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ON DIVERGENCE OF SINC-APPROXIMATIONS EVERYWHERE ON $(0,\pi)$

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Abstract. Some properties of sinc-approximations of continuous functions on a segment are studied.

For the first time, the sinc-approximations arose in the work of Pleynet. Later, in connection with developments in signals coding theory, E. Borel and E. T. Whittaker introduced the notions of a *cardinal function* and a *truncated cardinal function*, the restriction of which to $[0,\pi]$ looks like this:

$$\begin{aligned} L_n(f,x) &= \sum_{k=0}^{n} \frac{\sin(nx-k\pi)}{nx-k\pi} f\Big(\frac{k\pi}{n}\Big) \\ &= \sum_{k=0}^{n} \frac{(-1)^k \sin nx}{nx-k\pi} f\Big(\frac{k\pi}{n}\Big) = \sum_{k=0}^{n} l_{k,n}(x) f\Big(\frac{k\pi}{n}\Big). \end{aligned} \tag{1}$$

By the present time, the problem of sinc-approximation has been studied rather deeply for functions analytic on the real line and decaying exponentially at infinity (see, e.g., [1]–[5]). The most complete survey of the results obtained in this direction before 1993, together with numerous important applications of sinc-approximations, can be found in [6]. An interesting historical survey of investigations on this topic is contained in [7].

Also, a series of publications had their origin in what is called the *sampling theorem* or sometimes, the Whittaker–Kotel′nikov–Shannon discretization theorem [8, 9, 6, 11]; in those studies, various representations of entire functions were obtained in the form of series in sincs with interpolation nodes satisfying some "uniform distribution" conditions (see, e.g., [12]–[14]). Starting with the well-known paper [15] by Kramer, the relationship between the sampling theorems and the Lagrange interpolation with nodes in the Sturm–Liouville problem spectrum has been studied; see, e.g., [16].

The sinc-approximations are widely applied for constructing numerical methods of mathematical physics and approximation theory for functions of one or several variables [17]–[21], in the theory of quadrature formulas [6, 22] and wavelet transformations [8]–[11], [23, 24]. In the paper [25], an analog of the sampling theorem that employs the Hermite type interpolation was obtained.

The authors of [26] succeeded in proving approximability for less smooth functions. However, they were forced to modify the operator (1) somewhat. In [26], it was established that if a uniformly continuous and bounded functions f on $\mathbb{R}$ belongs to the Dini–Lipchitz class and satisfies the condition $f(x) = O(|x|^{-\delta})$ as $x \to \pm\infty$ for some

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$\delta > 0$ uniformly on $\mathbb{R}$, then for any $m \in \mathbb{N}$ and any $0 < a < 1$, we have

$$\lim_{W\to\infty} \sum_{k=-\infty}^{\infty} f\Big(\frac{k}{W}\Big)\Big\{\frac{\sin\pi(\frac{a(Wx-k)}{m})}{\pi\frac{a(Wx-k)}{m}}\Big\}^m \frac{\sin\pi(Wx-k)}{\pi(Wx-k)} = f(x).$$

Moreover, the same authors obtained Jackson type theorems for such approximations as $W \to \infty$. An interesting criterion for uniform convergence on the axis of the Whittaker functions themselves was presented in [27]. For all functions of class $F^p = \{f \in L^p(\mathbb{R}) \cap C(\mathbb{R}); \widehat{f} \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})\}$ (here $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$, and $\widehat{f}$ stands for the Fourier transform of f), we have the following estimate on the real axis:

$$\Big|f(x) - \sum_{k=-\infty}^{+\infty} f\Big(\frac{k}{w}\Big)\operatorname{sinc}(wx-k)\Big| \le \sqrt{\frac{2}{\pi}} \int_{|t|\ge\pi w} |\widehat{f}(t)|\,dt, \quad x \in \mathbb{R}.$$

Also in [27], some conditions sufficient for convergence in L^p were obtained for functions locally integrable in the Riemann sense on $\mathbb{R}$.

As far as I know, before the papers [28]–[35], approximation by such operators on a segment or on a bounded interval had only been studied for certain classes of analytic functions [6, 20] by reducing to the case of the axis with the help of a conformal mapping. In [33], an estimate from above was obtained for the best approximation of continuous functions vanishing at the ends of the segment $[0, \pi]$ by linear combinations of sincs.

The results of [32, 34], combined with the studies published in [31], describe the class of continuous functions on $[0, \pi]$ for which the sequence of values of the operators (1) converges to the function in question uniformly on any compact subset of the interval $(0, \pi)$. In [32], a criterion was obtained for convergence everywhere on $[0, \pi]$ of the values of the same operators to the function we wish to interpolate.

The information about a continuous function f needed for the description of the behavior of $L_n(f, \cdot)$ can be restricted to the values of f at the discrete nodes $\frac{k\pi}{n}$ that lie near the point where we study the approximation properties of (1). Therefore, the possibility of approximation of a specific function can easily be checked with the help of a computer. Note that if $f \in C[0, \pi]$, then, as was established in [35], already for $f \equiv 1$ we have

$$\Big|1 - L_n\Big(1, \frac{\pi}{2n}\Big)\Big| = \frac{4-\pi}{2\pi} + O\Big(\frac{1}{n}\Big),$$

which shows that the values of the operators (1) cannot converge uniformly on $[0, \pi]$.

I know of no publications containing results of a negative nature, i.e., describing the structure of the sets where the sinc-approximations diverge for functions in various classes. The logarithmic order of growth for the Lebesgue constants of the processes (1) (see [31]) is also typical of the trigonometrical Fourier series and the classical algebraic Lagrange interpolation processes involving the matrices of nodes of Chebyshov and Jacobi ($\alpha > -1, \beta > -1$). Nevertheless, it is well known that these approximative construction behave differently. For example, Grünwald [36] and Marcinkiewicz [37] independently constructed continuous functions on $[-1, 1]$ such that their Lagrange–Chebyshev interpolation processes diverge everywhere. In [38], Privalov gave an example of a continuous function for which the interpolation Lagrange polynomials corresponding to the Jacobi matrix ($\alpha > -1, \beta > -1$) diverge a.e. on $[-1, 1]$. At the same time, the Carleson theorem states that the trigonometric Fourier series of any function of class L^2 converges to it almost everywhere.

In the present paper, we use the results of [32] to show that the behavior of the sinc-approximations (1) on the class of continuous functions is close to the properties of the classical Lagrange–Chebyshev interpolation polynomials, rather than to those of the Fourier series.

Unfortunately, the methods that allow one to obtain results of a negative nature in the case of the classical algebraic and trigonometric interpolation, or in the case of Fourier series, cannot be applied to the study of approximative properties of the sinc-approximations (1). The reason is that the operators (1) do not possess some properties shared by the algebraic and trigonometric interpolation polynomials, or by the Fourier partial sums. For example, for $1 \le m < n$ we have $L_n(L_m(f,\cdot),x) \not\equiv L_m(f,x)$, or $L_n(1,x) \not\equiv 1$.

We denote by $C_0[0,\pi] = \{f : f \in C[0,\pi],\ f(0) = f(\pi) = 0\}$ the space of functions continuous on $[0,\pi]$ and vanishing at the ends of that segment, with the Chebyshev norm $\|f\| = \max_{x\in[0,\pi]} |f(x)|$.

Theorem 1. *There exists $f \in C_0[0,\pi]$ such that*

$$\limsup_{n\to\infty} |L_n(f,x)| = \infty$$

for all $x \in (0,\pi)$.

The results of [40, Chapter IV, §4] together with [31, Corollary to Theorem 2] show that for the classes of sufficiently smooth continuous functions, i.e., for the Dini–Lipschitz class, the trigonometric Fourier series converge everywhere on the interval $(0,\pi)$ and the same is true for the truncated cardinal Whittaker functions (1). The following question arises naturally: is it true that the convergence of one of these approximative constructions at a fixed point implies the same property for the other construction?

In the present paper we show that the two sets of continuous functions for which the sinc-approximations (1) and the Fourier series or Fourier integrals converge at a given point do not include each other.

We denote by $\sigma_n(f,x)$ the partial sum of the trigonometric Fourier series and by $\widetilde{\sigma}_n(f,x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t+x)\frac{\sin nt}{t}\,dt$ the simple Fourier integral; see [41].

Theorem 2. *There exists a function $g \in C_0[0,\pi]$ and a point $x_0 \in [0,\pi]$ such that*

$$\limsup_{n\to\infty} |L_n(g,x_0)| = \infty, \tag{2}$$

$$\lim_{n\to\infty} |\sigma_n(g,x_0) - g(x_0)| = 0. \tag{3}$$

There exists a function $f \in C[0,\pi]$ and a point $x_1 \in [0,\pi]$ such that

$$\limsup_{n\to\infty} |\sigma_n(f,x_1)| = \infty, \tag{4}$$

$$\lim_{n\to\infty} |L_n(f,x_1) - f(x_1)| = 0. \tag{5}$$

Corollary 1. *There exists a function $\widetilde{g} \in C(\mathbb{R})$ with $\operatorname{supp}\widetilde{g} \subset [0,\pi]$ and a point $x_0 \in [0,\pi]$ such that*

$$\limsup_{n\to\infty} |L_n(\widetilde{g},x_0)| = \infty, \tag{6}$$

$$\lim_{n\to\infty} |\widetilde{\sigma}_n(\widetilde{g},x_0) - \widetilde{g}(x_0)| = 0. \tag{7}$$

There exists a function $\widetilde{f} \in C(\mathbb{R})$ with $\operatorname{supp}\widetilde{f} \subset [-2\pi,2\pi]$ and a point $x_1 \in [0,\pi]$ such that

$$\limsup_{n\to\infty} |\widetilde{\sigma}_n(\widetilde{f},x_1)| = \infty, \tag{8}$$

$$\lim_{n\to\infty} |L_n(\widetilde{f},x_1) - \widetilde{f}(x_1)| = 0. \tag{9}$$

Before proving the above theorems, we present some notation and facts to be used below.

For any natural number n and any $x \in [0,\pi]$, we denote by $p = p(x,n)$ the integer satisfying

$$\frac{\pi p}{n} \le x < \frac{\pi(p+1)}{n}. \tag{10}$$

Also, we denote $x_{k,n} = \frac{k\pi}{n}$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$. When we say "uniformly inside the interval $(0,\pi)$", we mean "uniformly on any compact subset of $(0,\pi)$".

Theorem 3 (see [32, Theorem 1])**.** *Let $f \in C[0,\pi]$, and let two sequences of positive numbers γ_n and ε_n satisfy*

$$\gamma_n = o(1), \quad \lim_{n\to\infty} \frac{\gamma_n}{\omega(f,\frac{\pi}{n})} = \infty, \quad \varepsilon_n = \frac{1}{\pi}\exp -\Big\{\frac{\gamma_n}{\omega(f,\frac{\pi}{n})} + 1\Big\}.$$

Given $n \in \mathbb{N}$ and $x \in [0,\pi]$, we denote

$$m_1 = \Big[\frac{k_1}{2}\Big] + 1, \quad m_2 = \Big[\frac{k_2}{2}\Big],$$

where k_1 and k_2 are determined by the inequalities

$$\frac{\pi(k_1-1)}{n} < x - \varepsilon_n \le \frac{\pi k_1}{n}, \quad \frac{\pi k_2}{n} < x + \varepsilon_n \le \frac{\pi(k_2+1)}{n}.$$

Then, everywhere on $[0,\pi]$ and uniformly inside $(0,\pi)$, we have

$$\lim_{n\to\infty} \left| L_n(f,x) - f(x) - \frac{\sin nx}{2\pi} \sum_{m=m_1}^{m_2}{}' \frac{f(\frac{\pi(2m+1)}{n}) - 2f(\frac{2\pi m}{n}) + f(\frac{\pi(2m-1)}{n})}{p-2m} \right| = 0, \tag{11}$$

where the prime at the sum means that the term with zero denominator is absent, and the function f is extended to $\mathbb{R}$ by putting $f(x) = 0$ for $x \notin [0,\pi]$. If $m_2 < m_1$, then the sum in (11) is equal to zero.

Remark 1. This result is a criterion for convergence of sinc-approximations at a point, because it provides a necessary and sufficient condition of approximation by the operators (1) at a point.

Theorem 4 (see [32, Theorem 6] or [31, Theorem 2])**.** *If f is a continuous function on the segment $[0,\pi]$, then for all $x \in [0,\pi]$ we have the following equiconvergence formula:*

$$\lim_{n\to\infty} \left(f(x) - L_n(f,x) - \frac{1}{2}\sum_{k=0}^{n-1} (f(x_{k+1,n}) - f(x_{k,n}))\, l_{k,n}(x) \right) = 0, \tag{12}$$

where $l_{k,n}(x) = \frac{(-1)^k \sin(nx)}{nx - k\pi}$. The convergence in (12) is pointwise on $[0,\pi]$ and uniform inside the interval $(0,\pi)$.

Lemma 1. *If $f \in C_0[0,\pi]$, then the convergence in (12) is uniform in $x \in [0,\pi]$.*

Proof of Lemma 1. We change the independent variable: $t = \frac{x+\pi}{2}$, $x = 2t - \pi$, and consider the new function

$$\breve{f}(t) = \begin{cases} f(2t-\pi) & \text{if } t \in [\frac{\pi}{2},\pi], \\ 0 & \text{if } t \in [0,\frac{\pi}{2}). \end{cases}$$

Since f is continuous and $f(0)=f(\pi)=0$, we see that $\check{f}\in C_0[0,\pi]$. Observe that for $x\in[0,\pi]$ and $t\in[\frac{\pi}{2},\pi]$ we have

$$\begin{aligned}L_n(f,x)&=\sum_{k=0}^{n}f(x_{k,n})\frac{(-1)^k\sin nx}{n(x-x_{k,n})}=\sum_{k=0}^{n}\check{f}\Big(\frac{x_{k,n}+\pi}{2}\Big)\frac{(-1)^k\sin n(2t-\pi)}{n(2t-\pi-\frac{k\pi}{n})}\\&=\sum_{k=0}^{n}\check{f}\Big(\frac{(k+n)\pi}{2n}\Big)\frac{(-1)^{k+n}\sin 2nt}{n(2t-\frac{(n+k)\pi}{n})}\\&=\sum_{m=n}^{2n}\check{f}\Big(\frac{m\pi}{2n}\Big)\frac{(-1)^m\sin 2nt}{2n(t-\frac{m\pi}{2n})}\\&=\sum_{m=0}^{2n}\check{f}(t_{m,2n})\frac{(-1)^m\sin 2nt}{2n(t-t_{m,2n})}=L_{2n}(\check{f},t).\end{aligned}$$

We use Theorem 4:

$$\begin{aligned}&\lim_{n\to\infty}\max_{x\in[0,\frac{\pi}{2}]}\Big|f(x)-L_n(f,x)-\frac{1}{2}\sum_{k=0}^{n-1}(f(x_{k+1,n})-f(x_{k,n}))l_{k,n}(x)\Big|\\&=\lim_{n\to\infty}\max_{t\in[\frac{\pi}{2},\frac{3\pi}{4}]}\Big|\check{f}(t)-L_{2n}(\check{f},t)-\frac{1}{2}\sum_{m=n}^{2n-1}(\check{f}(t_{m+1,2n})-\check{f}(t_{m,2n}))l_{m,2n}(t)\Big|\\&=\lim_{n\to\infty}\max_{t\in[\frac{\pi}{2},\frac{3\pi}{4}]}\Big|\check{f}(t)-L_{2n}(\check{f},t)-\frac{1}{2}\sum_{m=0}^{2n-1}(\check{f}(t_{m+1,2n})-\check{f}(t_{m,2n}))l_{m,2n}(t)\Big|=0.\end{aligned}$$

Arguing similarly, or using the substitution $z=\pi-x$, we can establish the relation

$$\lim_{n\to\infty}\max_{x\in[\frac{\pi}{2},\pi]}\Big|f(x)-L_n(f,x)-\frac{1}{2}\sum_{k=0}^{n-1}(f(x_{k+1,n})-f(x_{k,n}))l_{k,n}(x)\Big|=0.$$

Thus, we see that (12) is uniform in $x\in[0,\pi]$. Lemma 1 is proved. □

Remark 2. If $f\in C[0,\pi]\setminus C_0[0,\pi]$, then the convergence in (12) is no longer uniform in $x\in[0,\pi]$.

Now we want to show that if $f\in C_0[0,\pi]$, then the convergence in (11) is uniform on the entire segment $[0,\pi]$. Also, to obtain results of a negative nature, it will be convenient to pass to an equivalent "global" version of (11), where summation goes over all indices $1\le m\le\big[\frac{n}{2}\big]$.

Lemma 2. *Let $f\in C_0[0,\pi]$. For any $n\in\mathbb{N}$ and any $x\in[0,\pi]$, let p be defined as in* (10)*. Then, uniformly on $[0,\pi]$, we have*

$$\lim_{n\to\infty}\Big|L_n(f,x)-f(x)-\frac{\sin nx}{2\pi}\sum_{m=1}^{[\frac{n}{2}]}{}'\frac{f(\frac{\pi(2m+1)}{n})-2f(\frac{2\pi m}{n})+f(\frac{\pi(2m-1)}{n})}{p-2m}\Big|=0,\tag{13}$$

where the prime at the sum sign means that the term with zero denominator is absent.

Proof of Lemma 2 2. After the extension $f(x)=0$ for $x\notin[0,\pi]$, the function $f\in C_0[0,\pi]$ becomes continuous on $\mathbb{R}$, with preservation of the modulus of continuity. Denote

$$\psi_{k,n}=f(x_{k+1,n})-f(x_{k,n}),\quad k\in\mathbb{Z},\ \ n\in\mathbb{N}.\tag{14}$$

We fix an arbitrary $x\in[0,\pi]$. In view of (10), $x\in[x_{p,n},x_{p+1,n})$. Then $x=x_{p,n}+\alpha(x_{p+1,n}-x_{p,n})=x_{p,n}+\frac{\alpha\pi}{n}$, where $\alpha=\alpha(x,n)\in[0,1)$ (in case $x=\pi$ we get $p(\pi,n)=n$, $\alpha=0$), and $x-x_{k,n}=\frac{p-k+\alpha}{n}\pi$.

Using (14), for all $x \in [0,\pi]$ we can write

$$\begin{aligned}\Big|\sum_{k\,:\,|p-k|\ge 3}\frac{(-1)^k\psi_{k,n}}{p-k+\alpha}-\sum_{k\,:\,|p-k|\ge 3}\frac{(-1)^k\psi_{k,n}}{p-k}\Big| \\ \le \omega\Big(f,\frac{\pi}{n}\Big)\sum_{k\,:\,|p-k|\ge 3}\frac{\alpha}{|p-k|(|p-k|-1)}\le \omega\Big(f,\frac{\pi}{n}\Big).\end{aligned} \tag{15}$$

We split the sum in (12) as follows:

$$\frac{1}{2}\sum_{k=0}^{n-1}(f(x_{k+1,n})-f(x_{k,n}))l_{k,n}(x)=\frac{1}{2}\sum_{\substack{k\,:\,0\le k\le n-1\\ |p-k|\ge 3}}\psi_{k,n}l_{k,n}(x)+\frac{1}{2}\sum_{\substack{k\,:\,0\le k\le n-1\\ |p-k|<3}}\psi_{k,n}l_{k,n}(x).$$

Now, with the help of the triangle inequality, from (14) and (15) we get

$$\begin{aligned}\Big|\frac{1}{2}\sum_{k=0}^{n-1}(f(x_{k+1,n})-f(x_{k,n}))l_{k,n}(x)-\frac{\sin nx}{2\pi}\sum_{k=0}^{n-1}{}'\frac{(-1)^k\psi_{k,n}}{p-k}\Big| \\ \le \frac{1}{2\pi}\Big|\sum_{k\,:\,|p-k|\ge 3}\frac{(-1)^k\psi_{k,n}}{p-k+\alpha}-\sum_{k\,:\,|p-k|\ge 3}\frac{(-1)^k\psi_{k,n}}{p-k}\Big| \\ +\frac{1}{2\pi}\sum_{k\,:\,|p-k|<3}\Big|\psi_{k,n}l_{k,n}(x)\Big|+\frac{1}{2\pi}\sum_{k\,:\,|p-k|<3}{}'\frac{|\psi_{k,n}|}{|p-k|}\le\frac{9}{2\pi}\omega\Big(f,\frac{\pi}{n}\Big),\end{aligned} \tag{16}$$

where, as before, the prime means the absence of the term with zero denominator.

We group the summands pairwise. At most 3 terms that have no pair will be estimated as in (14). Thus, we arrive at the inequality

$$\Big|\frac{\sin nx}{2\pi}\sum_{k=0}^{n-1}{}'\frac{(-1)^k\psi_{k,n}}{p-k}-\frac{\sin nx}{2\pi}\sum_{m=1}^{[\frac{n}{2}]}{}'\Big\{\frac{\psi_{2m,n}}{p-2m}-\frac{\psi_{2m-1,n}}{p-2m+1}\Big\}\Big|\le\frac{3}{2\pi}\omega\Big(f,\frac{\pi}{n}\Big).$$

Combining this with (14) and (16), we get

$$\begin{aligned}\Big|\frac{1}{2}\sum_{k=0}^{n-1}(f(x_{k+1,n})-f(x_{k,n}))l_{k,n}(x)-\frac{\sin nx}{2\pi}\sum_{m=1}^{[\frac{n}{2}]}{}'\frac{\psi_{2m,n}-\psi_{2m-1,n}}{p-2m}\Big| \\ \le\frac{1}{2\pi}\sum_{m=1}^{[\frac{n}{2}]}{}'\Big|\frac{\psi_{2m-1,n}}{(p-2m)(p-2m+1)}\Big|+\frac{13}{2\pi}\omega\Big(f,\frac{\pi}{n}\Big)\le\frac{7}{\pi}\omega\Big(f,\frac{\pi}{n}\Big).\end{aligned} \tag{17}$$

Estimate (17) is uniform in $x\in[0,\pi]$. However, it should not be forgotten that, for each n, we need to choose $p=p(x,n)$ appropriately in (17). Recalling Lemma 1, we arrive at (13). Lemma 2 is proved. □

Lemma 3. *Let $M>0$ and let $f\in C_0[0,\pi]$ belong to the Lipschitz class $\mathrm{Lip}_M\,1$ of order 1 with constant M. Then, uniformly for all $0\le\rho\le n$, $n\in\mathbb{N}$, we have*

$$\Big|\sum_{m=1}^{[\frac{n}{2}]}{}'\frac{f(\frac{\pi(2m+1)}{n})-2f(\frac{2\pi m}{n})+f(\frac{\pi(2m-1)}{n})}{\rho-2m}\Big|\le\frac{2\pi M}{n}(\ln n+1).$$

As before, the prime means that the term with zero denominator is absent, and $f(x)=0$ for $x\notin[0,\pi]$.

Proof of Lemma 3. Since $f \in \mathrm{Lip}_M 1$, for any $0 \le \rho \le n$, $n \in \mathbb{N}$, we can write

$$\begin{aligned}
&\Big|\sum_{m=1}^{[\frac{n}{2}]}{}' \frac{f(\frac{\pi(2m+1)}{n}) - 2f(\frac{2\pi m}{n}) + f(\frac{\pi(2m-1)}{n})}{\rho - 2m}\Big| \\
&\qquad \le \sum_{m=1}^{[\frac{n}{2}]}{}' \frac{|f(\frac{\pi(2m+1)}{n}) - f(\frac{2\pi m}{n})| + |f(\frac{2\pi m}{n}) - f(\frac{\pi(2m-1)}{n})|}{|\rho - 2m|} \\
&\qquad \le \sum_{m=1}^{[\frac{n}{2}]}{}' \frac{2M\frac{\pi}{n}}{|\rho - 2m|} \le \frac{2\pi M}{n}(\ln n + 1),
\end{aligned}$$

as required. □

Remark 3. If $f \in C_0[0,\pi] \cap \mathrm{Lip}_M 1$, then Lemmas 2 and 3 show that

$$\lim_{n\to\infty} \|L_n(f,\cdot) - f\|_{C_0[0,\pi]} = 0.$$

Proof of Theorem 1. Let a sequence $\{\nu_q\}_{q=1}^{\infty}$ of positive integers be such that

$$\frac{1}{\sqrt{\ln \nu_1}} \le 1, \quad \frac{1}{\sqrt{\ln \nu_{q+1}}} \le \frac{1}{3\sqrt{\ln \nu_q}}, \quad q = 1,2,3,\dots. \tag{18}$$

We shall consider a sequence of collections $\{n_{j,q}\}_{j=0,q=1}^{\nu_q-1,\infty}$ of odd positive integers and a sequence of collections $\{\phi_{j,q}(x)\}_{j=0,q=1}^{\nu_q-1,\infty}$ of functions of class $C_0[0,\pi]$ with certain properties. First, we choose a sequence $\{n_{0,q}\}_{q=1}^{\infty}$ of odd integers that are so large that the inequalities

$$n_{0,q} \ge \nu_q^2 \quad \text{and} \quad \frac{\pi}{(n_{0,q}+1)^2} \ge \frac{\pi e^{-n_{0,q}}}{n_{0,q}} \tag{19}$$

are fulfilled simultaneously for all $q = 1,2,3,\dots$.

Next, for any $q \in \mathbb{N}$ we denote

$$E(0,q) = \Big(\bigcup_{m=1}^{[\frac{n_{0,q}}{2}]} \Big\{\frac{(2m \pm e^{-n_{0,q}})\pi}{n_{0,q}}\Big\}\Big) \cup \Big(\bigcup_{\widetilde{m}=1}^{\frac{n_{0,q}-1}{2}} \Big\{\frac{(2\widetilde{m} \pm e^{-n_{0,q}})\pi}{n_{0,q}+1}\Big\}\Big) \cup \Big[0, \frac{\pi}{\nu_q}\Big] \cup \{\pi\}.$$

Also, for each q and $j = 0$ we define a continuous function, more precisely, a function of class $C_0[0,\pi]$, as follows:

$$\phi_{0,q}(x) = \begin{cases} -\frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{0,q}} \notin E(0,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{0,q}+1} \notin E(0,q), \\ & \text{where } 1 \le m < \frac{n_{0,q}}{4\nu_q}, 1 \le \widetilde{m} < \frac{(n_{0,q}+1)}{4\nu_q}, \\ \frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{0,q}} \notin E(0,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{0,q}+1}, \notin E(0,q), \\ & \text{where } \frac{n_{0,q}}{4\nu_q} < m \le [\frac{n_{0,q}}{2}], \frac{(n_{0,q}+1)}{4\nu_q} \le \widetilde{m} < \frac{n_{0,q}+1}{2}, \\ 0 & \text{if } x \in E(0,q), \\ \text{linear} & \text{on the remaining intervals of the segment } [0,\pi]. \end{cases} \tag{20}$$

This definition is possible because the numbers $n_{0,q}$ and $n_{0,q}+1$ are relatively prime.

Being a broken line with finitely many links, each function $\phi_{0,q}$ belongs to the Lipschitz class of order 1 with some constant $M_\phi(0,q)$. Consequently, Lemma 3 allows us to choose an odd number $n_{1,q}$ for each collection so that the inequalities

$$n_{1,q} \ge 2n_{0,q}e^{n_{0,q}}$$

are fulfilled for all q simultaneously ((19) implies that $\frac{\pi}{(n_{1,q}+1)^2} \geq \frac{\pi e^{-n_{1,q}}}{n_{1,q}}$). Also, we require that the following inequalities be true for all $0 \leq \widetilde{p} \leq n_{1,q}+1$ and $0 \leq p \leq n_{1,q}$ (i.e., for any $x \in [0,\pi]$, in accordance with the notation (10); here the tilde means that the corresponding quantity is defined for $n := n+1$):

$$\left| \sum_{m=1}^{[\frac{n_{1,q}}{2}]}{}' \frac{\phi_{0,q}(\frac{\pi(2m+1)}{n_{1,q}}) - 2\phi_{0,q}(\frac{2\pi m}{n_{1,q}}) + \phi_{0,q}(\frac{\pi(2m-1)}{n_{1,q}})}{p-2m} \right| \leq 1,$$

$$\left| \sum_{\widetilde{m}=1}^{\frac{n_{1,q}+1}{2}}{}' \frac{\phi_{0,q}(\frac{\pi(2\widetilde{m}+1)}{n_{1,q}+1}) - 2\phi_{0,q}(\frac{\pi(2\widetilde{m})}{n_{1,q}+1}) + \phi_{0,q}(\frac{\pi(2\widetilde{m}-1)}{n_{1,q}+1})}{\widetilde{p}-2\widetilde{m}} \right| \leq 1.$$

Now, for all $q \in \mathbb{N}$ we consider the sets

$$\begin{aligned} E(1,q) = \Bigg(\Bigg(& \bigcup_{m=1}^{[\frac{n_{0,q}}{2}]} \Big[\frac{(2m - e^{-n_{0,q}})}{n_{0,q}}\pi, \frac{(2m + e^{-n_{0,q}})}{n_{0,q}}\pi \Big] \Bigg) \\ & \cup \Bigg(\bigcup_{\widetilde{m}=1}^{\frac{n_{0,q}-1}{2}} \Big[\frac{(2\widetilde{m} - e^{-n_{0,q}})}{n_{0,q}+1}\pi, \frac{(2\widetilde{m} + e^{-n_{0,q}})}{n_{0,q}+1}\pi \Big] \Bigg) \Bigg) \cup \Big[\frac{\pi}{\nu_q}, \frac{2\pi}{\nu_q} \Big] \\ & \cup \Bigg(\bigcup_{m=1}^{[\frac{n_{1,q}}{2}]} \Big\{ \frac{(2m \pm e^{-n_{1,q}})\pi}{n_{1,q}} \Big\} \Bigg) \cup \Bigg(\bigcup_{\widetilde{m}=1}^{\frac{n_{1,q}-1}{2}} \Big\{ \frac{(2\widetilde{m} \pm e^{-n_{1,q}})\pi}{n_{1,q}+1} \Big\} \Bigg) \cup \{0\} \cup \{\pi\}, \end{aligned}$$

and introduce the following functions of class $C_0[0,\pi]$:

$$\phi_{1,q}(x) = \begin{cases} -\frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{1,q}} \notin E(1,q), \text{ and } x = \frac{2\widetilde{m}\pi}{n_{1,q}+1} \notin E(1,q) \\ & \text{where } 1 \leq m < \frac{3n_{1,q}}{4\nu_q}, \ \ 1 \leq \widetilde{m} < \frac{3(n_{1,q}+1)}{4\nu_q}, \\ \frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{1,q}} \notin E(1,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{1,q}+1} \notin E(1,q), \\ & \text{where } \frac{3n_{1,q}}{4\nu_q} < m \leq [\frac{n_{1,q}}{2}], \frac{3(n_{1,q}+1)}{4\nu_q} \leq \widetilde{m} < \frac{n_{1,q}+1}{2}, \\ 0, & \text{if } x \in E(1,q), \\ \text{linear} & \text{on the remaining intervals of the segment } [0,\pi]. \end{cases}$$

This definition is possible because the integers $n_{1,q}$ and $n_{1,q}+1$ are relatively prime.

Every function $\sum_{\alpha=0}^{1} \phi_{\alpha,q}(x)$ is Lipschitz of order 1 with some constant $M_\phi(1,q)$ $(q = 1,2,3,\dots)$. By Lemma 3, the next odd $n_{2,q}$ in each collection can be chosen so that the following inequalities are fulfilled:

$$n_{2,q} \geq \max\left(\left(\sum_{s=0}^{1} n_{s,q} \right)^2, 2n_{1,q}e^{n_{1,q}} \right)$$

((19) implies that $\frac{\pi}{(n_{2,q}+1)^2} \geq \frac{\pi e^{-n_{2,q}}}{n_{2,q}}$), and, for all $0 \leq \widetilde{p} \leq n_{2,q}+1$ and $0 \leq p \leq n_{2,q}$,

$$\left| \sum_{m=1}^{[\frac{n_{2,q}}{2}]}{}' \frac{\sum_{\alpha=0}^{1} \phi_{\alpha,q}(\frac{\pi(2m+1)}{n_{2,q}}) - 2\sum_{\alpha=0}^{1} \phi_{\alpha,q}(\frac{2\pi m}{n_{2,q}}) + \sum_{\alpha=0}^{1} \phi_{\alpha,q}(\frac{\pi(2m-1)}{n_{2,q}})}{p-2m} \right| \leq 1,$$

$$\left| \sum_{\widetilde{m}=1}^{\frac{n_{2,q}+1}{2}}{}' \frac{\sum_{\alpha=0}^{1} \phi_{\alpha,q}(\frac{\pi(2\widetilde{m}+1)}{n_{2,q}+1}) - 2\sum_{\alpha=0}^{1} \phi_{\alpha,q}(\frac{\pi(2\widetilde{m})}{n_{2,q}+1}) + \sum_{\alpha=0}^{1} \phi_{\alpha,q}(\frac{\pi(2\widetilde{m}-1)}{n_{2,q}+1})}{\widetilde{p}-2\widetilde{m}} \right| \leq 1.$$

After that, for each $q \in \mathbb{N}$ we construct the set

$$E(2,q) = \bigcup_{s=0}^{1} \Bigg(\Bigg(\bigcup_{m=1}^{[\frac{n_{s,q}}{2}]} \Big[\frac{(2m - e^{-n_{s,q}})}{n_{s,q}} \pi, \frac{(2m + e^{-n_{s,q}})}{n_{s,q}} \pi \Big] \Bigg) \cup \Bigg(\bigcup_{\widetilde{m}=1}^{\frac{n_{s,q}-1}{2}} \Big[\frac{(2\widetilde{m} - e^{-n_{s,q}})}{n_{s,q}+1} \pi, \frac{(2\widetilde{m} + e^{-n_{s,q}})}{n_{s,q}+1} \pi \Big] \Bigg) \Bigg) \cup \Big[\frac{2\pi}{\nu_q}, \frac{3\pi}{\nu_q} \Big] \cup \Bigg(\bigcup_{m=1}^{[\frac{n_{2,q}}{2}]} \Big\{ \frac{(2m \pm e^{-n_{2,q}})\pi}{n_{2,q}} \Big\} \Bigg) \cup \Bigg(\bigcup_{\widetilde{m}=1}^{\frac{n_{2,q}-1}{2}} \Big\{ \frac{(2\widetilde{m} \pm e^{-n_{2,q}})\pi}{n_{2,q}+1} \Big\} \Bigg) \cup \{0\} \cup \{\pi\}$$

and the function

$$\phi_{2,q}(x) = \begin{cases} -\frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{2,q}} \notin E(2,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{2,q}+1} \notin E(2,q), \\ & \text{where } 1 \le m < \frac{5n_{2,q}}{4\nu_q}, \ \ 1 \le \widetilde{m} < \frac{5(n_{2,q}+1)}{4\nu_q}, \\ \frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{2,q}} \notin E(2,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{2,q}+1} \notin E(2,q), \\ & \text{where } \frac{5n_{2,q}}{4\nu_q} < m \le [\frac{n_{2,q}}{2}], \frac{5(n_{2,q}+1)}{4\nu_q} \le \widetilde{m} < \frac{n_{2,q}+1}{2}, \\ 0 & \text{if } x \in E(2,q), \\ \text{linear} & \text{on the remaining intervals of the segment } [0,\pi]. \end{cases}$$

Next we continue to construct collections of odd numbers $\{n_{j,q}\}_{j=0,q=1}^{\nu_1-1,\infty}$ and functions $\{\phi_{j,q}(x)\}_{j=0,q=1}^{\nu_1-1,\infty}$ of class $C_0[0,\pi]$. Assuming that the members $\{n_{\alpha,q}\}_{\alpha=0}^{j-1}$ of each collection with number $q \in \mathbb{N}$ and the functions $\{\phi_{\alpha,q}(x)\}_{\alpha=0}^{j-1}$ have already been chosen, we construct $n_{j,q}$ and $\phi_{j,q}(x)$. Here, for each collection with number $q = 1,2,3,\ldots$, the index ranges over $1 \le j \le \nu_q - 1$.

First, we choose $n_{j,q}$ for each $q = 1,2,3,\ldots$. The function $\sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(x)$ is Lipschitz of order 1 with some constant $M_\phi(j-1,q)$. By Lemma 3, the odd number $n_{j,q}$ can be chosen so large that

$$n_{j,q} \ge \max\Bigg(\Bigg(\sum_{s=0}^{j-1} n_{s,q} \Bigg)^2, 2n_{j-1,q} e^{n_{j-1,q}} \Bigg) \tag{21}$$

(note that (21) and (19) imply $\frac{\pi}{(n_{j,q}+1)^2} \ge \frac{\pi e^{-n_{j,q}}}{n_{j,q}}$), and, for all $0 \le \widetilde{p} \le n_{j,q}+1$ and $0 \le p \le n_{j,q}$, we have

$$\Bigg| \sum_{m=1}^{[\frac{n_{j,q}}{2}]}{}' \frac{\sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(\frac{\pi(2m+1)}{n_{j,q}}) - 2\sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(\frac{2\pi m}{n_{j,q}}) + \sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(\frac{\pi(2m-1)}{n_{j,q}})}{p - 2m} \Bigg| \le 1, \tag{22}$$

$$\Bigg| \sum_{\widetilde{m}=1}^{\frac{n_{j,q}+1}{2}}{}' \frac{\sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(\frac{\pi(2\widetilde{m}+1)}{n_{j,q}+1}) - 2\sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(\frac{\pi(2\widetilde{m})}{n_{j,q}+1}) + \sum_{\alpha=0}^{j-1} \phi_{\alpha,q}(\frac{\pi(2\widetilde{m}-1)}{n_{j,q}+1})}{\widetilde{p} - 2\widetilde{m}} \Bigg| \le 1. \tag{23}$$

Next, for every $q = 1, 2, 3, \dots$ we consider the set

$$E(j,q) = \bigcup_{s=0}^{j-1} \Bigg(\Bigg(\bigcup_{m=1}^{[\frac{n_{s,q}}{2}]} \Big[\frac{(2m - e^{-n_{s,q}})}{n_{s,q}} \pi, \frac{(2m + e^{-n_{s,q}})}{n_{s,q}} \pi \Big] \Bigg)$$
$$\cup \Bigg(\bigcup_{\widetilde{m}=1}^{\frac{n_{s,q}-1}{2}} \Big[\frac{(2\widetilde{m} - e^{-n_{s,q}})}{n_{s,q}+1} \pi, \frac{(2\widetilde{m} + e^{-n_{s,q}})}{n_{s,q}+1} \pi \Big] \Bigg) \Bigg) \cup \Big[\frac{\pi j}{\nu_q}, \frac{\pi(j+1)}{\nu_q} \Big]$$
$$\cup \Bigg(\bigcup_{m=1}^{[\frac{n_{j,q}}{2}]} \Big\{ \frac{(2m \pm e^{-n_{j,q}})\pi}{n_{j,q}} \Big\} \Bigg) \cup \Bigg(\bigcup_{\widetilde{m}=1}^{\frac{n_{j,q}-1}{2}} \Big\{ \frac{(2\widetilde{m} \pm e^{-n_{j,q}})\pi}{n_{j,q}+1} \Big\} \Bigg) \cup \{0\} \cup \{\pi\}.$$

After that, for every $q = 1, 2, 3, \dots$ we define a function of class $C_0[0,\pi]$ by

$$\phi_{j,q}(x) = \begin{cases} -\frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{j,q}} \notin E(j,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{j,q}+1} \notin E(j,q), \\ & \text{where } 1 \le m < \frac{(2j+1)n_{j,q}}{4\nu_q}, 1 \le \widetilde{m} < \frac{(2j+1)(n_{j,q}+1)}{4\nu_q}, \\ \frac{1}{\sqrt{\ln \nu_q}} & \text{if } x = \frac{2m\pi}{n_{j,q}} \notin E(j,q) \text{ and } x = \frac{2\widetilde{m}\pi}{n_{j,q}+1} \notin E(j,q), \\ & \text{where } \frac{(2j+1)n_{j,q}}{4\nu_q} < m \le [\frac{n_{j,q}}{2}], \frac{(2j+1)(n_{j,q}+1)}{4\nu_q} \le \widetilde{m} < \frac{n_{j,q}+1}{2}, \\ 0 & \text{where } x \in E(j,q), \\ \text{linear} & \text{on the remaining intervals of the segments } [0,\pi]. \end{cases} \tag{24}$$

This definition is possible because the numbers $n_{j,q}$ and $n_{j,q}+1$ are relatively prime.

Having chosen the collections $\{n_{j,q}\}_{j=0,q=1}^{\nu_q-1,\infty}$ and the functions $\{\phi_{j,q}(x)\}_{j=0,q=1}^{\nu_q-1,\infty}$ of class $C_0[0,\pi]$, we see that, by the definitions (20) and (24), the $C_0[0,\pi]$-norm of the sum

$$f_q(x) = \sum_{j=0}^{\nu_q-1} \phi_{j,q}(x), \quad q = 1, 2, 3, \dots, \tag{25}$$

is equal to

$$\|f_q\| = \frac{1}{\sqrt{\ln \nu_q}}, \quad q = 1, 2, 3, \dots. \tag{26}$$

This follows from the fact that the supports of the terms in (25) are disjoint, by the definition of the sets $E(j,q)$ and relations (20) and (24). In the sequence $\{f_q\}_{q=1}^{\infty}$ we select a subsequence $\{f_{q_i}\}_{i=1}^{\infty}$. We put $f_{q_1} = f_1$; if $f_{q_1}, f_{q_2}, \dots, f_{q_{i-1}}$ have already been chosen, then the next element of the subsequence is taken so that the following four inequalities are fulfilled simultaneously:

$$\Bigg| \sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\sum_{\mu=1}^{i-1} \Big(f_{q_\mu}\big(\frac{\pi(2m+1)}{n_{j,q_i}}\big) - 2f_{q_\mu}\big(\frac{2\pi m}{n_{j,q_i}}\big) + f_{q_\mu}\big(\frac{\pi(2m-1)}{n_{j,q_i}}\big) \Big)}{p - 2m} \Bigg| \le 1 \tag{27}$$

for all $0 \le j \le \nu_{q_i} - 1$, $0 \le p \le n_{j,q_i}$;

$$\Bigg| \sum_{\widetilde{m}=1}^{\frac{n_{j,q_i}+1}{2}}{}' \frac{\sum_{\mu=1}^{i-1} \Big(f_{q_\mu}\big(\frac{\pi(2\widetilde{m}+1)}{n_{j,q_i}+1}\big) - 2f_{q_\mu}\big(\frac{2\pi\widetilde{m}}{n_{j,q_i}+1}\big) + f_{q_\mu}\big(\frac{\pi(2\widetilde{m}-1)}{n_{j,q_i}+1}\big) \Big)}{\widetilde{p} - 2\widetilde{m}} \Bigg| \le 1 \tag{28}$$

for all $0 \le j \le \nu_{q_i} - 1$, $0 \le \widetilde{p} \le n_{j,q_i} + 1$;

$$\Bigg| \sum_{m=1}^{[\frac{n_{j,q_{i-1}}}{2}]}{}' \frac{\sum_{s=q_i}^{\infty} \Big(f_s\big(\frac{\pi(2m+1)}{n_{j,q_{i-1}}}\big) - 2f_s\big(\frac{2\pi m}{n_{j,q_{i-1}}}\big) + f_s\big(\frac{\pi(2m-1)}{n_{j,q_{i-1}}}\big) \Big)}{p - 2m} \Bigg| \le 1 \tag{29}$$

for all $0 \le j \le \nu_{q_{i-1}} - 1$, $0 \le p \le n_{j,q_{i-1}}$; and

$$\left| \sum_{\widetilde{m}=1}^{\frac{n_{j,q_{i-1}}+1}{2}}{}' \frac{\sum_{s=q_i}^{\infty}\Big(f_s\big(\frac{\pi(2\widetilde{m}+1)}{n_{j,q_{i-1}}+1}\big) - 2f_s\big(\frac{2\pi\widetilde{m}}{n_{j,q_{i-1}}+1}\big) + f_s\big(\frac{\pi(2\widetilde{m}-1)}{n_{j,q_{i-1}}+1}\big)\Big)}{\widetilde{p} - 2\widetilde{m}} \right| \le 1 \tag{30}$$

for all $0 \le j \le \nu_{q_{i-1}} - 1$, $0 \le \widetilde{p} \le n_{j,q_{i-1}} + 1$.

Note that in what follows we shall show that inequalities (29) and (30) remain true if summation in the corresponding numerators is restricted to any subsequence of indices, rather than to all indices $s \ge q_i$.

Now we justify the possibility of the choice of f_{q_i} satisfying (27)–(29).

1°. By (20), (24), and (25), each function $\sum_{\mu=1}^{i-1} f_{q_\mu}$ is a broken line with finitely many links; therefore, it belongs to the Lipschitz class of order 1 with some constant M_{i-1}. Consequently, for any $0 \le p \le n_{j,q}$, the absolute value of the numerator of every term in the sum

$$\left| \sum_{m=1}^{[\frac{n_{j,q}}{2}]}{}' \frac{\sum_{\mu=1}^{i-1}\Big(f_{q_\mu}\big(\frac{\pi(2m+1)}{n_{j,q}}\big) - 2f_{q_\mu}\big(\frac{2\pi m}{n_{j,q}}\big) + f_{q_\mu}\big(\frac{\pi(2m-1)}{n_{j,q}}\big)\Big)}{p - 2m} \right|$$

admits the estimate

$$\begin{aligned} &\left| \sum_{\mu=1}^{i-1}\Big(f_{q_\mu}\Big(\frac{\pi(2m+1)}{n_{j,q}}\Big) - 2f_{q_\mu}\Big(\frac{2\pi m}{n_{j,q}}\Big) + f_{q_\mu}\Big(\frac{\pi(2m-1)}{n_{j,q}}\Big)\Big)\right| \\ &\qquad \le \left| \sum_{\mu=1}^{i-1}\Big(f_{q_\mu}\Big(\frac{\pi(2m+1)}{n_{j,q}}\Big) - f_{q_\mu}\Big(\frac{2\pi m}{n_{j,q}}\Big)\Big)\right| \\ &\qquad\qquad + \left| \sum_{\mu=1}^{i-1}\Big(f_{q_\mu}\Big(\frac{2\pi m}{n_{j,q}}\Big) - f_{q_\mu}\Big(\frac{\pi(2m-1)}{n_{j,q}}\Big)\Big)\right| \le 2\frac{M_{i-1}\pi}{n_{j,q}}. \end{aligned}$$

As a consequence,

$$\begin{aligned} &\left| \sum_{m=1}^{[\frac{n_{j,q}}{2}]}{}' \frac{\sum_{\mu=1}^{i-1}\Big(f_{q_\mu}\big(\frac{\pi(2m+1)}{n_{j,q}}\big) - 2f_{q_\mu}\big(\frac{2\pi m}{n_{j,q}}\big) + f_{q_\mu}\big(\frac{\pi(2m-1)}{n_{j,q}}\big)\Big)}{p - 2m} \right| \\ &\qquad \le 2\frac{M_{i-1}\pi}{n_{j,q}} 2 \sum_{m=1}^{[\frac{n_{j,q}}{2}]}{}' \left|\frac{1}{p-2m}\right| \le 2\frac{M_{i-1}\pi}{n_{j,q}} 2\Big(\frac{1}{2}\ln\frac{n_{j,q}}{2} + 1\Big) = 2\frac{M_{i-1}\pi}{n_{j,q}}\Big(\ln\frac{n_{j,q}}{2} + 2\Big). \end{aligned}$$

Combining this inequality with (18), (19), and (21), we see that the choice of q_i satisfying (27) is indeed possible.

Similarly, we can show that for all $0 \le \widetilde{p} \le n_{j,q} + 1$ we have

$$\begin{aligned} &\left| \sum_{\widetilde{m}=1}^{\frac{n_{j,q}+1}{2}}{}' \frac{\sum_{\mu=1}^{i-1}\Big(f_{q_\mu}\big(\frac{\pi(2\widetilde{m}+1)}{n_{j,q}+1}\big) - 2f_{q_\mu}\big(\frac{2\pi\widetilde{m}}{n_{j,q}+1}\big) + f_{q_\mu}\big(\frac{\pi(2\widetilde{m}-1)}{n_{j,q}+1}\big)\Big)}{\widetilde{p} - 2\widetilde{m}} \right| \\ &\qquad \le 2\frac{M_{i-1}\pi}{n_{j,q}+1}\Big(\ln\frac{n_{j,q}+1}{2} + 2\Big). \end{aligned}$$

Together with (18), (19), and (21), this ensures the possibility of choosing q_i so as to satisfy inequality (28).

2°. Relations (18), (25), and (26) imply that

$$\begin{aligned}
&\left|\sum_{m=1}^{[\frac{n_{j,q}}{2}]}{}' \frac{\sum_{s=q+1}^{\infty}\Big(f_s\big(\frac{\pi(2m+1)}{n_{j,q}}\big)-2f_s\big(\frac{2\pi m}{n_{j,q}}\big)+f_s\big(\frac{\pi(2m-1)}{n_{j,q}}\big)\Big)}{p-2m}\right| \\
&\qquad \le \sum_{m=1}^{[\frac{n_{j,q}}{2}]}{}' \frac{4\frac{1}{\sqrt{\ln\nu_{q+1}}}\sum_{s=0}^{\infty}3^{-s}}{|p-2m|} \\
&\qquad \le 6\frac{1}{\sqrt{\ln\nu_{q+1}}}2\Big(1+\frac{1}{2}\sum_{l=1}^{[\frac{n_{j,q}}{4}]}\frac{1}{l}\Big) \\
&\qquad \le 12\frac{1}{\sqrt{\ln\nu_{q+1}}}\Big(\ln\Big[\frac{n_{j,q}}{2}\Big]+1\Big),
\end{aligned}$$

for all $0 \le p \le n_{j,q}$, and that

$$\begin{aligned}
&\left|\sum_{\widetilde{m}=1}^{\frac{n_{j,q}+1}{2}}{}' \frac{\sum_{s=q+1}^{\infty}\Big(f_s\big(\frac{\pi(2\widetilde{m}+1)}{n_{j,q}+1}\big)-2f_s\big(\frac{2\pi\widetilde{m}}{n_{j,q}+1}\big)+f_s\big(\frac{\pi(2\widetilde{m}-1)}{n_{j,q}+1}\big)\Big)}{\widetilde{p}-2\widetilde{m}}\right| \\
&\qquad \le 12\frac{1}{\sqrt{\ln\nu_{q+1}}}\Big(\ln\frac{n_{j,q}+1}{2}+1\Big)
\end{aligned}$$

for all $0 \le \widetilde{p} \le n_{j,q}+1$.

Together with (18), (19), and (21), these inequalities show that q_i can be chosen so as to satisfy (29) and (30). It should be mentioned that if summation in the numerators on the left in (29) and (30) is taken over some subsequence of indices (rather than over all $s \ge q_i$), then these inequalities will remain valid.

By (18), (20), (24), (25), and the Weierstrass criterion, the series

$$f(x) = \sum_{i=1}^{\infty} f_{q_i}(x) \tag{31}$$

represents a continuous function of class $C_0[0,\pi]$.

We take an arbitrary $x \in (0,\pi)$ and define numbers p and $\widetilde{p}$ as in (10) (the tilde $\sim$ means that the corresponding quantity is defined for $n := n+1$). The number j will be chosen so that

$$x \in \Big[\frac{\pi j}{\nu_{q_i}}, \frac{\pi(j+1)}{\nu_{q_i}}\Big) \tag{32}$$

for each $i \in \mathbb{N}$. We estimate the number of nodes $\frac{2m\pi}{n_{j,q_i}}$ with even indices falling into the segment $[\frac{\pi j}{\nu_{q_i}}, \frac{\pi(j+1)}{\nu_{q_i}}]$:

$$l(j,q_i) \le \frac{n_{j,q_i}}{2\nu_{q_i}} + 1. \tag{33}$$

Similarly, for the number of nodes $\frac{2\widetilde{m}\pi}{n_{j,q_i}+1}$ with even indices falling into the segment $[\frac{\pi j}{\nu_{q_i}}, \frac{\pi(j+1)}{\nu_{q_i}}]$, we have

$$\widetilde{l}(j,q_i) \le \frac{n_{j,q_i}+1}{2\nu_{q_i}} + 1. \tag{34}$$

Also, by (24), every function ϕ_{j,q_i} vanishes at the nodes with even indices that belong

to $E(j,q_i)$. By (21), the maximal number $r(j,q_i)$ of such nodes in the set $E(j,q_i)\setminus[\frac{\pi j}{\nu_{q_i}},\frac{\pi(j+1)}{\nu_{q_i}}]$ can be estimated as follows:

$$\begin{aligned} r(j,q_i) &\le 2\frac{\sum_{s=0}^{j-1}\sum_{m=1}^{[\frac{n_{s,q_i}}{2}]}\frac{2\pi e^{-n_{s,q_i}}}{n_{s,q_i}}}{\frac{2\pi}{n_{j,q_i}}} \\ &= 2\frac{\sum_{s=0}^{j-1}\sum_{m=1}^{\frac{n_{s,q_i}-1}{2}}\frac{e^{-n_{s,q_i}}}{n_{s,q_i}}}{\frac{1}{n_{j,q_i}}} \le n_{j,q_i}\sum_{s=0}^{j-1}e^{-n_{s,q_i}}. \end{aligned}$$

Using (19), (21), and the fact that

$$\sum_{s=0}^{j-1}e^{-n_{s,q_i}} = e^{-n_{0,q_i}}\sum_{s=0}^{j-1}e^{-(n_{s,q_i}-n_{0,q_i})} \le e^{-n_{0,q_i}}\frac{e}{e-1} \le 2e^{-n_{0,q_i}} \le 2e^{-\nu_{q_i}^2},$$

we obtain the inequality

$$r(j,q_i) \le 2n_{j,q_i}e^{-\nu_{q_i}^2}. \tag{35}$$

Similar arguments show that the number $\widetilde{r}(j,q_i)$ of zeros of the function $\sin(n_{j,q_i}+1)x$ with even indices that lie in $E(j,q_i)\setminus[\frac{\pi j}{\nu_{q_i}},\frac{\pi(j+1)}{\nu_{q_i}}]$ satisfies the estimate

$$\widetilde{r}(j,q_i) \le 2(n_{j,q_i}+1)e^{-\nu_{q_i}^2}. \tag{36}$$

Now we want to find a lower bound for the sum

$$\begin{aligned} &\left|\sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}\frac{f\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2f\left(\frac{2\pi m}{n_{j,q_i}}\right)+f\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\ &\quad\ge \left|\sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}\frac{f_{q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2f_{q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+f_{q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\ &\qquad - \left|\sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}\frac{\sum_{\mu=1}^{i-1}\left(f_{q_\mu}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2f_{q_\mu}\left(\frac{2\pi m}{n_{j,q_i}}\right)+f_{q_\mu}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)\right)}{p-2m}\right| \\ &\qquad - \left|\sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}\frac{\sum_{\mu=i+1}^{\infty}\left(f_{q_\mu}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2f_{q_\mu}\left(\frac{2\pi m}{n_{j,q_i}}\right)+f_{q_\mu}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)\right)}{p-2m}\right|. \end{aligned} \tag{37}$$

If the number p defined as in (10) is even, then, as before, one term is missing in each sum in (37), namely, the term with zero denominator.

Note that, by (24), $\phi_{\alpha,q_i}(x_{k,n_{j,q_i}})=0$ if $0\le j<\alpha\le\nu_{q_i}-1$, $0\le k\le n_{j,q_i}$, $i\in\mathbb{N}$. Therefore, for every $x\in(0,\pi)$ there exists $j=j(x,i)$ such that, by (20), (24), (25), and

(32), the first sum in (37) can be estimated from below as follows:

$$
\begin{aligned}
&\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{f_{q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2f_{q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+f_{q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad=\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\sum_{\alpha=0}^{\nu_{q_i}-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2\sum_{\alpha=0}^{\nu_{q_i}-1}\phi_{\alpha,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+\sum_{\alpha=0}^{\nu_{q_i}-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad=\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\sum_{\alpha=0}^{j}\phi_{\alpha,q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2\sum_{\alpha=0}^{j}\phi_{\alpha,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+\sum_{\alpha=0}^{j}\phi_{\alpha,q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad=\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\phi_{j,q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2\phi_{j,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+\phi_{j,q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right. \\
&\qquad\left.+\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad\geq 2\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\phi_{j,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)}{2m-p} \\
&\qquad-\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right|.
\end{aligned}
$$

Here, by (19), (20), (24), and (25), for estimating the first sum from below, we can take only part of the terms whose indices correspond to the modes lying from one side of x. We choose the side on which there are at least as many terms as on the opposite side; by (20) and (33), this part has at least $[\frac{n_{j,q_i}}{4}]-l(j,q_i)$ nonnegative summands. This yields the inequality

$$
\begin{aligned}
&\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{f_{q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2f_{q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+f_{q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad\geq 2\frac{1}{\sqrt{\ln\nu_{q_i}}}\left(\sum_{m=l(j,q_i)}^{[\frac{n_{j,q_i}}{4}]}\frac{1}{2m}\right)-2\frac{1}{\sqrt{\ln\nu_{q_i}}}\sum_{m\,:\,\frac{2\pi m}{n_{j,q_i}}\in E(j,q_i)\setminus[\frac{\pi j}{\nu_{q_i}},\frac{\pi(j+1)}{\nu_{q_i}}]}\left|\frac{1}{2m-p}\right| \\
&\qquad-\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right)-2\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{2\pi m}{n_{j,q_i}}\right)+\sum_{\alpha=0}^{j-1}\phi_{\alpha,q_i}\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right|.
\end{aligned}
$$

By (24), every function ϕ_{j,q_i} vanishes at the nodes with even indices that lie in $E(j,q_i)$. Subtracting an upper bound for the second sum from the first sum, we use the fact that, by (35), the number of indices m such that $\frac{2\pi m}{n_{j,q_i}}\in E(j,q_i)\setminus[\frac{\pi j}{\nu_{q_i}},\frac{\pi(j+1)}{\nu_{q_i}}]$ does not exceed

$r(j,q_i)$. Now we can employ (22), (25), (26), (33), and (35) to get the estimate

$$\begin{aligned}
&\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{f_{q_i}\big(\frac{\pi(2m+1)}{n_{j,q_i}}\big)-2f_{q_i}\big(\frac{2\pi m}{n_{j,q_i}}\big)+f_{q_i}\big(\frac{\pi(2m-1)}{n_{j,q_i}}\big)}{p-2m}\right| \\
&\qquad \geq 2\frac{1}{\sqrt{\ln\nu_{q_i}}}\left(\sum_{m=l(j,q_i)+r(j,q_i)}^{[\frac{n_{j,q_i}}{4}]}\frac{1}{2m}\right)-1 \geq \frac{1}{\sqrt{\ln\nu_{q_i}}}\ln\frac{[\frac{n_{j,q_i}}{4}]}{\frac{n_{j,q_i}}{2\nu_{q_i}}+1+2n_{j,q_i}e^{-\nu_{q_i}^2}}-1 \\
&\qquad \geq \frac{1}{\sqrt{\ln\nu_{q_i}}}\ln\frac{1-\frac{4}{n_{j,q_i}}}{2+\frac{4\nu_{q_i}}{n_{j,q_i}}+8\nu_{q_i}e^{-\nu_{q_i}^2}}\nu_{q_i}-1.
\end{aligned}$$

By (19) and (21), for sufficiently large $i\in\mathbb{N}$ we obtain

$$\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{f_{q_i}\big(\frac{\pi(2m+1)}{n_{j,q_i}}\big)-2f_{q_i}\big(\frac{2\pi m}{n_{j,q_i}}\big)+f_{q_i}\big(\frac{\pi(2m-1)}{n_{j,q_i}}\big)}{p-2m}\right| \geq \frac{1}{2\sqrt{\ln\nu_{q_i}}}\ln\nu_{q_i}. \tag{38}$$

Similarly, for every $x\in(0,\pi)$ there is $j=j(x,i)$ for which, by (19), (20), (23), (24), (25), (26), (34), and (36), for sufficiently large $i\in\mathbb{N}$ the first sum in (37) admits a similar lower estimate:

$$\left|\sum_{\widetilde{m}=1}^{\frac{n_{j,q_i}+1}{2}}{}' \frac{f_{q_i}\big(\frac{\pi(2\widetilde{m}+1)}{n_{j,q_i}+1}\big)-2f_{q_i}\big(\frac{\pi(2\widetilde{m})}{n_{j,q_i}+1}\big)+f_{q_i}\big(\frac{\pi(2\widetilde{m}-1)}{n_{j,q_i}+1}\big)}{\widetilde{p}-2\widetilde{m}}\right| \geq \frac{1}{2\sqrt{\ln\nu_{q_i}}}\ln\nu_{q_i}. \tag{39}$$

Using (27), (29), (37), and (38) (note that inequality (29) can only become stronger after extracting the subsequence $\{f_{q_i}\}_{i\in\mathbb{N}}$ from the sequence $\{f_q\}_{q\in\mathbb{N}}$), we see that for every $x\in(0,\pi)$ and every sufficiently large $i\in\mathbb{N}$ we can find $0\leq j\leq\nu_{q_i}-1$ such that

$$\left|\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]}{}' \frac{f\big(\frac{\pi(2m+1)}{n_{j,q_i}}\big)-2f\big(\frac{2\pi m}{n_{j,q_i}}\big)+f\big(\frac{\pi(2m-1)}{n_{j,q_i}}\big)}{p-2m}\right| \geq \frac{1}{2\sqrt{\ln\nu_{q_i}}}\ln\nu_{q_i}-2. \tag{40}$$

Similarly, relations (28), (30), and (39) lead to the fact that for every $x\in(0,\pi)$ and every sufficiently large $i\in\mathbb{N}$ there exists $0\leq j\leq\nu_{q_i}-1$ such that (40) is true, and

$$\left|\sum_{\widetilde{m}=1}^{\frac{n_{j,q_i}+1}{2}}{}' \frac{f\big(\frac{\pi(2\widetilde{m}+1)}{n_{j,q_i}+1}\big)-2f\big(\frac{2\pi\widetilde{m}}{n_{j,q_i}+1}\big)+f\big(\frac{\pi(2\widetilde{m}-1)}{n_{j,q_i}+1}\big)}{\widetilde{p}-2\widetilde{m}}\right| \geq \frac{1}{2\sqrt{\ln\nu_{q_i}}}\ln\nu_{q_i}-2. \tag{41}$$

Consider a sequence of positive numbers $0<a_n\leq\frac{1}{4}$. If an integer k satisfies the inequality $(2n+1)a_n\leq k\leq(n+1)-(2n+1)a_n$, then, for any $n\in\mathbb{N}$, all $\frac{a_n\pi}{n}$-neighborhoods of the nodes $\frac{k\pi}{n}$ and all $\frac{a_n\pi}{n+1}$-neighborhoods of the nodes $\frac{k\pi}{n+1}$ have no common points lying in $E_n=\big[\frac{2\pi na_n}{n+1},\frac{\pi(n+1-2na_n)}{n+1}\big]$.

For each $i=3,4,5,\dots$ and $0\leq j<\nu_{q_i}$, we put

$$a_{n_{j,q_i}}=\frac{1}{2}\left(\frac{1}{\sqrt{\ln\nu_{q_i}}}\right)^{\frac{1}{2}}. \tag{42}$$

Now, if

$$x\notin\bigcup_{k=0}^{n_{j,q_i}}\left(\frac{k\pi}{n_{j,q_i}}-\frac{a_{n_{j,q_i}}\pi}{n_{j,q_i}},\frac{k\pi}{n_{j,q_i}}+\frac{a_{n_{j,q_i}}\pi}{n_{j,q_i}}\right)=D_{n_{j,q_i}}, \tag{43}$$

where $0 \le j \le \nu_{q_i} - 1$, $i \in \mathbb{N}$, then

$$|\sin(n_{j,q_i}x)| \ge \left(\frac{1}{\sqrt{\ln \nu_{q_i}}}\right)^{\frac{1}{2}}, \quad 0 \le j \le \nu_{q_i} - 1, \ \ i \in \mathbb{N}. \tag{44}$$

On the other hand, if

$$x \notin \bigcup_{k=0}^{n_{j,q_i}+1} \left(\frac{k\pi}{n_{j,q_i}+1} - \frac{a_{n_{j,q_i}}\pi}{n_{j,q_i}+1}, \frac{k\pi}{n_{j,q_i}+1} + \frac{a_{n_{j,q_i}}\pi}{n_{j,q_i}+1}\right) = \widetilde{D}_{n_{j,q_i}}, \tag{45}$$

where $0 \le j \le \nu_{q_i} - 1$, $i \in \mathbb{N}$, then

$$|\sin(n_{j,q_i}+1)x| \ge \left(\frac{1}{\sqrt{\ln \nu_{q_i}}}\right)^{\frac{1}{2}}, \quad 0 \le j \le \nu_{q_i} - 1, \ \ i \in \mathbb{N}. \tag{46}$$

We denote

$$E_{n_{j,q_i}} = \left[\frac{2\pi n_{j,q_i} a_{n_{j,q_i}}}{n_{j,q_i}+1}, \frac{\pi(n_{j,q_i}+1-2n_{j,q_i}a_{n_{j,q_i}})}{n_{j,q_i}+1}\right], \ \ 0 \le j \le \nu_{q_i} - 1, \ \ i = 3,4,5,\dots. \tag{47}$$

From (43)–(47) it follows that if $x \in E_{n_{j,q_i}} \subset [0,\pi] \setminus (D_{n_{j,q_i}} \cap \widetilde{D}_{n_{j,q_i}})$, then at least one of the inequalities (44) and (46) is fulfilled. By (40), (41), (44), and (46) for $x \in E_{n_{j,q_i}}$ we have at least one of the following two relations:

$$\begin{aligned}
&\left|\frac{\sin n_{j,q_i}x}{2\pi} \sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]} \frac{f\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right) - 2f\left(\frac{2\pi m}{n_{j,q_i}}\right) + f\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad = \frac{|\sin n_{j,q_i}x|}{2\pi}\left|\sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]} \frac{f\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right) - 2f\left(\frac{2\pi m}{n_{j,q_i}}\right) + f\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \\
&\quad \ge \frac{1}{2\pi}\left(\frac{1}{\sqrt{\ln \nu_{q_i}}}\right)^{\frac{1}{2}}\left(\frac{\frac{1}{\sqrt{\ln \nu_{q_i}}}}{2}\ln \nu_{q_i} - 2\right) = \frac{1}{4\pi}\left(\ln \nu_{q_i}\right)^{\frac{1}{4}} - \frac{\left(\frac{1}{\sqrt{\ln \nu_{q_i}}}\right)^{\frac{1}{2}}}{\pi}
\end{aligned}$$

or

$$\begin{aligned}
&\frac{|\sin(n_{j,q_i}+1)x|}{2\pi}\left|\sideset{}{'}\sum_{\widetilde{m}=1}^{\frac{n_{j,q_i}+1}{2}} \frac{f\left(\frac{\pi(2\widetilde{m}+1)}{n_{j,q_i}+1}\right) - 2f\left(\frac{2\pi\widetilde{m}}{n_{j,q_i}+1}\right) + f\left(\frac{\pi(2\widetilde{m}-1)}{n_{j,q_i}+1}\right)}{\widetilde{p}-2\widetilde{m}}\right| \\
&\quad \ge \frac{1}{2\pi}\left(\frac{1}{\sqrt{\ln \nu_{q_i}}}\right)^{\frac{1}{2}}\left(\frac{\frac{1}{\sqrt{\ln \nu_{q_i}}}}{2}\ln \nu_{q_i} - 2\right) = \frac{1}{4\pi}\left(\ln \nu_{q_i}\right)^{\frac{1}{4}} - \frac{\left(\frac{1}{\sqrt{\ln \nu_{q_i}}}\right)^{\frac{1}{2}}}{\pi}.
\end{aligned}$$

Consequently, by (19), (21), and (47), for every $x \in E_{n_{j,q_i}}$ there exists $0 \le j \le \nu_{q_i} - 1$ and $i_0 \in \mathbb{N}$ such that for all $i > i_0$ we have at least one of the inequalities

$$\frac{|\sin n_{j,q_i}x|}{2\pi}\left|\sideset{}{'}\sum_{m=1}^{[\frac{n_{j,q_i}}{2}]} \frac{f\left(\frac{\pi(2m+1)}{n_{j,q_i}}\right) - 2f\left(\frac{2\pi m}{n_{j,q_i}}\right) + f\left(\frac{\pi(2m-1)}{n_{j,q_i}}\right)}{p-2m}\right| \ge \frac{\left(\ln \nu_{q_i}\right)^{\frac{1}{4}}}{8\pi} \tag{48}$$

or

$$\frac{|\sin(n_{j,q_i}+1)x|}{2\pi}\left|\sideset{}{'}\sum_{\widetilde{m}=1}^{\frac{n_{j,q_i}+1}{2}} \frac{f\left(\frac{\pi(2\widetilde{m}+1)}{n_{j,q_i}+1}\right) - 2f\left(\frac{2\pi\widetilde{m}}{n_{j,q_i}+1}\right) + f\left(\frac{\pi(2\widetilde{m}-1)}{n_{j,q_i}+1}\right)}{\widetilde{p}-2\widetilde{m}}\right| \ge \frac{\left(\ln \nu_{q_i}\right)^{\frac{1}{4}}}{8\pi}. \tag{49}$$

Relations (10), (18), (19), (21), (32), (42), and (47) show that for every $x \in (0,\pi)$ there exists $i_x \in \mathbb{N}$ starting with which we have $x \in E_{n_{j(x,i),q_i}}$ and at least one of inequalities (48) and (49) holds true. Consequently, for every $x \in (0,\pi)$ we can find

a sequence of positive integers $\{n_i\}_{i=1}^{\infty}$ tending to infinity, and a sequence of integers $\{p_i\}_{i=1}^{\infty}$ (determined by $\{n_i\}$ in accordance with (10)) such that

$$\lim_{n\to\infty}\left|\frac{\sin n_i x}{2\pi}\sum_{m=1}^{[\frac{n_i}{2}]}{}' \frac{f\big(\frac{\pi(2m+1)}{n_i}\big)-2f\big(\frac{2\pi m}{n_i}\big)+f\big(\frac{\pi(2m-1)}{n_i}\big)}{p_i-2m}\right|=\infty.$$

Now, since the function (13) is bounded, relation (31) implies that

$$\lim_{i\to\infty}|L_{n_i}(f,x)|=\infty,$$

i.e., the processes $L_n(f,x)$ diverge unboundedly everywhere on $(0,\pi)$.

Theorem 1 is proved. □

Proof of Theorem 2. The existence of x_0 and g satisfying (2) and (3) follows from Theorem 1 and the Carleson theorem [39]. For the role of f we can take the function constructed by Fejér (see, e.g., [40, Chapter I, §45, (45.2)]). Then, putting $x_1=0$, we obtain (4). The interpolational property of the operators (1) implies that $L_n(f,0)=f(0)$, $n=1,2,3,\dots$, which yields (5). Since Fejér's function is continuous, Theorem 2 is proved. □

Proof of Corollary 1. The validity of (6)–(9) follows from Lemma 1 in [41, Chapter 16, §16.3], where the equiconvergence of the Fourier partial sums and the Fourier integrals was established, and Theorem 2. In the latter theorem, as g we take the function constructed in Theorem (1), extending it by the formula

$$\widetilde{g}(x)=\begin{cases} g(x) & \text{if } x\in[0,\pi],\\ 0 & \text{if } x\in\mathbb{R}\setminus[0,\pi]\end{cases}$$

and as f we take Fejér's function extended as follows:

$$\widetilde{f}(x)=\begin{cases} f(x), & \text{if } x\in[-\pi,\pi],\\ 0, & \text{if } x\in\mathbb{R}\setminus[-2\pi,2\pi],\\ \text{linear} & \text{on the remaining intervals of the segment } [-2\pi,2\pi].\end{cases}$$

Corollary 1 is proved. □

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