Алгебра и анализ
Том 22 (2010), № 5

St. Petersburg Math. J.
Vol. 22 (2011), No. 5, Pages 803–812
S 1061-0022(2011)01169-0
Article electronically published on June 28, 2011

ON INTRINSIC ISOMETRIES TO EUCLIDEAN SPACE

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Abstract. Compact metric spaces that admit intrinsic isometries to the Euclidean d-space are considered. Roughly, the main result states that the class of such spaces coincides with the class of inverse limits of Euclidean d-polyhedra.

§1. Introduction

The *intrinsic isometries* are defined in §2; this is a variation of the notion of a *path isometry*, i.e., a map that preserves the lengths of curves. Any intrinsic isometry is a path isometry; the converse is not true in general.

The following statement is a reason for me to prefer intrinsic isometry.

Starting Proposition 1.1. *Let $\mathcal{X}$ be a compact metric space that admits an intrinsic isometry to the d-dimensional Euclidean space (further denoted by $\mathbb{E}^d$). Then $\dim \mathcal{X} \le d$, where $\dim$ denotes the Lebesgue covering dimension.*

This statement is proved in §3. A similar statement for path isometry fails; see Example 4.2. Furthermore, also the Hausdorff dimension cannot be bounded. For example, the $\mathbb{R}$-tree admits an intrinsic isometry to $\mathbb{R}$ and it contains compact subsets of arbitrarily large Hausdorff dimension.

Here are some known results on length spaces that admit *intrinsic isometry* to $\mathbb{E}^d$.

Theorem 1.2. *Let $\mathcal{R}$ be a d-dimensional Riemannian space and $f : \mathcal{R} \to \mathbb{E}^d$ a short map.*[1] *Then, given $\varepsilon > 0$, there is an intrinsic isometry $\imath : \mathcal{R} \to \mathbb{E}^d$ such that*

$$|f(x)\,\imath(x)|_{\mathbb{E}^d} < \varepsilon$$

for any $x \in \mathcal{R}$.

In particular, any Riemannian d-space admits an intrinsic isometry to $\mathbb{E}^d$.

For path isometries, this theorem was proved in [7, 2.4.11], and the same proof works for intrinsic isometries. Applying this theorem, one can show that any limit of an increasing sequence of Riemannian metrics on a fixed d-dimensional manifold admits an intrinsic isometry to $\mathbb{E}^d$. (The proof is similar to the "if"-part of the main theorem.) In particular, any sub-Riemannian metric on a d-dimensional manifold admits an intrinsic isometry to $\mathbb{E}^d$.

Theorem 1.3. *Let $\mathcal{P}$ be a Euclidean polyhedron and $f : \mathcal{P} \to \mathbb{E}^d$ a short map. Then, given $\varepsilon > 0$, there is a piecewise linear intrinsic isometry $\imath : \mathcal{P} \to \mathbb{E}^d$ such that*

$$|f(x)\,\imath(x)|_{\mathbb{E}^d} < \varepsilon$$

for any $x \in \mathcal{P}$.

2010 *Mathematics Subject Classification.* Primary 53C23.

Key words and phrases. Intrinsic isometry, path isometry, Riemannian metric, polyhedron, pro-Euclidean space.

[1] That is, a 1-Lipschitz map.

Corollary 1.4. *Any d-dimensional Euclidean polyhedron admits a piecewise linear intrinsic isometry to $\mathbb{E}^d$.*

This corollary was proved in [8] for dimensions not exceeding 4, but a slight modification of the proof works in all dimensions; see [10]. The 2-dimensional case of this statement was proved in [10]. Later, the proof was extended to all dimensions; see [1]. The proof involves a *piecewise linear analog of the Nash–Kuiper theorem.* That statement was proved in [6], but had been remaining unnoticed for many years and was reproved independently in [2].

A necessary and sufficient condition. Now we describe the main result of the paper.

A compact metric space $\mathcal{X}$ is called a *pro-Euclidean space of rank at most d* if it can be presented as an *inverse limit* $\mathcal{X} = \varprojlim \mathcal{P}_n$ (see §2) of a sequence of Euclidean d-polyhedra $\mathcal{P}_n$.

Theorem 1.5 (Main Theorem)**.** *A compact metric space $\mathcal{X}$ admits an intrinsic isometry to $\mathbb{E}^d$ if and only if $\mathcal{X}$ is a pro-Euclidean space of rank at most d.*

The proof is straightforward, and the statement is more interesting than the proof. It is rare when inverse limits help to solve a natural problem in metric geometry; the only other example I know is the characterization of the homogeneous locally compact metric spaces given in [3].

Note that the statement in Theorem 1.2 (in the compact case) is equivalent to the fact that any compact Riemannian d-space is a pro-Euclidean space of rank at most d. The latter can be obtained directly from the following exercise. In particular, the main theorem provides an alternative proof to Theorem 1.2 in the compact case.

Exercise 1.6. Show that any compact Riemannian space admits a Lipschitz approximation by Euclidean polyhedra.

A nonexample. Recall that a *Minkowski space* is a finite-dimensional real vector space with metric induced by a norm.

Proposition 1.7. *Let Ω be an open subset of a Minkowski d-space $\mathbb{M}^d$. If Ω admits an intrinsic[2] isometry to $\mathbb{E}^m$, then $d \le m$ and $\mathbb{M}^d$ is isometric to $\mathbb{E}^d$.*

In particular, the condition in Theorem 1.1 on Lebesgue's dimension is not sufficient.

I am grateful to A. Akopyan, D. Burago, S. Ivanov, and the anonymous referee for helpful letters and discussions.

§2. Preliminaries

Standard definitions. Given a metric space $\mathcal{X}$ and two points $x, x' \in \mathcal{X}$, we denote by $|xx'| = |xx'|_{\mathcal{X}}$ the distance from x to x' in $\mathcal{X}$.

A *length space* is a metric space such that for any two points x, x' the distance $|xx'|$ coincides with the infimum of the lengths of curves connecting x and x'.

A map $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is said to be *short* if for any $x, x' \in \mathcal{X}$ we have

$$|f(x)f(x')|_{\mathcal{Y}} \le |xx'|_{\mathcal{X}}.$$

A length space $\mathcal{P}$ is called a *Euclidean d-polyhedron* if there is a finite triangulation of $\mathcal{P}$ such that each simplex is isometric to a simplex in $\mathbb{E}^d$.

[2]In fact, the same is true for a path isometry.

Inverse limit. Consider an *inverse system* $(\mathcal{X}_n)_{n=0}^{\infty}$ of compact metric spaces and short maps $\varphi_{m,n} : \mathcal{X}_m \to \mathcal{X}_n$ for $m \geq n$; i.e.,

(1) $\varphi_{m,n} \circ \varphi_{k,m} = \varphi_{k,n}$ for any triple $k \geq m \geq n$, and
(2) for any n, the map $\varphi_{n,n}$ is the identity map of $\mathcal{X}_n$.

A compact metric space $\mathcal{X}$ is called the *inverse limit of the system* $(\varphi_{m,n}, \mathcal{X}_n)$ (we write $\mathcal{X} = \varprojlim \mathcal{X}_n$) if its underlying space consists of all sequences $x_n \in \mathcal{X}_n$ such that $\varphi_{m,n}(x_m) = x_n$ for all $m \geq n$ and for two such sequences (x_n) and (x'_n) the distance is defined by

$$|(x_n)\,(x'_n)|_{\mathcal{X}} = \lim_{n\to\infty} |x_n\, x'_n|_{\mathcal{X}_n}.$$

If $\mathcal{X} = \varprojlim \mathcal{X}_n$, then the maps $\psi_n : \mathcal{X} \to \mathcal{X}_n$ defined by $\psi_n : (x_i)_{i=0}^{\infty} \mapsto x_n$ are called *projections*. Clearly $\psi_n = \varphi_{m,n} \circ \psi_m$ for all $m \geq n$.

Comments. The above definition is equivalent to that of the usual inverse limit in the category with the class of objects formed by compact metric spaces and the class of morphisms formed by short maps.

Note that the inverse limit is not always defined, and if it is defined, then the result is compact by definition. (In principle, the category of compact metric spaces can be extended so that the limit of any inverse system is well defined.)

It is easily seen that the inverse limit of length spaces is a length space.

In general, the inverse limit of a system of spaces differs from its Gromov–Hausdorff limit. For example, consider the inverse system $\mathcal{X}_n = [0,1]$ with the maps $\varphi_{m,n}(x) \equiv 0$. The inverse limit of this system is isometric to the one-point space, while the Gromov–Hausdorff limit is isometric to $[0,1]$. Nevertheless, it is easy to show that if for any $\varepsilon > 0$ the images of $\varphi_{m,n}$ form an ε-net in X_n for all sufficiently large m and n, then $\mathcal{X} = \varprojlim \mathcal{X}_n$ is isometric to the Gromov–Hausdorff limit.

Intrinsic isometries and pullback metrics. Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ a continuous map. Given two points $x, x' \in \mathcal{X}$, a sequence of points $x = x_0, x_1, \dots, x_n = x'$ is called an ε-chain from x to x' if $|x_{i-1}x_i| \leq \varepsilon$ for all $i > 0$. Set

$$\operatorname{pull}_{f,\varepsilon}(x,x') = \inf\left\{\sum_{i=1}^{n} |f(x_{i-1})f(x_i)|_{\mathcal{Y}}\right\},$$

where the infimum is taken along all ε-chains $(x_i)_{i=0}^{n}$ from x to x'.

Clearly, $\operatorname{pull}_{f,\varepsilon}$ is a premetric[3] on $\mathcal{X}$, and $\operatorname{pull}_{f,\varepsilon}(x,x')$ is monotone nonincreasing in ε. Thus, the (possibly infinite) limit

$$\operatorname{pull}_f(x,x') = \lim_{\varepsilon\to 0} \operatorname{pull}_{f,\varepsilon}(x,x')$$

is well defined. The premetric $\operatorname{pull}_f : \mathcal{X}\times\mathcal{X} \to [0,\infty]$ will be called the *pullback metric* for f.

A map $f : \mathcal{X} \to \mathcal{Y}$ between length spaces $\mathcal{X}$ and $\mathcal{Y}$ is an *intrinsic isometry* if

$$|xx'|_{\mathcal{X}} = \operatorname{pull}_f(x,x')$$

for any $x, x' \in \mathcal{X}$.

Any intrinsic isometry is a short map. Moreover, it is easily seen that any intrinsic isometry preserves the lengths of curves. The converse fails; see §4.

Proposition 2.1. *Let $\mathcal{X}$ be a compact (or even proper[4]) metric space. Then existence of an intrinsic isometry $f : \mathcal{X} \to \mathcal{Y}$ implies that $\mathcal{X}$ is a length space.*

[3]That is, it satisfies the triangle inequality, it is symmetric and nonnegative, and $\operatorname{pull}_{f,\varepsilon}(x,x) = 0$, but it might happen that $\operatorname{pull}_{f,\varepsilon}(x,x') = 0$ for $x \neq x'$.

[4]That is, all closed bounded sets in $\mathcal{X}$ are compact.

The proof is left to the reader. Note that this is not true for general $\mathcal{X}$. Consider two points connected by countably many unit intervals $\mathbb{I}_n$ and one interval of length $\frac{1}{2}$; we equip the resulting space with the natural intrinsic metric. Let us remove the interval of length $\frac{1}{2}$ from our space. The metric on the remaining space $\mathcal{X}$ is not intrinsic. Next, construct a map $f : \mathcal{X} \to \mathbb{R}$ so that the restriction $f_n = f|_{\mathbb{I}_n}$ is an intrinsic isometry, $f_n(0) = 0$, $f_n(1) = \frac{1}{2}$, and the $f_n(x)$ converge uniformly to $\frac{x}{2}$. It is easily seen that $f : \mathcal{X} \to \mathbb{R}$ is an intrinsic isometry.

Proposition 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, let $\mathcal{X}$ be compact, and let a continuous map $f : \mathcal{X} \to \mathcal{Y}$ be such that*

$$\sup_{x,x'\in\mathcal{X}} \operatorname{pull}_f(x,x') < \infty.$$

Then, given $\varepsilon > 0$, there is $\delta = \delta(f,\varepsilon) > 0$ such that for any short map $h : \mathcal{X} \to \mathcal{Y}$ such that

$$|f(x)\,h(x)|_{\mathcal{Y}} < \delta \quad \text{for any} \quad x \in \mathcal{X}$$

we have

$$\operatorname{pull}_f(x,x') < \operatorname{pull}_h(x,x') + \varepsilon$$

for any $x, x' \in \mathcal{X}$.

The proof is a direct application of Lemma 2.3.

For a compact metric space $\mathcal{X}$, we denote by $\operatorname{pack}_\varepsilon \mathcal{X}$ the maximal number of points in $\mathcal{X}$ at a distance exceeding ε from each other. Clearly, $\operatorname{pack}_\varepsilon \mathcal{X}$ is finite for any $\varepsilon > 0$.

Lemma 2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, let $\mathcal{X}$ be compact, and let $f, h : \mathcal{X} \to \mathcal{Y}$ be two continuous maps.*

Assume that $|f(x)h(x)| < \delta$ for any $x \in \mathcal{X}$. Then for any $x, x' \in \mathcal{X}$ we have

$$\operatorname{pull}_{f,\varepsilon}(x,x') \le \operatorname{pull}_{h,\varepsilon}(x,x') + 4\cdot\delta\cdot\operatorname{pack}_\varepsilon \mathcal{X}.$$

Proof. Suppose $\operatorname{pull}_{h,\varepsilon}(x,x') < \ell$, i.e., suppose there is an ε-chain $\{x_i\}_{i=0}^n$ from x to x' such that

$$(\star) \qquad \sum_{i=1}^n |h(x_{i-1})\,h(x_i)|_{\mathcal{Y}} < \ell.$$

Since $|h(x_i)f(x_i)| < \delta$, it follows that

$$\operatorname{pull}_{f,\varepsilon}(x,x') \le \sum_{i=1}^n |f(x_{i-1})\,f(x_i)|_{\mathcal{Y}} < \sum_{i=1}^n |h(x_{i-1})\,h(x_i)|_{\mathcal{Y}} + 2\cdot n\cdot\delta.$$

Assume n is the smallest number for which there is an ε-chain satisfying $(\star)$. It suffices to show that

$$n < 2\cdot\operatorname{pack}_\varepsilon \mathcal{X}.$$

If $n \ge 2\cdot\operatorname{pack}_\varepsilon \mathcal{X}$, there are indices i and j such that $j - i > 1$ and $|x_i x_j| \le \varepsilon$. We remove all elements x_k with $i < k < j$ from the chain; i.e., we consider the new ε-chain

$$x = x_0, \dots, x_{i-1}, x_i, x_j, x_{j+1}, \dots, x_n = x'.$$

By the triangle inequality in $\mathcal{Y}$, the new chain satisfies $(\star)$, so that n is not the smallest number, a contradiction. □

Proposition 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces, let $\mathcal{X}$ be compact, and let $\imath : \mathcal{X} \to \mathcal{Y}$ be an intrinsic isometry.*

Then, given $\varepsilon > 0$, there is $\delta = \delta(\imath,\varepsilon) > 0$ such that, for any connected set $W \subset \mathcal{X}$,

$$\operatorname{diam} \imath(W) < \delta \quad \Longrightarrow \quad \operatorname{diam} W < \varepsilon.$$

Proof. Suppose the contrary; then there is a sequence of connected subsets $W_n \subset \mathcal{X}$ such that $\operatorname{diam} \imath(W_n) \to 0$ as $n \to \infty$ but $\operatorname{diam} W_n > \varepsilon$. Thus, there are two sequences of points $x_n, x'_n \in W_n$ with $|x_n x'_n| \geq \varepsilon$. We pass to a subsequence of n such that $W_n \to W$ in the Hausdorff sense and $x_n \to x$, $x'_n \to x'$. We obtain a closed connected subset $W \subset \mathcal{X}$ with two distinct points x and x' such that $\imath(W) = p$ for some $p \in \mathcal{Y}$.

Since W is connected, for any $\varepsilon > 0$ there is an ε-chain $(x_i)_{i=0}^n$ from x to x' such that $\imath(x_i) = p$ for all i. Thus, we have $\operatorname{pull}_{\imath,\varepsilon}(x, x') = 0$ for any $\varepsilon > 0$, i.e., $\operatorname{pull}_\imath(x, x') = 0$, a contradiction. □

§3. Proofs

Proof of the Starting Proposition (1.1). Given $\varepsilon > 0$, choose $\delta = \delta(\imath, \varepsilon)$ as in Proposition 2.4. Since $\dim \mathbb{E}^d = d$, there is a finite open covering $\{U_i\}_{i=1}^n$ of $\imath(\mathcal{X})$ with multiplicity at most $d+1$ and such that $\operatorname{diam} U_i < \delta$ for each i.

Consider the covering $\{V_\alpha\}$ of $\mathcal{X}$ by connected components of $\imath^{-1}(U_i)$ for all i. By Proposition 2.1, $\mathcal{X}$ is a length space. In particular, all sets V_α are open. By Proposition 2.4, we have $\operatorname{diam} V_\alpha < \varepsilon$. Clearly, the multiplicity of $\{V_\alpha\}$ is at most $d+1$. Thus, the claim follows. □

Proof of the "if" part in the Main Theorem 1.5. Let $\mathcal{X}$ be a pro-Euclidean space of rank not exceeding d. Assume $(\mathcal{P}_n)_{n=0}^\infty$ is a sequence of d-dimensional Euclidean polyhedra and that

$$\varphi_{m,n} : \mathcal{P}_m \to \mathcal{P}_n$$

is an inverse system of short maps such that $\mathcal{X} = \varprojlim \mathcal{P}_n$. Let $\psi_n : \mathcal{X} \to \mathcal{P}_n$ be the projections.

By Theorem 1.3, given $\varepsilon_{n+1} > 0$ and a piecewise linear intrinsic isometry $\imath_n : \mathcal{P}_n \to \mathbb{E}^d$, there is a piecewise linear intrinsic isometry $\imath_{n+1} : \mathcal{P}_{n+1} \to \mathbb{E}^d$ such that

$$|\imath_{n+1}(x)\ \imath_n \circ \varphi_{n+1,n}(x)| < \varepsilon_{n+1}$$

for any $x \in \mathcal{P}_n$. It remains to show that the sequence ε_n can be chosen in such a way that $\imath_n \circ \psi_n$ converges to an intrinsic isometry $\imath : \mathcal{X} \to \mathbb{E}^d$.

We choose $\varepsilon_{n+1} > 0$ so that

$$\varepsilon_{n+1} < \tfrac{1}{2} \min\left\{\varepsilon_n, \delta(\imath_n, \tfrac{1}{n})\right\},$$

where $\delta(\imath_n, \frac{1}{n})$ is as in Proposition 2.2. Since, clearly, $\sum_i \varepsilon_i < \infty$, the following limit exists:

$$\imath = \lim_{n\to\infty} \imath_n \circ \psi_n, \quad \imath : \mathcal{X} \to \mathbb{E}^d.$$

Obviously, $\imath$ is short. Next, for any $x \in \mathcal{X}$ we have

$$|\imath(x)\ \imath_n \circ \psi_n(x)| < \sum_{i=n+1}^\infty \varepsilon_i < \delta(\imath_n, \tfrac{1}{n}).$$

Thus, by Proposition 2.2,

$$\operatorname{pull}_\imath(x, x') + \tfrac{1}{n} > \operatorname{pull}_{\imath_n \circ \psi_n}(x, x') \geq |\psi_n(x)\, \psi_n(x')|_{\mathcal{P}_n}.$$

Since $|\psi_n(x)\, \psi_n(x')|_{\mathcal{P}_n} \to |xx'|_{\mathcal{X}}$ as $n \to \infty$, the map $\imath : \mathcal{X} \to \mathbb{E}^d$ is an intrinsic isometry. □

Proof of the "only if" part in the Main Theorem 1.5. We shall give a construction of a polyhedron $\mathcal{P}$ associated with an intrinsic isometry $\imath : \mathcal{X} \to \mathbb{E}^d$ and a tiling of $\mathbb{E}^d$ by coordinate a-cubes. (The space $\mathcal{P}$ will be glued out of a-cubes.) The construction will be done in such a way that if a tiling τ' is a subdivision of a tiling τ, then for the corresponding polyhedra $\mathcal{P}'$ and $\mathcal{P}$ there will be a natural intrinsic isometry $\mathcal{P}' \to \mathcal{P}$. Thus,

we shall construct the required inverse system of polyhedra out of nested subdivisions of $\mathbb{E}^d$.

We set $a_n = \frac{1}{2^n}$ and $r_n = \frac{1}{10} \cdot a_n$. Fixing n for a while, consider the tiling of $\mathbb{E}^d$ by coordinate a_n-cubes. We want to construct a Euclidean polyhedron $\mathcal{P}_n$ associated with this tiling.

The image $\imath(\mathcal{X})$ is covered by a finite number of such a_n-cubes, say $\{\square_n^i\}$. For each $\square_n^i$, consider all connected components $\{W_n^{ij}\}$ of

$$B_{r_n}\big(\imath^{-1}(\square_n^i)\big) \subset \mathcal{X},$$

where $B_r(S)$ denotes the r-neighborhood of a set S.

By Proposition 2.1, $\mathcal{X}$ is a length space. In particular, each set W_n^{ij} is open and contains a ball of radius r_n. Thus, for fixed i, the collection of open sets $\{W_n^{ij}\}$ is finite. Therefore, the set of all $\{W_n^{ij}\}$ for all $\{\square_n^i\}$ forms a finite open cover of $\mathcal{X}$. For each W_n^{ij}, we make an isometric copy $\square_n^{ij}$ of $\square_n^i$ and fix an isometry $\imath_n^{ij} : \square_n^{ij} \to \square_n^i$. The Euclidean polyhedron $\mathcal{P}_n$ is glued from the $\square_n^{ij}$ by the following rule: we glue $\square_n^{i_1j_1}$ to $\square_n^{i_2j_2}$ along $(\imath_n^{i_2j_2})^{-1} \circ \imath_n^{i_1j_1}$ if and only if $W_n^{i_1j_1} \cap W^{i_2j_2} \neq \varnothing$. (The map $(\imath_n^{i_2j_2})^{-1} \circ \imath_n^{i_1j_1}$ sends one of the faces of $\square_n^{i_1j_1}$ isometrically to a face of $\square_n^{i_2j_2}$.)

The resulting polyhedron $\mathcal{P}_n$ admits a natural piecewise linear intrinsic isometry $\imath_n : \mathcal{P}_n \to \mathbb{E}^d$, defined by $\imath_n(x) = \imath_n^{ij}(x)$ for $x \in \square_n^{ij}$. Next, there is a uniquely determined intrinsic isometry $\varphi_{m,n} : \mathcal{P}_m \to \mathcal{P}_n$ for $m \ge n$ that satisfies $\imath_m = \imath_n \circ \varphi_{m,n}$ and

$$\varphi_{m,n}(\square_m^{i'j'}) \subset \square_n^{ij} \subset \mathcal{P}_n \;\Rightarrow\; W_m^{i'j'} \subset W_n^{ij} \subset \mathcal{X}.$$

We set $\psi_n : \mathcal{X} \to \mathcal{P}_n$ to be the intrinsic isometry that is uniquely determined by $\imath_n \circ \psi_n = \imath$ and

$$\psi_n(x) \in \square_n^{ij} \subset \mathcal{P}_n \quad\Rightarrow\quad x \in W_n^{ij} \subset \mathcal{X}.$$

Clearly, together with $\varphi_{m,n}$, the $\mathcal{P}_n$ form an inverse system, and $\psi_n = \varphi_{m,n} \circ \psi_m$ for any pair $m \ge n$.

In order to prove that $\mathcal{X} = \varprojlim \mathcal{P}_n$, it only remains to show that

$$(\star) \qquad |xx'|_{\mathcal{X}} \le \lim_{n\to\infty} |\psi_n(x)\,\psi_n(x')|$$

for all $x, x' \in \mathcal{X}$.

Given a subset $K \subset \mathcal{P}_n$, we denote by $K^* \subset \mathcal{X}$ the union of all $W_n^{ij} \subset \mathcal{X}$ such that $\square_n^{ij} \cap K \neq \varnothing$. Clearly, if K is connected, then so is K^*. Moreover, $\imath(K^*) \subset B_{r_n}(\imath_n(K))$. Thus, Proposition 2.4 shows that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(\star\star) \qquad r_n + \operatorname{diam} K < \delta \quad\Longrightarrow\quad \operatorname{diam} K^* < \varepsilon.$$

Suppose $(\star)$ does not hold; then we can choose $x, x' \in \mathcal{X}$ and $\varepsilon, \ell > 0$ so that

$$(\star\star\star) \qquad \operatorname{pull}_{\imath,\varepsilon}(x, x') > \ell > |\psi_n(x)\,\psi_n(x')|_{\mathcal{P}_n}$$

for all n. In particular, for all n there is a path $\gamma_n : [0,1] \to \mathcal{P}_n$ from $\psi_n(x)$ to $\psi_n(x')$ with length less than ℓ. Choose $\delta = \delta(\imath, \varepsilon)$ as in Proposition 2.4. Let $0 = t_0 < t_1 < \dots < t_m = 1$ be such that

$$(\star\star\star\star) \qquad \operatorname{diam} \gamma([t_{i-1}, t_i]) < \tfrac{\delta}{2}.$$

Clearly, we may assume that $m \le 2 \cdot \lceil \frac{\ell}{\delta} \rceil$. For each t_i we take a point $x_i \in \gamma(t_i)^* \subset \mathcal{X}$; clearly,

$$(\star\star\star\star\star) \qquad |\imath(x_i)\;\imath_n \circ \gamma(t_i)|_{\mathbb{E}^d} < 2 \cdot a_n.$$

Note that $x_{i-1}, x_i \in \gamma([t_{i-1}, t_i])^*$. Now, $(\star\star\star\star)$ and $(\star\star)$ imply that

$$|x_{i-1}x_i| < \operatorname{diam} \gamma_n([t_{i-1}, t_i])^* < \varepsilon$$

for all large n. Thus, the x_i form an ε-chain from x to x', and ($\star\star\star\star\star$) implies that

$$\begin{aligned}\operatorname{pull}_{\imath,\varepsilon}(x,x') &\le \sum_{i=1}^{m} |\imath(x_{i-1})\,\imath(x_i)| \\ &< \sum_{i=1}^{m} |\imath_n\circ\gamma_n(t_{i-1})\;\imath_n\circ\gamma_n(t_i)| + 4\cdot a_n\cdot\lceil\tfrac{\ell}{\delta}\rceil \\ &< \ell + 4\cdot a_n\cdot\lceil\tfrac{\ell}{\delta}\rceil,\end{aligned}$$

which contradicts ($\star\star\star$) if n is sufficiently large. □

Remark 1. In the inverse system $(\varphi_{m,n},\mathcal{P}_n)$ constructed above, the images of $\varphi_{m,n}$ form a $\sqrt{d}\cdot a_n$-net in $\mathcal{P}_n$. It follows that the space $\mathcal{X}$ is isometric to the Gromov–Hausdorff limit of $\mathcal{P}_n$ (see also §2).

Proof of Proposition 1.7. The inequality $d\le m$ follows from the Starting Proposition 1.1. In the proof of the second part, we use the following two statements.

(1) Assume $\imath:\Omega\subset\mathbb{M}^d\to\mathbb{E}^m$ is an intrinsic isometry; then it is a Lipschitz map for a Euclidean structure on Ω. Thus, by Rademacher's theorem (see [12, 3.1.6]), the differential $d_p\imath$ is well defined for almost all $p\in\Omega$.

(2) For any curve $\gamma(t)$ with natural parameter in a metric space, for almost all values of the parameter t_0 we have

$$|\gamma(t_0)\,\gamma(t_0+\varepsilon)| = \varepsilon + o(\varepsilon);$$

see [4, 2.7.5].

Let $\|*\|$ denote the norm that induces the metric on $\mathbb{M}^d$. Fixing u with $\|u\|=1$, consider the pencil of lines of the form $p+u\cdot t$ in Ω. The two statements above imply that $|d_p\imath(v)|\overset{\text{a.e.}}{=}\|v\|$. Hence, we obtain the parallelogram identity

$$2\cdot\big(\|v\|^2+\|w\|^2\big) = \|v+w\|^2+\|v-w\|^2$$

for any two vectors v and w. Thus, the norm $\|*\|$ is Euclidean. □

§4. About path isometries

In this section we relate the notion of an intrinsic isometry defined in §2 to the more common (but less natural) notion of a path and weak path isometry.

Definition 4.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two length spaces. A map $\imath:\mathcal{X}\to\mathcal{Y}$ is called

1) a path isometry if for any path $\gamma:[0,1]\to\mathcal{X}$ we have

$$\operatorname{length}\gamma = \operatorname{length}\imath\circ\gamma;$$

2) a weak path isometry if for any rectifiable path $\gamma:[0,1]\to\mathcal{X}$ we have

$$\operatorname{length}\gamma = \operatorname{length}\imath\circ\gamma.$$

As was noted in §2, any intrinsic isometry is a path isometry (and therefore, a weak path isometry). Later we shall show that the converse fails. Similar counterexamples for weak path isometries are much simpler: one can take a left-invariant sub-Riemannian metric d on the Heisenberg group H; then factoring by the center gives a weak path isometry $(H,d)\to\mathbb{E}^2$ (which is not a path isometry and, thus, not an intrinsic isometry).

Example 4.2. There is a length space $\mathcal{X}$ and a path isometry $f:\mathcal{X}\to\mathbb{R}$ such that $f^{-1}(0)$ is a nontrivial connected subset.

Moreover, in such an example the Lebesgue covering dimension of $f^{-1}(0)$ can be made arbitrarily large.

In particular, an analog of Proposition 1.1 fails for path isometries.

The following construction was suggested by D. Burago; it is based on two ideas: 1) the construction in [5, 3.1], and 2) the construction of a *pseudoarc* given in [9] (see also the survey [11] and the references therein). In fact, in the first part of the construction, $f^{-1}(0)$ will be homeomorphic to a pseudoarc, and in the second part $f^{-1}(0)$ will be homeomorphic to a product of pseudoarcs.

Proof. The space $\mathcal{X}$ will be a completion $\bar\Gamma$ of a certain metric graph[5] Γ.

First, we describe the construction of f modulo the construction of Γ. Set $\grave\Gamma = \bar\Gamma \setminus \Gamma$. Consider $f : \bar\Gamma \to \mathbb{R}$, where $f(x)$ is the distance from x to $\grave\Gamma$. Then f is a path isometry on Γ and $f(\grave\Gamma) = 0$. To finish that proof, we need to construct Γ in such a way that

(i) $\grave\Gamma$ is connected and contains more than one point;
(ii) f is a path isometry on the entire $\bar\Gamma$ (not only on Γ).

Construction of Γ. For two real intervals $\mathbb{I}$ and $\mathbb{J}$, a continuous map h of $\mathbb{I}$ onto $\mathbb{J}$ is said to be *ε-crooked* if for any $t_1 < t_2$ in $\mathbb{I}$ there are $t_1 < t_2' < t_1' < t_2$ such that $|h(t_i') - h(t_i)| \le \varepsilon$ for $i \in \{1, 2\}$. The existence of an ε-crooked map for any given $\mathbb{I}$ and $\varepsilon > 0$ can be proved easily by induction on $n = \lceil \frac{1}{\varepsilon} \cdot \operatorname{length} \mathbb{J} \rceil$.

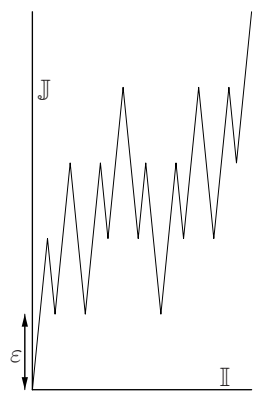

FIGURE 1. Graph of an ε-crooked map.

We fix a sequence of real intervals $\mathbb{J}_n$ with short $\frac{1}{2^n}$-crooked maps $h_n : \mathbb{J}_n \to \mathbb{J}_{n-1}$. The topological inverse limit $\mathbb{J}_\infty = \varprojlim \mathbb{J}_n$ is a compact connected set that has no non-trivial paths.

We can think of $\mathbb{J}_n$ as of a (linear) metric graph with length of each edge at most $\frac{1}{2^n}$. We construct a graph Γ from the disjoint union $\bigsqcup_n \mathbb{J}_n$ by joining each vertex v of $\mathbb{J}_n$ to a vertex of $\mathbb{J}_{n-1}$ that is closest to $h_n(v)$ by an edge of length $\frac{1}{2^n}$. Then $\grave\Gamma$ is homeomorphic to $\mathbb{J}_\infty$; this yields (i).

Let Γ_n denote the finite subgraph of Γ formed by all vertices in $\mathbb{J}_1, \mathbb{J}_2, \dots, \mathbb{J}_n$. Note that there is a short map $\Gamma_n \to \Gamma_{n-1}$ that is identical on Γ_{n-1}. It follows that, for any path $\alpha : [0,1] \to \bar\Gamma$, the total length of $\alpha \setminus \grave\Gamma$ is at least $|\alpha(0)\,\alpha(1)|_{\bar\Gamma}$, and (ii) follows.

Second part. We construct a graph $\Gamma^{(m)}$ so as to make $\grave\Gamma^{(m)}$ homeomorphic to the product of m copies of $\grave\Gamma$.

We do this in the case where $m = 2$; the other cases are similar. The set of vertices of $\Gamma^{(2)}$ is the disjoint union $\bigsqcup_n (\operatorname{Vert} \mathbb{J}_n \times \operatorname{Vert} \mathbb{J}_n)$, where $\operatorname{Vert} \mathbb{J}_n$ denotes the set of vertices

[5]That is, a locally finite graph with intrinsic metric such that each edge is isometric to a real interval.

of $\mathbb{J}_n$. We connect two vertices $(x,y) \in \operatorname{Vert}\mathbb{J}_n \times \operatorname{Vert}\mathbb{J}_n$ and $(x',y') \in \operatorname{Vert}\mathbb{J}_k \times \operatorname{Vert}\mathbb{J}_k$ if and only if the pairs (x,x') and (y,y') were connected in Γ; the length of this edge will be the maximum of the lengths of the edges xx' and yy' (we assume that a vertex is connected to itself by an edge of length 0).

Clearly, there is a homeomorphism $\grave{\Gamma}^{(2)} \to \grave{\Gamma} \times \grave{\Gamma}$. Note that there are two short coordinate projections $\varsigma_1, \varsigma_2 : \Gamma^{(2)} \to \Gamma$. Thus, for any path $\alpha : [0,1] \to \bar{\Gamma}^2$, the total length of $\alpha \setminus \grave{\Gamma}$ is at least $\max_i |\varsigma_i \circ \alpha(0)\, \varsigma_i \circ \alpha(1)|$. This ensures that $\grave{\Gamma}^{(2)}$ is bi-Lipschitz equivalent to the product $\grave{\Gamma} \times \grave{\Gamma}$. $\square$

§5. Comments and open questions

A length space $\mathcal{M}$ is called a *Minkowski d-polyhedron* if there is a finite triangulation of $\mathcal{M}$ such that each simplex is isometric to a simplex in a Minkowski space. Accordingly, a compact metric space $\mathcal{X}$ is called a *pro-Minkowski space* of rank not exceeding d if it can be presented as an inverse limit of Minkowski d-polyhedra.

Question 5.1. Is it true that any length space with Lebesgue's covering dimension d is a pro-Minkowski space of rank d?

Or even more specifically:

Question 5.2. Is it true that any metric space homeomorphic to a disk is a pro-Minkowski space of rank 2?

One can reformulate this philosophically: *Is there any essential difference between the Finsler metric and the general metric on an n-manifold?* This question was asked by D. Burago; it served as the original motivation for this paper (see also [5, Theorem 1]).

If we remove the restriction on dimension, then the answer to the above question is YES. Namely, the following exercise can be solved by using the Kuratowski embedding $x \mapsto \operatorname{dist}_x$.

Exercise 5.3. Show that any compact length space is an inverse limit of Minkowski polyhedra $\mathcal{M}_n$ with $\dim \mathcal{M}_n \to \infty$.

Question 5.4. Is it true that any path isometry from a closed Euclidean ball to Euclidean space is an intrinsic isometry?

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Received 10/FEB/2010

Translated by THE AUTHOR