by $p_2$ the second-by-priority vertex of $D(\alpha)$. Then $q \leq p_2$, because $q$ is a vertex of $D(\alpha)$ and $q < p_1$. Therefore, either $q = p_2$, or $q < p_2$.

2.1. Consider the case where $q = p_2$. Since $vq$ is an SR-word, its composition $|v| + |q|$ is not rooted, i.e., $q \notin M^-(\alpha)$. From Remark 6.2 we see that if the vertex $q$ in $D(\alpha)$ is single, then in this graph the total multiplicity of the vertices adjacent to $q$ is not equal to 3.

Conversely, consider a pair $(v, q)$ (where $v$ is an RR1-word and $q$ is a letter that is less than the first letter $p_1$ of $v$) for which the assertions of case 2.1 are valid. With the help of Corollary 6.5 we show that $vq \in S(1,1,0)$. Since $q = p_2$, we have $r(vq) = q$. Therefore, property 1) of Corollary 6.5 is fulfilled.

Thus, $q$ is a vertex of $D(\alpha)$, and if $2q \leq \alpha$, then, in the graph $D(\alpha)$, the total multiplicity of the vertices adjacent to $q$ is not equal to 3. Therefore, by Remark 6.3 we obtain $q \notin M^-(\alpha)$, i.e., property 2) of Corollary 6.5 holds. Since $q = p_2$, property 1) of Lemma 6.3 is not fulfilled. Therefore, property 3) of Corollary 6.5 also holds. By Corollary 6.5, $vq \in S(1,1,0)$.

2.2. Consider the case where $q < p_2$. Denote by $v_2$ the ending of the word $v$ that begins with the right occurrence of the letter $p_2$. Then $v_2q$ is a regular proper subword of the SR-word $vq$, and so it is an RR-word. Moreover, it has a unique occurrence of its leading letter $p_2$. Therefore, $v_2q$ is an RR1-word. In particular, it has rooted composition, i.e., the element $|v_2| + |q|$ is a root, or $q \in M^-(|v_2|)$. Therefore, the graph $D(|v_2q|)$ is connected.

a) Assume that $|q| \leq |v_2|$. Since $q \in M^-(|v_2|)$, it follows that the vertex $q$ in the graph $D(|v_2|)$ is single and the total multiplicity in this graph of the vertices adjacent to $q$ is equal to 3, i.e., either in the graph $D(|v_2|)$ there are no double vertices, this graph contains $z^*$ and the three vertices adjacent to $z^*$, and $q = z^*$, or the vertex $q$ in $D(|v_2|)$ has an adjacent double vertex and an adjacent single vertex. In both cases, the graph $D(|v_2q|)$ is not an interval.

Only for $\alpha \in \mathbb{Rt}_{1,3}$ the graph $D(|v_2|)$ is not an interval. Therefore $\alpha \in \mathbb{Rt}_{1,3}$. By Lemma 6.2 for any vertex of $D(|v_2|)$ (different from $p_2$) the total multiplicity of the vertices adjacent to it is the same as in the graph $D(\alpha)$. Since $q < p_2$ and, by assumption, $q$ is a vertex of $D(|v_2|)$, we conclude that in the graph $D(\alpha)$, as in the graph $D(|v_2|)$, the total multiplicity of the vertices adjacent to $q$ is equal to 3, i.e., $q \in M^-(\alpha)$, which is impossible, because the composition of the word $vq$ as an SR-word is not rooted.

Thus, $|q| \leq |v_2|$. So, $q \in M^-(|v_2|)$, $q$ has an adjacent vertex in the graph $D(|v_2|)$. Therefore, property a) is fulfilled.

b) $q \notin M^-(\alpha)$, because the composition of the SR-word $vq$ is not rooted.

c) The word $v_2q$, being a proper subword of the SR-word $vq$, is reduced. It has the composition $|v_2| + |q|$. Hence, we obtain property c).

d) Since $vq \in S(1,1,0)$, it follows that, by Corollary 6.5 properties 1)–4) of Lemma 6.3 cannot be fulfilled simultaneously. Since $q < p_2$, property 1) of Lemma 6.3 is fulfilled. Therefore, properties 2)–4) of Lemma 6.3 cannot hold simultaneously, i.e., if the vertex $p_2$ in the graph $D(\alpha)$ is double, $p_1 \notin O(\alpha)$, and the least of the vertices adjacent to $p_2$ is the vertex $p'_2$ of the interval $[p_2, p_1]$ then $|p'_2| \leq |v_2q|$.

Consider the following case: the vertex $p_2$ in the graph $D(\alpha)$ is double, $p_1 \notin O(\alpha)$, and the least of the vertices adjacent to $p_2$ is the vertex $p'_2$ of the interval $[p_2, p_1]$. In this case, $v_2 = [p_2, x^+_{2}(\alpha), x^{-}_{1}(\alpha) \in (p_2, p_1), p'_2 \in (p_2, x^{-}_{1}(\alpha))]$.

If $p_2 \neq x^-_{2}(\alpha)$, then $p_1 \notin O(\alpha)$ implies that $p'_2 \leq [p_2, x^+_{2}(\alpha)] \leq |v_2q|$, which contradicts
Since the vertex \(D\) of the interval \([vq, q\alpha]\), let \(q\) be a vertex of the graph \(D(\alpha)\) (here \(\alpha = |v|\)), let \(q\) be less than the second-by-priority letter \(p_2\) of \(v\), and suppose that properties a)–d) of case 2.2 occur. We use Corollary 6.5 to show that \(vq \in S(1, 1, 0)\).

Since the ending \(vq\) of \(v\) that begins with the right occurrence of the letter \(p_2\) is an RR1-word, property 1) of Remark 6.6 is fulfilled for the word \(vq\). From property a) and the inequality \(q < p_2\), we obtain property 2) of Remark 6.6 for \(vq\). Therefore, by Remark 6.6 \(vq\) is an RR1-word, i.e., property 1) of Corollary 6.5 is fulfilled for the word \(vq\). Property b) implies property 2) of Corollary 6.5.

If the vertex \(p_2\) in the graph \(D(\alpha)\) is double, \(p_1 \notin O(\alpha)\), and the least of the vertices adjacent to \(p_2\) is the vertex \(p_1\) of the interval \([p_2, p_1]\), then property d) implies that \(|p_2^2| = |x_1(\alpha)| \leq |p_2| + |q| = |vq|\), and properties 2)–4) of Lemma 6.3 cannot be fulfilled simultaneously, and thus, property 3) in Corollary 6.5 occurs. Therefore, by Corollary 6.5 \(vq \in S(1, 1, 0)\).

If the root \(\alpha\) is not simple in case 2.2, Lemma 4.2 shows that \(\alpha \in Rt_{1, 1} \cup \alpha \in Rt_{1, 2} \cup \alpha \in Rt_{1, 3}\).

2.2.1. In the case where \(\alpha \in Rt_{1, 1}\), from Lemma 4.2 we have \(|vq| = |p_2, x_1(\alpha)|\), whence we conclude that property a) of case 2.2 is equivalent to property 1).

We show that if property 1) occurs, then properties b) and 2) are equivalent. If \(2|q| \leq \alpha\), then properties b) and 2) are fulfilled. Moreover, \(|q| \leq \alpha\). Therefore, it suffices to consider the case where \(q\) is a single vertex of \(D(\alpha)\).

Property 1) implies that the vertex \(q\) of \(D(\alpha)\) has an adjacent double vertex in \(D(\alpha)\). Therefore, in \(D(\alpha)\), the total multiplicity of the vertices adjacent to \(q\) is either 2 or 3. Since the vertex \(q\) in the graph \(D(\alpha)\) is single, the condition \(q \notin M(\alpha)\) is equivalent to the fact that the total multiplicity of the vertices adjacent to \(q\) is not equal to 3. This implies that if property 1) holds, then properties b) and 2) are equivalent.

Since \(\alpha \in Rt_{1, 1}\), property 1) implies that the graph \(D(|vq| + |q|)\) contains no double vertices. This fact and Lemma 4.2 imply that if property 1) holds, then properties c) and 3) are equivalent.

2.2.2. Let \(\alpha \in Rt_{1, 2}\). Then the graph \(D(\alpha)\) has at least five vertices, \(p_2 = x_1(\alpha)\), and Lemma 4.2 implies that \(|vq| = |x_2(\alpha), x_1(\alpha)|\).

Assume that conditions a)–d) of case 2.2 are satisfied. Then from condition a) we obtain \((q, x_1(\alpha)) = |x_2(\alpha), x_1(\alpha)|\). Moreover, \(p_2 = x_1(\alpha)\) is the greatest vertex of the interval \([vq, q]\). The fact and property c) imply that \(\max\{q, x_1(\alpha)\} > \max(x_1(\alpha), x_1(\alpha))\).

Conversely, let \((q, x_1(\alpha)) = |x_2(\alpha), x_1(\alpha)|\) and \(\max\{q, x_2(\alpha)\} > \max(x_1(\alpha), x_1(\alpha))\). Then, from the relation \((q, x_1(\alpha)) = |x_2(\alpha), x_1(\alpha)|\) we obtain properties a) and b); the inequality \(\max\{q, x_2(\alpha)\} > \max(x_1(\alpha), x_1(\alpha))\) implies property c). Property d) is fulfilled, because the vertex \(p_2\) in the graph \(D(\alpha)\) is not double.

2.2.3. Consider the case where \(\alpha \in Rt_{1, 3}\). Assume that conditions a)–d) of case 2.2 are satisfied. Since \(q\) is a vertex of \(D(\alpha)\) but not of \(D(|vq|)\), and it has an adjacent vertex in \(D(|vq|)\), the relation \(\alpha \in Rt_{1, 3}\) and Lemma 4.2 show that \(q\) coincides with the vertex of the interval \((p_2, p_1)\) adjacent to \(p_2\). Since \(q < p_1\), property 1) holds.

Since \(q \in (p_1, p_2)\), the vertex \(q\) has exactly two adjacent vertices in \([p_1, p_2]\). Moreover, \([p_1, p_2] \leq \alpha\). Therefore, \(q\) has at least two adjacent vertices in the graph \(D(\alpha)\) and the total multiplicity of the vertices adjacent to \(q\) is at least 2.

The vertex \(q\) is single in the graph \(D(\alpha)\), because \(q\) coincides with the vertex of the interval \([p_2, p_1]\) that is adjacent to \(p_2\), and \(\alpha \in Rt_{1, 3}\). Therefore, in the graph \(D(\alpha)\) the total multiplicity of the vertices adjacent to \(q\) does not exceed 3. Consequently,
from property b) it follows that, in the graph $D(\alpha)$, the total multiplicity of the vertices adjacent to $q$ is equal to 2. Since $q \in (p_2, p_1)$ and $|p_2, p_1| \leq \alpha$, property 2) holds.

Since $\alpha \in \text{Rt}_{1,3}$ and $q$ coincides with the vertex of the interval $[p_2, p_1]$ adjacent to $p_2$, Lemma 4.2 shows that the vertices of the graph $D(\{v_2\})$ are precisely $p_2$ and the vertices $x$ of $D(\alpha)$ that satisfy the condition $p_2 \in (x, q)$. Moreover, in $D(\{v_2q\})$, $q$ is an end vertex, and the leading vertex $(p_2)$ of this graph is single and adjacent to $q$. This fact and property c) imply property 3).

Conversely, assume that conditions 1)–3) are satisfied. Since $\alpha \in \text{Rt}_{1,3}$, Lemma 4.2 and property 1) show that the vertices of the graph $D(\{v_2\})$ are precisely $p_2$ and the vertices $x$ of the graph $D(\alpha)$ that satisfy the condition $p_2 \in (x, q)$. Moreover, the vertex $q$ is adjacent to $p_2$. This implies property a).

Since $\alpha \in \text{Rt}_{1,3}$, we have $2|p_2| \leq \alpha$. Therefore, property 2) implies that the total multiplicity of the vertices in $D(\alpha)$ that are adjacent to $q$ is equal to 2. Since $\alpha \in \text{Rt}_{1,3}$, property 1) shows that the vertex $q$ is single. This yields property b). Since the vertices of the graph $D(\{v_2\})$ are precisely $p_2$ and the vertices $x$ of the graph $D(\alpha)$ that satisfy $p_2 \in (x, q)$, property 3) implies property c). Property d) is fulfilled because $2|p_2| \leq \alpha$. $

\S 7. The set $S(1, 2, 0)$

Remark 7.1. By definition, $S(1, 2, 0)$ is precisely the set of SR-words $u = vw$, where $v$ and $w$ are regular words, $w \sqsubset v$, and the word $v$ has two occurrences of the leading letter of the word $u$, while $w$ has none of them.

Corollary 7.2. $S(1, 2, 0)$ is precisely the set of SR-words $vq$, where $v$ is an RR2-word and $q$ is a letter that is less than the leading letter $p_1$ of $v$.

Proof. Indeed, consider an arbitrary word $u$ in $S(1, 2, 0)$ and its representation $u = vw$ as in Remark 7.1. Since $v$ has the occurrence of the leading letter $p_1$ of the word $u$, it follows that $v$ is not empty and $p_1$ is its leading letter. Since the word $v$ is regular, it begins with its leading letter $p_1$. Since $w$ has no occurrences of $p_1$ and $w \sqsubset v$, we see that $w$ is a letter such that $q < p_1$.

Since $w$ is not empty, we conclude that $v$ is a regular proper subword of the SR-word $vw$, so that it is reduced; moreover, it has two occurrences of its leading letter $p_1$.

Conversely, if $v$ is a reduced word with two occurrences of its leading letter $p_1$, $q$ is a letter such that $q < p_1$, and $vq$ is an SR-word, then Remark 7.1 shows that $vq \in S(1, 2, 0)$.

Corollary 7.3. $S(1, 2, 0)$ is precisely the set of words $vq$ for which

1) $v$ is an RR2-word;
2) $q$ is a letter that is less than the leading letter $p_1$ of $v$;
3) $q \notin M^+(\{v\})$;
4) the word $v_2q$ is reduced (here $v_2$ is the ending of $v$ that begins with the right occurrence of $p_1$).

Proof. Consider an arbitrary word $u$ in $S(1, 2, 0)$. We denote by $q$ the last letter of $u$ and by $v$ the beginning of $u$ obtained by removing the last letter. Then $u = vq$ and, by Corollary 7.2 properties 1) and 2) hold.

Since the composition of $vq$, as an SR-word, is not rooted, property 3) also holds.

The word $v_2q$ begins with the only occurrence of its leading letter $p_1$, so that it is regular. Being a regular proper subword of the SR-word $vq$, the word $v_2q$ is reduced, i.e., property 4) holds.

Conversely, let conditions 1)–4) be satisfied for a word $u = vq$. Since the word $v$ is regular (property 1)), condition 2) shows that the word $vq$ is regular. Property 3) implies
Therefore, by Corollary 7.3, the graph \( D(q) \) contains no double vertices. This fact and property 4) of Corollary 7.3 imply \( S(1,2,0) \).

Lemma 7.4. The set \( S(1,2,0) \) consists precisely of the words \( u = vq \) for which \( v = \text{red}(\alpha) \) and

1. \( \alpha \) is a root whose graph is such that its leading vertex \( p_1 \) is double;
2. \( q \) is a vertex of the graph \( D(\alpha) \) such that \( q < p_1 \);
3. \( |q| \leq |v_2| \), but \( q \) has an adjacent vertex in the graph \( D(|v_2|) \) (here \( v_2 \) is the ending of \( v \) that begins with the right occurrence of the letter \( p_1 \));
4. if the vertex \( q \) in the graph \( D(\alpha) \) is single, then it has no adjacent single vertices;
5. \( q \) is greater (under the order \( \leq p_1 \)) than any other vertex of \( D(|v_2|) \).

Proof. Consider \( u \in S(1,2,0) \). We denote by \( q \) the last letter of \( u \) and by \( v \) the beginning of \( u \) obtained by removing the last letter. Then \( u = vq \), and property 1) of Corollary 7.3 yields 1).

2. By property 4) of Corollary 7.3 the element \( |v_2| + |q| \) is a root. Therefore, \( q \) has an adjacent vertex in \( D(|v_2|) \). Consequently, \( q \) has an adjacent vertex in the graph \( D(|v|) \) as well. If \( |q| \leq |v| \), then \( q \in M^{-}(|v|) \), which contradicts property 3) of Corollary 7.3.

Therefore, \( |q| < |v| \) This fact and property 2) of Corollary 7.3 imply statement 2).

3. By property 4) of Corollary 7.3 \( q \in M^{-}(|v_2|) \). Moreover, by Lemma 4.10 the graph \( D(|v_2|) \) is an interval. This yields 3).

4. Since \( q \) is a vertex of \( D(\alpha) \) (see 2)), using 3) and considering the cases of Lemma 4.10 item by item, we see that \( q \) has an adjacent double vertex in \( D(\alpha) \).

Assume that the vertex \( q \) in the graph \( D(\alpha) \) is single. Since \( q \) has an adjacent double vertex, it follows that the total multiplicity of the vertices adjacent to \( q \) is either 2 or 3. Furthermore, \( q \notin M^{-}(\alpha) \). Therefore, the total multiplicity of the vertices adjacent to \( q \) is equal to 2. This proves statement 4).

5. Since the graph \( D(|v_2|) \) is an interval, from 3) we deduce that the graph \( D(|v_2q|) \) contains no double vertices. This fact and property 4) of Corollary 7.3 imply 5).

Conversely, consider a pair (\( \alpha, q \)) and the word \( v = \text{red}(\alpha) \) for which statements 1)–5) are true; we use Corollary 7.3 to show that \( vq \in S(1,2,0) \). From 1) and 2) we obtain properties 1) and 2) of Corollary 7.3 respectively.

Since \( q \) is a vertex of \( D(\alpha) \) (see 2)), using 3) and considering the cases of Lemma 4.10 we conclude that \( q \) has a double adjacent vertex in \( D(\alpha) \).

Assume that the vertex \( q \) in \( D(\alpha) \) is single. Since \( q \) has a double adjacent vertex, the total multiplicity of the vertices adjacent to \( q \) is equal either to 2 or to 3. Using 4) we deduce that the total multiplicity of the vertices adjacent to \( q \) is equal to 2. Therefore, property 3) of Corollary 7.3 holds.

By Lemma 4.10 the graph \( D(|v_2|) \) is an interval. This fact and statement 3) imply that the graph \( D(|v_2q|) \) has no double vertices. Moreover, 1) and 2) show that \( v_2 \) is an RR1-word. Next, statement 5) and Remark 5.6 imply property 4) of Corollary 7.3. Therefore, by Corollary 7.3 \( vq \in S(1,2,0) \).

Theorem 7.5. \( S(1,2,0) \) is precisely the set of words \( vq \) for which \( v = \text{red}(\alpha) \), \( \alpha \in \text{Rt}_2 \), and \( q \) is a letter that is less than the leading letter \( p_1 \) of the word \( v \). Moreover, one of the following statements 1–3 is true.

1. \( \alpha \in \text{Rt}_2^1 \). In this case,
   a) \( q \in O(\alpha) \) and \( q \) is not the least among the vertices adjacent to \( z^* \);
   b) \( q > \max(z^*,k) \), where \( k \) is an end vertex of the graph \( D(\alpha) \) such that the interval \( (z^*,k] \) contains the least of the vertices adjacent to \( z^* \).
2. \( \alpha \in \text{Rt}_{2,2} \). In this case, either \( q = x_1^+(\alpha) \) and the graph \( D(\alpha) \) contains no adjacent single vertices, or \( q = o_2 > \max(p_1, x_1^+(\alpha)) \).

3. \( \alpha \in \text{Rt}_{2,3} \). In this case, either \( o_1 = m \) and \( q \in O(\alpha) \), or \( o_1 \neq m = q \). Here \( m \) is the vertex of the interval \([o_1, p_1]\) that is closest to \( p_1 \) and satisfies \( m > \max(p_1, x_1^+(\alpha)) \).

Proof. It suffices to compute the composition of the word \( v_2 \) with the help of Lemma 1.16. We have \([v_2] = [x^*, k] \) for \( \alpha \in \text{Rt}_{2,1} \), \([v_2] = [o_1, x_1^+(\alpha)] \) for \( \alpha \in \text{Rt}_{2,2} \), and \([v_2] = (m, x_1^+(\alpha)) \) for \( \alpha \in \text{Rt}_{2,3} \). Then we apply Lemma 8.4.

\[8. \text{The sets } S(2, 1, 0) \text{ and } S(2, 2, 0)\]

Denote \( S(2, 0) = S(2, 1, 0) \cup S(2, 2, 0) \).

Remark 8.1. By definition, the words from \( S(2, 0) \) are precisely the SR-words \( u = v^2w \), where \( v \) and \( w \) are regular words, \( w \not\equiv v \), and \( w \) has no occurrences of the leading word of \( u \).

Lemma 8.2. The words in \( S(2, 0) \) are precisely the SR-words \( u = v^2q \), where \( v \) is a regular word and \( q \) is a letter, and \( q < p_1 \) (\( p_1 \) is the first letter of \( v \)).

Proof. For \( u \in S(2, 0) \), we consider the representation \( u = v^2w \) as in Remark 8.1. Since \( w \) has no occurrences of the leading letter \( p_1(u) \) of \( u \), we conclude that \( p_1(u) \) coincides with the leading letter \( p_1(v) \) of \( v \). Since \( v \) is regular, it begins with its leading letter, i.e., \( p_1 = p_1(v) \). Since \( w \not\equiv v \) and \( w \) has no occurrences of the letter \( p_1(v) \), it follows that \( w \) is a letter and \( w < p_1(u) = p_1(v) = p_1 \).

Conversely, let \( v \) be a regular word, let \( q \) be a letter, let \( q < p_1 \), where \( p_1 \) is the first letter of \( v \), and let \( v^2q \) be an SR-word. Then \( q \) is a regular word that has no occurrences of the leading letter of the word \( v^2q \) and \( q \not\equiv v \). Therefore, by Remark 8.1, \( v^2q \in S(2, 0) \).

Lemma 8.3. The words in \( S(2, 0) \) are precisely the words \( u = v^2q \), where \( v \) is a regular word and \( q \) is a letter; moreover, \( vq \) is an RR-word.

Proof. For \( u \in S(2, 0) \), we consider the representation \( u = v^2q \) as in Lemma 8.2. Since the word \( v \) is regular, it is not empty. Then \( vq \) is a proper regular subword of the SR-word \( v^2q \); thus, it is an RR-word.

Conversely, let \( v \) be a regular word and \( q \) a letter; moreover, let \( vq \) be an RR-word. Since \( vq \) is an RR-word, it is regular. Since \( v \) and \( vq \) are regular words and \( v > vq \), the word \( v^2q \) is regular by Corollary 5.2.

Since \( q \) is a letter and the word \( v \) is not empty (it is regular), the composition of the word \( v^2q \) is not rootet (the graph \( D([v^2q]) \) has double vertices and at most one single vertex).

The word \( v \) is regular and \( q \) is a letter. Therefore, \( l(v^2q) = vq \), where \( a \) is a beginning of the word \( v \). If \( a \neq 1 \), then \( va \leq v, v \leq a \) (a nonempty beginning of any word is not less than the word itself), and thus \( va \leq a \), which contradicts Lemma 5.1. Consequently, \( a = 1 \) and \( l(v^2q) = vq = v \). Being a regular subword of the RR-word \( vq \), \( v \) is an RR-word. Therefore, \( l(v^2q) \) is an RR-word.

The word \( vq \) is regular and \( v \neq 1 \). Therefore, \( r(v^2q) = wvq \), where \( w \) is an ending of \( v \) such that \( w \neq v \). Assume that \( w \neq 1 \). Then \( vq < wvq \) (Lemma 5.1). \( wvq < w \) (a proper beginning of a word is greater than the word itself), and \( w < v \) (Lemma 5.1). Therefore, \( vq < w < v \), so that \( q \) is a beginning of \( w \). But \( w \) is an ending of \( v \) and \( w \neq v \). Therefore, \( w = 1 \) and \( r(v^2q) = vq \). Consequently, \( r(v^2q) \) is an RR-word.

Combining the proved above with Corollary 5.11 we see that \( v^2q \) is an SR-word. Moreover, \( q \) is less than the first letter of \( v \) (the word \( vq \) is regular). Thus, by Lemma 8.2, \( v^2q \in S(2, 0) \).
A regular word \( v \) of length at least two is said to be prolonged if the word \( \bar{v} \) obtained from \( v \) by removing the last letter is regular.

**Corollary 8.4.** The relation \( u = \bar{v}v \) determines a one-to-one correspondence between the words \( u \) in \( S(2,0) \) and the prolonged reduced words \( v \). Here, \( \bar{v} \) is the word obtained from \( v \) by elimination of the last letter.

**Theorem 8.5.** 1. The relation \( u = \bar{v}v \) determines a one-to-one correspondence between the words \( u \) in \( S(2,1,0) \) and the reduced words \( v \) of length at least 2. Here, \( \bar{v} \) is the word obtained from \( v \) by elimination of the last letter.

2. The relation \( u = \text{red}(\alpha - |q|) \text{red}(\alpha) \) determines a one-to-one correspondence between the words \( u \) in \( S(2,1,0) \) and the nonsimple roots \( \alpha \). Here, \( q \) is the last letter of the word \( \text{red}(\alpha) \); this letter can be computed starting with the root \( \alpha \) and using Corollary 8.4.

**Proof.** For \( \alpha \in \text{Rt}_1 \) the word \( v \) begins with the only occurrence of its leading letter. Therefore, any nonempty beginning of it is regular. Being a regular beginning of the RR-word \( v \), this beginning is an RR-word. Consequently, any RR1-word different from a letter is prolonged. Now we apply Corollary 8.4.

**Lemma 8.6.** Let \( v = \text{red}(\alpha) \) be an RR2-word, and let \( v_2 \) be the ending of \( v \) that begins with the right occurrence of the leading letter \( p_1 \) of \( v \). Then the word \( v \) is not prolonged if and only if the least (under the order \( \leq p_1 \)) of the single vertices of \( D(\alpha) \) that have a double adjacent vertex is the greatest (under the order \( \leq p_1 \)) among the end vertices of the interval \( D(|v_2|) \).

**Proof.** Since, by Lemma 4.13, the graph \( D(|v_2|) \) contains only one of the single vertices of \( D(\alpha) \) that have a double adjacent vertex, we see that \( D(|v_2|) \) is an interval.

Consider a word \( v_1 \) such that \( v = v_1v_2 \). By Lemma 3.1, we have \( v_2 \prec v_1 \), i.e., \( v_1 = ax_1b, \quad v_2 = ax_2c, \quad x_2 < x_1 \) for some words \( a, b, c \) and letters \( x_1, x_2 \). By Lemma 3.7, \( x_2 \) is the least (under the order \( \leq p_1 \)) among the single vertices of the graph \( D(\alpha) \) that have a double adjacent vertex. Since the word \( a \) is not empty (it begins with the leading letter of \( v \)), it follows that the word \( v \) is not prolonged if and only if \( c = 1 \) or, equivalently, the reduced word of composition \( v_2 \) ends with the letter \( x_2 \), which is true if and only if the vertex \( x_2 \) in the interval \( D(|v_2|) \) is an end vertex and is the greatest (under the order \( \leq p_1 \)) among the end vertices of this interval.

Examining the cases of Lemma 3.2 and using Lemma 8.6, we obtain the following statement.

**Corollary 8.7.** Let \( v = \text{red}(\alpha) \) be an RR2-word, and let \( q \) be its last letter.

1. For \( \alpha \in \text{Rt}_{2,1} \), \( q \) is an end vertex such that the interval \( [z^*, q] \) contains the least of the vertices adjacent to \( z^* \). In this case, the word \( v \) is prolonged if and only if in the graph \( D(\alpha) \), the least of the vertices adjacent to \( z^* \) has a single adjacent vertex.

2. For \( \alpha \in \text{Rt}_{2,2} \), the word \( v \) is prolonged if and only if the leading vertex \( p_1 \) of the graph \( D(\alpha) \) is different from \( x_2^+(\alpha) \) and \( \max[\alpha_1, p_1] < \max[p_1, x_2^+(\alpha)] \). For \( \alpha \in \text{Rt}_{2,2} \), if the word \( v \) is prolonged, then \( q = x_2^+(\alpha) \).

3. For \( \alpha \in \text{Rt}_{2,3} \), the word \( v \) is prolonged if and only if the graph \( D(\alpha) \) has adjacent single vertices. For \( \alpha \in \text{Rt}_{2,3} \), if the word \( v \) is prolonged, then \( q = x_1^+(\alpha) \).

**Theorem 8.8.** 1. The relation \( u = \bar{v}v \) determines a one-to-one correspondence between the words \( u \) in \( S(2,2,0) \) and the prolonged reduced words \( v \) described in Corollary 8.7 (\( \bar{v} \) is the word obtained from \( v \) by removing the last letter).

2. The relation \( u = \text{red}(\alpha - |q|) \text{red}(\alpha) \) determines a one-to-one correspondence between the words \( u \) in \( S(2,2,0) \) and the compositions \( \alpha \) of the prolonged reduced words described
In Corollary 8.7, here \( q \) is the last letter of the word \( \text{red}(\alpha) \), which is calculated starting with the root \( \alpha \) in accordance with Corollary 8.7.

§9. THE SET \( S(1, 1, 1) \)

In the sequel, \( P \) is the set of pairs \((v, w)\) of RR1-words with one and the same leading letter and such that \( w \not\sim v \).

Remark 9.1. The relation \( u = vw \) determines a one-to-one correspondence between the elements \( u \) in the set \( S(1, 1, 1) \) and the pairs \((v, w) \in P \) satisfying the condition that \(|v| + |w|\) is not a root.

Lemma 9.2. For \((v, w) \in P\), the element \(|v| + |q|\) is a root (here \( q \) is the last letter of the word \( w \)).

Proof. The RR-word \( w \) has rooted composition. Therefore, the graph \( D(|w|) \) is connected, so that the vertex \( q \) has an adjacent vertex in the graph \( D(|\bar{w}|) \) (here \( \bar{w} \) is the word obtained from \( w \) by removing the last letter). Moreover, \(|\bar{w}| \leq |v|\). Consequently, \( q \) has an adjacent vertex in the graph \( D(|v|) \).

If \( q \) is not a vertex of \( D(|v|) \), then \(|v| + |q|\) is a root. Thus, we may assume that \( q \) is a vertex of \( D(|v|) \).

Suppose \( q \) is a double vertex of \( D(|v|) \). Since the word \( w = \bar{w}q \) has rooted composition, \( q \) is not a double vertex of \( D(|\bar{w}|) \). Moreover, \(|\bar{w}| \leq |v|\). Therefore, \(|w| \leq |v|\). ByLemma 4.8, we obtain \( w \not\sim v \), which contradicts the relation \( w \not\sim v \).

Thus, \( q \) is a single vertex of the graph \( D(|\bar{w}|) \). If \( q \) has no occurrences in the word \( \bar{w} \), then \(|w| \leq |v|\) and, by Lemma 4.8, we obtain \( w \not\sim v \), which contradicts the relation \( w \not\sim v \).

Thus, \( q \) is a single vertex of the graph \( D(|\bar{w}|) \). Moreover, the element \(|w| = |\bar{w}| + |q|\) is a root. Therefore, either \( q = z^* \) and \( q \) has three adjacent vertices in \( D(|\bar{w}|) \), or the vertex \( q \) of \( D(|\bar{w}|) \) has an adjacent single vertex \( y \) and an adjacent double vertex. Moreover, \(|\bar{w}| \leq |v|\) and \( q \) is a single vertex of \( D(|v|) \). Therefore, in the second case the vertex \( y \) in \( D(|v|) \) is also single (because the double vertices of \( D(|v|) \) form a connected subgraph). Consequently, in any case \(|v| + |q|\) is a root. \( \square \)

Corollary 9.3. For an arbitrary word \( u = vw \) in \( S(1, 1, 1) \) (here \( v, w \in P \)), its last letter \( q \) lies in \( M^+|v| + |q|\).

Lemma 9.4. For \((v, w) \in P\), let \( q \) be the last letter of \( w \). Then the number of occurrences of \( q \) in \( v \) is smaller precisely by \( 1 \) than that in \( v \).

Proof. Since \(|\bar{w}| \leq |v|, |w| = |\bar{w}| + |q|\), we have \(|v| = |w| − |q|\). Therefore, \( d_q(v) \geq d_q(w) − 1 \), where \( d_q(v) \) and \( d_q(w) \) are the numbers of occurrences of \( q \) in \( v \) and \( w \), respectively.

Assume that the assertion of the lemma does not hold, i.e., \( d_q(v) \neq d_q(w) − 1 \). Then \( d_q(v) \geq d_q(w) \). Since \( d_q(w) \geq 1 \), we have \( d_q(v) \geq 1 \), i.e., the word \( v \) has at least one occurrence of \( q \). Denote by \( t \) the beginning of \( v \) that ends with the last occurrence of \( q \).

Since \( w \not\sim v \) and \( d_q(v) \geq d_q(w) \), the word \( t \) has length greater than that of \( w \). This fact and \( w \not\sim v \) imply that \( w \not\sim t \). Consequently, \( w < t \).

Therefore, the word \( t \) is regular (it begins with a unique occurrence of its leading letter). Moreover, \( t \) is a subword of the RR-word \( v \). Consequently, \( t \) is an RR-word. In particular, it has rooted composition. Moreover, \(|w| \leq |t|\). Now, by Lemma 4.8, we obtain \( w \geq t \), which contradicts the inequality \( w < t \) proved above. \( \square \)

Lemma 9.5. For \((v, w) \in P\), the pair \((\alpha, q)\), where \( q \) is the last letter of \( w \) and \( \alpha = |v| + |q| \), possesses the following properties.
1) $\alpha$ is a positive root whose graph is such that its leading vertex $p_1$ is single and the set $M^+(\alpha)$ has at least two elements different from $p_1$;

2) $q$ is an element of the set $M^+(\alpha)$ that is different from $p_1$ and from the last letter of the reduced word of composition $\alpha$.

Proof. By Lemma 9.2, $\alpha$ is a root. Since $(v, w) \in P$, the word $w$ is regular and has length at least two. Therefore, $w$ begins with its leading letter $p_1$, and its last letter $q$ is less than $p_1$. Consequently, the vertex $p_1$ in the graph $D(\alpha)$ is leading and single. Since $\alpha = |v| + |q|$ and the word $v$ has rooted composition, we obtain $q \in M^+(\alpha)$. Since $q < p_1$, we have $q \not= p_1$.

We denote by $a$ the reduced word of composition $\alpha$ and by $q'$ the last letter of $a$. It remains to show that $q' \not= q$. Suppose the contrary: $q' = q$. Then the word $a$ ends with $q$. Moreover, $a$ begins with the only occurrence of its leading letter. Therefore, the word $\bar{a}$ obtained from $a$ by removing the last letter is regular. Being a regular proper subword of the RR-word $a$, it is an RR-word. But the RR-words are determined by their compositions. Therefore, $\bar{a} = v$. Consequently, $a = vq$. Also, $w \sqsubseteq v$, whence $w \sqsubseteq a$. Thus, $w < a$. Since $|w| \leq |a|$, Lemma 9.8 yields $w \geq a$, which contradicts the inequality $w < a$ proved above.

We use the notation Base for the set of all pairs $(\alpha, q)$ satisfying properties 1) and 2) of Lemma 9.5.

Remark 9.6. For an arbitrary root $\beta$, any single vertex $q$ of the graph $D(\beta)$ such that $q \in M^+(\beta)$ is an end vertex in $D(\beta)$.

Lemma 9.7. An RR1-word cannot have the subword $q^2$, where $q$ is a letter.

Proof. Any RR1-word $a = bq_2c$ begins with its leading letter $p_1$ and does not have other occurrences of this letter. Therefore, $q < p_1$ and $b$ begins with the letter $p_1$. Then $|bq_2|$ is a root, $q$ is a double vertex of the graph $D(|bq_2|)$, and $q \in M^+(|bq_2|)$. The double vertex $q$ of $D(|bq_2|)$ is a single and not an end vertex in the graph $D(|bq|)$.

The nonempty beginnings $b$ and $bq$ of the RR1-word $a$ have rooted composition. Therefore, $q \in M^+(|bq|)$. What we have proved above contradicts Remark 9.6.

Lemma 9.8. Let $(\alpha, q) \in \text{Base}$, let $p_1$ be the leading vertex of the graph $D(\alpha)$, and let $v$ and $w$ be RR1-words with common leading letter $p_1$. Let $|v| = \alpha - |q|$. Then $w \sqsubseteq v$ is equivalent to the fact that $w$ is the beginning of the reduced word $a$ of composition $\alpha$ that ends with the right occurrence of the letter $q$.

Proof. Recall that $(\alpha, q) \in \text{Base}$ if and only if the vertex $p_1$ in the graph $D(\alpha)$ is single, $q < p_1$, $\alpha - |q|$ is a root, and the word $a = \text{red}(\alpha)$ does not end with $q$.

Since $\alpha - |q|$ is a root, $q$ has an occurrence in $a$. Denote by $b$ and $c$ two words such that $a = bq_c$, $|q| \not\leq |c|$.

Since $a$ is an RR-word, it is regular. Since the leading vertex $p_1$ of the graph $D(\alpha)$ is single, the word $a$ has only one occurrence of $p_1$. For this reason, the regular word $a$ begins with the only occurrence of its leading letter $p_1$. Moreover, $a = bq_c$ and $q < p_1$. Therefore, the word $b$ begins with the only occurrence of its leading letter $p_1$. In particular, $b$ is nonempty and $p_1 \leq |b|$.

Since $a$ does not end with the letter $q$, the word $c$ is not empty. Denote by $m$ the first letter of $c$. Since $q \in M^+(|a|)$, by Lemma 9.7 we have $q < m$.

Since $a$ is an RR1-word and $q \in M^+(|a|)$, Corollary 14.6 shows that the word $bc$ obtained from $a$ by removing the right occurrence of the letter $q$ is an RR1-word. Since it has the same composition as the RR-word $v$, we have $v = bc$. The relation $v = bc$ implies that the word $bm$ is a beginning of $v$. 

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Let \( w = bq \). Since \( bm \) is a beginning of \( v \) and \( q < m \), we have \( w = bq \bowtie v \).

Conversely, let \( w \bowtie v \), and let \( \tilde{w} \) be a word such that \( w = \tilde{w}q \). From \( \tilde{w}q \bowtie v \) it follows that \( \tilde{w} \) is a beginning of \( v \).

Assume that \( w \) is shorter than the word \( bq \). Since \( \tilde{w} \) is a beginning of the word \( v = bc \), we conclude that \( \tilde{w} \) is a beginning of \( b \) different from \( b \). Therefore, \( |w| < |bq| \). Hence \( |w| < |bq| \). Since \( w \) and \( bq \) are RR1-words with one and the same leading letter \( p_1 \), Lemma 9.8 shows that \( bq < w \).

Since \( w \bowtie v \), the word \( w \) is not longer than the word \( b \), and \( b \) is a beginning of \( v \), we have \( w \bowtie b \). Therefore, \( w < b \) and \( w < bq \). Hence \( w < bq \), which contradicts the inequality \( bq < w \).

Thus, \( w = \tilde{w}q \) is not shorter than the word \( bq \), i.e., \( \tilde{w} \) is not shorter than \( b \). Since \( \tilde{w} \) is a beginning of the word \( v = bc \), we have \( \tilde{w} = bd \), where \( d \) is a beginning of \( c \). Therefore, \( bdq \) is a beginning of the RR1-word \( a = bqc \). In particular, \( bdq \) is an RR-word. The word \( w = bdq \) is also an RR-word and has the same composition as \( bdq \). Therefore, \( bdq = bdq \), \( dq = qd \), and \( d = q^{k} \) for some nonnegative integer \( k \). If \( k \geq 1 \), then the RR1-word \( bdq \) ends with two occurrences of the same letter, which contradicts Lemma 9.7. Therefore, \( k = 0 \), \( d = 1 \), and \( w = bdq = bq \). □

**Corollary 9.9.** From Lemmas 9.5 and 9.8, we conclude that, associating the pair \((\alpha, q)\), where \( q \) is the last letter of the word \( w \) and \( \alpha = |v| + |q| \), with a pair \((v, w)\) \( P \), we obtain a one-to-one correspondence between the sets \( P \) and \( \text{Base} \). Moreover, the preimage of a pair \((\alpha, q)\) \( \text{Base} \) is \((v, w)\), where \( v = \text{red}(\alpha - |q|) \) and \( w \) is the beginning of the word \( \text{red}(\alpha) \) that ends with the right occurrence of the letter \( q \).

**Corollary 9.10.** From Remark 9.1 and Corollary 9.9, it follows that the set \( S(1, 1, 1) \) consists precisely of the words \( vw \), where \( v = \text{red}(\alpha - |q|) \), \( w \) is the beginning of the word \( \text{red}(\alpha) \) that ends with the right occurrence of the letter \( q \), \((\alpha, q) \in \text{Base} \), and the element \( |v| + |w| \) is not a root.

**Lemma 9.11.** Suppose \((v, w) \in P \) and \( |v| + |w| \) is a root. Then:

1. the graphs \( D(|v| + |w|) \) and \( D(\alpha) \) have one and the same vertices (here \( \alpha = |v| + |q| \) and \( q \) is the last letter of \( w \));
2. the leading vertex \( p_1 \) of the graph \( D(|v| + |w|) \) is double in it;
3. \( q \) is an end vertex in the graph \( D(|v| + |w|) \);
4. the graph \( D(|v| + |w|) \) contains the \(*\)-center and the three vertices adjacent to it;
5. if the vertex \( q \) is not adjacent to the \(*\)-center, then all nonend vertices of \( D(|v| + |w|) \) are double in this graph.

**Proof.** 1) Since \(|v| = \alpha - |q| \), we have \( \alpha - |q| \leq |v| \leq \alpha \). Moreover, \(|q| \leq |w| \leq \alpha \). Therefore, \( \alpha \leq |v| + |w| \leq 2\alpha \), and the graphs \( D(|v| + |w|) \) and \( D(\alpha) \) have one and the same vertices.

2) Since \(|p_1| \leq |v|, |p_1| \leq |w|, \) and \( |v| + |w| \) is a root, the vertex \( p_1 \) in the graph \( D(|v| + |w|) \) is double.

3) By Lemma 9.4, the letter \( q \) has an odd number of occurrences in any word of composition \(|v| + |w| \). Moreover, \(|v| + |w| \) is a root. Therefore, \( q \) has precisely one occurrence in an arbitrary word of composition \(|v| + |w| \). Therefore, by Lemma 9.4, \( q \) does not have occurrences in an arbitrary word of composition \(|v| \). Moreover, \( \alpha = |v| + |q| \). Consequently, \( q \) is a single vertex of the graph \( D(\alpha) \). But \( q \in M^+(\alpha) \) \((\alpha, q) \in \text{Base} \). Therefore, \( q \) is an end vertex of the graph \( D(\alpha) \). Now statement 1) implies that \( q \) is an end vertex of the graph \( D(|v| + |w|) \).

4) The graph \( D(|v| + |w|) \) has double vertices (for example, \( p_1 \), because \( p_1 \leq |v| \) and \( p_1 \leq |w| \)). Therefore, the graph \( D(|v| + |w|) \) contains the \(*\)-center and the three vertices adjacent to it.
5) Consider the case where the vertex $q$ is not adjacent to the $*$-center. We have $z^* \leq |v| + |w|$ (statement 4)). Now statement 3) implies that the interval $[z^*, q)$ consists precisely of all nonend vertices of the graph $D(|v| + |w|)$.

Moreover, from statement 2) it follows that $p_1$ is not an end vertex in the graph $D(|v| + |w|)$. Therefore, $p_1 \in [z^*, q)$. In particular, the interval $[z^*, q)$ is not empty. Consequently, there is a vertex $y$ of the graph $D(|v| + |w|)$ such that $[z^*, y) = [z^*, q)$.

Since $|v| = \alpha - |q|$, $q \neq y$, and $|y| \leq \alpha$, we have $|y| \leq |v|$. Since $p_1 \leq |w|$, $q \leq |w|$, and the graph $D(|w|)$ is connected ($|w|$ is a root), we have $[p_1, q] \leq |w|$. Moreover, $y \in [p_1, q]$. Therefore, $|y| \leq |w|$. The inequalities $|y| \leq |v|$ and $|y| \leq |w|$ imply that $2|y| \leq |v| + |w|$, i.e., the vertex $y$ is double in the graph $D(|v| + |w|)$. Therefore, all the vertices of the interval $[z^*, y)$ are double in the graph $D(|v| + |w|)$, i.e., all nonend vertices of $D(|v| + |w|)$ are double in this graph.

\[\Box\]

**Lemma 9.12.** For $(\alpha, q) \in \text{Base}$, the property that $|v| + |w|$ is a root is equivalent to the following properties 1)–4). Here $v = \text{red}(\alpha) - |q|$ and $w$ is the beginning of the word $\text{red}(\alpha)$ that ends with the right occurrence of the letter $q$.

1) The graph $D(\alpha)$ contains the $*$-center and the three vertices adjacent to it.

2) The leading vertex $p_1$ of the graph $D(\alpha)$ is single and is not an end vertex.

3) $q$ is the least (under the order $\leq_{\Gamma}$) among the end vertices of the graph $D(\alpha)$.

4) If the vertex $q$ is not adjacent to the $*$-center, then an arbitrary vertex $x$ of $D(\alpha)$ is double in it if and only if $x \in [z^*, p_1)$, $\max(x, p_1) > \max(p_1, q)$.

**Proof.** Assume that $(\alpha, q) \in \text{Base}$ and the element $|v| + |w|$ is a root. We prove properties 1)–4).

1) By statements 1) and 4) of Lemma 9.11 the graph $D(\alpha)$ has $*$-center.

2) Since $p_1 \leq |v|$, $p_1 \leq |w|$, and $|v| + |w|$ is a root, it follows that the vertex $p_1$ is double in the graph $D(|v| + |w|)$ and is single in the graphs $D(|v|)$ and $D(|w|)$.

Since $(\alpha, q) \in \text{Base}$, we have $q \neq p_1$. Moreover, $|v| = \alpha - |q|$. Therefore, the vertex $p_1$ has one and the same multiplicity in the graphs $D(v)$ and $D(\alpha)$. Moreover, the vertex $p_1$ is single in $D(|v|)$. Hence it is also single in $D(\alpha)$.

From statements 1) and 2) of Lemma 9.11 we deduce that $p_1$ is not an end vertex of $D(\alpha)$.

3) By statement 3) of Lemma 9.11 $q$ is an end vertex of $D(\alpha)$.

Consider an arbitrary end vertex $x$ of $D(\alpha)$ different from $q$. Since $|v| = \alpha - |q|$, we have $|x| \leq |v|$.

Assume that $x \leq p_1$. Lemma 4.13 implies that $|x| \leq |w|$. But in this case $2|x| \leq |v| + |w|$, i.e., $x$ is a double end vertex in the graph $D(|v| + |w|)$, which is impossible. Hence, we obtain property 3).

4) Consider the case where the vertex $q$ is not adjacent to the $*$-center. Let $x$ be an arbitrary vertex in $[z^*, p_1)$.

Since the vertex $q$ in the graph $D(\alpha)$ is an end vertex and is not adjacent to $z^*$, but the vertex $p_1$ of $D(\alpha)$ is not an end vertex, we have $p_1 \in [z^*, q)$. For this reason, $p_1 \in (x, q)$, and $\max(p_1, q) < \max(x, p_1)$ if and only if $q \leq p_1$. In the case $x \leq p_1$, $z^*$ is a root, and $|x| \leq |w|$, i.e., $x$ is a double end vertex in the graph $D(|v| + |w|)$, which is impossible. Hence, we obtain property 3).

Since the vertex $q$ in the graph $D(\alpha)$ is an end vertex and is not adjacent to $z^*$, but the vertex $p_1$ of $D(\alpha)$ is not an end vertex, we have $p_1 \in [z^*, q)$. For this reason, $p_1 \in (x, q)$, and $\max(p_1, q) < \max(x, p_1)$ if and only if $q \leq p_1$. In the case $x \leq p_1$, $z^*$ is a root, and $|x| \leq |w|$, i.e., $x$ is a double end vertex in the graph $D(|v| + |w|)$, which is impossible. Hence, we obtain property 3).

Now, for an arbitrary double vertex $x$ of $D(\alpha)$ we have $2|x| \leq \alpha$, and property 2) implies that $x \in [z^*, p_1)$. This and the said above yield the inequality $\max(p_1, q) < \max(x, p_1) \Leftrightarrow 2|x| \leq \alpha$. Conversely, if $x \in [z^*, p_1)$ and $\max(p_1, q) > \max(p_1, q)$, then we have $2|x| \leq \alpha$. Property 4) is proved.
Conversely, suppose properties 1)–4) are fulfilled for a pair \((\alpha, q) \in \text{Base}\). We show that the element \(|v| + |w|\) is a root. Since \((\alpha, q) \in \text{Base}\), the element \(\alpha - |q|\) is a root.

First, we consider the case where \(q \in O(\alpha)\). Since \(\alpha - |q|\) is a root, the graph \(D(\alpha)\) has no double vertices. By property 1), the graph \(D(\alpha)\) has at least four vertices.

Consider the subcase where the graph \(D(\alpha)\) has precisely four vertices. From property 2) we have \(p_1 = z^*\). Now property 3) implies that \(\alpha\) is the sum of the vertex \(z^*\) and the vertices adjacent to it; \(|v| = \alpha - |q|\) is the sum of the vertex \(z^*\) and the vertices adjacent to it except for \(q\); \(|w| = |q| + |z^*|\); and \(|v| + |w|\) is a root the graph of which has precisely four vertices and has a double vertex.

Now we consider the subcase where the graph \(D(\alpha)\) has at least five vertices. Let \(o\) be the end vertex of \(D(\alpha)\) that is adjacent to \(z^*\) and is different from \(q\), and let \(x\) be an end vertex of \(D(\alpha)\) that is not adjacent to \(z^*\). Then \(p_1 \in [z^*, x]\) (by property 2), the vertex \(p_1\) in the graph \(D(\alpha)\) is not an end vertex). Property 3) yields \(q < o\) and \(\max(q, p_1) < \max(p_1, x)\).

Since the graph \(D(\alpha)\) has no double vertices, \(\{q, o, x\}\) is the set of its end vertices, and \(x \notin O(\alpha)\), we have \(\alpha = |q| + |o| + |z^*, x|\). Therefore, \(|v| = \alpha - |q| = |o| + |z^*, x| = |o, x|\).

Now, by Corollary 4.11 \(|w| = |q, m|\), where \(m\) is the vertex of the interval \((p_1, x)\) that is closest to \(p_1\) and satisfies \(\max(q, p_1) < \max(p_1, m)\) (the inequality \(\max(q, p_1) < \max(p_1, x)\) shows that such a vertex \(m\) exists).

By property 3), \(\max(q, p_1) > \max(p_1, x)\). Therefore, \(x \neq m\). Thus, \(|v| + |w| = |o, x| + |q, m|\) is a root (because \(O(\alpha) = \{o, q\}, x \neq m\).

Now we consider the case where \(q \notin O(\alpha)\). Since, by property 3), the vertex \(q\) in the graph \(D(\alpha)\) is an end vertex, \(q\) is not adjacent to the \(+\)-center. Also, by property 2), \(p_1\) is not an end vertex in \(D(\alpha)\). Therefore, \(p_1 \in [z^*, q]\). Property 3) implies that \(\max(p_1, q) < \max(p_1, k)\) for \(k \in O\).

Denote by \(m\) the vertex of the interval \([z^*, p_1]\) such that \(m\) is most distant from \(p_1\) and satisfies the inequality \(\max(m, p_1) < \max(p_1, q)\).

Since \(q\) is an end vertex in \(D(\alpha)\) and \(q \notin O(\alpha)\), the vertices of the graph \(D(\alpha)\) are precisely the vertices of the graph \(O \cup [z^*, q]\). Property 4) implies that the double vertices of \(D(\alpha)\) are vertices of the interval \([z^*, m]\). Therefore, \(\alpha = \sum O + |z^*, q| + |z^*, m|\). Moreover, \(|v| = \alpha - |q|\) and, by Corollary 4.11 \(|w| = |m, q|\). Thus, \(|v| + |w| = \alpha - |q| + |m, q| = \sum O + |z^*, q| + |z^*, q|\) is a root.

**Theorem 9.13.** The words in \(S(1, 1, 1)\) are in one-to-one correspondence with the pairs \((\alpha, q)\) from Base for which at least one of the conditions 1)–4) of Lemma 9.12 is not fulfilled. The correspondence is given by the relation \(u = vw\), where \(v\) is the reduced word of composition \(\alpha - |q|\) and \(w\) is the beginning of the reduced word of composition \(\alpha\) that ends with the right occurrence of the letter \(q\).

\[\text{§10. The set } S(1, 2, 1)\]

**Remark 10.1.** By definition, \(S(1, 2, 1)\) is precisely the set of SR-words \(u\) such that in the representation \(u = v^k w\), where \(w \sqsubset v\), we have \(k = 1\), the word \(v\) has precisely two occurrences of the leading letter of \(u\), and \(w\) has only one occurrence of this letter.

We denote by \(P_{2,1}\) the set of pairs \((v, w)\) of RR-words with common leading letter \(p_1\) such that the word \(v\) has two occurrences of \(p_1\), the word \(w\) has one occurrence of \(p_1\), and \(w \sqsubset v\).

We use Remark 10.1 and Corollary 5.11 to deduce the following statement.

**Corollary 10.2.** \(S(1, 2, 1)\) is precisely the set of words \(vw\), where \((v, w) \in P_{2,1}\) and \(r(vw)\) is an RR-word.
Proof. For \( u \in S(1,2,1) \), we consider the representation \( u = vw \) from Remark 10.1. Since the words \( v \) and \( w \) are regular, we have \( v \neq 1 \neq w \). Therefore, \( v \) and \( w \) are regular proper SR-words of \( vw \), so that they are RR-words. Therefore, \((v, w) \in \bar{P}_{2,1}\).

By definition, the word \( r(vw) \) is a regular proper subword of \( vw \). Since \( vw \) is an SR-word, it follows that \( r(vw) \) is an RR-word.

Conversely, for an arbitrary pair \((v, w) \in \bar{P}_{2,1}\) such that \( r(vw) \) is an RR-word we show that \( vw \in S(1,2,1) \). Since \( w \not\in v \), we have \( w < v \) and thus, by Corollary 5.2, the word \( vw \) is regular. Since the word \( vw \) has three occurrences of its leading letter, we conclude that its composition is not rooted.

Since \( v \) and \( w \) are RR-words, the word \( v \) is regular and \( w \neq 1 \). Therefore, \( v \neq vw \) and \( v \) is a beginning of the word \( l(vw) \). Denote by \( \bar{w} \) the beginning of the word \( w \) obtained from \( w \) by removing the last letter. Since \( w \neq 1 \), it follows that \( l(vw) \) is a beginning of the word \( vw \). Since \( v \) is a beginning of the word \( l(vw) \), we have \( l(vw) = va \), where \( a \) is a beginning of \( \bar{w} \). Since \( w \not\in v \), \( \bar{w} \) is a beginning of \( v \). Since \( \bar{a} \) is a beginning of \( \bar{w} \), it follows that \( a \) is a beginning of \( v \).

Suppose \( a \neq 1 \). The word \( w \) begins with the only occurrence of its leading letter \( p_1 \). Therefore, the nonempty beginning \( a \) of the word \( w \) begins with the only occurrence of the letter \( p_1 \). Consequently, the word \( a \) is regular. Moreover, \( v \leq a \) (\( a \) is a nonempty beginning of the word \( v \)). Thus, the words \( v, a, \) and \( va = l(vw) \) are regular and \( v \leq a \), which contradicts Corollary 5.2. Therefore, \( a = 1 \) and \( l(vw) = va = v \) is an RR-word.

Since \( r(vw) \) is an RR-word, from the above argument and Corollary 5.1 we conclude that \( vw \) is an RR-word. Since \((v, w) \in \bar{P}_{2,1}\), Remark 10.1 shows that \( vw \in S(1,2,1) \). \( \square \)

For an arbitrary RR-word \( v \), we denote by \( v_1 \) and \( v_2 \) regular words with a common leading letter that satisfy the condition \( v = v_1 v_2 \). Denote by \( Q \) the set of pairs \((\alpha, q)\), where \( \alpha \in \mathbb{R}_2 \) and \( q \in M^-(|v_1|) \) is a letter that is less than the leading vertex \( p_1 \) of the graph \( D(\alpha) \); let \( v_1 \) be the longest beginning of the word \( v = \text{red}(\alpha) \) that has only one occurrence of the letter \( p_1 \). For any root \( \gamma \) and any vertex \( q \) of the graph \( D(\gamma) \), we denote by \( w_q(\gamma) \) the beginning of the word \( \text{red}(\gamma + |q|) \) that ends with the right occurrence of the letter \( q \).

**Lemma 10.3.** Associating the pair \((|v|, q)\), where \( q \) is the last letter of the word \( w \), with a pair \((v, w) \in \bar{P}_{1,2}\), we obtain an injective mapping of \( \bar{P}_{1,2} \) onto \( Q \). Under this mapping, the preimage of a pair \((\alpha, q)\) is the pair \((v, w_q(|v_1| + |q|))\), where \( v = \text{red}(\alpha) \) and \( v_1 \) is the longest beginning of the word \( v \) that has only one occurrence of the leading letter \( \alpha \).

**Proof.** Let \((v, w) \in \bar{P}_{1,2}\), and let \( p_1 \) and \( q \) be the first and last letters of \( w \). The words \( v \) and \( w \) begin with \( p_1 \), and \( w \not\in v \). Therefore, the word \( w \) has length at least two. Moreover, \( w \) is regular. Therefore, \( q < p_1 \). Since \( w \not\in v \), it follows that either \( w = v_1 q \), or \( w \not\in v_1 \). In both cases, \( q \in M^-(|v_1|) \) (in the second case, by Lemma 5.2). Therefore, \((v, q) \in Q\), i.e., there exists a mapping \( f : \bar{P}_{1,2} \to Q \) that takes a pair \((v, w) \in \bar{P}_{1,2}\) to the pair \((|v|, q)\), where \( q \) is the last letter of \( w \).

Consider \((\alpha, q) \in Q\), \( v = \text{red}(\alpha) \). If the word \( \text{red}(|v_1| + |q|) \) ends with \( q \), then we have \( w_q(|v_1| + |q|) = v_1 q \). If the word \( \text{red}(|v_1| + |q|) \) does not end with \( q \), then \( w_q(|v_1| + |q|) \not\in v_1 \) by Corollary 5.9. In both cases, \((v, w_q(|v_1| + |q|)) \in \bar{P}_{1,2}\) and \( f(v, w_q(|v_1| + |q|)) = (v, q) \). Therefore, the mapping \( f \) is surjective.

Let \((v, w) \in \bar{P}_{1,2}\), and let \( q \) be the last letter of \( w \). Since \( w \not\in v \), we have either \( w = v_1 q \) or \( w \not\in v_1 \). If \( w = v_1 q \), then the word \( \text{red}(|v_1| + |q|) \) ends with the letter \( q \), i.e., \( w_q(|v_1| + |q|) = v_1 q = w \). If \( w \neq v_1 q \), then \( w \not\in v_1 \) and, by Corollary 5.9, \( w = w_q(|v_1| + |q|) \). Therefore, the mapping \( f \) is injective. \( \square \)

**Theorem 10.4.** I. \( S(1,2,1) = \{\text{red}(\alpha)w_q(|v_1| + |q|) : (\alpha, q) \in Q\} \).

II. One of the following statements 1–3 is true.
1. \( \alpha \in \text{Rt}_{2,1} \), \( q \) is the least of the vertices adjacent to \( z^* \), and \( w_q([v_1] + |q|) = z^* q \).

2. \( \alpha \in \text{Rt}_{2,2} \), and one of the following alternative statements 2.1 or 2.2 is fulfilled.

2.1. \( q = o_1 \). In this case, \( w_q([v_1] + |q|) = [o_1, x] \), where \( x \) is the vertex of the interval \( (p_1, x^+_1(\alpha)) \) closest to \( p_1 \) and such that the second-by-priority vertex of the interval \([o_1, x] \) lies in the interval \([o_1, p_1] \).

2.2. The graph \( D(\alpha) \) contains not all vertices of the graph \( D_n \), and \( q \) is a unique vertex not lying in the graph \( D(\alpha) \) but having an adjacent vertex in it. In this case, one of the following alternative statements 2.2.1 or 2.2.2 is fulfilled.

2.2.1. \( o_2 > \max(p_1, q) \) (here \( o_2 \in O \), \( o_2 > o_1 \)) and \( w_q([v_1] + |q|) = [z^*, q] \).

2.2.2. \( o_2 = \max(p_1, q) \), \( w_q([v_1] + |q|) = [o_2, q] \), and \( w_q([v_1] + |q|) = v_1 q \).

3. \( \alpha \in \text{Rt}_{2,3} \). In this case, one of the alternative statements 3.1 or 3.2 is fulfilled.

3.1. \( q = x^+_1(\alpha) \) and \( w_q([v_1] + |q|) = [m, x^+_1(\alpha)] \), where \( m \) is the vertex of the interval \([z^*, p_1] \) most distant from \( p_1 \) and such that the second-by-priority vertex of the interval \([m, x^+_1(\alpha)] \) lies in the interval \([p_1, x^+_1(\alpha)] \).

3.2. \( q = m, w_q([v_1] + |q|) = [o_1, m] + [o_2, x^+_2(\alpha)] \), and \( w_q([v_1] + |q|) = v_1 q \).

Proof. I. For \( u \in S(1,2,1) \), by Corollary 10.2, we have \( u = vw \), where \( (v, w) \in \tilde{P}_{2,1} \). By Lemma 10.3 \( ([v], q) \in Q \) and \( w = w_q([v_1] + |q|) \), where \( q \) is the last letter of \( w \). Therefore, \( S(1,2,1) \subseteq \{ \text{red}(\alpha)w_q([v_1] + |q|) : (\alpha, q) \in Q \} \). From 10.2 it is clear that to prove the reverse inclusion it suffices to check the relation \( r(vw) = w = \text{red}(v) \), \( w = w_q([v_1] + |q|) \), \( ([v], q) \in Q \). This relation follows from the inequality \( v_2 \leq w_q([v_1] + |q|) \) for \( v = \text{red}(v) \), \( ([v], q) \in Q \). Below, we prove this inequality for all leaves of the tree of alternatives in statement II.

II. The set \( \{ \text{Rt}_{2,1}, \text{Rt}_{2,2}, \text{Rt}_{2,3} \} \) is a partition of the set \( \text{Rt}_{2} \).

1. For \( \alpha \in \text{Rt}_{2,1} \), from Lemma 4.4 we obtain \( |v_2| = [z^*, k] \), where \( k \) is an end vertex of the graph \( D(\alpha) \) such that the interval \([z^*, k] \) contains the least vertex \( m_1 \) among the vertices adjacent to \( z^* \). Since \( |v| = |v_1| + |v_2| \), it follows that \( |v_1| \) is the largest interval that consists of vertices of \( D(\alpha) \) and satisfies the condition \( |m_1| \leq |v_1| \). Therefore, \( M^{-}(|v_1|) = \{ m_1 \} \). Thus, \( q = m_1, |v_1| + |q| \) is a graph that contains \( z^* \) and the three vertices adjacent to it and has no double vertices. Now Corollary 4.4 shows that \( w_q([v_1] + |q|) = z^* q \) is a beginning of the word \( v_2 \). Therefore, \( v_2 \leq w_q([v_1] + |q|) \).

2. For \( \alpha \in \text{Rt}_{2,2} \), from Lemma 4.4 we obtain \( |v_2| = [o_1, x^+_2(\alpha)] \). Consequently, \( |v_1| = [o_2, x^+_1(\alpha)] \) and \( M^{-}(|v_1|) = \{ o_1, x_0(\alpha) \} \), where \( x_0(\alpha) \) is a unique vertex that does not lie in the graph \( D(\alpha) \) but has an adjacent vertex in it.

Since \( \alpha \in \text{Rt}_{2,2} \), we have \( \max(o_1, p_1) < \max(p_1, x^+_1(\alpha)) \). Therefore, there exists a vertex \( x \) of the interval \([p_1, x^+_1(\alpha)] \) closest to \( p_1 \) and such that \( \max(o_1, p_1) < x \). Denote \( a = \text{red}([z^*, x]) \).

Since \( [o_1, x] \leq |v_2| = [o_1, x^+_2(\alpha)] \), Corollary 4.4 shows that \( ao_1 \) is a beginning of the word \( v_2 \) by the choice of the vertex \( x \).

2.1. For \( q = o_1 \), we see that \( |v_1| + |q| = |o_1| + [o_2, x^+_1(\alpha)] \) is a graph without double vertices, which consists of all vertices of the graph \( D(\alpha) \). Since \( \max(o_1, p_1) < \max(p_1, x^+_1(\alpha)) \), from Corollary 4.4 using the choice of the vertex \( x \), we conclude that \( w_q([v_1] + |q|) = [o_1, x] \). Therefore, \( w_q([v_1] + |q|) = [o_1, x_0(\alpha)] \) if \( [o_1, x] \leq [o_1, x^+_1(\alpha)] \) if \( [o_1, x] \leq [o_1, x^+_2(\alpha)] \). Since \( \max(\ast, q) \leq \max([o_1, p_1] < \max(p_1, x^+_1(\alpha)) \leq \max(p_1, q) \), we have \( \max(\ast, q) \leq w_q([v_1] + |q|) \leq |v_1| + |q| = [o_2, q] \).

2.2. For \( q = x_0(\alpha) \), we have \( |v_1| + |q| = [o_2, q] \). Since \( \max(\ast, q) \leq \max(o_1, p_1 \leq \max(o_1, x^+_1(\alpha)) \leq \max(p_1, q) \), we have \( \max(\ast, q) \leq w_q([v_1] + |q|) \leq |v_1| + |q| = [o_2, q] \).

2.2.1. For \( o_2 > \max(p_1, q) \), we have \( w_q([v_1] + |q|) = [z^*, q] \). Since \( \max(\ast, x) \leq w_q([v_1] + |q|) \) \( = [z^*, q] \), Corollary 4.4 and the choice of the vertex \( x \) show that \( az \) is a beginning of the word \( w_q([v_1] + |q|) \). Since \( ao_1 \) is a beginning of the word \( v_2 \) and \( o_1 \leq \max(o_1, p_1) < x \), we have \( v_2 \leq ao_1 < w_q([v_1] + |q|) \).
2.2.2. For \( \alpha_2 < \max(p_1, q) \), we have \(|w_q(|v_1| + |q|)| = |\alpha_2, q|\). Since \( \alpha_1 \prec \alpha_2 \), the choice of the vertex \( x \) shows that any element of the set \( \{\alpha_2, x\} \) is greater than the second-by-priority vertex of the interval \([\alpha^+, x)\). Therefore, by Corollary 4.14 \( \gamma \) is a beginning of the word \( w_o(|v_1| + |q|) \), and \( y \) is the least of the elements of the set \( \{\alpha_2, x\} \).

Since \( \alpha_1 \prec y \) and \( \alpha_2 \) is a beginning of the word \( v_2 \), we have \( v_2 \prec \alpha_2 \prec w_q(|v_1| + |q|) \).

3. For \( \alpha \in R_{l, 3} \), from Lemma 11.2 we obtain \( |v_2| = [m, x_1^+(\alpha)] \), where \( m \) is the vertex of the interval \([\alpha^+, p_1]\) that is most distant from \( p_1 \) and satisfies the condition that the second-by-priority vertex of the interval \([m, x_1^+(\alpha)]\) lies in the interval \((p_1, x_1^-(\alpha))\).

Then \(|v_1| = |\alpha_1, m| + [\alpha_2, z^+, x_2^+(\alpha)]\). Therefore, \( M^- = \{x_1^-(\alpha), m\} \).

3.1. \( q = x_1^-(\alpha) \). Then \(|v_1| + |q| = |\alpha_1, m| + [\alpha_2, z^+, x_2^+(\alpha)]\), and, by the choice of the vertex \( m \) and Corollary 4.14, we obtain \(|w_q(|v_1| + |q|)| = [m, x_1^-(\alpha)]\). Therefore, \(|w_q(|v_1| + |q|)| = [m, x_1^+(\alpha)] \leq |m, x_1^+(\alpha)| = |v_2|\), and, by Corollary 4.8, \( v_2 \leq w_q(|v_1| + |q|) \).

3.2. \( q \neq x_1^-(\alpha) \). Then \( m \neq p_1, q = m \), and the vertex \( q \) in the graph \( D(|v_1| + |q|) \) is double. Therefore, \(|v_1| + |q| = |\alpha_1, m| + [\alpha_2, x_2^+(\alpha)]\).

Since \( \alpha \in R_{l, 3} \), we have \( \max(p_1, x_2^+(\alpha)) \leq \max(p_1, x_1^+(\alpha)) \). Moreover, \( p_1 \in (m, x_2^+(\alpha)) \). Therefore, the word \( \text{red}([\alpha_1, m] + [\alpha_2, x_2^+(\alpha)]) \) ends with the letter \( m \).

Consequently, this word coincides with \( \text{red}([\alpha_1, m] + [\alpha_2, x_2^+(\alpha)]) \), and \( |w_q(|v_1| + |q|)| = |v_1| + |m| \). Since \( v_1 \) is an RR1-word, we have \( |w_q(|v_1|)| = v_1 |m| \). By 33, Lemma 3.1, \( v_2 \prec v_1 \). Therefore, \( v_2 \prec v_1 \).

\[\Box\]

\section{The sets \( S(1, 2, 2), S(2, 1, 1) \) and the main results}

By definition, \( S(1, 2, 2) \) is the set of all SR-words \( u \) such that

1. the word \( l(u) \) has precisely two occurrences of the leading letter \( p_1 \) of \( u \);
2. the word \( l(u) \) is not a beginning of \( u \);
3. the word \( \bar{l}(u) \) has precisely two occurrences of the letter \( p_1 \).

By Corollary 5.6, \( S(1, 2, 2) \) is the set of all SR-words of the form \( u = vw \), where \( v \) and \( w \) are regular words having two occurrences of the leading letter \( p_1 \) of \( u \) and such that \( w \not\preceq v \).

Since \( v \) and \( w \) are regular proper subwords of the SR-word \( u \), it follows that \( v \) and \( w \) are RR-words. They have two occurrences of their leading letter \( p_1 \) each. Let \( v_1, v_2, w_1 \), and \( w_2 \) be RR-words such that \( v = v_1 v_2 \) and \( w = w_1 w_2 \).

\textbf{Lemma 11.1.} For any word \( u = vw \) in \( S(1, 2, 2) \), where \( v \) and \( w \) are regular words each of which has two occurrences of the leading letter \( p_1 \) for \( u \) and \( w \not\preceq v \), the following conditions are fulfilled:

1. \( v_1 = w_1 \) and \( w_2 \not\preceq v_2 \);
2. the roots \( |v_2| \) and \( |w_2| \) are not comparable.

\textbf{Proof.} 1. Since \( w \not\preceq v \) and each of the words \( v \) and \( w \) has two occurrences of \( p_1 \), condition 1) is fulfilled.

2. By condition 1), \( w_2 \not\preceq v_2 \). Therefore, the length of \( v_2 \) is not less than that of \( w_2 \). Assume that the assertion does not hold. Then \( |w_2| \leq |v_2| \). Since \( w_2 \not\preceq v_2 \), we can apply Lemma 9.2 to conclude that \( |v_2| + |q| \) is a root (\( q \) is the last letter of \( w_2 \)). Moreover, from Lemma 4.16 it follows that the graph \( D(|v_2|) \) is an interval. Therefore, \( |q| \not\leq |v_2| \), so that \( |q| \not\leq |w_2| \not\leq |v_2| \), a contradiction.

\[\Box\]

\textbf{Lemma 11.2.} The set \( S(1, 2, 2) \) is empty.

\textbf{Proof.} Let \( u \in S(1, 2, 2) \); let \( v \) and \( w \) be regular words each of which has two occurrences of the leading letter \( p_1 \) of \( u \), and let \( u = vw, w \not\preceq v \). By Lemma 11.1 we have \( v_1 = w_1 \). Denote \( \alpha_1 = |v_1| \). Let \( \min_n, \min_w \) be the least (under the order \( \leq p_1 \)) of the single vertices of the graph \( D(|v|) \) (respectively, the graph \( D(|w|) \)) that have a double adjacent vertex.
The structure of RR2-words shows that \( \text{min}_w \) is a unique vertex of \( D_n \) that does not lie in \( D(|v_1|) \) but has an adjacent vertex in it. Similarly, \( \text{min}_w \) is a unique vertex of \( D_n \) that does not lie in \( D(|w_1|) \) but has an adjacent vertex in it. Moreover, \( v_1 = w_1 \). Therefore, \( \text{min}_v = \text{min}_w \); we denote \( \text{min} = \text{min}_v \).

If \( \text{min} \in O \), then \(|v_2| = |x^*_2(v)|\) and \(|w_2| = |x^*_2(w)|\); thus, the roots \( v_2 \) and \( w_2 \) are comparable, which contradicts Lemma 11.1. Therefore, \( \text{min} \notin O \).

The graph \( D(|v_1|) \) contains all single vertices of the graph \( D(|v|) \) that have an adjacent double vertex except for \( \text{min} \). Therefore, the graph \( D(|v_1| + |\text{min}|) \) contains all single vertices of \( D(|v|) \) that have an adjacent double vertex. Since \( \text{min} \notin O \), the double vertices of \( D(|v|) \) are not end vertices of the graph \( O \cup \{\text{min}\} \). Similarly, the double vertices of \( D(|w|) \) are not end vertices of the graph \( O \cup \{\text{min}\} \). Consequently, \( [z^*, x^*_2(v)] = [z^*, x^*_2(w)] \).

Let \( I \) be the set of all vertices of the interval \([z^*, x^*_2] \cup \{\text{min}\} \) that do not exceed the vertex \( \text{min} \) in the order \( \leq p \). The set \( I \) is an interval and \([p_1, \text{min}] \subseteq I \). Since \( \text{min} \) is an end vertex of the interval \([z^*, x^*_2] \cup \{\text{min}\} \) and \( \text{min} \in I \subseteq [z^*, x^*_2] \cup \{\text{min}\} \), it follows that \( \text{min} \) is an end vertex of the interval \( I \). Since \( \text{min} \neq p_1 \), the interval \( I \) contains at least two vertices. Let \( m \) denote the end vertex of \( I \) different from \( \text{min} \).

Since \( \text{min} \notin O \), the structure of RR2-words shows that \(|v_2| \) is an interval in which \( m \) is an end vertex, and \(|\text{min}| \leq |v_2| \). Similarly, \(|w_2| \) is an interval in which \( m \) is an end vertex, and \(|\text{min}| \leq |w_2| \). Since \( m \neq \text{min} \), it follows that the roots \(|v_2| \) and \(|w_2| \) are comparable, which contradicts Lemma 11.1.

\[ \square \]

**The set \( S(2,1,1) \)**

By definition, \( S(2,1,1) \) is the set of SR-words \( u \) such that

1. the word \( l(u) \) has precisely one occurrence of the leading letter \( p_1 \) of \( u \);
2. the word \( (l(u))^2 \) is a beginning of \( u \), but \( (l(u))^3 \) is not;
3. the ending of the word \( u \) obtained by removing the beginning \( (l(u))^2 \) from \( u \) has precisely one occurrence of the letter \( p_1 \).

**Remark 11.3.** By Corollary 5.6, \( S(2,1,1) \) is the set of SR-words \( u = a^2b \), where \( a \) and \( b \) are regular words with a common leading letter that have a unique occurrence of this letter and are such that \( b \mathcal{B} a \).

**Lemma 11.4.** 1. For \( u \in S(2,1,1) \), \( r(u) \) is not a prolonged RR2-word and \( \overline{r}(u) = \overline{r}(r(u)) \).

2. If an RR2-word \( v \) is not prolonged, then \( \overline{r}(v)v \in S(2,1,1) \) and \( r(\overline{r}(v)v) = v \).

**Proof.** 1. Consider an arbitrary word \( u \) in \( S(2,1,1) \). By Remark 11.3 there exist uniquely determined regular words \( a \) and \( b \) with a common leading letter \( p_1 \) that have only one occurrence of this letter and satisfy \( u = a^2b \), \( b \mathcal{B} a \).

Since \( b \mathcal{B} a \), we have \( b < a \), so that the word \( ab \) is regular. Moreover, the word \( a^2b \) has precisely three occurrences of the letter \( p_1 \) and its ending \( ab \) begins with the letter \( p_1 \) and has precisely two occurrences of this letter. Therefore, \( r(u) = ab \).

Being a regular proper subword of the SR-word \( u \), \( ab \) is an RR-word. Since \( ab \) has exactly two occurrences of the leading letter \( p_1 \), we conclude that \( ab \) is an RR2-word. Moreover, \( b \mathcal{B} a \). Therefore, the RR2-word \( r(u) = ab \) is not prolonged. Moreover, \( \overline{\mathcal{B}}(r(u)) = a = \overline{\mathcal{B}}(u) \).

2. Consider an arbitrary nonprolonged RR2-word \( v \). Denote \( v_1 = \overline{r}(v) \). Since \( v_1 \) is a beginning of \( v \) and \( v \neq v_1 \), we have \( v < v_1 \). Therefore, the word \( u = v_1v \) is regular. Since the word \( u \) has three occurrences of the leading letter \( p_1 \), its composition is not rooted.

By [33] Lemma 3.1, we have \( v_2 \prec v_1 \), where \( v_2 = r(v) \). Therefore, \( v_1 = zx_1c \) and \( v_2 = zx_2d \), where \( z \), \( c \), and \( d \) are words, and \( x_1 \) and \( x_2 \) are letters such that \( x_2 < x_1 \).
Since the regular word \( v \) is not prolonged, we have \( d = 1 \) and \( v_2 = zz_2 \). Then \( v_1^2z \) is the beginning of the word \( v_1v \) obtained from \( v_1v \) by removing the last letter. Therefore, \( l(u) \) is a beginning of the word \( v_1^2z \). Moreover, \( z \) is a beginning of the word \( v_1 \). Therefore, \( l(v_1v) = v_1 \) is an RR-word. Also, \( r(u) = v \) is an RR-word. By Corollary 5.11 the above implies that \( v_1v \) is an SR-word. Moreover, \( u = v_1^2v_2 \) and \( v_2 = zz_2 \) by \( z_1c = v_1 \). Consequently, \( u \in S(2,1,1) \) by Remark 11.3. Since the words \( \tilde{r}(v) \) and \( v \) are regular and have common leading letter, and the word \( \tilde{r}(v) \) has exactly one occurrence of this letter, we have \( r(\tilde{r}(v)v) = v \). □

Lemma 11.5. An arbitrary RR2-word \( a \) is not prolonged if and only if

1) the least vertex \( m^- \) in the order \( \leq_{p_1} \) among the single vertices of the graph \( D(|a|) \) that have a double adjacent vertex has no single adjacent vertices (\( p_1 \) is the first letter of the word \( a \));

2) if \( m^- \in O \), then the second-by-priority vertex of the interval \([m^-, x^+_2 (|a|)]\) lies in the interval \([m^-, p_1]\).

Proof. Let \( a \) be an arbitrary nonprolonged word; let \( a_1 \) and \( a_2 \) be words with one occurrence of the letter \( p_1 \) and such that \( a = a_1a_2 \).

Since \( a \) is an RR2-word, from [33] Lemma 3.1 it follows that \( a_1 = dx_1b \) and \( a_2 = dx_2c \), where \( d \), \( b \), and \( c \) are words, \( x_1 \) and \( x_2 \) are letters, and \( x_2 < x_1 \). By [33] Lemma 3.7, \( x_2 = m^- \). Since the regular word \( a \) is not prolonged, we have \( c = 1 \) and \( a_2 = dx_2 = dm^- \).

1. Assume that property 1) is not fulfilled. Let \( y \) be a single vertex of the graph \( D(|a|) \) adjacent to \( m^- \). Suppose \( |y| \leq |a_1| \). Then \( m^- \in [p_1, y] \leq |a_1| \), which is incorrect. Thus, \( |y| \leq |a_1| \), i.e., \( |y| \leq |a_2| = |dm^-| \).

All vertices of the graph \( D(|d|) \) are double in the graph \( D(|a|) \), and the vertex \( y \) is single in \( D(|a|) \). Therefore, \( |y| \leq |d| \). Moreover, \( y \neq m^- \). Consequently, \( |y| \leq |c| \), so that the word \( c \) is nonempty. We denote by \( \tilde{c} \) and \( \tilde{a}_2 \) the words obtained from \( c \) (respectively, from \( a_2 \)) by removing the last letter.

Then \( \tilde{a}_2 = dx_2\tilde{c} < dx_1b = a_1 \). Therefore, the word \( a_1\tilde{a}_2 \) is regular. Since the word \( \tilde{a}_2 \) is nonempty, we have \( a_1\tilde{a}_2 < a_1 \); thus, the word \( a_1^2\tilde{a}_2 \) is regular. Being a proper regular subword of the SR-word \( a_1^2\tilde{a}_2 \), \( a_1^2\tilde{a}_2 \) is an RR-word, and, thus, it has rooted composition, which is impossible because \( 3|p_1| \leq |a_1^2\tilde{a}_2| \). This contradiction shows that property 1) is fulfilled.

2. If \( m^- \in O \), then the structure of RR2-words for the word \( a \) shows that \( |a_2| = [m^-, x^+_2 (|a|)] \). Since \( a_2 = dm^- \), it follows that an RR1-word of composition \([m^-, x^+_2 (|a|)] \) ends with the letter \( m^- \), which implies property 2).

Conversely, let \( a \) be an RR2-word satisfying 1) and 2). Since \( a \) is an RR2-word, in accordance with [33] Lemma 3.1 we have \( a_1 = dx_1b \) and \( a_2 = dx_2c \), where \( d \), \( b \), and \( c \) are words, \( x_1 \) and \( x_2 \) are letters, and \( x_2 < x_1 \). By [33] Lemma 3.7, we have \( x_2 = m^- \). From properties 1) and 2) it follows that the word \( a_2 \) ends with the letter \( m^- \). Therefore, \( c = 1 \), \( a_2 = dm^- \), and the word \( a \) is not prolonged. □

Lemma 11.6. Associating the word \( r(u) \) with every word \( u \) in \( S(2,1,1) \), we obtain an injective mapping onto the set of nonprolonged RR2-words. The inverse mapping takes any nonprolonged RR2-word \( v \) to the word \( \tilde{r}(v)v \) (see Lemma 11.3).

Now we state the main results of the paper.

Theorem 11.7. The set of SR-words of the Lie algebra \( D^+_n \) is partitioned into the sets \( S(1,1,0), S(1,2,0), S(2,1,0), S(2,2,0), S(1,1,1), S(1,2,1), \) and \( S(2,1,1) \) described in statements 6.7, 7.5, 8.5, 8.8, 9.13, 10.4, and 11.0.
Theorem 11.8. The reduced Gröbner–Shirshov basis of the Lie algebra $D_4^+$ is precisely the set of relations of the form $[u] = 0$, where $u$ is an SR-word and $[u]$ is the element of the free Lie algebra obtained from $u$ by the regular arrangement of brackets.

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