EXTREMAL PROPERTIES OF SPHERICAL SEMIDESIGNS

N. O. KOTELINA AND A. B. PEVNYĬ

ABSTRACT. For every even \( t \geq 2 \) and every set of vectors \( \Phi = \{\phi_1, \ldots, \phi_m\} \) on the sphere \( S^{n-1} \), the notion of the \( t \)-potential \( P_t(\Phi) = \sum_{i,j=1}^{m} [\langle \phi_i, \phi_j \rangle]^t \) is introduced. It is proved that the minimum value of the \( t \)-potential is attained at the spherical semidesigns of order \( t \) and only at them. The first result of this type was obtained by B. B. Venkov. The result is extended to the case of sets \( \Phi \) that do not lie on the sphere \( S^{n-1} \). For the V. A. Yudin potentials \( U_k(\Phi) \), \( k = 2, 4, \ldots, t \), it is shown that they attain the minimal value at the spherical semidesigns of order \( t \) and only at them.

§1. NOTATION AND PRELIMINARY INFORMATION

We use the dot product \( \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \) of vectors \( x, y \in \mathbb{R}^n \) and the norm \( \|x\| = \sqrt{\langle x, x \rangle} \).

Let

\[ S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \]

be the unit sphere in \( \mathbb{R}^n \). In the sequel, we denote by \( t \) an even number, \( t \geq 2 \).

Definition 1. A set of vectors \( \Phi = \{\phi_1, \phi_2, \ldots, \phi_m\} \subset S^{n-1} \) is called a spherical semidesign of order \( t \) (or, simply, a spherical \( t \)-semidesign) if there is a constant \( A_t > 0 \) such that the Waring identity

\[ \sum_{i=1}^{m} [\langle \phi_i, x \rangle]^t = A_t \|x\|^t, \quad x \in \mathbb{R}^n, \]

is valid.

The constant \( A_t \) can be found by successive applications of the Laplace operator \( \Delta \) to both sides of equation (1). As a result, we obtain \( A_t = cm \), where

\[ c = \frac{(t-1)!!}{n(n+2) \cdots (n+t-2)}. \]

We also prove that a spherical semidesign of order \( t \) is a spherical semidesign of all orders \( k = 2, 4, \ldots, t \). The spherical \( p \)-designs were defined by Delsarte, Goethals, and Seidel in 1977. The same definition was used by many other authors [2, 4, 11, 12].

Definition 2. A set of points \( \Phi = \{\varphi_1, \ldots, \varphi_m\} \subset S^{n-1} \) is called a spherical \( p \)-design if the identity

\[ \frac{1}{\sigma_n} \int_{S^{n-1}} Q(x) \, dS = \frac{1}{m} \sum_{i=1}^{m} Q(\varphi_i) \]

is valid for all polynomials \( Q(x) \) of degree at most \( p \).

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Here \( \sigma_n = 2\pi^{n/2}/\Gamma\left(\frac{n}{2}\right) \) is the area of the sphere \( S^{n-1} \), the order \( p \) can be odd, and the polynomial \( Q(x) \) is not necessarily homogeneous.

It is easy to establish a relationship between Definitions 1 and 2. If \( \Phi = \{\varphi_1, \ldots, \varphi_m\} \) is a spherical semidesign of even order \( t \), then

\[
\Phi_{2m} = \{\varphi_1, \ldots, \varphi_m, -\varphi_1, \ldots, -\varphi_m\}
\]

is a spherical design of order \( t + 1 \).

Conversely, let the set \( \Phi_{2m} \subset S^{n-1} \) be symmetric, i.e., for each vector \( \varphi \in \Phi_{2m} \) we have \(-\varphi \in \Phi_{2m}\). Then \( \Phi_{2m} \) can be represented in the form

\[
\Phi_{2m} = \Phi \cup -\Phi, \quad \Phi \cap -\Phi = \emptyset.
\]

If \( \Phi_{2m} \) is a spherical \((t + 1)\)-design, then \( \Phi \) is a spherical semidesign of order \( t \).

It should be noted that there exist nonsymmetric spherical designs. Interesting examples were presented in the papers [11, 12] by Yudin.

\[\] §2. THE INEQUALITY OF B. B. VENKOV

We consider an arbitrary set \( \Phi = \{\varphi_1, \ldots, \varphi_m\} \) of vectors on the sphere \( S^{n-1} \). By the \( t \)-potential of this set we mean the quantity

\[
P_t(\Phi) = \sum_{i=1}^{m} \sum_{j=1}^{m} \left[ \langle \varphi_i, \varphi_j \rangle \right]^t.
\]

For \( t = 2 \), the potential \( P_2(\Phi) \) is called a frame. This potential was introduced in the papers [7, 8].

**Theorem 1** (B. B. Venkov [1]). For every set \( \Phi = \{\varphi_1, \ldots, \varphi_m\} \subset S^{n-1} \), we have

\[
P_t(\Phi) \geq cm^2,
\]

where the constant \( c \) is defined by (2). Equality occurs in (5) on the spherical \( t \)-semidesigns and only on them.

In [1], only spherical sets \( \Phi \) were considered, and in this case, the minimum value of \( P_t(\Phi) \) is attained at spherical \( t \)-designs and only at them. In the case of arbitrary sets \( \Phi \) on the sphere, the minimal value of \( P_t(\Phi) \) is attained at spherical semidesigns of order \( t \). In the present paper, we generalize Theorem 1 to the case of nonunit vectors (see Theorem 2 below).

**Example 1.** Consider the set \( \Phi = \{\varphi_1, \ldots, \varphi_6\} \) of six vertices of an icosahedron inscribed in \( S^2 \) such that \( \Phi \) contains no opposite vectors. We calculate the 4-potential \( P_4(\Phi) \).

Since \( \varphi_i \in S^2 \), we obtain \( ||\varphi_i||^2 = 1 \), \( i = 1, \ldots, 6 \). By Haantjes’s theorem [6], the following relations are valid for the vertices of the icosahedron:

\[
\langle \varphi_i, \varphi_j \rangle = \pm \frac{1}{\sqrt{5}} \quad \text{for} \quad i \neq j.
\]

Therefore, \( P_4(\Phi) = 6 \cdot 1 + 30 \cdot \frac{1}{25} = \frac{36}{5} \). At the same time, for \( n = 3, \ t = 4, \) and \( m = 6 \), we have

\[
cm^2 = \frac{1 \cdot 3}{3 \cdot 5} \cdot 6^2 = \frac{36}{5}.
\]

Since \( P_4(\Phi) = cm^2 \), \( \Phi \) is a 4-semidesign. The set \( \Phi_{12} = \{\varphi_1, \ldots, \varphi_6, -\varphi_1, \ldots, -\varphi_6\} \) of all 12 vertices of the icosahedron is a spherical 5-design.
§3. Nonspherical semidesigns

Now we consider $t$-semidesigns in $\mathbb{R}^n$ containing vectors with different lengths. Let $t$ be even, $t \geq 2$.

**Definition 3.** A set of nonzero vectors $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ in $\mathbb{R}^n$ is called a $t$-semidesign if there is $A_t > 0$ such that

\begin{equation}
\sum_{i=1}^{m} |\langle \varphi_i, x \rangle|^t = A_t \|x\|^t, \quad x \in \mathbb{R}^n.
\end{equation}

**Lemma 1.** Let $t \geq 4$, and let $\Phi$ be a $t$-semidesign. Then the set of vectors

$$\theta_i = \|\varphi_i\|^2/(t-2) \varphi_i, \quad i = 1, \ldots, m,$$

is a $(t-2)$-semidesign with the constant

$$A_{t-2} = A_t \frac{t+n-2}{t-1}.$$

**Proof.** Applying the Laplace operator $\Delta$ to both sides of equation (6), we obtain

$$t(t-1) \sum_{i=1}^{m} |\langle \varphi_i, x \rangle|^{t-2} \|\varphi_i\|^2 = A_t t(t+n-2) \|x\|^{t-2}, \quad x \in \mathbb{R}^n.$$

This implies the claim. \hfill \square

We put $s = t/2$. Applying the operator $\Delta$ $s - 1$ times, we arrive at the identity

$$\sum_{i=1}^{m} |\langle \varphi_i, x \rangle|^2 \|\varphi_i\|^{s-2} = A_2 \|x\|^2, \quad x \in \mathbb{R}^n.$$

Applying the operator $\Delta$ yet another time, we obtain

$$2 \sum_{i=1}^{m} \|\varphi_i\|^t = A_2 \cdot 2n.$$

Hence $A_2 = \mu/n$, where

\begin{equation}
\mu = \sum_{i=1}^{m} \|\varphi_i\|^t.
\end{equation}

The recurrence relation in the lemma and the equation $A_2 = \mu/n$ imply that $A_t = c \mu$, where $c$ is defined by (2).

§4. An inequality for the $t$-potential

The $t$-potential $P_t(\Phi)$ is defined as above by formula (4).

**Theorem 2.** For every set $\Phi$ of vectors in the space $\mathbb{R}^n$, we have

\begin{equation}
P_t(\Phi) \geq c \left( \sum_{i=1}^{m} \|\varphi_i\|^t \right)^2,
\end{equation}

where the constant $c$ is defined by (2). Equality in (8) is attained at the $t$-semidesigns and only at them.
For the proof, we need a certain technique involving a dot product in the space of homogeneous polynomials \( P_{n,t} \) of degree \( t \).

For polynomials \( f, g \in P_{n,t} \),
\[
f(x) = \sum_{|i|=t} a(i)x^i, \quad g(x) = \sum_{|i|=t} b(i)x^i,
\]
where \( i = (i_1, \ldots, i_n) \) is a vector with nonnegative integral components (multi-index), \(|i| = i_1 + \cdots + i_n\), and \( x^i = x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \), we define the dot product
\[
[f, g] = \sum_{|i|=t} \frac{a(i)b(i)}{c(i)}, \quad \text{where} \quad c(i) = \frac{t!}{i_1!i_2!\cdots i_n!}.
\]
Supplied with this dot product, \( P_{n,t} \) becomes a Hilbert space. For every \( \varphi \in \mathbb{R}^n \), we consider the polynomial
\[
\rho_\varphi(x) = [\langle \varphi, x \rangle]^t = [\varphi_1 x_1 + \cdots + \varphi_n x_n]^t.
\]
It is easy to verify that for every \( \varphi \in \mathbb{R}^n \) and \( f \in P_{n,t} \) we have
\[
[\rho_\varphi, f] = f(\varphi);
\]
i.e., \( \rho_\varphi(x) \) is the reproducing kernel of the space \( P_{n,t} \). We also consider the polynomial \( \omega_t(x) = \|x\|^t \) (as above, \( t \) is an even number, \( t \geq 2 \)). In [1] and [2], the following relation was proved:
\[
[\omega_t, \omega_t] = \frac{1}{c},
\]
where \( c \) is a constant of the form (2).

**Proof.** The proof of Theorem 2 follows the idea indicated by B. B. Venkov in [1]. Using the notation \( \mu \) introduced in [7], we can represent the required inequality in the form \( P_t(\Phi) \geq c\mu^2 \). By the inequality
\[
W := \left[ \sum_{i=1}^m \rho_{\varphi_i} - c\mu \omega_t, \sum_{j=1}^m \rho_{\varphi_j} - c\mu \omega_t \right] \geq 0,
\]
we obtain
\[
W = W_1 - 2c\mu W_2 + c^2\mu^2 \cdot [\omega_t, \omega_t].
\]
By (9), we have
\[
W_1 = \sum_{i,j=1}^m [\rho_{\varphi_i}, \rho_{\varphi_j}] = \sum_{i,j=1}^m \rho_{\varphi_j}(\varphi_i) = \sum_{i,j=1}^m [\langle \varphi_i, \varphi_j \rangle]^t = P_t(\Phi),
\]
\[
W_2 = \sum_{i=1}^m [\rho_{\varphi_i}, \omega_t] = \sum_{i=1}^m \omega_t(\varphi_i) = \sum_{i=1}^m \|\varphi_i\|^t = \mu.
\]
Consequently, using (10), we see that
\[
W = P_t(\Phi) - 2c\mu^2 + c^2\mu^2 \frac{1}{c} = P_t(\Phi) - c\mu^2.
\]
Since \( W \geq 0 \), we have \( P_t(\Phi) \geq c\mu^2 \), which is equivalent to (3).

Now, we study when equality is attained.

Let the set \( \Phi \) be a \( t \)-semidesign. In (10), we put \( x = \varphi_j \) and use the fact that \( A_t = c\mu \). We obtain
\[
\sum_{i=1}^m [\langle \varphi_i, \varphi_j \rangle]^t = c\mu\|\varphi_j\|^t, \quad j = 1, \ldots, m.
\]
Performing summation in (11), we obtain $P_t(\Phi) = c_\mu^2$.

Conversely, suppose equality in (8) occurs for some set $\Phi$. Then the quantity $W$ is equal to zero. The definition of $W$ shows that

$$\sum_{i=1}^{m} \rho_{\phi_i}(x) - c_\mu \omega_t(x) = 0, \quad x \in \mathbb{R}^n;$$

i.e., relation (6) is satisfied with the constant $A_t = c_\mu$. Therefore, $\Phi$ is a $t$-semidesign. The theorem is proved. \(\square\)

§5. HEHENBAUER POLYNOMIALS AND A THEOREM OF V. A. YUDIN

We fix a positive integer $n \geq 3$ and consider the weight

$$w_n(u) = (1 - u^2)^{(n-3)/2}, \quad u \in [-1, 1].$$

Let $G_k(u)$ be the Hehenbauer polynomials (ultraspherical polynomials) of degree $k$. They are orthogonal with weight $w_n(u)$,

$$(12) \quad \int_{-1}^{1} G_k(u) G_s(u) w_n(u) \, du = 0, \quad k \neq s$$

(see [9]). For $k = 0$, we have $G_0(u) \equiv 1$. For $k \geq 1$, the polynomials $G_k(u)$ contain only even or only odd powers of $u$ depending on whether $k$ is even or odd. The addition formula

$$(13) \quad G_k(\langle x, y \rangle) = \sum_{j=1}^{N(k)} Y_j(x) Y_j(y), \quad x, y \in S^{n-1}$$

is valid, where the $Y_j, j = 1, \ldots, N(k)$, are spherical functions of order $k$ (see [10]).

Let $\Phi^* = \{\varphi_1^*, \ldots, \varphi_m^*\} \subset S^{n-1}$ be a spherical $t$-semidesign, where $t$ is even, $t \geq 2$.

In [12], Yudin proved a criterion for the sphericity of designs in terms of Hehenbauer polynomials.

Since our definition of a spherical $t$-semidesign differs from that in [12], it is necessary to change the statement of the theorem.

**Theorem 3** (V. A. Yudin). A set $\Phi^* \subset S^{n-1}$ is a spherical $t$-semidesign if and only if

$$\sum_{i=1}^{m} G_k(\langle \varphi_i^*, x \rangle) = 0, \quad x \in S^{n-1}, \quad k = 2, 4, \ldots, t.$$ 

§6. YUDIN POTENTIALS

For every set $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ on the sphere $S^{n-1}$, we define the Yudin potential

$$U_k(\Phi) = \sum_{i=1}^{m} \sum_{j=1}^{m} G_k(\langle \varphi_i, \varphi_j \rangle), \quad k = 1, 2, \ldots, t.$$ 

The motivation for the term “potential” is the fact that the expressions $U_k(\Phi)$ are nonnegative. Indeed, by the addition formula (13), we have

$$U_k(\Phi) = \sum_{s=1}^{N(k)} \sum_{i=1}^{m} Y_s(\varphi_i) \sum_{j=1}^{m} Y_s(\varphi_j) = \sum_{s=1}^{N(k)} \left( \sum_{i=1}^{m} Y_s(\varphi_i) \right)^2 \geq 0.$$ 

Thus, for every set $\Phi \subset S^{n-1}$, the Yudin potentials $U_k(\Phi)$ are nonnegative for all $k = 1, 2, \ldots, t.$
In the case where \( n = 3 \), so that the Hehenbauer polynomials become Legendre polynomials, the following theorem was stated in [13]. In the proof of the “if” part, it is convenient to use Venkov’s theorem.

**Theorem 4.** Let \( t \) be an even number with \( t \geq 2 \). A set \( \Phi^* = \{ \varphi_1^*, \ldots, \varphi_m^* \} \) on the sphere \( S^{n-1} \) is a spherical \( t \)-semidesign if and only if

\[
U_k(\Phi^*) = 0, \quad k = 2, 4, \ldots, t. \tag{15}
\]

**Proof.** The “only if” part. For a spherical \( t \)-semidesign \( \Phi^* \), the Yudin theorem implies equations (14). We substitute \( x = \varphi_j^* \) in these equations and sum over \( j = 1, \ldots, m \). We obtain

\[
\sum_{i=1}^m \sum_{j=1}^m G_k \left( \langle \varphi_i^*, \varphi_j^* \rangle \right) = 0, \quad k = 2, 4, \ldots, t, \tag{16}
\]

which is equivalent to (15).

The “if” part. We use Theorem 1. Assume that relations (15) are valid. We calculate the usual \( t \)-potential

\[
P_t(\Phi^*) = \sum_{i,j=1}^m \left[ \langle \varphi_i^*, \varphi_j^* \rangle \right]^t
\]

and verify that it is equal to \( cm^2 \), where \( c \) is defined by formula (2).

The polynomial \( u^t \) can be expanded in even Hehenbauer polynomials as follows: \( u^t = \sum_{k=0,2,\ldots,t} d(k)G_k(u) \). We substitute \( u = \langle \varphi_i^*, \varphi_j^* \rangle \) and sum over \( i, j = 1 : m \). Then

\[
P_t(\Phi^*) = \sum_{k=0,2,\ldots,t} d(k) \sum_{i,j=1}^m G_k \left( \langle \varphi_i^*, \varphi_j^* \rangle \right). \tag{17}
\]

By (16) and the identity \( G_0(u) = 1 \), we arrive at the formula

\[
P_t(\Phi^*) = d(0)m^2.
\]

To calculate \( d(0) \), we multiply \( u^t \) by \( G_0 \), using the dot product \( (\ , \ ) \) in the space \( L_2(-1, 1) \) with weight \( w_n(u) \),

\[
d(0) = \frac{(u^t, G_0)}{(G_0, G_0)} = \frac{\int_{-1}^1 u^t w_n(u) \, du}{\int_{-1}^1 w_n(u) \, du} = \frac{I(t)}{I(0)}. \]

For even \( t \), the integral \( I(t) \) is calculated as follows:

\[
I(t) = 2 \int_0^1 u^t w_n(u) \, du = \int_0^1 x^{(t-1)/2}(1-x)^{(n-3)/2} \, dx = \frac{\Gamma \left( \frac{t+1}{2} \right) \Gamma \left( \frac{n-1}{2} \right)}{\Gamma \left( \frac{t+n}{2} \right)}. \]

For \( t = 0 \), we obtain the integral \( I(0) \). As a result, we have

\[
d(0) = \frac{\Gamma \left( \frac{t+1}{2} \right) \Gamma \left( \frac{n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{t+n}{2} \right)} = \frac{(t-1)!!}{n(n+2) \cdots (n+t-2)} = c,
\]

where \( c \) is as in (2). From (17), we conclude that \( P_t(\Phi^*) = cm^2 \), and, by Venkov’s theorem, \( \Phi^* \) is a spherical \( t \)-semidesign. The theorem is proved. \( \square \)

**Remark 1.** For a symmetric spherical \( t \)-semidesign of the form

\[
\Phi_{2m}^* = \{ \varphi_1^*, \ldots, \varphi_m^*, -\varphi_1^*, \ldots, -\varphi_m^* \},
\]

the “only if” part of the theorem can be refined, namely, we have

\[
U_k(\Phi_{2m}^*) = 0 \quad \text{for all} \quad k = 1, 2, 3, \ldots, t.
\]
References


