THE QUASINORMED NEUMANN–SCHATTEN IDEALS
AND EMBEDDING THEOREMS FOR THE GENERALIZED
LIONS–PEETRE SPACES OF MEANS

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Abstract. For the spaces $\varphi(X_0, X_1)_{p_0, p_1}$, which generalize the spaces of means introduced by Lions and Peetre to the case of functional parameters, necessary and sufficient conditions are found for embedding when all parameters (the function $\varphi$ and the numbers $1 \leq p_0, p_1 \leq \infty$) vary. The proof involves a description of generalized Lions–Peetre spaces in terms of orbits and co-orbits of von Neumann–Schatten ideals (including quasinormed ideals).

The first interpolation theorem for linear operators, proved by M. Riesz in [17], states practically that if a bounded linear operator $T$ sends simultaneously $L_{p_0}$ to $L_{q_0}$ and $L_{p_1}$ to $L_{q_1}$, where $p_0 \leq q_0$ and $p_1 \leq q_1$, then $T$ sends $L_p$, where $p_0 < p < p_1$, to the space $L_q$ such that the partition of the interval $[1/q_0, 1/q_1]$ by $1/q$ is similar to the partition of $[1/p_0, 1/p_1]$ by $1/p$. We also have the corresponding multiplicative estimate for the norm of the operator $T$ we interpolate. Several decades later it became clear that this theorem can be refined considerably. It turned out that $T$ actually sends $L_p$ to the Lorentz space $L_{q,p}$, which is smaller than $L_q$ if $p_0 < q_0$ or $p_1 < q_1$. Moreover, the space $L_{q,p}$ cannot be reduced if we consider all operators that map $\{L_{p_0}, L_{p_1}\}$ to $\{L_{q_0}, L_{q_1}\}$. The latter result belongs to Dikarev and Matsaev [4].

Thus, the problem of existence of the smallest target Banach space arose, for a given domain space $E$ and all bounded linear operators that map one couple $\{E_0, E_1\}$ of Banach spaces to another couple $\{F_0, F_1\}$ of Banach spaces. It turned out that such a space exists indeed; Aronszajn and Gagliardo [1] called this space the interpolation orbit of $E$ with respect to bounded linear operators mapping $\{E_0, E_1\}$ to $\{F_0, F_1\}$, denoting it by

$$\text{Orb}(E, \{E_0, E_1\} \to \{F_0, F_1\})$$

The Riesz theorem was extended by Thorin to arbitrary $L_p$; i.e., the conditions $p_0 \leq q_0$ and $p_1 \leq q_1$ were dropped. The sharp theorem in the general case appeared only in the 1980s. From the very beginning, that theorem was stated in the framework of the Lorentz spaces $L_{p,q}$. These spaces reflect the subtler behavior of measurable functions, and the family of all Lorentz spaces contains all $L_p$. It was proved in [11] that if a linear operator $T$ sends $L_{p_0,s_0}$ to $L_{q_0,t_0}$, and $L_{p_1,s_1}$ to $L_{q_1,t_1}$ for some $1 \leq p_0, p_1, q_0, q_1, s_0, s_1, t_0, t_1 \leq \infty$, then $T$ sends $L_{p,s}$ to $L_{q,t}$, where

$$\frac{1}{p} = (1 - \theta)/p_0 + \theta/p_1, \quad \frac{1}{q} = (1 - \theta)/q_0 + \theta/q_1,$$
$$\frac{1}{t} = 1/s + (1 - \theta)(1/t_0 - 1/s_0)_+ + \theta(1/t_1 - 1/s_1)_+,$$

for all $0 < \theta < 1$. (As usual $x_+ = \max(x, 0)$.)

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Since $L_p$ is a special case of the Lorentz space ($L_p = L_{p,p}$), we find, in particular, that if $T$ sends $L_{p_0}$ to $L_{q_0}$ and $L_{p_1}$ to $L_{q_1}$, then $T$ sends $L_p$ to $L_{q,r}$, where $p$ and $q$ are as above, and

$$1/r = (1 - \theta)/\min(p_0, q_0) + \theta/\min(p_1, q_1).$$

Moreover, it is impossible to refine $L_{q,r}$, i.e.,

$$\text{Orb}(L_{p,s}, \{L_{p_0,s_0}, L_{p_1,s_1}\} \to \{L_{q_0,t_0}, L_{q_1,t_1}\}) = L_{q,r},$$

for function spaces on $\mathbb{R}^n$.

If $p_0 \geq q_0$ and $p_1 \geq q_1$, then we see that the Riesz–Thorin theorem itself turns out to be sharp, i.e.,

$$\text{Orb}(L_p, \{L_{p_0}, L_{p_1}\} \to \{L_{q_0}, L_{q_1}\}) = L_q.$$ 

Note that in the case of function spaces defined on a finite interval, this result had been known before (see [5]).

The spaces $L_{p,s}$ with $p_0 < p < p_1$ are remarkable for their position between $L_{p_0}$ and $L_{p_1}$. They are situated strictly inside the family of all intermediate spaces between $L_{p_0}$ and $L_{p_1}$. In particular, any of $L_{p,s}$ can be inserted into an interpolation scale connecting $L_{p_0}$ and $L_{p_1}$. That is why at the early stages of the theory practically all interpolation theorems were obtained for these spaces.

The consideration of spaces in the vicinity of the end-point spaces $L_{p_0}$ and $L_{p_1}$ turned out to be possible much later. Typically, quasiconcave functions on $(0, \infty)$ play the role of the first parameter of such spaces. The Lorentz spaces $L_{p,s}$ or $L_p$ have $t^{1/p}$ as the function parameter. Possible irregularities in the behavior of arbitrary quasiconcave functions make the study of these spaces difficult. Nevertheless, the difficulties were overcome successfully by Janson in [6], and full range analogs $(X_0, X_1)_{\rho,r}$ of the Lorentz spaces with a quasiconcave function parameter $\rho$ were constructed and studied for arbitrary Banach couples $\{X_0, X_1\}$.

It turned out that the description of the target space for $(X_0, X_1)_{\rho,r}$ in sharp interpolation theorems requires involving new spaces $\varphi(X_0, X_1)_{r_0,r_1}$, where $1 \leq r_0, r_1 \leq \infty$, and $\varphi(s,t)$ is a positive, degree one homogeneous, and monotone increasing function of two positive variables $s$ and $t$. Such functions are in one-to-one correspondence with quasiconcave functions $\rho(t)$ ($\rho(t) = \varphi(1,t)$ and $\varphi(s,t) = s^\rho(t/s)$). In particular (see [12]), for any bounded linear operator $T$ sending $\{L_{p_0}, L_{p_1}\}$ to $\{L_{q_0}, L_{q_1}\}$ we have

$$T : (L_{p_0}, L_{p_1})_{p,\infty} \to (L_{q_0}, L_{q_1})_{r_0,r_1},$$

where $\varphi(1,t) = \rho(t)$ and

$$1/r_0 = (1/q_0 - 1/p_0)_+, \quad 1/r_1 = (1/q_1 - 1/p_1)_+.$$

Moreover,

$$\text{Orb}((L_{p_0}, L_{p_1})_{p,\infty}, \{L_{p_0}, L_{p_1}\} \to \{L_{q_0}, L_{q_1}\}) = \varphi(L_{q_0}, L_{q_1})_{r_0,r_1},$$

which means that the interpolation theorem stated above is sharp.

Thus, the spaces $(X_0, X_1)_{\rho,r}$ give rise to $\varphi(X_0, X_1)_{r_0,r_1}$, like the $L_p$ give rise to Lorentz spaces. The analogy is emphasized by the fact that if $\varphi(1,t) = \rho(t)$, then

$$(X_0, X_1)_{\rho,r} = \varphi(X_0, X_1)_{r,r}.$$ 

In general, $\varphi(X_0, X_1)_{r_0,r_1}$ do not reduce to classical spaces. However, if $X_0 = L_{p_0}$ and $X_1 = L_{p_1}$, then

$$\varphi(L_{p_0}, L_{p_1})_{p_0,p_1} = L_N,$$

where $L_N$ is the Orlicz space corresponding to $N(t)$ such that

$$N^{-1}(t) = \varphi(t^{1/p_0}, t^{1/p_1})$$

(see [12]).
Thus, we can view \( \varphi(X_0, X_1)_{p_0,p_1} \) as a special extension of Orlicz spaces to arbitrary Banach couples. Actually, for weighted \( L_p \) we have

\[
\varphi(L_{p_0}(w_0), L_{p_1}(w_1))_{p_0,p_1} = \varphi(L_{p_0}(w_0), L_{p_1}(w_1)),
\]
where \( 1 \leq p_0, p_1 \leq \infty \), and \( \varphi(E_0, E_1) \) is the Calderón–Lozanovskii construction for Banach lattices \( E_0 \) and \( E_1 \) (for the definition of the Calderón–Lozanovskii construction, see Theorem A in the Introduction). By the way, the latter equation explains the notation \( \varphi(X_0, X_1)_{p_0,p_1} \).

We are interested in the study of \( \varphi(X_0, X_1)_{p_0,p_1} \) not only because of sharp interpolation theorems. The theory of spaces of smooth functions involves analogs of Besov spaces for which the condition on the modulus of continuity is expressed in terms of a weighted Orlicz space. We trace the analogy between the Besov spaces of this type and \( \varphi(X_0, X_1)_{p_0,p_1} \) for appropriate Sobolev spaces \( X_0 \) and \( X_1 \). The study of the exact relationship between these constructions looks very promising.

The topic of the present paper is the study of \( \varphi(X_0, X_1)_{p_0,p_1} \). The main result is a criterion for the embedding

\[
\varphi(X_0, X_1)_{p_0,p_1} \subset \psi(X_0, X_1)_{q_0,q_1},
\]
where \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \). It turns out that some nontrivial embeddings are possible.

The first criterion of \( \psi \) for \( p_0 \geq q_0, p_1 \geq q_1 \), or \( p_0 \leq q_0, p_1 \leq q_1 \) was obtained in \( [8] \) and \( [15] \). In the general case, we have only a sufficient condition (see \( [8] \)); the results of the present paper show that this sufficient condition is far from being necessary.

It is of interest to compare \( \varphi(X_0, X_1)_{p_0,p_1} \) with the well-studied \( \psi(X_0, X_1)_{r,r} = (X_0, X_1)_{\sigma,r} \), where \( \sigma(t) = \psi(1,t) \). In this case we really meet the situations where

\[
\varphi(X_0, X_1)_{p_0,p_1} \subset \psi(X_0, X_1)_{r,r} \quad \text{or} \quad \varphi(X_0, X_1)_{r,r} \subset \psi(X_0, X_1)_{p_0,p_1},
\]
with \( p_0 < r < p_1 \). These cases were not covered by the results of \( [15] \).

The reduction of \( \psi \) to Hilbert couples works fine in the present paper, as it did in \( [8] \) and \( [15] \). We take advantage of the orbital description of \( \varphi(H_0, H_1)_{p_0,p_1} \), where \( \{H_0, H_1\} \) is a Hilbert couple. In the present paper we also use a supplementary description of \( \varphi(H_0, H_1)_{p_0,p_1} \) as a co-orbit with respect to the von Neumann–Schatten ideals (for general co-orbits with respect to operator ideals, see, e.g., \( [13] \)). Moreover, the study of \( \psi \) with arbitrary \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \) requires consideration and an orbital description of \( \varphi(H_0, H_1)_{p_0,p_1} \) for \( 0 < p_0, p_1 \leq \infty \). Here we use quasinormed von Neumann–Schatten ideals.

The paper is organized as follows. In the Introduction we present the necessary definitions and notation used in the paper. We consider the definition of \( \varphi(X_0, X_1)_{p_0,p_1} \) for \( 1 \leq p_0, p_1 \leq \infty \), and a new definition of \( \varphi(X_0, X_1)_{p_0,p_1} \) for \( 0 < p_0, p_1 \leq \infty \), and discuss the simplest properties of these spaces. §1 contains the proofs of the main theorems on description of \( \varphi(H_0, H_1)_{p_0,p_1} \) in terms of co-orbits for \( 1 \leq p_0, p_1 \leq \infty \), and in terms of orbits for \( 0 < p_0, p_1 \leq \infty \). §2 contains embedding theorems based on the descriptions of \( \varphi(H_0, H_1)_{p_0,p_1} \) obtained in §1. We find a criterion for the embedding \( \psi \) in terms of spaces conjugate to some quasinormed Orlicz sequence spaces.

**Introduction**

**Generalities on interpolation.** For basic information on interpolation of linear operators we refer to \( [2] \) and \( [13] \). Here we recall the definitions and notation that we intend to use in the paper.

A positive function \( \varphi(s,t) \) of two positive variables \( s \) and \( t \) is called an interpolation function if \( \varphi(s,t) \) is homogeneous of degree one and monotone increasing in \( s \) and \( t \). The function \( \rho(t) = \varphi(1,t) \), which determines \( \varphi(s,t) \) uniquely, is said to be quasiconcave.
(it is equivalent to a monotone increasing concave function). Recall that two positive functions \( f(x) \) and \( g(x) \) are equivalent on their common domain if for all \( x \) we have
\[
cf(x) \leq g(x) \leq Cf(x)
\]
for some positive constants \( c \) and \( C \). We denote this equivalence by \( f \asymp g \) or \( f(x) \asymp g(x) \).

Suppose that \( \varphi \in \Phi_0 \), which means that \( \varphi(1, t) \to 0 \) and \( \varphi(t, 1) \to 0 \) as \( t \to 0 \). Let \( \{t_n\} \) be a balanced sequence corresponding to \( \rho(t) \) (see [3]). In the terminology of [6], a balanced sequence is a filling and uniformly sparse sequence for the function \( \rho \). We do not reproduce these definitions here because we are not going to use them in what follows. We shall be concerned with a specific balanced sequence. We build a sequence \( \{t_n\} \) by induction with the help of the Oskolkov–Janson construction (see, e.g., [4])

\[
t_0 = 1 \quad \text{and} \quad \min \left( \frac{\rho(t_{n+1})}{\rho(t_n)}, \frac{t_{n+1}\rho(t_n)}{t_n\rho(t_{n+1})} \right) = q > 1.
\]

The set of \( n \) for which we construct \( t_n \) is an interval \( \mathcal{N} \) in the set \( \mathbb{Z} \) of integers. If \( \mathcal{N} = \mathbb{Z} \), then \( \rho(t) \) or the corresponding interpolation function \( \varphi(s, t) \) is said to be nondegenerate.

As usual, we denote by \( \{X_0, X_1\} \) a couple of Banach spaces. If \( x \in X_0 + X_1, \, s > 0, \, t > 0 \), then we define

\[
K(s, t; \{X_0, X_1\}) = \inf_{x = x_0 + x_1} s\|x_0\|_{X_0} + t\|x_1\|_{X_1},
\]

where the infimum is taken over all representations of \( x \) as a sum of \( x_0 \in X_0 \) and \( x_1 \in X_1 \).

Obviously, the function \( K(s, t, \ldots) \) is an interpolation function. Moreover, it is concave. We also denote by \( K(t, x; \{X_0, X_1\}) \) the function \( K(1, t; \{X_0, X_1\}) \). It is known as the \( K \)-functional.

Let \( \varphi \in \Phi_0 \). The main property of any balanced sequence of \( \rho(t) = \varphi(1, t) \) is the equivalence (see [13])

\[
K(s, t, \{\rho(t_n)\}, \{l_{p_0}, l_{p_1}(t_n^{-1})\}) \asymp \varphi(s, t)
\]

for all \( 1 \leq p_0, p_1 \leq \infty \), where, as usual, \( l_p \) denotes the standard sequence space with \( 0 < p \leq \infty \), considered on an interval \( \mathcal{N} \subset \mathbb{Z} \) (we drop the interval in the notation). We denote by \( l_p(t_n^{-1}) \) the weighted space of sequences \( \{x_n\} \) such that \( \{x_n t_n^{-1}\} \in l_p \), with the natural quasinorm.

### Definition of the generalized spaces of means

Suppose that \( \{X_0, X_1\} \) is a couple of Banach spaces, \( \rho(t) \) is a quasiconcave function such that \( \varphi \in \Phi_0 \), and \( 1 \leq p_0, p_1 \leq \infty \). The space \( \varphi(X_0, X_1)_{p_0,p_1} \) is defined to be the space of elements \( x \in X_0 + X_1 \) such that

\[
x = \sum_{n \in \mathcal{N}} \rho(t_n)x_n \quad \text{(convergence in } X_0 + X_1),
\]

where \( x_n \in X_0 \cap X_1 \), \( \{\|x_n\|_{X_0}\} \in l_{p_0} \), and \( \{t_n\|x_n\|_{X_1}\} \in l_{p_1} \) (see [13]).

The norm on \( \varphi(X_0, X_1)_{p_0,p_1} \) is defined naturally. For the function \( \varphi(s, t) = s^{1-\theta}t^\theta \), where \( 0 < \theta < 1 \), these spaces were introduced by Lions and Peetre and were called the spaces of means (see [9]). In this case, we can take \( t_n = 2^n \) as a balanced sequence, independently of \( 0 < \theta < 1 \).

The definition of \( \varphi(X_0, X_1)_{p_0,p_1} \) turns out to be independent of the choice of a balanced sequence \( \{t_n\} \). This follows from the theorem below, which also gives us the description of \( \varphi(X_0, X_1)_{p_0,p_1} \) in terms of the \( K \)-functional.

### Theorem A (see [12])

For all \( \varphi \in \Phi_0 \) and \( 1 \leq p_0, p_1 \leq \infty \), the space \( \varphi(X_0, X_1)_{p_0,p_1} \) consists of all \( x \in X_0 + X_1 \) such that

\[
\{K(w_k, x, \{X_0, X_1\})\} \in \varphi(l_{p_0}, l_{p_1}(w_k^{-1})),
\]
where \( \{w_k\} \) is a balanced sequence of \( K(t,x,\{X_0, X_1\}) \). (Recall that \( \varphi(E_0,E_1) \) denotes the Calderón–Lozanovskii construction.)

The Calderón–Lozanovskii construction \( \varphi(E_0,E_1) \), where \( E_0 \) and \( E_1 \) are quasi-Banach lattices of measurable functions, is defined to be the space of all measurable functions \( f \) such that \( |f| = \varphi(|f_0|,|f_1|) \), where \( f_0 \in E_0 \) and \( f_1 \in E_1 \). In particular, if \( E_0 = L_{p_0}^0 \) and \( E_1 = L_{p_1}^1 \) for \( 0 < p_0, p_1 \leq \infty \), we obtain \( \varphi(L_{p_0}^0,L_{p_1}^1) = L_N^\varphi \), where \( L_N^\varphi \) is the Orlicz space generated by the function \( N(t) \) such that \( N(\varphi(t^{1/p_0},t^{1/p_1})) = t \). In the case of weighted sequence spaces, we have \( \varphi(l_p(\alpha_n),l_p(\beta_n)) = l_p(\varphi^*(\alpha_n,\beta_n)) \), where \( \varphi^*(s,t) = 1/\varphi(1/s,1/t) \). (See, e.g., [2] or [13].)

Theorem A is a source of the main definition of the paper, namely, the definition of \( \varphi(X_0, X_1)_{p_0,p_1} \) for \( 0 < p_0, p_1 \leq \infty \).

**Definition 1.** Suppose that \( \{X_0, X_1\} \) is a Banach couple, \( \varphi \in \Phi_0 \), and \( 0 < p_0, p_1 \leq \infty \). The space \( \varphi(X_0, X_1)_{p_0,p_1} \) is defined to be the space of \( x \in X_0 + X_1 \) such that

\[
\{K(w_k, x, \{X_0, X_1\})\} \in \varphi(l_{p_0}, l_{p_1}(w_k^{-1}))
\]

where \( \{w_k\} \) is a balanced sequence of \( K(t,x,\{X_0, X_1\}) \).

It should be noted that the interpolation property of \( \varphi(X_0, X_1)_{p_0,p_1} \) or even the property for this set to be a vector space are not immediate. We shall deduce these properties and the \( K \)-monotonicity of \( \varphi(X_0, X_1)_{p_0,p_1} \) from the orbital description of \( \varphi(H_0, H_1)_{p_0,p_1} \) for Hilbert couples \( \{H_0, H_1\} \), obtained in §1.

In what follows we shall also use the criterion, found in [13], for the embedding \( \varphi(1,t_n) \) with \( 1 \leq q_0 \leq p_0 \leq \infty \) and \( 1 \leq q_1 \leq p_1 \leq \infty \). Under these conditions, \( \varphi(1,t_n) \) is equivalent to \( \varphi(1,t_n) \in \psi(l_{q_0}, l_{r_1}(t_n^{-1})) \), where \( 1/r_0 = 1/q_0 - 1/p_0 \) and \( 1/r_1 = 1/q_1 - 1/p_1 \), and \( t_n \) is a balanced sequence of \( \varphi(1,t) \).

**Orbits and co-orbits with respect to the von Neumann–Schatten ideals.** If both spaces in a Banach couple are Hilbert spaces, i.e., \( X_0 = H_0 \) and \( X_1 = H_1 \), then \( \{H_0, H_1\} \) is called a Hilbert couple.

Suppose that \( H \) and \( G \) are Hilbert spaces, and \( p < \infty \). We denote by \( S_p(H \to G) \) the space of linear operators that map \( H \) to \( G \) and satisfy

\[
\text{tr}(T^*T)^{p/2} < \infty,
\]

where \( \text{tr} \) denotes the trace of an operator in \( H \). If \( p = \infty \), then it is convenient to denote by \( S_\infty(H \to G) \) the space of all bounded linear operators acting from \( H \) to \( G \).

We consider operators that map the Hilbert couple \( \{H_0, H_1\} \) to the Hilbert couple \( \{G_0, G_1\} \) and belong to \( S_{p_0}(H_0 \to G_0) \cap S_{p_1}(H_1 \to G_1) \).

The interpolation orbit of an element \( a \in H_0 + H_1 \) with respect to the ideal \( S_{p_0}(H_0 \to G_0) \cap S_{p_1}(H_1 \to G_1) \) is defined to be the set

\[
\text{Orb}(a, S_{p_0}(H_0 \to G_0) \cap S_{p_1}(H_1 \to G_1)) \subset G_0 + G_1
\]

that consists of all elements \( Ta \) with \( T \in S_{p_0}(H_0 \to G_0) \cap S_{p_1}(H_1 \to G_1) \).

In the paper [8], it was proved that if \( \varphi \) is a nondegenerate interpolation function and \( 1 \leq p_0, p_1 \leq \infty \), then

\[
\varphi(H_0, H_1)_{p_0,p_1} = \text{Orb}(a_\varphi, S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)),
\]

where \( \{F_0, F_1\} \) is an arbitrary Hilbert couple and \( a_\varphi \in F_0 + F_1 \) is such that \( \varphi(s,t) \asymp K(s,t, a_\varphi, \{F_0, F_1\}) \).

One of the main results of the present paper is the proof of relation (4) for all \( 0 < p_0, p_1 \leq \infty \).
Obviously, if \( \mathbf{1} \) is valid for \( 0 < p_0, p_1 \leq \infty \), then \( \varphi(H_0, H_1)_{p_0, p_1} \) is a vector space for \( 0 < p_0, p_1 \leq \infty \). Moreover, \( \mathbf{1} \) shows that \( \varphi(H_0, H_1)_{p_0, p_1} \) is a \( K \)-monotone interpolation space, i.e., that \( x \in \varphi(H_0, H_1)_{p_0, p_1} \) and

\[
K(t, y, \{H_0, H_1\}) \leq K(t, x, \{H_0, H_1\})
\]

imply that \( y \in \varphi(H_0, H_1)_{p_0, p_1} \). Indeed, \( x \in \varphi(H_0, H_1)_{p_0, p_1} \) means \( x = T(a_\varphi) \) for some \( T \in S_{p_0}(F_0 \rightarrow H_0) \cap S_{p_1}(F_1 \rightarrow H_1) \), by \( \mathbf{1} \). By the Sadaev theorem (see \( \mathbf{10} \)), condition (5) means that there exists a bounded linear operator \( U : \{H_0, H_1\} \rightarrow \{H_0, H_1\} \) such that \( y = U(x) \). Thus, \( y = UT(a_\varphi) \). Hence, \( y \in \varphi(H_0, H_1)_{p_0, p_1} \), because \( UT \in S_{p_0}(F_0 \rightarrow H_0) \cap S_{p_1}(F_1 \rightarrow H_1) \) and \( \mathbf{1} \) is true.

The same properties are valid for the space \( \varphi(X_0, X_1)_{p_0, p_1} \) for each Banach couple \( \{X_0, X_1\} \). We discuss in detail the \( K \)-monotonicity only. The vector space structure of \( \varphi(X_0, X_1)_{p_0, p_1} \) can be established similarly. By the \( K \)-monotonicity of \( \varphi(H_0, H_1)_{p_0, p_1} \), there exists a lattice \( E \) of sequences such that \( x \in \varphi(H_0, H_1)_{p_0, p_1} \) is equivalent to \( \{K(2^n, x, \{H_0, H_1\})\} \in E \) for each Hilbert couple \( \{H_0, H_1\} \). This implies easily that for all quasiconcave functions \( \psi(t) \) the conditions \( \{\psi(2^n)\} \in E \) and \( \{\psi(u_m)\} \) for \( (t, p_0, \psi(t)) \) is a balanced sequence of \( \psi \), are equivalent. Now, by the lattice structure of \( E \), the \( K \)-monotonicity of \( \varphi(X_0, X_1)_{p_0, p_1} \) follows readily for each Banach couple. Note that the case of \( 0 \leq p_0, p_1 \leq \infty \) was considered in \( \mathbf{7} \) without any appeal to Hilbert couples.

Finally, we recall the definition of the co-orbits with respect to the Neumann–Schatten ideals (see \( \mathbf{13} \)).

Suppose that \( \{F_0, F_1\} \) is a Hilbert couple, \( 1 \leq p_0, p_1 \leq \infty \), and \( \varphi \) is a nondegenerate interpolation function. The co-orbit \( \text{Corb} \varphi(S_{p_0}(H_0 \rightarrow F_0) \cap S_{p_1}(H_1 \rightarrow F_1)) \) is defined to be the space of \( x \in H_0 + H_1 \) such that

\[
\sup \|T(x)\|_{\varphi(F_0, F_1)_{1,1}} < \infty,
\]

where the supremum is taken over the unit ball of \( S_{p_0}(H_0 \rightarrow F_0) \cap S_{p_1}(H_1 \rightarrow F_1) \). (We employ the usual notation \( p_0' \) and \( p_1' \) for the conjugate exponents of \( p_0 \) and \( p_1 \), i.e., \( 1/p_0' = 1 - 1/p_0 \), \( 1/p_1' = 1 - 1/p_1 \).)

\section{1. Description of Generalized Spaces of Means in Terms of Orbits and Co-orbits}

Recall that a Banach couple \( \{X_0, X_1\} \) is said to be \( K_0 \)-abundant (see \( \mathbf{2} \)) if for each interpolation function \( \psi \in \Phi_0 \) there exists \( x \in X_0 + X_1 \) such that \( \psi(s, t) \simeq K(s, t, x, \{X_0, X_1\}) \). It can be shown (see \( \mathbf{2} \)) that it suffices to find the corresponding \( x \in X_0 + X_1 \) for some nonlinear power function \( s^{1-\theta} t^\theta \) only.

\textbf{Theorem 1.} If \( \varphi(s, t) \) is a nondegenerate interpolation function and \( 1 \leq p_0, p_1 \leq \infty \), then

\[\varphi(H_0, H_1)_{p_0, p_1} = \text{Corb} \varphi(S_{p_0}(H_0 \rightarrow F_0) \cap S_{p_1}(H_1 \rightarrow F_1)),\]

where \( \{F_0, F_1\} \) is a \( K_0 \)-abundant Hilbert couple, and \( p_0', p_1' \) are the conjugate exponents for \( p_0 \) and \( p_1 \).

\textbf{Proof.} Let \( x \in \text{Corb} \varphi(S_{p_0}(H_0 \rightarrow F_0) \cap S_{p_1}(H_1 \rightarrow F_1)) \). In particular, this implies that the orbit of \( x \) with respect to the operators \( T \in S_{p_0}(H_0 \rightarrow F_0) \cap S_{p_1}(H_1 \rightarrow F_1) \) is contained in \( \varphi(F_0, F_1)_{p_0', p_1'} \). This orbit is equal to \( \chi(F_0, F_1)_{p_0', p_1'} \) by the results of \( \mathbf{8} \), where \( \chi(s, t) \) denotes \( K(s, t, x, \{H_0, H_1\}) \). Thus, we obtain the embedding

\[\chi(F_0, F_1)_{p_0', p_1'} \subset \varphi(F_0, F_1)_{1,1} \]
for a $K_0$-abundant couple $\{F_0, F_1\}$. It is easily seen that this particular embedding implies similar embeddings for all Hilbert couples. The criterion mentioned above (see [15]) shows that the embeddings for all Hilbert couples are equivalent to $\{\chi(1, w_k) \in \varphi(l_{r_k}, l_r(w_k^{-1}))\}$, where $1/r_0 = 1 - 1/p_0'$, $1/r_1 = 1 - 1/p_1'$, and $\{w_k\}$ is a balanced sequence of $\chi$. Hence, $x \in \varphi(H_0, H_1)_{p_0,p_1}$ by Theorem A.

Now, let $x \in \varphi(H_0, H_1)_{p_0,p_1}$, and let $T \in S_{p_0'}(H_0 \to F_0) \cap S_{p_1'}(H_1 \to F_1)$. Then, by the orbital description of $\varphi(H_0, H_1)_{p_0,p_1}$, there exists $U \in S_{p_0'}(\widehat{F_0} \to H_0) \cap S_{p_1'}(\widehat{F_1} \to H_1)$ such that $x = U(a_\varphi)$, where $K(s,t,a_\varphi, \{\widehat{F_0}, \widehat{F_1}\}) \approx \varphi(s,t)$. Therefore $T(x) = TU(a_\varphi)$, and

$$||TU||_{S_{p_0}(\widehat{F_0} \to F_0)} \leq ||U||_{S_{p_0}(\widehat{F_0} \to H_0)} ||T||_{S_{p_0'}(H_0 \to F_0)}.$$ 

Similarly,

$$||TU||_{S_{p_1}(\widehat{F_1} \to F_1)} \leq ||U||_{S_{p_1}(\widehat{F_1} \to H_1)} ||T||_{S_{p_1'}(H_1 \to F_1)}.$$ 

Applying the orbital description of $\varphi(F_0, F_1)_{1,1}$, i.e., $\varphi(F_0, F_1)_{1,1} = \text{Orb}(a_\varphi, S_1(\widehat{F_0} \to F_0) \cap S_1(\widehat{F_1} \to F_1))$, we get

$$||T(x)||_{\varphi(F_0, F_1)_{1,1}} \leq C \max(||T||_{S_{p_0'}(H_0 \to F_0)}, ||T||_{S_{p_1'}(H_1 \to F_1)}),$$

where $C$ is independent of $T$. Hence, $x \in \text{Corb}(\varphi, S_{p_0'}(H_0 \to F_0) \cap S_{p_1'}(H_1 \to F_1))$.

This completes the proof of Theorem 1.

**Theorem 2.** Suppose $0 < p_0, p_1 \leq \infty$ and $\varphi \in \Phi_0$; then

$$\text{Orb}(a_\varphi, S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)) = \varphi(H_0, H_1)_{p_0,p_1},$$

where $K(t, a_\varphi, \{F_0, F_1\}) \approx \rho(t) = \varphi(1, t)$.

**Proof.** To simplify the notation, we write Orb instead of $\text{Orb}(a_\varphi, S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1))$. Recall that this theorem was already proved for $1 \leq p_0, p_1 \leq \infty$; see [8]. We shall consider the case of $1/2 \leq p_0, p_1 \leq \infty$ in detail.

Let $x \in \text{Orb}$, which means that $x = T_0 a_\varphi$, where $T \in S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)$. We “split” $T$, i.e., represent $T$ as a product $TT'$, where $T'' \in S_{2p_0}(F_0 \to G_0) \cap S_{2p_1}(F_1 \to G_1)$ and $T'' \in S_{2p_0}(G_0 \to H_0) \cap S_{2p_1}(G_1 \to H_1)$, for some Hilbert couple $\{G_0, G_1\}$. Such a factorization is really possible (e.g., see [13]).

The image of $a_\varphi$ under the operator $T''$ is contained in $\varphi(G_0, G_1)_{2p_0,2p_1}$, by the orbital description of generalized spaces of means for $1 \leq 2p_0, 2p_1 \leq \infty$. Furthermore, $x \in \varphi_x(H_0, H_1)_{2p_0,2p_1}$, where $\varphi_x(s,t) = K(s,t, a_\varphi, \{G_0, G_1\})$, and $\xi = T''(a_\varphi)$, because $x = T''(\xi)$. Hence,

$$x \in \bigcup_{\xi \in \varphi(G_0, G_1)_{2p_0,2p_1}} \varphi_x(H_0, H_1)_{2p_0,2p_1}.$$ 

Clearly, the above arguments can be reversed. Therefore,

$$\text{Orb} = \bigcup_{\xi \in \varphi(G_0, G_1)_{2p_0,2p_1}} \varphi_x(H_0, H_1)_{2p_0,2p_1}.$$ 

Thus, in order to prove that $\text{Orb} = \varphi(H_0, H_1)_{p_0,p_1}$ for $1/2 \leq p_0, p_1 \leq \infty$, we need to verify the identity

$$\bigcup_{\xi \in \varphi(G_0, G_1)_{2p_0,2p_1}} \varphi_x(H_0, H_1)_{2p_0,2p_1} = \varphi(H_0, H_1)_{p_0,p_1}$$

for $1/2 \leq p_0, p_1 \leq \infty$ provided that it is valid for $1 \leq p_0, p_1 \leq \infty$. 

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If we denote by $\Psi(\varphi, p_0, p_1)$ the set of all quasiconcave functions $\psi \in \Phi_0$ such that 
$\{\psi(u_m)\} \in \varphi(l_{p_0}, l_{p_1}(u_{m}^{-1}))$, where $u_m$ is a balanced sequence of $\psi$, then for all Hilbert couples is equivalent to 

$$\Psi(\varphi, p_0, p_1) = \bigcup_{\varphi_\xi \in \Psi(\varphi, 2p_0, 2p_1)} \Psi(\varphi_\xi, 2p_0, 2p_1).$$

Indeed, it is easily seen that $x \in \varphi(X_0, X_1)_{p_0, p_1}$ is equivalent to $K(t, x, \{H_0, H_1\}) \in \Psi(\varphi, p_0, p_1)$. Observing that the set of functions $K(t, x, \{H_0, H_1\})$ with respect to all regular Hilbert couples coincides with $\Phi_0$ up to equivalence, we establish the desired result.

Thus we conclude that

$$\Psi(\varphi, \bar{p}_0, \bar{p}_1) = \bigcup_{\varphi_\xi \in \Psi(\varphi, 2\bar{p}_0, 2\bar{p}_1)} \Psi(\varphi_\xi, 2\bar{p}_0, 2\bar{p}_1)$$

for $1 \leq \bar{p}_0, \bar{p}_1 \leq \infty$.

We take $\lambda > 0$ and apply the power transformation $\varphi \mapsto \varphi^{(\lambda)} = \varphi^\lambda(s^{1/\lambda}, t^{1/\lambda})$ to relation (9).

Then

$$(\Psi(\varphi, \bar{p}_0, \bar{p}_1))^{(\lambda)} = \bigcup_{\varphi_\xi \in \Psi(\varphi, 2\bar{p}_0, 2\bar{p}_1)} (\Psi(\varphi_\xi, 2\bar{p}_0, 2\bar{p}_1))^{(\lambda)}.$$

We claim that if $\psi \in \Psi(\varphi, \bar{p}_0, \bar{p}_1)$, then $\psi^{(\lambda)}$ belongs to $\Psi(\varphi^{(\lambda)}, \bar{p}_0/\lambda, \bar{p}_1/\lambda)$. Indeed, $\psi \in \Psi(\varphi, \bar{p}_0, \bar{p}_1)$ means that $\{\psi(u_m)\} \in \varphi(l_{\bar{p}_0}, l_{\bar{p}_1}(u_{m}^{-1}))$, i.e., $\psi(u_m) = \varphi(\alpha_0^\lambda u_m, \alpha_1^\lambda u_m)$, where $\alpha_0^\lambda \in l_{\bar{p}_0}$ and $\alpha_1^\lambda \in l_{\bar{p}_1}$. Hence,

$$\psi^{(\lambda)}(u_m^\lambda) = \varphi^{(\lambda)}((\alpha_0^\lambda)^\lambda, (\alpha_1^\lambda)^\lambda u_m^\lambda),$$

where, clearly, $\{((\alpha_0^\lambda)^\lambda)\} \in l_{\bar{p}_0}/\lambda$ and $\{((\alpha_1^\lambda)^\lambda)\} \in l_{\bar{p}_1}/\lambda$, which yields

$$\{\psi^{(\lambda)}(u_m^\lambda)\} \in \varphi^{(\lambda)}(l_{\bar{p}_0}/\lambda, l_{\bar{p}_1}/\lambda(u_{m}^{-1})).$$

Since $u_m$ is a balanced sequence of $\psi$, we conclude that $\{u_m^\lambda\}$ is a balanced sequence of $\psi^{(\lambda)}$. Consequently,

$$(\Psi(\varphi, \bar{p}_0, \bar{p}_1))^{(\lambda)} = \Psi(\varphi^{(\lambda)}, \bar{p}_0/\lambda, \bar{p}_1/\lambda).$$

We choose $\lambda$ so that $\bar{p}_0/\lambda \geq 1$ and $\bar{p}_1/\lambda \geq 1$. Then (10) implies that

$$\Psi(\varphi^{(\lambda)}, p_0, p_1) = \bigcup_{\varphi_\xi \in \Psi(\varphi, 2\bar{p}_0, 2\bar{p}_1)} \Psi(\varphi_\xi^{(\lambda)}, 2\bar{p}_0/\lambda, 2\bar{p}_1/\lambda).$$

As has already been shown, the condition $\varphi_\xi \in \Psi(\varphi, 2\bar{p}_0, 2\bar{p}_1)$ is equivalent to $\varphi_\xi^{(\lambda)} \in \Psi(\varphi^{(\lambda)}, 2\bar{p}_0/\lambda, 2\bar{p}_1/\lambda)$. Hence,

$$\Psi(\varphi^{(\lambda)}, p_0, p_1) = \bigcup_{\varphi_\xi^{(\lambda)} \in \Psi(\varphi^{(\lambda)}, 2\bar{p}_0/\lambda, 2\bar{p}_1/\lambda)} \Psi(\varphi_\xi^{(\lambda)}, 2\bar{p}_0/\lambda, 2\bar{p}_1/\lambda).$$

Since $2\bar{p}_0/\lambda = 2p_0$ and $2\bar{p}_1/\lambda = 2p_1$, we obtain

$$\Psi(\varphi^{(\lambda)}, p_0, p_1) = \bigcup_{\varphi_\xi^{(\lambda)} \in \Psi(\varphi^{(\lambda)}, 2p_0, 2p_1)} \Psi(\varphi_\xi^{(\lambda)}, 2p_0, 2p_1).$$
Note that the transformation \( \varphi \mapsto \varphi^{(\lambda)} \) is a bijection of \( \Phi_0 \); therefore, after a “change of variables”, we see that

\[
\Psi(\varphi, p_0, p_1) = \bigcup_{\psi \in \Psi(\varphi, 2p_0, 2p_1)} \Psi(\psi, 2p_0, 2p_1),
\]

which yields the desired identity for orbits.

Now we can apply the same arguments to the relation \( \text{Orb} = \varphi(H_0, H_1)_{p_0, p_1} \), proved already for \( 1/2 \leq p_0, p_1 \leq \infty \), concluding that \( \text{Orb} = \varphi(H_0, H_1)_{p_0, p_1} \) for \( 1/4 \leq p_0, p_1 \leq \infty \), and so on. Thus, \( \text{Orb} = \varphi(H_0, H_1)_{p_0, p_1} \) for all \( 0 < p_0, p_1 \leq \infty \).

The proof of Theorem 2 is complete. \( \square \)

§2. Embedding theorems for generalized spaces of means

The main result of this section is the following theorem.

**Theorem 3.** Let \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \), and let \( \varphi \) and \( \psi \) be nondegenerate interpolation functions; then the embedding

\[
\varphi(H_0, H_1)_{p_0, p_1} \subset \psi(H_0, H_1)_{q_0, q_1}
\]

for all Hilbert couples is equivalent to

\[
\{ \psi^*(1, u_m^{-1}) \} \in \varphi(l_{s_0}, l_{s_1}(u_m^{-1}))^*,
\]

where \( \{ u_m \} \) is a balanced sequence of \( \psi \), and \( 1/s_0 = 1 - (1/q_0 - 1/p_0) \), \( 1/s_1 = 1 - (1/q_1 - 1/p_1) \). (As usual, * denotes the conjugate space.)

The proof of this theorem, based on the orbital descriptions of \( \varphi(H_0, H_1)_{p_0, p_1} \) and \( \psi(H_0, H_1)_{q_0, q_1} \), obtained in §1, requires some auxiliary facts from the general interpolation theory.

Let \( E \) be a quasi-Banach lattice of measurable functions. For \( \alpha > 0 \), let \( E^\alpha \) denote the quasi-Banach lattice of measurable functions \( x \) such that \( |x|^{1/\alpha} \in E \), equipped with the natural quasinorm \( \| x \|_{E^\alpha} = \| x^{1/\alpha} \|_E \).

In particular,

\[
(l_2)^{1/2} = l_4, \quad (l_2(u_m^{-1}))^{1/2} = l_4(u_m^{-1/2}), \quad (l_1(\psi^*(1, u_m^{-1})))^{1/2} = l_2(\psi^*(1, u_m^{-1/2})).
\]

**Lemma 1.** Suppose that \( \{ E_0, E_1 \} \) is a couple of Banach lattices of measurable functions, \( x \in E_0 + E_1 \), and \( x \geq 0 \); then

\[
K(t, x, \{ E_0, E_1 \})^{1/2} \leq K(t^{1/2}, x^{1/2}, \{ E_0^{1/2}, E_1^{1/2} \}).
\]

**Proof.** It is well known that for Banach lattices of measurable functions we have

\[
K(t, x, \{ E_0, E_1 \}) \leq \inf_A \| x_{\chi_A} \|_{E_0} + t \| x(1 - \chi_A) \|_{E_1} \leq \inf_A \max_A (\| x_{\chi_A} \|_{E_0}, t \| x(1 - \chi_A) \|_{E_1}),
\]

where the infimum is taken over all measurable subsets \( A \) of the domain of functions under consideration, and \( \chi_A \) is the characteristic function of \( A \).

Therefore,

\[
K(t^{1/2}, x^{1/2}, \{ E_0^{1/2}, E_1^{1/2} \}) \geq \inf_A \max_{A'} (\| x_{\chi_{A'}} \|_{E_0^{1/2}}, t^{1/2} \| x^{1/2}(1 - \chi_{A'}) \|_{E_1^{1/2}})
\]

\[
\geq (\inf_A \max_{A'} (\| x_{\chi_{A'}} \|_{E_0}, t \| x(1 - \chi_{A'}) \|_{E_1}))^{1/2}.
\]

**Lemma 2.** Suppose \( \varphi \in \Phi_\lambda, 0 < s_0 \leq \infty, \) and \( 0 < s_1 \leq \infty \). Then

\[
(\varphi(l_2, l_2(u_m^{-1})))^{s_0, s_1} = \varphi^{(1/2)}(l_4, l_4(u_m^{-1})))^{2s_0, 2s_1}
\]

for an arbitrary positive sequence \( u_m \).
Proof. If \( y \in (\varphi(l_2, l_2(u_{m-1}^-)))_{s_0, s_1} \), then \( |y| = x^{1/2} \), where \( x \in \varphi(l_2, l_2(u_{m-1}^-))_{s_0, s_1} \). By definition, \( \{K(w_k, x, \{l_2, l_2(u_{m-1}^-)\})\} \in \varphi(l_{s_0}, l_{s_1}(w_k^{-1})) \), where \( w_k \) denotes a balanced sequence of \( K(t, x, \{l_2, l_2(u_{m-1}^-)\}) \). Hence,

\[
\{K^{1/2}(w_k, x, \{l_2, l_2(u_{m-1}^-)\})\} \in (\varphi(l_{s_0}, l_{s_1}(w_k^{-1})))^{1/2} \equiv (l_{s_0}, l_{s_1}(w_k^{-1})).
\]

By Lemma 1, we conclude that

\[
K^{1/2}(w_k, x, \{l_2, l_2(u_{m-1}^-)\}) \in (\varphi(l_{s_0}, l_{s_1}(w_k^{-1})))^{1/2} = (l_{s_0}, l_{s_1}(w_k^{-1})).
\]

Furthermore, Lemma 1 shows that

\[
K(t, y, \{l_4, l_4(u_{m-1}^-)\}) \simeq K^{(2)}(t, x, \{l_2, l_2(u_{m-1}^-)\}),
\]

so that \( w_k^{1/2} \) is a balanced sequence of \( K(t, y, \{l_4, l_4(u_{m-1}^-)\}) \). Consequently, \( (12) \) means that \( |y| = x^{1/2} \in \varphi(1/2)(l_4, l_4(u_{m-1}^-))_{s_0, s_1} \).

\[\square\]

**Lemma 3.** Suppose that \( u_m \) is a balanced sequence of a quasiconcave function; then, up to equivalence, the set of all functions \( \rho \in \Phi_0 \) such that \( \rho(t) = K(t, x, \{l_{p_0}, l_{p_1}(u_{m-1})\}) \) for some \( x \in l_{p_0} + l_{p_1}(u_{m-1}) \) is independent of \( 1 \leq p_0, p_1 \leq \infty \).

Proof. We actually show that, up to equivalence, the set in question coincides with the subset of \( \Phi_0 \) consisting of concave piecewise linear functions that have breaks at the points \( u_m \). Obviously, the latter set is independent of \( p_0 \) and \( p_1 \).

Note that each balanced sequence is equivalent to a subsequence of a geometrical sequence, say, \( 2^n \). Therefore, each couple \( \{l_{p_0}, l_{p_1}(u_{m-1})\} \) is naturally isomorphic to a subcouple of \( \{l_{p_0}, l_{p_1}(2^{-n})\} \).

Recall that

\[
K(2^k, x, \{l_{p_0}, l_{p_1}(2^{-n})\}) \equiv \left( \sum_{n=-\infty}^{k} |x_n|^{p_0} \right)^{1/p_0} + 2^k \left( \sum_{n=k+1}^{\infty} |x_n|^{2^{-n}p_1} \right)^{1/p_1}
\]

(see, e.g., \[10\]). Therefore, each \( K(t, x, \{l_{p_0}, l_{p_1}(u_{m-1})\}) \) is equivalent to a piecewise linear concave function with breaks at \( u_m \).

Conversely, if a piecewise linear concave function \( \rho \in \Phi_0 \) has breaks at \( u_m \), then we can choose a subsequence \( u_{m_n} \) of \( u_m \) such that \( u_{m_n} \) is a balanced sequence of \( \rho \). In order to prove this, observe that \( u_m \) is a filling sequence of \( \rho \). Thus, we need to choose a uniformly sparse subsequence keeping the filling property. For this, we can use an analog of the inductive construction \[2\] applied to the restriction of the function \( \rho(t) \) to the points \( u_m \). Then we obtain a subsequence \( u_{m_n} \) such that

\[
u_{m_0} = u_0, \min \left( \frac{\rho(u_{m+1})}{\rho(u_m)} \right) \geq q > 1,
\]

and

\[
\min \left( \frac{\rho(u_m)}{\rho(u_{m+n})} \right) < q
\]

for all \( m \) between \( m_n \) and \( m_{n+1} \).

The first condition implies that \( u_{m_n} \) is a uniformly sparse sequence of \( \rho \), and the second condition ensures that \( u_{m_n} \) is still a filling sequence of \( \rho \). (See \[13\] for the necessary details.)

Thus, \( u_{m_n} \) is a balanced sequence of \( \rho \), which implies (see \[13\]) that

\[
\rho(t) \simeq K(t, \{\rho(u_m^-)\}, \{l_{p_0}, l_{p_1}(u_{m_n})\})
\]

for all \( 1 \leq p_0, p_1 \leq \infty \).

The proof of Lemma 3 is complete. \[\square\]
Proof of Theorem 3. We regard $\varphi(H_0, H_1)_{p_0, p_1}$ as an orbit $\text{Orb}(a_{\varphi}, S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1))$ of $a_{\varphi} \in F_0 + F_1$ such that $\varphi(s, t) = K(s, t, a_{\varphi}, \{F_0, F_1\})$, and we view $\psi(H_0, H_1)_{q_0, q_1}$ as a co-orbit $\text{Corb}(\psi, S_{q_0}(H_0 \to \overline{F}_0) \cap S_{q_1}(H_1 \to \overline{F}_1))$, where $\{\overline{F}_0, \overline{F}_1\}$ is a $K_0$-abundant Hilbert couple. In [6] or [13] it was shown that the generalized Lorentz space $A_\psi(\overline{F}_0, \overline{F}_1)$ (equal to $\psi(\overline{F}_0, \overline{F}_1, 1, 1)$) is the co-orbit of $l_1(\psi^*(1, u_m^{-1})) = \psi(l_2, l_1(u_m^{-1}))_{1, 1, 1}$, where $u_m$ is a balanced sequence of $\psi$, with respect to the bounded linear operators mapping $\{\overline{F}_0, \overline{F}_1\}$ to $\{l_2, l_2(u_m^{-1})\}$. Thus, we see that $\psi(H_0, H_1)_{q_0, q_1}$ is the co-orbit $\text{Corb}(\psi, S_{q_0}(H_0 \to l_2) \cap S_{q_1}(H_1 \to l_2(u_m^{-1})))$.

Hence, if $\varphi(H_0, H_1)_{p_0, p_1} \subset \psi(H_0, H_1)_{q_0, q_1}$, then for each $U \in S_{p_0}(F_0 \to H_0) \cap S_{p_1}(F_1 \to H_1)$ and each $V \in S_{q_0}^s(H_0 \to l_2) \cap S_{q_1}^s(H_1 \to l_2(u_m^{-1}))$ we have $\text{VU}(a_{\varphi}) \in \psi(l_2, l_2(u_m^{-1}))_{1, 1, 1}$.

We claim that $\varphi(l_2, l_2(u_m^{-1}))_{s_0, s_1}$ is contained in $\psi(l_2, l_2(u_m^{-1}))_{1, 1, 1}$. Indeed, Theorem 2 allows us to represent $\varphi(l_2, l_2(u_m^{-1}))_{s_0, s_1}$ as the orbit $\text{Orb}(a_{\varphi}, S_{s_0}(F_0 \to l_2) \cap S_{s_1}(F_1 \to l_2(u_m^{-1})))$, where $1/s_0 = 1/p_0 + 1/q_0$, $1/s_1 = 1/p_1 + 1/q_1$. Hence, for each $x \in \varphi(l_2, l_2(u_m^{-1}))_{s_0, s_1}$ there exists $T \in S_{s_0}(F_0 \to l_2) \cap S_{s_1}(F_1 \to l_2(u_m^{-1}))$ such that $x = T(a_{\varphi})$. It is well known (see, e.g., [13]) that we can choose a Hilbert couple $(G_0, G_1)$ and operators $\tilde{U} \in S_{p_0}(F_0 \to G_0) \cap S_{p_1}(F_1 \to G_1)$ and $\tilde{V} \in S_{q_0}^s(G_0 \to l_2) \cap S_{q_1}^s(G_1 \to l_2(u_m^{-1}))$ such that $T = \tilde{V}\tilde{U}$.

Thus,

$$\varphi(l_2, l_2(u_m^{-1}))_{s_0, s_1} \subset \psi(l_2, l_2(u_m^{-1}))_{1, 1, 1}.$$

It is easily seen that our arguments can be reversed; therefore, $\varphi(H_0, H_1)_{p_0, p_1} \subset \psi(H_0, H_1)_{q_0, q_1}$, for all Hilbert couples is equivalent to (13).

Clearly, (13) is equivalent to

$$\varphi(l_2, l_2(u_m^{-1}))_{s_0, s_1} \subset \psi(l_2, l_2(u_m^{-1}))_{1, 1, 1}$$

which is equivalent to

$$\varphi^{(1/2)}(l_4, l_4(u_m^{-1/2}))_{2s_0, 2s_1} \subset \psi^{(1/2)}(l_4, l_4(u_m^{-1/2}))_{2, 2}$$

by Lemma 2.

Since the set of $K$-functionals is independent of $p_0$ and $p_1$ by Lemma 3, the latter embedding is equivalent to a similar embedding for the couple $\{l_2s_0, l_2s_1(u_m^{-1/2})\}$, i.e.,

$$\varphi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2}))_{2s_0, 2s_1} \subset \psi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2}))_{2, 2}.$$

Our assumption $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ yields $2s_0 \geq 1$ and $2s_1 \geq 1$. Therefore, we are able to describe $\varphi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2}))_{2s_0, 2s_1}$ and $\psi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2}))_{2, 2}$.

In [12] it was shown that

$$\varphi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2}))_{2s_0, 2s_1} = \varphi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2})).$$

Recall that $u_m$ is a balanced sequence of $\psi$, so that $u_m^{1/2}$ is a balanced sequence of $\psi^{(1/2)}$, and $\psi^{(1/2)}$ is nondegenerate together with $\psi$. Therefore,

$$\psi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2}))_{2, 2} = \psi^{(1/2)}(l_2, l_2(u_m^{-1/2}))_{2, 2} = \psi^{(1/2)}(l_2, l_2(u_m^{-1/2})).$$

Thus, we arrive at the equivalent embedding

$$\varphi^{(1/2)}(l_2s_0, l_2s_1(u_m^{-1/2})) \subset \psi^{(1/2)}(l_2, l_2(u_m^{-1/2}))$$

or

$$\varphi(l_2, l_2(u_m^{-1}))_{s_0, s_1} \subset \psi(l_1, l_1(u_m^{-1})).$$

Consequently, (11) is equivalent to

$$\varphi(l_2, l_2(u_m^{-1})) \subset \psi(l_1, l_1(u_m^{-1})) = l_1(\psi^*(1, u_m^{-1})).$$
This means that for each \( \{x_m\} \in \varphi(l_{s_0}, l_{s_1}(u_m^{-1})) \) we have
\[
\sum_m |x_m| \psi^*(1, u_m^{-1}) < \infty,
\]
or \( \{\psi^*(1, u_m^{-1})\} \in (\varphi(l_{s_0}, l_{s_1}(u_m^{-1})))^* \).

This completes the proof of Theorem 3. \( \square \)

In [15] it was noted that (11) is equivalent to the conjugate embedding
\[
\psi^*(H_0, H_1)_{q'_0, q'_1} \subset \varphi^*(H_0, H_1)_{p'_0, p'_1}.
\]

Thus, we obtain yet another criterion of (11).

**Corollary.** Suppose that \( 1 \leq p_0, p_1, q_0, q_1 \leq \infty \) and that \( \varphi, \psi \) are nondegenerate interpolation functions; then the embedding
\[
\varphi(H_0, H_1)_{p_0, p_1} \subset \psi(H_0, H_1)_{q_0, q_1}
\]
for all Hilbert couples \( \{H_0, H_1\} \) is equivalent to
\[
\{\rho(t_n)\} = \{\varphi(1, t_n)\} \in \psi^*(l_{s_0}, l_{s_1}(t_n))^*,
\]
where \( \{t_n\} \) is a balanced sequence of \( \rho(t) = \varphi(1, t) \), and \( 1/s_0 = 1 - (1/q_0 - 1/p_0) \), \( 1/s_1 = 1 - (1/q_1 - 1/p_1) \).

In conclusion, we see that the inclusion \( \varphi(H_0, H_1)_{p_0, p_1} \subset \psi(H_0, H_1)_{q_0, q_1} \) for all Hilbert couples \( \{H_0, H_1\} \) is equivalent to the inclusion \( \varphi(X_0, X_1)_{p_0, p_1} \subset \psi(X_0, X_1)_{q_0, q_1} \) for all Banach couples (see [15]). Thus, we get a criterion for (1).

**References**


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