MONODROMY ZETA-FUNCTION OF A POLYNOMIAL
ON A COMPLETE INTERSECTION, AND NEWTON POLYHEDRA

G. G. GUSEV

Abstract. For a generic (polynomial) one-parameter deformation of a complete intersection, its monodromy zeta-function is defined. Explicit formulas for this zeta-function in terms of the corresponding Newton polyhedra are obtained in the case where the deformation is nondegenerate with respect to its Newton polyhedra. This result is employed to obtain a formula for the monodromy zeta-function at the origin of a polynomial on a complete intersection, which is an analog of the Libgober–Sperber theorem.

§1. Introduction

Let $F_0, F_1, \ldots, F_k$ be a set of functions on $\mathbb{C}^n$ defined as polynomials in $n$ complex variables $z = (z_1, z_2, \ldots, z_n)$. Consider the family of varieties

$$V_c = \{z \in \mathbb{C}^n \mid F_0(z) = c, F_i(z) = 0, \ i = 1, 2, \ldots, k\},$$

where $c \in \mathbb{C}$ is a complex parameter. This family provides a fibration over the punctured neighborhood of the origin in the parameter space with the fiber $V_c$ over a point $c$ (see below). In this paper we obtain a formula for the monodromy zeta-function of the above fibration in terms of the Newton polyhedra of the polynomials $F_0, F_1, \ldots, F_k$. This result can be viewed as a global analog of [3, Theorem 2.2] and an analog of [5, Theorem 5.5], where the monodromy zeta-function at infinity is calculated. In §2 we consider the case where $F_0(z) = z^n$, so that the fibration corresponds to a polynomial deformation of a set of polynomials in $n - 1$ variables $z_1, z_2, \ldots, z_{n-1}$. The general case is deduced from this special one in §3. The study is partially motivated by the results of D. Siersma and M. Tibar (6).

Let $A = \mathbb{C}^n \setminus Y$ be the complement to an arbitrary algebraic hypersurface $Y \subset \mathbb{C}^n$. Let $Z = \{z \in \mathbb{C}^n \mid F_i = 0, \ i = 1, 2, \ldots, k\} \cap A$. We denote by $\mathbb{D}_r$ and $\mathbb{D}_r^*$ the closed disk in $\mathbb{C}$ of radius $r$ centered at the origin and the punctured disk $\mathbb{D}_r^* := \mathbb{D}_r \setminus \{0\}$, respectively. From [11, Theorem 5.1] it follows that there exists a finite set $B \subset \mathbb{C}$ such that the restriction $F = F_0|_Z$ of the function $F_0$ is a topological fibration over $\mathbb{C} \setminus B$. In particular, the map $F|_{F^{-1}(\mathbb{D}_r^*)} \ (F|_{F^{-1}(\mathbb{C} \setminus \mathbb{D}_d)})$ is a fibration for any sufficiently small $\delta$ (for any sufficiently large $d$). Consider the restriction of this fibration to the cycle $\{c \cdot \exp(2\pi it) \mid t \in [0, 1]\}$, where $|c|$ is sufficiently small (large, respectively). Consider the monodromy transformation $h_{F,0} : Z_c \to Z_c \ (h_{F,\infty} : Z_c \to Z_c)$ of the fiber $Z_c$ over the point $c$ of the resulting fibration.

2010 Mathematics Subject Classification. Primary 14Q15, 14D05; Secondary 58K15, 58K10, 32S20.
Key words and phrases. Deformations of polynomials, monodromy zeta-function, Newton polyhedron.

Partially supported by the grants RFBR-10-01-00678, RFBR-08-01-00110-a, RFBR and SU HSE 09-01-12185-off-m, and NOSH-8462.2010.1.

©2012 American Mathematical Society
The zeta-function of an arbitrary transformation $h: X \to X$ of a topological space $X$ is the rational function
\[ \zeta_h(t) = \prod_{i \geq 0} (\det(\Id - th_i|_{H_i^c(X; \mathbb{C})}))(-1)^i, \]
where $H_i^c(X; \mathbb{C})$ denotes the $i$th homology group with closed support.

**Definition 1.** The monodromy zeta-function (at the origin) of the function $F_0$ on the set $Z$ is the zeta-function of the transformation $h_{F,0}, \zeta_{F_0,Z}(t) := \zeta_{h_{F,0}}(t)$. The monodromy zeta-function at infinity of the function $F_0$ on the set $Z$ is the zeta-function of the transformation $h_{F,\infty}, \zeta_{F_0,\infty}(t) := \zeta_{h_{F,\infty}}(t)$.

Let $S_1, S_2, \ldots, S_n \subset \mathbb{R}^n$ be a collection of convex bodies. We denote by $S_1 S_2 \ldots S_n$ their Minkovski mixed volume (see, e.g., [8]). If $S_j = \emptyset$ for some $j$, we put $S_1 S_2 \ldots S_n = 0$. For a homogeneous polynomial $T(x_1, x_2, \ldots, x_k) = \sum \alpha_{i_1i_2\ldots i_n} x_{i_1} x_{i_2} \ldots x_{i_n}$ of degree $n$, we define $T(S_1 S_2 \ldots S_k)$ as $\sum \alpha_{i_1i_2\ldots i_n} S_1 S_2 \ldots S_n$.

In this paper we obtain a formula for the zeta-function $\zeta_{F_0, V}(t)$, $V = \{z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0\}$ for a generic set of polynomials $F_0, F_1, \ldots, F_k$ in terms of the integer mixed volumes of the faces of their Newton polyhedra $\Delta_0, \Delta_1, \ldots, \Delta_k$.

**§2. Zeta-function of a polynomial deformation**

In this section we study the case where $F_0(z) = z_n$. Consider the set of deformations $f_{i,\sigma}(z_1, \ldots, z_{n-1}) := F_i(z_1, \ldots, z_{n-1}, \sigma)$ of the functions $f_i := f_{i,0}$ on the set $\mathbb{C}^{n-1}$, $i = 1, 2, \ldots, k$, where $\sigma \in \mathbb{C}$ is the deformation parameter. The fiber over the point $c$ of the deformation provided by the function $F_0$ on the set $\{F_1 = F_2 = \cdots = F_k = 0\}$ is $\{f_{1,c} = f_{2,c} = \cdots = f_{k,c} = 0\} \times \{c\}$. This fact motivates the following definition.

**Definition 2.** Consider $V = \{z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0\}$. The zeta-function $\zeta_{z_n,V}(t)$ ($\zeta_{z_n, \infty}(t)$) will be called the monodromy zeta-function (at infinity) of the deformation $\{f_{i,\sigma} \mid i = 1, 2, \ldots, k\}$.

**2.1. Formulas for the zeta-function of a deformation.** Consider the representation $F_i = \sum_{k \in \mathbb{Z}^n} F_{i,k} z^k$, where the $F_{i,k} \in \mathbb{C}$, $k \in \mathbb{Z}^n$, are the coefficients of the polynomial $F_i$ and $k = (k_1, k_2, \ldots, k_n)$ are the coordinates in the space $\mathbb{R}^n$ that correspond to the variables $(z_1, z_2, \ldots, z_n)$. Let $\Delta_i = \Delta(F_i)$ denote the Newton polyhedron of the polynomial $F_i$, $i = 1, 2, \ldots, k$, i.e., the convex hull of the set $\{k \in \mathbb{Z}^n \mid F_{i,k} \neq 0\}$. A subset $I$ of the set $\{1, 2, \ldots, n\}$ will be called an index set. Denote $\mathbb{R}^I = \{k \in \mathbb{R}^n \mid k_i = 0, i \notin I\}$. Let $j_1^I < j_2^I < \cdots < j_{k(I)}^I$ be the elements of the set $\{j \in \{1, 2, \ldots, k\} \mid \Delta_j \cap \mathbb{R}^I \neq \emptyset\}$. We put $\Delta_I = \Delta_{j_1^I} \cap \mathbb{R}^I$, $i = 1, 2, \ldots, k(I)$ and $F_I = \sum_{k \in \Delta_I} F_{j_1^I,k} z^k$.

An integer covector is said to be primitive if it is not a multiple of another integer covector. We denote by $Z^I$ the set of primitive covectors in the dual space $(\mathbb{R}^I)^*$. For a convex set $S \subset \mathbb{R}^I$ and a covector $\alpha \in Z^I$, let $S^\alpha$ be the subset of $S$ formed by the points where the function $\alpha|_S$ attains its minimal value: $S^\alpha = \{x \in S \mid \alpha(x) = \min(\alpha|_S)\}$. For an arbitrary polynomial $P = \sum_{k \in \Delta} P_k z^k$ with the Newton polyhedron $\Delta \subset \mathbb{R}^I$ and a covector $\alpha \in Z^I$, we denote by $P^\alpha$ the polynomial $\sum_{k \in \Delta^\alpha} P_k z^k$. For an index set $I$ containing $n$, let $Z^n_+ \subset Z^I$ be the subset of covectors $\alpha = \cdots + \alpha_n dk_n$ with strictly positive last component: $\alpha_n > 0$ (strictly negative last component: $\alpha_n < 0$).
**Definition 3.** Consider a covector \( \alpha \in \mathbb{Z}^{1,2,\ldots,m} \). We say that a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \alpha \)-nondegenerate with respect to its Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \) if the 1-forms \( dF_i^\alpha \), \( i = 1, 2, \ldots, k \), are linearly independent at all the points of the set
\[ \{ z \in (\mathbb{C}^*)^n \mid F_1^\alpha(z) = F_2^\alpha(z) = \cdots = F_k^\alpha(z) = 0 \}. \]

We say that a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \sigma \)-nondegenerate (at infinity) with respect to its Newton polyhedra if for each index set \( I \) containing \( n \) and each covector \( \alpha \in \mathbb{Z}_I^I \) (\( \alpha \in \mathbb{Z}_I^I \)), the system of polynomials \( F_1^I, F_2^I, \ldots, F_k^I \) is \( \alpha \)-nondegenerate with respect to its Newton polyhedra.

Finally, a system of polynomials \( F_1, F_2, \ldots, F_k \) is said to be nondegenerate with respect to its Newton polyhedra if for each index set \( I \) and each \( \alpha \in \mathbb{Z}^I \) the system of polynomials \( F_1^I, F_2^I, \ldots, F_k^I \) is \( \alpha \)-nondegenerate.

For each index set \( I \subset \{1, 2, \ldots, n\} \) containing \( n \), we define the following rational functions:
\[
\zeta^I_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t) = \prod_{\alpha \in \mathbb{Z}_I^I} (1 - t^{\alpha(\frac{\partial}{\partial k_n})})^l Q_k^I(t^{\Delta_{\alpha, \alpha}^I, \Delta_{\alpha, \alpha}^I, \ldots, \Delta_{\alpha, \alpha}^I}),
\]
\[
\zeta^I_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t) = \prod_{\alpha \in \mathbb{Z}_I^I} (1 - t^{-\alpha(\frac{\partial}{\partial k_n})})^l Q_k^I(t^{\Delta_{\alpha, \alpha}^I, \Delta_{\alpha, \alpha}^I, \ldots, \Delta_{\alpha, \alpha}^I}),
\]
where \( l = |I| - 1 \), \( \frac{\partial}{\partial k_n} \) is the vector in \( \mathbb{R}^I \) whose only nonzero coordinate is \( k_n = 1 \), and \( Q_k^I(x_1, x_2, \ldots, x_k) := [\prod_{l=1}^k \frac{x_l}{1+x_l}]^t \), where \([\cdot]_t\) denotes the degree \( t \) homogeneous part of the power series under consideration. In particular, \( Q_k^I \equiv 0 \) for \( l > 0 \) and \( Q_k^I \equiv 1 \).

**Theorem 1.** Suppose a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \alpha \)-nondegenerate with respect to its Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \). Then

\[
\zeta_{z_n, V \cap (\mathbb{C}^*)^n}(t) = \zeta^{(1,2,\ldots,n)}_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t),
\]
\[
\zeta_{z_n, V}(t) = \prod_{I: n \in I \subset \{1, 2, \ldots, n\}} \zeta^I_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t),
\]
where \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \).

**Theorem 2.** Suppose a system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \alpha \)-nondegenerate at infinity with respect to its Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \). Then

\[
\zeta_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t) = \prod_{I: n \in I \subset \{1, 2, \ldots, n\}} \zeta^{I, \infty}_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t),
\]
\[
\zeta_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t) = \prod_{I: n \in I \subset \{1, 2, \ldots, n\}} \zeta^{I, \infty}_{\Delta_1, \Delta_2, \ldots, \Delta_k}(t),
\]
where \( V = \{ z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0 \} \).

**Remark 1.** For \( k = 1 \), equation (1) implies
\[
\zeta_{z_n, V \cap (\mathbb{C}^*)^n}(t) = \prod_{\alpha \in \mathbb{Z}_I^I} (1 - t^{\alpha(\frac{\partial}{\partial k_n})})^l (-1)^n(n-1)! \text{Vol}_{n-1}(\Delta^I_\alpha),
\]
where \( \text{Vol}_l(\cdot) \) denotes the \( l \)-dimensional integer volume, \( I_0 = \{1, 2, \ldots, n\} \). This relation is similar to formula (1) in [3 Theorem 2.2] for the zeta-function of a singularity deformation. In fact, let \( f_\sigma \) denote the germ at the origin of the deformation defined by \( f_\sigma(z_1, \ldots, z_{n-1}) = F_1(z_1, \ldots, z_{n-1}, \sigma) \). Using the equation in [3], we obtain
\[
\zeta_{f_\sigma \mid (\mathbb{C}^*)^{n-1}}(t) = \prod_{\alpha \in \mathbb{Z}_I^I} (1 - t^{\alpha(\frac{\partial}{\partial k_n})})^l (-1)^n(n-1)! \text{Vol}_{n-1}(\Delta^0_{I_0, \alpha}),
\]
where \( \mathbb{Z}_I^I \) is the subset of covectors in \( \mathbb{Z}_I^I \) whose components are all strictly positive. Hence, the local zeta-function
Theorem 3 as a deformation in the parameter \( 514 \) G. G. GUSEV bundle over a variety, introduced by S. M. Gusein-Zade and D. Siersma in \([2]\). Let \( X \) be a compact complex analytic variety, and let \( W \) imply that \( \Lambda \) is a line bundle over \( X \). We reduce the calculation of the zeta-function to integration with respect to the Euler characteristic (see., e.g., \([7]\)), using the following localization principle.

2.2. Proofs of the theorems. We reduce the calculation of the zeta-function to integration with respect to the Euler characteristic (see., e.g., \([7]\)), using the following localization principle.

We recall the notion of the zeta-function as applied to a family of sections of a line bundle over a variety, introduced by S. M. Gusein-Zade and D. Siersma in \([2]\). Let \( W \) be a compact complex analytic variety, and let \( W_1 \) be the complement to a compact subvariety of \( W \). Let \( L \) be a line bundle over \( W \), and let \( q_\sigma \) be a family of sections of \( L \) analytic in \( \sigma \in \mathbb{C}_\sigma \). Let \( U \) be the subset of \( W_1 \times \mathbb{C}_\sigma \) given by \( q_\sigma(x) = 0 \). The restriction to \( U \) of the projection \( W_1 \times \mathbb{C}_\sigma \to \mathbb{C}_\sigma \) is a fibration over the punctured disk \( \mathbb{D}_\sigma^* \subset \mathbb{C}_\sigma \) for \(|\sigma| \ll 1 \). The zeta-function of a family of sections \( q_\sigma \) restricted to the set \( W_1 \) is the zeta-function of the monodromy transformation of the above fibration. We denote it by \( \zeta_{q_\sigma|W_1}(t) \).

The fibration \( L \) is trivial over a neighborhood of a point \( x \in W \). Therefore, using a fixed coordinate system, we can view the family of germs at the point \( x \) of sections \( q_\sigma \) as a deformation in the parameter \( \sigma \) of a function germ. We denote by \( \zeta_{q_\sigma|W_1,x}(t) \) the zeta-function of the germ at the point \( \sigma = 0 \) of the above deformation restricted to the set \( W_1 \) (see, e.g., \([3]\)).

Theorem 3 (\([2]\), “localization principle”). We have

\[
\zeta_{q_\sigma|W_1}(t) = \int_W \zeta_{q_\sigma|W_1,x}(t) \, d\chi.
\]

Using the Newton polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \) of the polynomials \( F_1, F_2, \ldots, F_k \), we construct a unimodular simplicial partition \( \Lambda \) of the dual space \((\mathbb{R}^n)^*\); we assume that this partition is sufficiently fine for the system \( \{\Delta_i\} \) in the sense of \([9]\). Consider the toroidal compactification \( X_\Lambda \) of the torus \((\mathbb{C}^*)^n\) that corresponds to the partition \( \Lambda \). Recall that the standard action of the torus \((\mathbb{C}^*)^n\) on itself uniquely extends to an action of the torus on the variety \( X_\Lambda \). The cones \( \lambda \in \Lambda \) of the partition are in one-to-one correspondence with the orbits \( T_\lambda \subset X_\Lambda \) of this action and the orbit \( T_\lambda \) is isomorphic to \((\mathbb{C}^*)^{n-\dim \lambda}\). Denote by \( X'_\lambda \) the complement in \( X_\Lambda \) to the torus \( T_{\{0\}} \cong (\mathbb{C}^*)^n \). Let \( \tilde{V} \) be the closure of the set \( V \cap T_{\{0\}} \subset X_\Lambda \), and let \( V' = \tilde{V} \cap X'_\lambda \). We prove the following statement.

Lemma 1. For a sufficiently fine partition \( \Lambda \), we have

\[
\zeta_{\zeta_{z_n,v\cap(\mathbb{C}^*)^n}}(t) = \int_{V'} \zeta_{\zeta_{z_n,v\cap(\mathbb{C}^*)^n},x}(t) \, d\chi,
\]

\[
\zeta_{\zeta_{\zeta_{z_n,v\cap(\mathbb{C}^*)^n}}}^\infty(t) = \int_{V'} \zeta_{\zeta_{\zeta_{z_n,v\cap(\mathbb{C}^*)^n}},x}(t) \, d\chi,
\]

\( 0 \leq t \leq 1 \).
where, for a germ at \( x \in V' \) of a meromorphic function \( f \) on the set \( \tilde{V} \) and for an open subset \( A \subset \tilde{V} \), the expression \( \zeta_f|_{A \times x}(t) \) \((\zeta_f^\infty|_{A \times x}(t)) \) denotes the local zeta-function (at infinity) of the germ at \( x \) of the function \( f \) restricted to \( A \).

**Proof.** We may assume the partition \( \Lambda \) to be a subdivision of the standard partition \( \Pi \) of the space \((\mathbb{R}^n)^*\) corresponding to the \( n \)-dimensional projective space: \( X_\Pi = \mathbb{CP}^n \supset (\mathbb{C}^*)^n \). Let \( p : X_\Lambda \rightarrow \mathbb{CP}^n \) be the map of the toric varieties induced by the refinement \( \Lambda \prec \Pi \). Consider the family of global sections \( s_\sigma, \sigma \in \mathbb{C}, \) of the fibration \( \mathcal{O}(1) \) over \( \mathbb{CP}^n \) that is defined by the condition \( s_\sigma|_{(\mathbb{C}^*)^n} = z_n - \sigma \). Denote \( \pi = p \circ \text{inj} \), where \( \text{inj} : \tilde{V} \hookrightarrow X_\Lambda \) is the inclusion map. Let \( S_\sigma = \pi^*(s_\sigma) \) be the family of sections of the bundle \( \pi^*(\mathcal{O}(1)) \) that is the pull-back of \( s_\sigma \). In a similar way, consider a family of sections \( s'_{\sigma}, \sigma \in \mathbb{C}, \) of the fibration \( \mathcal{O}(1) \) that is defined by the condition \( s'_\sigma|_{(\mathbb{C}^*)^n} = 1 - \sigma z_n \), and consider the pull-back \( S'_\sigma = \pi^*(s'_\sigma) \).

By simple reformulations, we can easily show that

\[
\zeta_{s_n|_{V \cap (\mathbb{C}^*)^n}}(t) = \zeta_{s_\sigma|_{V \cap (\mathbb{C}^*)^n}}(t), \quad \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n}}(t) = \zeta_{s'_\sigma|_{V \cap (\mathbb{C}^*)^n}}(t),
\]

Applying Theorem 3 to the families \( S_c \) and \( S'_c \), we obtain

\[
\begin{align*}
\zeta_{s_\sigma|_{V \cap (\mathbb{C}^*)^n}}(t) &= \int_\tilde{V} \zeta_{s_\sigma|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi = \int_\tilde{V} \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi, \\
\zeta_{s'_\sigma|_{V \cap (\mathbb{C}^*)^n}}(t) &= \int_\tilde{V} \zeta_{s'_\sigma|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi = \int_\tilde{V} \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi.
\end{align*}
\]

Moreover, it is easily seen that \( \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x}(t) = \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x}(t) = 1 \) for \( x \notin V' \). Therefore, using the multiplicative property of the integration, we get

\[
\begin{align*}
\int_\tilde{V} \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi &= \int_{V'} \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi, \\
\int_\tilde{V} \zeta_{s'_n|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi &= \int_{V'} \zeta_{s'_n|_{V \cap (\mathbb{C}^*)^n},x}(t) \, d\chi.
\end{align*}
\]

Let \( \Lambda_+ \subset \Lambda (\Lambda_- \subset \Lambda) \) be the subset of cones \( \lambda \in \Lambda \) generated by a set of primitive covectors \( \alpha_1, \alpha_2, \ldots, \alpha_l \) lying in \( \mathbb{Z}_{+1,2,\ldots,n} \) \(( \mathbb{Z}_{+1,2,\ldots,n} \setminus \mathbb{Z}_{+1,2,\ldots,n}^{+1} \), respectively). We may assume that \( \Lambda \) is so fine that \( \Lambda_- \cup \Lambda_+ = \Lambda \).

Consider an arbitrary point \( x_0 \in V' \). It is contained in the torus \( T_\lambda \) that corresponds to some \( l \)-dimensional cone \( \lambda \in \Lambda, l < n \). This cone lies on the border of an \( n \)-dimensional cone \( \lambda' \in \Lambda \). Denote by \( \alpha_1, \alpha_2, \ldots, \alpha_l \) the primitive integer covectors that generate the cone \( \lambda \). The cone \( \lambda' \) is generated by the covectors \( \alpha_1, \alpha_2, \ldots, \alpha_l \) and some covectors \( \alpha_{l+1}, \alpha_{l+2}, \ldots, \alpha_n \). Consider the coordinate system \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) corresponding to the set of covectors \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \). We have \( u_i(x_0) = 0, i \leq l, u_i(x_0) \neq 0, i > l \). We express the monomial \( z_n \) as a function \( F \) of the variables \( \mathbf{u} \):

\[
F(\mathbf{u}) = b \cdot u_1^{\alpha_1(\partial/\partial u_n)} u_2^{\alpha_2(\partial/\partial u_n)} \cdots u_l^{\alpha_l(\partial/\partial u_n)},
\]

where \( b(\mathbf{u}) = \prod_{j=l+1}^n u_j^{\alpha_j(\partial/\partial u_n)}, b(x_0) \in \mathbb{C}^* \). Now we are ready to calculate the values of the integrands \( \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x_0}(t) \) and \( \zeta_{s'_n|_{V \cap (\mathbb{C}^*)^n},x_0}(t) \) in the following two cases.

1. If \( \lambda \in \Lambda_+ \), then the value of the function \( F \) at the point \( x \) is not zero, so that \( \zeta_{s_n|_{V \cap (\mathbb{C}^*)^n},x_0}(t) = 1 \). Accordingly, assume that \( \lambda \in \Lambda_- \). Then the point \( x_0 \) is not a pole of the function \( F \) and therefore \( \zeta_{s'_n|_{V \cap (\mathbb{C}^*)^n},x_0}(t) = 1 \).

2. Assume that \( \lambda \in \Lambda \setminus \Lambda_- (\lambda \in \Lambda \setminus \Lambda_+) \). The system of polynomials \( F_1, F_2, \ldots, F_k \) is \( \sigma \)-nondegenerate (at infinity) with respect to its polyhedra \( \Delta_1, \Delta_2, \ldots, \Delta_k \).
Therefore, \( l + k \leq n \), and there is a coordinate system \((u_1, \ldots, u_l, w_{l+1}, \ldots, w_n)\) in a neighborhood of \(x_0\) such that \(u_i(x_0) = 0\), \(i = l + 1, \ldots, n\), and

\[
F_i = a_i u_1^{m_{i,1}} u_2^{m_{i,2}} \cdots u_l^{m_{i,l}} w_{n-i+1}^{m_{i,n-i}}, \quad i = 1, 2, \ldots, k,
\]

where \(m_{i,j} = \min(\alpha_j | \Delta_i)\) and \(a_i\) is a germ of an analytic function such that \(a_i(x_0) \neq 0\). Denote \(V_{x_0} = V \cap (C^*)^n \cap U\). By \((7)\), we have

\[
V_{x_0} = \{u_i \neq 0, i \leq l; w_i = 0, i > n - k\} \subset U.
\]

Hence,

\[
(8) \quad \zeta_{|V \cap (C^*)^n}, x_0(t) = \zeta_{|_{(u_i \neq 0, i \leq l)}, 0}(t) \quad (\zeta_{\infty |V \cap (C^*)^n}, x_0(t) = \zeta_{\infty |_{(u_i \neq 0, i \leq l)}, 0}(t)),
\]

where \(g\) is the germ of the function in the variables \((u_1, \ldots, u_l, w_{l+1}, \ldots, w_{n-k})\) that is given by the relation

\[
g = \prod_{j=1}^{l} u_j^{\alpha_j(\partial/\partial k_n)} \cdot b(u_1, \ldots, u_l, w_{l+1}, \ldots, w_{n-k}, 0, \ldots, 0).
\]

Using the Varchenko-type formula for meromorphic functions (see \([\Pi]\)), we calculate the right-hand side of \((8)\). For \(l = 1\), we obtain

\[
(9) \quad (\zeta_{\infty |V \cap (C^*)^n}, x_0(t) = 1 - t^{\alpha_1(\partial/\partial k_n)}. \quad (\zeta_{\infty |V \cap (C^*)^n}, x_0(t) = 1 - t^{\alpha_1(\partial/\partial k_n)}),
\]

Finally, the two zeta-functions in question are trivial if \(l > 1\).

Now we specify the only case where the function \(\zeta_{|V \cap (C^*)^n}, x_0(t) (\zeta_{\infty |V \cap (C^*)^n}, x_0(t))\) is not trivial. Namely, we assume that \(x_0 \in T_\lambda, \lambda \in \Lambda \setminus \Lambda_+ (\lambda \in \Lambda \setminus \Lambda_+)\) and \(\dim \lambda = 1\). Denote \(\alpha = \alpha_1\). The set \(T_\lambda \cap V'\) can be defined in the coordinates \((u_2, \ldots, u_{n+1})\) on the torus \(T_\lambda = \{u_1 = 0\}\) by the system of equations \(Q_1^{\alpha} = Q_2^{\alpha} = \cdots = Q_n^{\alpha} = 0\), where

\[
Q_i^{\alpha} = \sum_{k \in \Delta_i^{\alpha}} F_{i,k} u_2^{\alpha_2(k)} u_3^{\alpha_3(k)} \cdots u_n^{\alpha_n(k)}.
\]

Using the main results of \([9]\) and \([10]\), we obtain

\[
(10) \quad \chi(T_\lambda \cap V') = (n - 1)! Q_k^{n-1}(\Delta(Q_1^{\alpha}), \Delta(Q_2^{\alpha}), \ldots, \Delta(Q_n^{\alpha})),
\]

where \(\Delta(\cdot)\) denotes the Newton polyhedron of the Laurent polynomial under consideration. The covectors \(\alpha_2, \alpha_3, \ldots, \alpha_n\) determine an isomorphism of the integer lattices of the hyperplane \(\{\alpha = 0\} \subset \mathbb{R}^n_k\) and the space \(\mathbb{R}^{n-1}\), which contains the polyhedra \(\Delta(Q_i^{\alpha})\). Under this isomorphism, the polyhedra \(\Delta(Q_i^{\alpha})\) correspond to parallel shifts of the polyhedra \(\Delta_i\). Therefore, the corresponding mixed integer volumes coincide and

\[
(11) \quad Q_k^{n-1}(\Delta(Q_1^{\alpha}), \Delta(Q_2^{\alpha}), \ldots, \Delta(Q_n^{\alpha})) = Q_k^{n-1}(\Delta_1^{\alpha}, \Delta_2^{\alpha}, \ldots, \Delta_k^{\alpha}).
\]

Relations \((9), (10), (11)\) imply the following answers:

\[
(12) \quad \int_{T_\lambda \cap V'} \zeta_{|V \cap (C^*)^n}, x(t) d\chi = (1 - t^{\alpha(\partial/\partial k_n)}) (n-1)! Q_k^{n-1}(\Delta_1^{\alpha}, \Delta_2^{\alpha}, \ldots, \Delta_k^{\alpha}),
\]

\[
\int_{T_\lambda \cap V'} \zeta_{\infty |V \cap (C^*)^n}, x(t) d\chi = (1 - t^{\alpha(\partial/\partial k_n)}) (n-1)! Q_k^{n-1}(\Delta_1^{\alpha}, \Delta_2^{\alpha}, \ldots, \Delta_k^{\alpha}).
\]

We can multiply identities \((12)\) over all strata \(T_\lambda \subset X_\Lambda\) of dimension \((n - 1)\) corresponding to the tori \(\lambda \in \Lambda \setminus \Lambda_+ (\lambda \in \Lambda \setminus \Lambda_+)\), and apply \((6)\), obtaining the required formulas \((1)\) and \((3)\). Formulas \((2)\) and \((4)\) follow from \((1)\) and \((3)\) (respectively) by the multiplicative property of zeta-functions.
§3. ZETA-FUNCTION OF A POLYNOMIAL ON A COMPLETE INTERSECTION

In this section we obtain the general formula for the zeta-function at the origin of a polynomial $F_0 = \sum_{k \in \mathbb{Z}^n} F_{0,k}z^k$ on the set of common zeros of a set of polynomials $F_1, F_2, \ldots, F_k$. We use the notation and definitions introduced in \cite{2}. Let $\Delta_0$ be the Newton polyhedron of $F_0$. For an index set $I$, we denote $\Delta^I_0 = \Delta_0 \cap \mathbb{R}^I$, $F^I_0 = \sum_{k \in \Delta^I_0} F_{0,k}z^k$.

For each index set $I \subset \{1, 2, \ldots, n\}$, consider the following rational function:

\begin{equation}
\label{eq:13}
\zeta^I_{\Delta_0; \Delta_1, \ldots, \Delta_k}(t) := \prod_{\alpha \in \mathbb{Z}^I_{\Delta_0}} \left(1 - t^{m^I_{\Delta_0}(\alpha)}\right)^{-1} \tilde{Q}^I_{k(1)+1}(\Delta^I_0, \Delta^I_1, \ldots, \Delta^I_k(t)),
\end{equation}

where $m^I_{\Delta_0}(\alpha) = \min(\alpha|_{\Delta^I_0})$ is the minimal value of the covector $\alpha$ on the set $\Delta^I_0$, the symbol $\mathbb{Z}^I_{\Delta_0}$ stands for the set of covectors $\alpha \in \mathbb{Z}^I$ such that $\min(\alpha|_{\Delta^I_0}) > 0$ (for $\Delta^I_0 = \emptyset$, we put $\mathbb{Z}^I_{\Delta_0} = \emptyset$), and

\begin{equation}
\label{eq:14}
\tilde{Q}^I_{k+1}(x_0, x_1, \ldots, x_k) = Q^I_k(x_1, x_2, \ldots, x_k) - Q^I_{k+1}(x_0, x_1, \ldots, x_k)
\end{equation}

is a homogeneous polynomial of degree $l := |I| - 1$. The following statement is a consequence of Theorem 1 and some observations concerning the formula for the Euler characteristic of a nondegenerate complete intersection that was obtained in \cite{10}.

**Theorem 4.** Suppose that systems of polynomials $F_0, F_1, \ldots, F_k$ and $F_1, F_2, \ldots, F_k$ are nondegenerate with respect to their Newton polyhedra $\Delta_0, \Delta_1, \ldots, \Delta_k$ and $\Delta_1, \Delta_2, \ldots, \Delta_k$, respectively. Then

\begin{equation}
\label{eq:15}
\zeta_{F_0, V \cap (\mathbb{C}^*)^n}(t) = \zeta^I_{\Delta_0; \Delta_1, \ldots, \Delta_k}(t),
\end{equation}

\begin{equation}
\label{eq:16}
\zeta_{F_0, V}(t) = \prod_{I \subset \{1, \ldots, n\} : I \neq \emptyset} \zeta^I_{\Delta_0; \Delta_1, \ldots, \Delta_k}(t),
\end{equation}

where $V = \{z \in \mathbb{C}^n \mid F_1(z) = F_2(z) = \cdots = F_k(z) = 0\}$ is the set of common zeros of the system $F_1, F_2, \ldots, F_k$.

**Remark 2.** Consider the case where $k = 0$. Using (16) and (13), we can obtain the relation

\[ \zeta_{F_0, \mathbb{C}^n}(t) = \prod_{I \neq \emptyset} \prod_{\alpha \in \mathbb{Z}^I_{\Delta_0}} \left(1 - t^{m^I_{\Delta_0}(\alpha)}\right)^{-1} \text{Vol}_0(\Delta^I_0, \alpha) \]

(here we put $\text{Vol}_0(pt) = 1$). This is an analog of the Libgober–Sperber theorem (see \cite{4}) and (in a slightly different form) was obtained by Y. Matsui and K. Takeuchi (\cite{5}, §4).

**Proof of the theorem.** Note that formula (16) follows from (15) by the multiplicative property of zeta-functions. We prove (15).

Consider the system of polynomials $G_1, G_2, \ldots, G_{k+1}$ in $n + 1$ variables $(z, z_{n+1}) = (z_1, z_2, \ldots, z_{n+1})$ given by

\begin{equation}
\label{eq:17}
G_i(z_1, z_2, \ldots, z_{n+1}) = F_i(z_1, z_2, \ldots, z_n), \quad i = 1, 2, \ldots, k,
\end{equation}

\begin{equation}
\label{eq:18}
G_{k+1}(z_1, z_2, \ldots, z_{n+1}) = F_0(z_1, z_2, \ldots, z_n) - z_{n+1}.
\end{equation}

Consider the set $W = \{(z, z_{n+1}) \in \mathbb{C}^{n+1} \mid G_1(z) = G_2(z) = \cdots = G_{k+1}(z) = 0\}$. Since, obviously, the fibrations defined by the maps

\[ V \cap (\mathbb{C}^*)^n \cap F_0^{-1}(\mathbb{D}_\delta) \xrightarrow{F_0} \mathbb{D}_\delta \quad \text{and} \quad W \cap (\mathbb{C}^*)^{n+1} \cap \{0 < |z_{n+1}| \leq \delta\} \xrightarrow{z_{n+1}} \mathbb{D}_\delta \]

are isomorphic, we have

\begin{equation}
\label{eq:19}
\zeta_{F_0, V \cap (\mathbb{C}^*)^n}(t) = \zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t).
\end{equation}
The space $\mathbb{R}^n$ with the coordinates $(k_1, k_2, \ldots, k_n)$ is enclosed in a standard manner in the space $\mathbb{R}^{n+1}$ with the additional coordinate $k_{n+1}$ that corresponds to the variable $z_{n+1}$. For $i \leq k$, the Newton polyhedra of the polynomials $F_i$ and $G_i$ coincide: $\Delta(G_i) = \Delta_i$.

The Newton polyhedron of the polynomial $G_{k+1}$ is a cone of integer height 1 over the Newton polyhedron of the polynomial $F_0$, $\Delta(G_{k+1}) = \Delta_0$.

**Proposition 1.** For a system of polynomials $F_0, F_1, \ldots, F_k$ such that both this system itself and the system $F_1, F_2, \ldots, F_k$ are nondegenerate with respect to their Newton polyhedra, the system of polynomials $G_1, G_2, \ldots, G_{k+1}$ is also nondegenerate with respect to its Newton polyhedra.

**Proof.** Consider an arbitrary subset $I \subset \{1, 2, \ldots, (n+1)\}$ and an arbitrary covector $\alpha \in \mathbb{Z}^I$. For $n+1 \notin I$, obviously, the conditions of $\alpha$-nondegeneracy as applied to the system $\{G_i^I\}$ and to the system $\{F_i^I\}$ are equivalent. Assume that $n+1 \in I$. Denote $I' = I \setminus \{n+1\}$, $\alpha' = \alpha |_{\mathbb{R}^I}$. Extending the notation of Subsection 2.1 to the system of polynomials $G_1, G_2, \ldots, G_{k+1}$, we see that $k(I) = k(I') + 1$, $G^I_{i, \alpha}(z, z_{n+1}) = F^I_{i, \alpha'}(z)$ for $i \leq k(I')$. Three cases are possible.

1. $\alpha(\frac{\partial}{\partial k_{n+1}}) > \min(\alpha' |_{\Delta'_I})$. Then $(C\Delta_0 \cap \mathbb{R}^I) = \Delta_0', \alpha'$. $G^I_{i, \alpha}(z, z_{n+1}) = F^I_{i, \alpha'}(z)$.

2. $\alpha(\frac{\partial}{\partial k_{n+1}}) < \min(\alpha' |_{\Delta'_I})$. Then $(C\Delta_0 \cap \mathbb{R}^I) = \{\frac{\partial}{\partial k_{n+1}}\}$, $G^I_{i, \alpha} = -z_{n+1}$. $G^I_{i, \alpha}(z, z_{n+1}) = F^I_{i, \alpha'}(z) - z_{n+1}$.

Using the $\alpha'$-nondegeneracy of the systems $F_0, F_1, \ldots, F_k$ and $F_1, F_2, \ldots, F_k$, we verify easily that the 1-forms $dG^I_{i, \alpha}$, $i = 1, 2, \ldots, k(I)$, are linearly independent at the points of the set

$$\{(z, z_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid G_1(z, z_{n+1}) = G_2(z, z_{n+1}) = \cdots = G_{k+1}(z, z_{n+1}) = 0\}.$$

Proposition 1 shows that Theorem 1 is applicable to the polynomials $G_1, G_2, \ldots, G_{k+1}$:

$$
\zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t) = \prod_{\alpha \in \mathbb{Z}^I_{\alpha}} \left(1 - t^n \frac{\partial}{\partial k_{n+1}} \right)^n Q^n_{k+1}(\{C\Delta_0\}, \Delta_1^0, \ldots, \Delta_k^0),
$$

where $I_0 = \{1, 2, \ldots, n+1\}$. It is easily seen that in the above cases 1 and 2 the exponent $Q^n_{k+1}(\{C\Delta_0\}, \Delta_1^0, \ldots, \Delta_k^0)$ equals 0. Therefore,

$$
\zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t) = \prod_{\alpha \in \mathbb{Z}^I_{\alpha}} \left(1 - t^n \Delta_0(\alpha) \right)^n Q^n_{k+1}(\{C\Delta_0\}, \Delta_1^0, \ldots, \Delta_k^0),
$$

where $I_0 = \{1, 2, \ldots, n\}$. Now, formula 18 follows from 18, 20, and the identity

$$
n! Q^n_{k+1}(\{C\Delta_0\}, \Delta_1^0, \ldots, \Delta_k^0) = (n-1)! Q^{n-1}_{k+1}(\Delta_0^0, \Delta_1^0, \ldots, \Delta_k^0),
$$

which is a consequence of the following statement.

**Proposition 2.** Let $\Delta_0, \Delta_1, \ldots, \Delta_k$ be a set of integer polyhedra lying in a rational affine hyperplane $L \subset \mathbb{R}^{n+1}$. Let $C\Delta_0$ be the cone over $\Delta_0$ with vertex at some point $v \in \mathbb{R}^{n+1}$ that lies at the integer distance 1 from the hyperplane $L$. Then

$$
(n+1)! Q^n_{k+1}(C\Delta_0, \Delta_1, \ldots, \Delta_k) = n! Q^n_{k+1}(\Delta_0, \Delta_1, \ldots, \Delta_k).
$$

**Proof.** We choose an affine integer coordinate system $k = (k_1, k_2, \ldots, k_{n+1})$ in the space $\mathbb{R}^{n+1}$ in such a way that $L = \{k \in \mathbb{R}^n \mid k_{n+1} = 0\}$ and $v = (0, 0, \ldots, 1)$. Choose Laurent polynomials $F_0, F_1, \ldots, F_k$ in the variables $z = (z_1, z_2, \ldots, z_n)$ with fixed Newton polyhedra $\Delta_0, \Delta_1, \ldots, \Delta_k$ in such a way that the systems $F_0, F_1, \ldots, F_k$ and $F_1, F_2, \ldots, F_k$ are nondegenerate with respect to their Newton polyhedra.
are nondegenerate in the sense of [9] with respect to their Newton polyhedra. One can easily show (see Proposition 1) that the system of Laurent polynomials $G_1, G_2, \ldots, G_{k+1}$ in $n + 1$ variables defined in terms of the polynomials $\{F_i\}$ by formulas (17) is also nondegenerate in the sense of [9] with respect to its Newton polyhedra $\Delta_1, \ldots, \Delta_k, C(\Delta_0)$. We put

$$V = \{F_0 = F_1 = \cdots = F_k = 0\} \subset (\mathbb{C}^*)^n, $$
$$V_1 = \{F_1 = F_2 = \cdots = F_k = 0\} \subset (\mathbb{C}^*)^n, $$
$$W = \{G_1 = G_2 = \cdots = G_{k+1} = 0\} \subset (\mathbb{C}^*)^{n+1}. $$

Applying the results of [10], we find the Euler characteristics of the sets $V, V_1, W$:

$$(22) \quad \chi(V) = n! Q_{k+1}^n(\Delta_0, \Delta_1, \ldots, \Delta_k), \quad \chi(V_1) = n! Q_{k}^n(\Delta_1, \Delta_2, \ldots, \Delta_k), $$
$$(23) \quad \chi(W) = (n + 1)! Q_{k+1}^{n+1}(C\Delta_0, \Delta_1, \ldots, \Delta_k). $$

Consider the projection $p : (\mathbb{C}^*)^{n+1} \to (\mathbb{C}^*)^n$ to the coordinate hyperplane with the coordinates $(z_1, z_2, \ldots, z_n)$. Its restriction $p|_W$ provides an isomorphism between $W$ and $V_1 \setminus V$. Therefore,

$$\chi(W) = \chi(V_1) - \chi(V).$$

Applying this relation and using (22), (23), and (14), we get (21). □

REFERENCES


Moscow Institute of Physics and Technology, Independent University of Moscow, Russia
E-mail address: gusev@mccme.ru

Received 23/SEP/2009

Translated by THE AUTHOR