HÖLDER FUNCTIONS ARE OPERATOR-HÖLDER

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Abstract. Let $A$ and $B$ be selfadjoint operators in a separable Hilbert space such that $A - B$ is bounded. If a function $f$ satisfies the Hölder condition of order $\alpha$, $0 < \alpha < 1$, i.e., $|f(x) - f(y)| \leq L|x - y|^{\alpha}$, then $\|f(A) - f(B)\| \leq CL\|A - B\|^\alpha$, where $C$ is a constant, specifically, $C = 2^{1-\alpha} + 2\pi \sqrt{\frac{1}{2}}$. This result is a consequence of a general inequality in which the norm of $f(A) - f(B)$ is controlled in terms of the continuity modulus of $f$. Similar results are true for the quasicommutators $f(A)K - Kf(B)$, where $K$ is a bounded operator.

Introduction

In this paper, the following natural question is resolved. Is it true that a function $f$ satisfying a Hölder condition of order $\alpha$ gives rise to an “operator-Hölder” function? In other words, do arbitrary selfadjoint operators on a Hilbert space satisfy the inequality

\[(1) \quad \|f(A) - f(B)\| \leq \text{const} \cdot \|A - B\|^\alpha?\]

We mention at once that, for $\alpha = 1$, the question was answered in the negative (see [1]). So, it seemed natural to suppose a similar feature for every $\alpha$. However, for $\alpha < 1$ the answer is in the positive. Specifically, if $A$ and $B$ are selfadjoint operators and $f$ belongs to the Hölder class with exponent $\alpha < 1$, then

\[(1') \quad \|f(A) - f(B)\| \leq CL\|A - B\|^\alpha,\]

where $C = 2^{1-\alpha} + 2\pi \sqrt{\frac{1}{2}}$.

A similar result was obtained almost simultaneously by Aleksandrov and Peller (see [2]) by different methods. Inequality (1’) is a consequence of a more general estimate in which the norm of $f(A) - f(B)$ is controlled in terms of the modulus of continuity of $f$ (see also [3]). The paper consists of two sections. In §1 the main lemmas are discussed (partially, they were proved in [3] and [4]). Mostly, they yield certain norm estimates for Hadamard–Schur multipliers. The theorems are proved in §2.

§1. Main lemmas

Recall that the Hadamard product of two matrices $M = (m_{ij})$ and $A = (a_{ij})$ is the matrix $M \circ A = (m_{ij}a_{ij})$. A matrix $M$ is called an Hadamard–Schur multiplier if

$$\|M\|_H = \sup \{\|M \circ A\| : A : l^2 \to l^2, \|A\| \leq 1\} < \infty.$$ 

In what follows, finite or infinite matrices are treated as operators on $l^2$, $\|M\|$ is the operator norm, and $\|M\|_H$ denotes the multiplier norm. Let $\varphi$ be a function defined on $\mathbb{Z}_+$ and taking values in $\mathbb{Z}_+ \cup \{\infty\}$. Put

$$\sigma_\varphi = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \varphi(j) \leq i\}$$

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and denote \( n = \text{card}(\varphi(\mathbb{Z}_+)) \) (i.e., \( n \) is the number of different values of \( \varphi \)); we agree that \( n = \infty \) if \( \varphi(\mathbb{Z}_+) \) is infinite). Next, let \( M_\varphi = \chi_{\sigma_\varphi} \) be the characteristic function (matrix) of \( \sigma_\varphi \), i.e., \( M_\varphi \) consists of entries of the form

\[
m_{ij} = \begin{cases} 1, & \varphi(j) \leq i, \\ 0, & \varphi(j) > i. \end{cases}
\]

The following lemma was proved in [4].

**Lemma 1.** The matrix \( M_\varphi \) is an Hadamard–Schur multiplier if and only if \( n < \infty \). In this case

\[
c \log(n + 1) \leq \|M_\varphi\|_H \leq 1 + \log n,
\]

where \( c \) is a universal constant.

**Remark 2.** Lemma 1 is a generalization of the well-known Matsaev triangle projection theorem; see [5].

**Remark 3.** The claim of Lemma 1 will not change if we take

\[
\sigma_\varphi = \{ (i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \varphi(i) \leq j \}.
\]

We also note that the lemma remains true for functions defined on \( \mathbb{Z} \).

We describe a particular case to be used in what follows. Fix two integers \( k \) and \( r \) (\( r > 0 \)) and consider the following matrix \( M_r = (m_{ij}) \) (the “block-triangle projection” multiplier):

1) all entries of \( M_r \) outside the vertical strip \( k < j \leq k + 2r \) are zero;
2) the strip \( (m_{ij})_{k < j < k + 2r} \) is decomposed in square blocks of size \( 2r \times 2r \) such that
3) every block is the upper (or lower) triangular projection of the square matrix of 1’s.

**Corollary 4** (to Lemma 1). We have

\[
\|M_r\|_H \leq r + 1.
\]

For the proof, it suffices to define \( \varphi \) in the following way. If \( m \) is such that \( m2^r < i \leq (m + 1)2^r \), we put \( \varphi(i) = i - m2^r \). By Lemma 1, we have \( \|M_\varphi\|_H \leq r + 1 \). Clearly, \( \|M_r\|_H \leq \|M_\varphi\|_H \).

Obviously, Corollary 4 can be extended to strips of arbitrary length. The next lemma is an elementary generalization of the Bennet lemma [6]. For completeness, we prove this lemma in the vector case. Let \( M = (m_{ij}) \) be a scalar matrix, and \( V = (V_{ij}) \) a block matrix consisting of square blocks of the same size. We denote by \( V_j \) the block “column matrix”, i.e., \( V_j = J \circ V \), where \( J = (a_{ik})_{i, k \in \mathbb{Z}_+} \); \( a_{ik} = 1 \), for \( k = j, i \in \mathbb{Z} \), and \( a_{ik} = 0 \) for \( k \neq j \).

**Lemma 5.** \( \|M \circ V\| \leq \sup_i \left( \sum_j |m_{ij}|^2 \right)^{\frac{1}{2}} \sup_j \|V_j\| \).

**Proof.** Consider the family \( (P_j) \) of projections in \( l_2(\mathbb{Z}) \) corresponding to the block decomposition of the matrix \( V \) (\( \sum P_j = I, P_k P_j = 0 \) for \( k \neq j \)). For \( X \in l_2(\mathbb{Z}) \), put
\[ X_j = P_j X. \] Then
\[
\|(M \circ V) X\|^2 = \sum_i \left\| \sum_j m_{ij} V_j X_j \right\|^2 \leq \sum_i \left( \sum_j |m_{ij}| \|V_j X_j\| \right)^2
\]
\[
\leq \sum_i \left( \sum_j |m_{ij}|^2 \sum_j \|V_j X_j\|^2 \right) \leq \sup_i \left( \sum_j |m_{ij}|^2 \right) \sum_i \sum_j \|V_j X_j\|^2
\]
\[
= \sup_i \left( \sum_j |m_{ij}|^2 \right) \sum_j \|V_j X_j\|^2 \leq \sup_i \left( \sum_j |m_{ij}|^2 \right) \sum_j \|V_j\|^2 \|X_j\|^2
\]
\[
\leq \sup_i \left( \sum_j |m_{ij}|^2 \right) \sup_j \|V_j\|^2 \|X\|^2.
\]

Now we fix a positive number \( h > 0 \) and put
\[
T_h = \begin{pmatrix}
\frac{1}{h} & \frac{1}{h+1} & \frac{1}{h+2} & \cdots \\
& \frac{1}{h+1} & \frac{1}{h+2} & \cdots \\
& & \frac{1}{h+2} & \cdots \\
& & & \cdots
\end{pmatrix},
H_h = \begin{pmatrix}
\frac{1}{h} & \frac{1}{h+1} & \frac{1}{h+2} & \cdots \\
\frac{1}{h+1} & \frac{1}{h+2} & \frac{1}{h+3} & \cdots \\
\frac{1}{h+2} & \frac{1}{h+3} & \cdots \\
& & & \cdots
\end{pmatrix}.
\]

The following lemma was proved in \[3\].

**Lemma 6.**
\[ \|T_h\|_H = \frac{1}{h}, \quad \|H_h\|_H \leq \frac{1}{h}. \]

**Remark 7 (to Lemma 6).** \( T_h \) is a Toeplitz matrix:
\[ T_h = (m_{ij})_{i,j \geq 1}, \quad m_{ij} = \frac{1}{|j-i| + h} = c_{i-j}, \quad c_n = \frac{1}{h + |n|}, \quad n \in \mathbb{Z}. \]

Consider the matrix \( Y_h = (m_{ij})_{i,j \leq 0} \) whose entries are defined by the same formulas.

The matrix \( Y_h \) has the form
\[
Y_h = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{h} & \frac{1}{h+1} & \frac{1}{h+2} & \cdots \\
& \frac{1}{h} & \frac{1}{h+1} & \cdots \\
& & \frac{1}{h+1} & \cdots \\
& & & \cdots
\end{pmatrix},
\]
and \( \|Y_h\|_H = \frac{1}{h} \).

We shall need yet another obvious statement; we formulate it as a lemma.

**Lemma 8.** Let \((a_i)_i\) be a bounded sequence, and let \( M = (m_{ij}) \), where \( m_{ij} = a_i \). Then
\[ \|M\|_H \leq \sup_i |a_i|. \]

The next lemma was proved in \[3\].

**Lemma 9.** Let \( A \) and \( B \) be commuting selfadjoint operators in a separable Hilbert space such that \( A - B \) is bounded, and let \( f \) be a continuous function on the reals. Let \( \omega_f \) denote the modulus of continuity of \( f \). Then
\[ \|f(A) - f(B)\| \leq \omega_f(\|A - B\|). \]
§2. THEOREMS

**Theorem 1.** Let $A$ and $B$ be selfadjoint operators in a separable Hilbert space; suppose $A - B$ is bounded. Next, let $f$ be a continuous function on the real line whose modulus of continuity is $\omega_f$. Then

$$\|f(A) - f(B)\| \leq 2\omega_f\left(\frac{\|A - B\|}{2}\right) + 2\pi\sqrt{\frac{\omega_f}{2}}\sum_{r=0}^{\infty} \frac{r + 1}{2^r} \omega_f(\|A - B\|2^r).$$

**Proof.** We split the real line into infinitely many intervals of equal length by points $\{\lambda_i\}$ with $\lambda_{i+1} - \lambda_i = \|A - B\| = d$ and define a piecewise constant function $\varphi$ as follows: we put $\varphi(t) = \lambda_i$ for $\lambda_i - \frac{d}{2} \leq t < \lambda_i + \frac{d}{2}$. Next, consider the operators $A_1 = \varphi(A)$ and $B_1 = \varphi(B)$. The spectra of $A$ and $B$ are included in the set $\{\lambda_i\}$, and moreover, $A$ commutes with $A_1$ and $B$ commutes with $B_1$.

Since $\|A - A_1\| \leq \frac{d}{2}$ and $\|B - B_1\| \leq \frac{d}{2}$, by Lemma 9 we have

$$(3) \quad \|f(A) - f(A_1)\| \leq \omega_f\left(\frac{d}{2}\right), \quad \|f(B) - f(B_1)\| \leq \omega_f\left(\frac{d}{2}\right).$$

Thus,

$$(4) \quad \|f(A) - f(B)\| \leq 2\omega_f\left(\frac{\|A - B\|}{2}\right) + \|f(A_1) - f(B_1)\|.$$ 

Our goal is to estimate the norm of $f(A_1) - f(B_1)$. We denote by $P_i$ and $Q_j$ the orthogonal projections onto the eigenspaces of $A_1$ and $B_1$ corresponding, respectively, to the points $\lambda_i$ and $\lambda_j$. (If $\lambda_i$ is not an eigenvalue, we agree that the corresponding projection is zero.) Since $P_iA_1 = A_1P_i = \lambda_iP_i$ and $Q_jB_1 = \lambda_jQ_j$, we have

$$P_iQ_j = \frac{P_i(A_1 - B_1)Q_j}{\lambda_i - \lambda_j}, \quad i \neq j.$$ 

Therefore, $f(A_1) - f(B_1)$ can be represented as follows:

$$(5) \quad f(A_1) - f(B_1) = \sum_{i,j} \frac{f(\lambda_j) - f(\lambda_i)}{\lambda_j - \lambda_i} P_i(A_1 - B_1)Q_j.$$ 

(By definition, $\frac{f(\lambda_j) - f(\lambda_i)}{\lambda_j - \lambda_i} = 0$ if $f(\lambda_j) - f(\lambda_i) = 0$.) We observe that formula (5) represents $f(A_1) - f(B_1)$ as a double operator integral (see [7] and also [8]). On the other hand, (5) allows us to view $f(A_1) - f(B_1)$ as an operator matrix $F = M \circ T$, where

$$M = \left(\frac{f(\lambda_j) - f(\lambda_i)}{\lambda_j - \lambda_i}\right)_{i,j \in \mathbb{Z}}, \quad T = (P_i(A_1 - B_1)Q_j)_{i,j \in \mathbb{Z}} = (T_{ij})_{i,j \in \mathbb{Z}}.$$ 

Thus, the problem reduces to estimating the norm of the multiplier $M$. The passage from an estimation of the norm $\|f(A) - f(B)\|$ to that of the multiplier $M$ will be called quantization of the spectrum. \hfill \Box

In the sequel, we consider the upper-triangular projection of the matrix $F$, to be denoted by $F_{\text{up}}$. Since $\lambda_j - \lambda_i = d \cdot (j - i)$, the matrix $F_{\text{up}}$ has the form

$$F_{\text{up}} = \frac{1}{d} \left(\frac{f(\lambda_j) - f(\lambda_i)}{j - i} T_{ij}\right)_{i,j \in \mathbb{Z}, \ j > i}$$ 

(all other entries are equal to zero). Clearly,

$$(6) \quad \|f(A_1) - f(B_1)\| \leq 2\|F_{\text{up}}\|.$$
We want to represent \( F_{up} \) as a sum \( F_{up} = \sum_{r=0}^{\infty} F_r \) of matrices of a special form. The matrices \( F_r \) are constructed as follows. Fixing two indices \( i \) and \( j \) (\( j \geq i + 1 \)), we expand \( j - i \) in decreasing powers of 2:

\[
\begin{align*}
    j - i &= 2^r_1 + 2^r_2 + \cdots + 2^r_j + 2^r_{j+1} + \cdots + 2^r_r \\
    (7)
\end{align*}
\]

(we assume that the summand \( 2^r \) does occur in (7)). Accordingly,

\[
\frac{f(\lambda_j) - f(\lambda_i)}{j - i} = \frac{f(\lambda_{i+2^r_1}) - f(\lambda_{i+2^r_1})}{j - i} + \cdots + \frac{f(\lambda_{i+2^r_1+2^r_j+2^r_{j+1}}) - f(\lambda_{i+2^r_1+2^r_j+2^r_{j+1}})}{j - i} + \cdots + \frac{f(\lambda_{i+2^r_1+2^r_j+2^r_{j+1}}) - f(\lambda_{i+2^r_1+2^r_j+2^r_{j+1}})}{j - i}.
\]

By definition, we put

\[
F_r = \frac{1}{d} \left( \frac{f(\lambda_{i+2^r_1+2^r_j+2^r_{j+1}}) - f(\lambda_{i+2^r_1+2^r_j+2^r_{j+1}})}{j - i} T_{ij} \right)_{j > i, i \in \mathbb{Z}}.
\]

In more detail, the matrix entries of \( F_r \) are of the following nature. For the numbers \( i + 2^r_1 + \cdots + 2^r_j + 2^r_{j+1} \) and \( i + 2^r_1 + \cdots + 2^r_{j+1} \), there is \( k \geq 1 \) such that

\[
i + 2^r_1 + \cdots + 2^r_j + 2^r_{j+1} = i + k2^{r+1} - 2^r, \quad i + 2^r_1 + \cdots + 2^r_j = i + k2^{r+1} - 2^r + 1.
\]

Thus, for \( i \) (the row number) fixed, the nonzero entries of \( F_r \) have the form

\[
\frac{1}{d} \cdot \frac{f(\lambda_{i+k2^{r+1}-2^r}) - f(\lambda_{i+k2^{r+1}-2^r})}{j - i} T_{ij}, \quad j > i.
\]

Clearly, the numerators of the Hadamard multipliers (for \( i \) and \( k \) fixed) are the same for all values of \( j \) in the interval \( i + k2^{r+1} - 2^r \leq j \leq i + k2^{r+1} - 1 \) of "length" \( 2^r \). If \( j \) satisfies \( i + (k - 1)2^{r+1} \leq j \leq i + k2^{r+1} - 2^r - 1 \), then the summand \( 2^r \) does not occur in the dyadic expansion of \( j - i \) (\( i \) is fixed). Therefore, the corresponding entries of \( F_r \) fill strips of width \( 2^r \) parallel to the main diagonal. The distance between the strips is also equal to \( 2^r \). For \( r = 2 \), the nonzero entries of \( F_r \) are shown by dots in Figure 1. We split

\[
\frac{f(\lambda_{i+2k-2}) - f(\lambda_{i+2k-2})}{j - i} P_i T Q_j, \quad i + 2^{r+1}k - 2^r \leq j \leq i + 2^{r+1}k - 1.
\]

\[
F_r \text{ into blocks of height } 2^r \text{ and width } 2 \cdot 2^r: F_r = \frac{1}{d}(B_{pk})_{p \in \mathbb{Z}, k \geq 1}, \text{ where } B_{pk} = (m_{ij}),
\]

\[
p2^{r+1} + 1 \leq i \leq (p + 1)2^r, \quad (p - 1)2^r + k2^{r+1} + 1 \leq j \leq (p + 1)2^r + 2k2^r.
\]

The nonzero entries \( m_{ij} \) of \( B_{pk} \) are of the form

\[
m_{ij} = \frac{f(\lambda_{i+2k-2}) - f(\lambda_{i+2k-2})}{j - i} P_i T Q_j, \quad i + 2^{r+1}k - 2^r \leq j \leq i + 2^{r+1}k - 1.
\]
Each block can be represented as the union of two blocks $B_{pk}^{(1)}$ and $B_{pk}^{(2)}$ as follows:

$$B_{pk}^{(1)} = (m_{ij}), \quad p2^r + 1 \leq i \leq (p + 1)2^r, \quad (p - 1)2^r + 2k2^r + 1 \leq j \leq p2^r + 2k2^r,$$

and the nonzero entries $m_{ij}$ of $B_{pk}^{(1)}$ are of the form

$$m_{ij} = f(\lambda_i + (2k - 1)2^r) - f(\lambda_i + (2k - 2)2^r)P_iTQ_j, \quad p2^r + 1 \leq i \leq (p + 1)2^r, \quad i + 2^r(2k - 1) \leq j \leq p2^r + k2^r + 1.$$ 

That is, the block $B_{pk}^{(1)}$ is an upper-triangular square matrix (including the main diagonal) of size $2^r \times 2^r$. The complementary block $B_{pk}^{(2)} = (m_{ij})$, 

$$p2^r + 1 \leq i \leq (p + 1)2^r, \quad (p + 2k)2^r + 1 \leq j \leq (p + 2k + 1)2^r,$$

is also a square matrix of size $2^r \times 2^r$, which is strictly lower-triangular. Its nonzero entries are of the same form as those of $B_{pk}^{(1)}$.

Thus, the matrix $F_r$ is split into square blocks of size $2^r \times 2^r$. We represent it as the sum $F_r = F_r^{(1)} + F_r^{(2)}$ of two block matrices of the following form. To obtain $F_r^{(1)}$, we replace the blocks $B_{pk}^{(2)}$ in $F_r$ with zero blocks of the same size and leave the blocks $B_{pk}^{(1)}$ as they are. Similarly, to obtain $F_r^{(2)}$, we replace the blocks $B_{pk}^{(1)}$ in $F_r$ with zero blocks, leaving the $B_{pk}^{(2)}$ intact. Clearly, $F_r = F_r^{(1)} + F_r^{(2)}$. Our nearest goal is to estimate the norm of the operator $F_r^{(1)}$. We observe that there is a zero block between each two neighboring blocks of $F_r^{(1)}$. We enumerate the nonzero blocks as described below. For fixed $p \in \mathbb{Z}$ (the number of a horizontal strip), the nonzero blocks are situated in vertical strips with numbers $m = p + 2k - 1$, $k \geq 1$. Each of these blocks has upper-triangular form. Their arrangement is clear from Figure 2 ($r = 2$).

Nonzero strips parallel to the main diagonal are enumerated by $k \geq 1$. In each strip (with $k$ fixed), we represent the blocks $B_{pk}^{(1)}$ in the form

$$B_{pk}^{(1)} = B_{m-(2k-1),k}^{(1)} = \frac{1}{2k - 1} B_{m-(2k-1),k}^{(1)} \cdot (2k - 1) = \frac{1}{2k - 1} D_{m-(2k-1),k}.$$
The matrix \( F^{(1)} \) takes the form
\[
\begin{pmatrix}
0 & D_{-11} & 0 & \frac{1}{3}D_{-12} & 0 & \frac{1}{5}D_{-13} & 0 & \cdots & \cdots & \cdots \\
0 & D_{01} & 0 & \frac{1}{3}D_{02} & 0 & \frac{1}{5}D_{03} & 0 & \cdots & \cdots & \cdots \\
0 & D_{11} & 0 & \frac{1}{3}D_{12} & 0 & \frac{1}{5}D_{13} & 0 & \cdots & \cdots & \cdots \\
0 & D_{21} & 0 & \frac{1}{3}D_{22} & 0 & \frac{1}{5}D_{23} & 0 & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
Clearly, it satisfies the assumptions of Lemma 5. Since \( \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \), we have
\[
\|F^{(1)}\| \leq \frac{\pi}{\sqrt{8}} \sup_m \|V^{(1)}_m\|,
\]
where \( V^{(1)}_m \), \( m \in \mathbb{Z} \), is the vertical strip cut out of the matrix \( F^{(1)} \) and consisting of blocks of size \( 2^r \times 2^r \); the nonzero blocks among them have been denoted by \( D_{pk} \), \( p = m - 2k + 1 \), \( k \geq 1 \). All blocks below the main diagonal are zero. For fixed \( k \geq 1 \) and \( p = m - 2k + 1 \), the block \( D_{pk} \) has the form \( D_{pk} = \frac{1}{m} F_{(m)}(m_{ij}) \), where
\[
m_{ij} = f(\lambda_{i+2k-1,2^r}) - f(\lambda_{i+2k-2,2^r})(2k-1)2^rT_{ij},
\]
\[
p2^r + 1 \leq i \leq (p + 1)2^r, \quad i + k2^{r+1} - 2^r \leq j \leq p2^r + 2^{r+1}k;
\]
and \( m_{ij} = 0 \) if \((p - 1)2^r + 1 + k2^r < j < i + k2^{r+1} - 2^r \).

We represent the matrix \( V^{(1)}_m \) as the Hadamard product \( V^{(1)}_m = V_m \circ T \), the entries of \( V_m \) being the Hadamard multipliers in (9). The quantities \( f(\lambda_{i+2k-1,2^r}) - f(\lambda_{i+2k-2,2^r}) \) do not depend on \( j \). Therefore, Lemma 8 is applicable, yielding
\[
\|V_m\| \leq \frac{1}{d2^r} \sup_{i,k} |f(\lambda_{i+2k-1,2^r}) - f(\lambda_{i+2k-2,2^r})| \cdot \|W_m\|_{\mathcal{H}},
\]
where \( W_m \) is the vertical strip consisting of triangular blocks \( G_{pk} \) whose nonzero entries are equal to \( \frac{(2k-1)2^r}{j-i} \):
\[
G_{pk} = \bigg(\frac{(2k-1)2^r}{j-i}\bigg)_{p2^r+1 \leq i \leq (p+1)2^r, \quad i + k2^{r+1} - 2^r \leq j \leq p2^r + 2^{r+1}k; \quad p = m - 2k + 1, \quad k \geq 1.}
\]
The entries below the diagonal are zero.

Since the indices of the points \( \lambda_{i+2k-1,2^r} \) and \( \lambda_{i+2k-2,2^r} \) differ by \( 2^r \), (10) leads to the inequality
\[
\|V_m\|_{\mathcal{H}} \leq \frac{\omega f(2d2^r)}{d2^r} \|W_m\|_{\mathcal{H}}, \quad d = \|A - B\|.
\]
We estimate the norm of the multiplier \( W_m \). First, we rewrite the nonzero block \( G_{pk} \), \( p = m - 2k + 1 \), in terms of \( m \) and \( k \). The nonzero entries \( \frac{(2k-1)2^r}{j-i} \) correspond to the following values of \( i \) and \( j \):
\[
(m - 2k + 1)2^r + 1 \leq i \leq (m - 2k + 2)2^r,
\]
\[
i + 2^{r+1}k - 2^r \leq j \leq (m + 1)2^r; \quad k \geq 1, \quad p = m - 2k + 1.
\]
The entries strictly below the diagonal are zero.

Since
\[
\frac{(2k-1)2^r}{j-i} = 1 - \frac{(j-i) - (2k-1)2^r}{j-i},
\]
the matrix \( W_m \) can be represented as the sum \( W_m = I_m + J_m \), where \( I_m \) is the multiplier of block-triangular projecting, as was described in Corollary 4 (it is supplemented with zero blocks).
The matrix $J_m$ also has the form of a vertical strip consisting of upper-triangular blocks of size $2^r \times 2^r$, but the entries of the upper triangle are $m_{ij} = (j-i) - (2k-1)2^r$. Corollary 4 leads to the estimate

$$\|I_m\|_H \leq r + 1.$$  

The nonzero blocks of the matrix $J_m$ (we denote them by $J_{mk}$) have the form

$$J_{mk} = \begin{pmatrix} 0 & \frac{1}{(2k-1)2^r + 1} & \frac{2}{(2k-1)2^r + 2} & \frac{3}{(2k-1)2^r + 3} & \cdots & \frac{2k-2}{2k-2} & \frac{2k-1}{2k-1} \\ 0 & 0 & \frac{1}{(2k-1)2^r + 1} & \frac{2}{(2k-1)2^r + 2} & \cdots & \frac{2k-2}{2k-2} & \frac{2k-1}{2k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (m-2k+2)2^r - 1 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{(2k-1)2^r + 1} \\ (m-2k+2)2^r & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} = J_{mk}.$$  

That is, all strips $J_m$ are similar, up to parallel translation along the main diagonal. Now, we represent the matrix $J_{mk}$ (a nonzero block) as a sum:

$$J_{mk} = \sum_{n=1}^{2^r-1} J_{mkn},$$  

where $J_{mkn} = (m_{ij})$,

$$\left(\begin{array}{c} (m-2k+1)2^r + 1 \leq i \leq (m-2k+2)2^r, \\
                        m2^r + 1 \leq j \leq (m+1)2^r. 
\end{array}\right)$$  

We have $m_{ij} = \frac{1}{2^r}$ for

$$\left(\begin{array}{c} (m-2k+1)2^r + 1 \leq i \leq (m-2k+2)2^r - n, \\
                        i + k2^{r+1} - 2^r + n \leq j \leq (m+1)2^r. 
\end{array}\right)$$  

For all other values of $i, j$, we have $m_{ij} = 0$. In other words,

$$J_{mkn} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{(2k-1)2^r + n} & \frac{1}{(2k-1)2^r + n + 1} & \cdots & \frac{1}{2k-2} \\ 0 & \cdots & 0 & \frac{1}{(2k-1)2^r + n} & \frac{1}{(2k-1)2^r + n + 1} & \cdots & \frac{2k-2}{2k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{(2k-1)2^r + n} \end{pmatrix}.$$  

Thus, the entire “block strip” $J_m$ is represented as the sum of $2^r - 1$ block strips $J_{mn}$, $J_m = \sum_{n=1}^{2^r-1} J_{mn}$. The nonzero blocks $J_{mkn}$ of the strip $J_{mn}$ are described by formulas (15) and (16). The lowest block corresponds to $k = 1$. 

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We return to Remark 7 (see §1). For \( n = 1, \ldots, 2^r - 1 \), we put \( h = \frac{1}{2^r + n} \) in the definition of the multiplier \( Y_h \), and consider the matrices \( Y_{nr} \):

\[
Y_{nr} = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots \\
\cdots & \frac{1}{2^r + n} & \frac{1}{2^r + n + 1} & \frac{1}{2^r + n + 2} \\
\cdots & \frac{1}{2^r + n + 1} & \frac{1}{2^r + n} & \frac{1}{2^r + n + 1} \\
\cdots & \frac{1}{2^r + n + 2} & \frac{1}{2^r + n + 1} & \frac{1}{2^r + n} \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 \\
\end{pmatrix}
\]

(\( n \) zero rows are added on the bottom; this does not change the multiplier norm). By Remark 7, \( \|Y_{nr}\|_H = \frac{1}{2^r + n} \).

Next, for every \( n = 1, \ldots, 2^r - 1 \), consider the multiplier \( I_{mn} \) of upper-triangular projecting (a generalized block-triangular projection), where the matrix \( I_{mn} \) is the “vertical strip” of width \( 2^r \) that is split into square blocks \( I_{mkn} \) of size \( 2^r \times 2^r \), of which the nonzero blocks are like \( J_{mkn} \) with the only difference that all nonzero entries of \( I_{mkn} \) are equal to 1. We agree that the lowest block corresponds to the index \( k = 1 \) in the block representation of \( J_{mn} \). The adjacent blocks are zero. That is, the “strip” \( I_{mn} \) has the following form (each block has \( n \) zero bottom rows):

\[
I_{mn} = \begin{pmatrix}
\vdots \\
0 \\
0 \ldots 0 & 1 & 1 \ldots 1 \\
0 \ldots 0 & 0 & 1 \ldots 1 \\
\vdots & \vdots & 0 & 0 \ldots \vdots \\
\vdots & \vdots & 0 & 0 \ldots \vdots \\
0 \ldots 0 & 0 & 0 \ldots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 \ldots 0 & 1 & 1 \ldots 1 \\
0 \ldots 0 & 0 & 1 \ldots 1 \\
\vdots & \vdots & 0 & 0 \ldots \vdots \\
\vdots & \vdots & 0 & 0 \ldots \vdots \\
0 \ldots 0 & 0 & 0 \ldots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 \ldots 0 & 0 & 0 \ldots 0 \\
\end{pmatrix}
\]

Moreover, \( I_{mn} \circ Y_{nr} = J_{mn} \), whence \( \|J_{mn}\|_H \leq \|I_{mn}\|_H \cdot \|Y_{mn}\|_H \).

Every nonzero block of the matrix \( I_{mn} \) has \( 2^r - n < 2^r \) nonzero diagonals. Therefore (see Corollary 4),

\[
\|J_{mn}\|_H \leq (r + 1) \frac{1}{2^r + n}.
\]
This leads to the estimate
\begin{equation}
\|J_m\|_H \leq \sum_{n=1}^{2^r-1} \frac{r + 1}{2^r + n} \leq (r + 1)^{2^r - 1 \over 2^r + 1} \leq r + 1.
\end{equation}

Taking (14) into account, we obtain
\[ \|W_m\|_H \leq 2(r + 1) \]
(because \(W_m = I_m + J_m\)). Thus, (12) turns into
\begin{equation}
\|V_m\|_H \leq \frac{2(r + 1)\omega_f(d2^r)}{d2^r}.
\end{equation}

Consequently, the norm of the vertical strip \(V^{(1)}_m = V_m \circ T\) cut out of the matrix \(F^{(1)}_r\) (see (8)) admits the following control in terms of the continuity modulus of \(f\):
\[ \|V^{(1)}_m\| \leq \frac{2(r + 1)\omega_f(2^r)}{d2^r} \|	ext{sup}_m\|V^{(2)}_m\| \leq \frac{2\omega_f(2^r)}{d2^r} \|A - B\| = 2d). \]

By (8), we obtain
\begin{equation}
\|F^{(1)}_r\| \leq \frac{\pi}{\sqrt{8}} \cdot \frac{4(r + 1)\omega_f(\|A - B\|2^r)}{2^r}.
\end{equation}

It remains to estimate the norm of \(F^{(2)}_r\). The corresponding blocks \(B^{(2)}_{pk}\) lie in the same vertical strips, but are shifted by \(2^r\) and have lower-triangular form (without the main diagonal). We represent the block \(B^{(2)}_{pk}\) in the form
\[ B^{(2)}_{pk} = \frac{1}{2^k} D_{m-2k,k}, \quad k \geq 1, \ p = m - 2k. \]

Since \(\sum_{k=1}^{\infty} \frac{1}{(2^k)^2} = \frac{\pi^2}{24}\), Lemma 5 implies the inequality
\[ \|F^{(2)}_r\| \leq \frac{\pi}{\sqrt{24}} \sup_m \|V^{(2)}_m\| < \frac{\pi}{\sqrt{8}} \sup_m \|V^{(2)}_m\|, \]
where \(V^{(2)}_m\) is a strip similar to \(V^{(1)}_m\).

The new blocks \(G_{pk}\) will be of lower-triangular form with the nonzero entries
\[ \frac{2k \cdot 2^r}{j - i} = \frac{2k \cdot 2^r - (j - i)}{j - i} + 1, \]
and the new blocks \(J_{mk}\) of the new strip \(J_m\) will look like this:
\[ J_{mk} = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
\frac{1}{2^k \cdot 2^r} & 0 & 0 & \ldots & 0 \\
\frac{1}{2^k \cdot 2^r - 2} & \frac{1}{2^k \cdot 2^r - 1} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
\frac{1}{2^k \cdot 2^r - 2} & \frac{1}{2^k \cdot 2^r - 1} & \ldots & \frac{1}{2^k \cdot 2^r - 1} & 0
\end{pmatrix}. \]

The norm of the new multiplier corresponding to \(J_{mn}\) is estimated as follows: \(\|J_{mn}\|_H \leq \frac{r + 1}{2^r + 1}\), independently of \(n\) (see Corollary 4 and Lemma 7). Thus, the operator \(F^{(2)}_r\) obeys (18). Consequently,
\begin{equation}
\|F_r\| \leq \frac{\pi}{\sqrt{8}} \cdot \frac{r + 1}{2^r} \omega_f(\|A - B\|2^r).
\end{equation}
Finally, by (4) and (6) we obtain
\begin{equation}
\|f(A) - f(B)\| \leq 2\omega_f \left( \frac{\|A - B\|}{2} \right) + 2\pi \sqrt{8} \sum_{r=0}^{\infty} \frac{r + 1}{2^r} \omega_f (\|A - B\| 2^r).
\end{equation}

If \( f \) belongs to the Hölder class of order \( \alpha, 0 < \alpha < 1 \), its continuity modulus satisfies the inequality \( \omega_f (\|A - B\| 2^r) \leq L \|A - B\| \alpha 2^r \). Therefore, the following theorem holds.

**Theorem 2.** Let \( A \) and \( B \) be selfadjoint operators on a separable Hilbert space; suppose \( A - B \) is bounded. If \( f \) is a Hölder function with \( 0 < \alpha < 1 \), then
\begin{equation}
\|f(A) - f(B)\| \leq CL \|A - B\| \alpha,
\end{equation}
where \( C \) is a universal constant, \( C = 2^{1-\alpha} + 2\pi \sqrt{8} \frac{1}{\alpha - 1} \).

**Remark on quasicommutators.** Almost without changes, the proofs of Theorems 1 and 2 can be extended to the case of quasicommutators.

A quasicommutator is an operator of the form \( f(A)K - Kf(B) \), where \( K \) is a nonzero bounded operator, \( A \) and \( B \) are selfadjoint operators, and \( f \) is a continuous function on the reals (its modulus of continuity is denoted by \( \omega_f \)). We assume that the operator \( AK - KB \) is bounded.

We split the real line by points \( \{\lambda_i\} \) in such a way that \( \lambda_{i+1} - \lambda_i = \frac{\|AK - KB\|}{\|K\|} = d \).

We have
\begin{equation}
|f(A)K - Kf(B)| \leq 2\|K\| \omega_f \left( \frac{d}{2} \right) + \|f(A_1)K - Kf(B_1)\|,
\end{equation}
\begin{equation}
|f(A_1)K - Kf(B_1)| = \sum_{i,j} \frac{f(\lambda_j) - f(\lambda_i)}{\lambda_j - \lambda_i} P_i(A_1K - KB_1)Q_j,
\end{equation}
where \( A_1 \) and \( B_1 \) are the same operators as before.

The matrix \( T \) in the representation \( F = M \circ T \) has the form
\[ T = (P_i(A_1K - KB_1)Q_j)_{i,j \in \mathbb{Z}} = (T_{ij})_{i,j \in \mathbb{Z}}. \]
The multiplier \( M \) remains the same as before. Thus, the preceding calculations lead to inequality (18), whence
\[ \|V_m^{(1)}\| \leq \frac{2(r + 1)\omega_f (d2^r)}{d2^r} \|T\|. \]
Since
\[
\|T\| \leq \|K\| \|A - A_1\| + \|K\| \|B - B_1\| + \|AK - KB\| \\
\leq \|K\| \cdot \frac{d}{2} + \|K\| \cdot \frac{d}{2} + \frac{\|AK - KB\|}{\|K\|} \|K\| \leq 2d \|K\|,
\]
we have
\[ \|V_m^{(1)}\| \leq \frac{4(r + 1)\omega_f (d2^r)}{2^r} \|K\| \]
and
\[ \|F_r\| \leq \frac{\pi \sqrt{8} (r + 1)\omega_f (\frac{\|AK - KB\|}{\|K\|} 2^r)}{2^r} \cdot \|K\|. \]
Finally,
\[
\|f(A)K - Kf(B)\| \\
\leq 2\|K\| \omega_f \left( \frac{\|AK - KB\|}{2\|K\|} \right) + 2\pi \sqrt{8} \|K\| \sum_{r=0}^{\infty} \frac{r + 1}{2^r} \omega_f \left( \frac{\|AK - KB\|}{\|K\|} 2^r \right).
\]
If \( f \) is Hölder continuous of order \( \alpha \), we have
\[
\|f(A)K - Kf(B)\| \leq 2L\|K\| \frac{\|AK - KB\|^\alpha}{2\alpha\|K\|^\alpha} \\
+ 2\pi\sqrt{8L} \sum_{r=0}^{\infty} \frac{r + 1}{2^r} \cdot \frac{\|AK - KB\|^\alpha}{\|K\|^\alpha} \cdot 2^{\alpha^2} \cdot \|K\|
\]
\[
= L\|K\|^{1-\alpha} \cdot \|AK - KB\|^\alpha \left(2^{1-\alpha} + 2\pi\sqrt{8} \sum_{r=0}^{\infty} \frac{r + 1}{2^r(1-\alpha)}\right)
\]
\[
= C \cdot L\|K\|^{1-\alpha} \|AK - KB\|^\alpha,
\]
where
\[
C = 2^{1-\alpha} + 2\pi\sqrt{8} \frac{1}{(1 - 2\alpha - 1)^2}.
\]

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