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MONODROMY ZETA-FUNCTION OF A POLYNOMIAL ON A COMPLETE INTERSECTION, AND NEWTON POLYHEDRA

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Abstract. For a generic (polynomial) one-parameter deformation of a complete intersection, its monodromy zeta-function is defined. Explicit formulas for this zeta-function in terms of the corresponding Newton polyhedra are obtained in the case where the deformation is nondegenerate with respect to its Newton polyhedra. This result is employed to obtain a formula for the monodromy zeta-function at the origin of a polynomial on a complete intersection, which is an analog of the Libgober–Sperber theorem.

§1. Introduction

Let $F_0, F_1, \dots, F_k$ be a set of functions on $\mathbb{C}^n$ defined as polynomials in n complex variables $\mathbf{z} = (z_1, z_2, \dots, z_n)$. Consider the family of varieties

$$V_c = \{\mathbf{z} \in \mathbb{C}^n \mid F_0(\mathbf{z}) = c,\ F_i(\mathbf{z}) = 0,\ i = 1, 2, \dots, k\},$$

where $c \in \mathbb{C}$ is a complex parameter. This family provides a fibration over the punctured neighborhood of the origin in the parameter space with the fiber V_c over a point c (see below). In this paper we obtain a formula for the monodromy zeta-function of the above fibration in terms of the Newton polyhedra of the polynomials $F_0, F_1, \dots, F_k$. This result can be viewed as a global analog of [3, Theorem 2.2] and an analog of [5, Theorem 5.5], where the monodromy zeta-function at infinity is calculated. In §2 we consider the case where $F_0(\mathbf{z}) = z_n$, so that the fibration corresponds to a polynomial deformation of a set of polynomials in $n-1$ variables $z_1, z_2, \dots, z_{n-1}$. The general case is deduced from this special one in §3. The study is partially motivated by the results of D. Siersma and M. Tibar ([6]).

Let $A = \mathbb{C}^n \setminus Y$ be the complement to an arbitrary algebraic hypersurface $Y \subset \mathbb{C}^n$. Let $Z = \{\mathbf{z} \in \mathbb{C}^n \mid F_i = 0,\ i = 1, 2, \dots, k\} \cap A$. We denote by $\mathbb{D}_r$ and $\mathbb{D}_r^*$ the closed disk in $\mathbb{C}$ of radius r centered at the origin and the punctured disk $\mathbb{D}_r^* := \mathbb{D}_r \setminus \{0\}$, respectively. From [11, Theorem 5.1] it follows that there exists a finite set $B \subset \mathbb{C}$ such that the restriction $F = F_0|_Z$ of the function F_0 is a topological fibration over $\mathbb{C} \setminus B$. In particular, the map $F|_{F^{-1}(\mathbb{D}_\delta^*)}$ ($F|_{F^{-1}(\mathbb{C}\setminus\mathbb{D}_d)}$) is a fibration for any sufficiently small δ (for any sufficiently large d). Consider the restriction of this fibration to the cycle $\{c \cdot \exp(2\pi i t) \mid t \in [0,1]\}$, where $|c|$ is sufficiently small (large, respectively). Consider the monodromy transformation $h_{F,0} : Z_c \to Z_c$ ($h_{F,\infty} : Z_c \to Z_c$) of the fiber Z_c over the point c of the resulting fibration.

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The *zeta-function* of an arbitrary transformation $h\colon X \to X$ of a topological space X is the rational function

$$\zeta_h(t) = \prod_{i\geq 0} (\det(\mathrm{Id} - th_*|_{H_i^c(X;\mathbb{C})}))^{(-1)^i},$$

where $H_i^c(X;\mathbb{C})$ denotes the ith homology group with closed support.

Definition 1. The *monodromy zeta-function* (*at the origin*) of the function F_0 on the set Z is the zeta-function of the transformation $h_{F,0}$, $\zeta_{F_0,Z}(t) := \zeta_{h_{F,0}}(t)$. The *monodromy zeta-function at infinity* of the function F_0 on the set Z is the zeta-function of the transformation $h_{F,\infty}$, $\zeta^{\infty}_{F_0,Z}(t) := \zeta_{h_{F,\infty}}(t)$.

Let $S_1, S_2, \dots, S_n \subset \mathbb{R}^n$ be a collection of convex bodies. We denote by $S_1S_2\dots S_n$ their Minkovski mixed volume (see, e.g., [8]). If $S_j = \varnothing$ for some j, we put $S_1S_2\dots S_n = 0$. For a homogeneous polynomial $T(x_1, x_2, \dots, x_k) = \sum \alpha_{i_1 i_2 \dots i_n} x_{i_1} x_{i_2} \dots x_{i_n}$ of degree n, we define $T(S_1, S_2, \dots, S_k)$ as $\sum \alpha_{i_1 i_2 \dots i_n} S_{i_1} S_{i_2} \dots S_{i_n}$.

Let $S_1, S_2, \dots, S_l \subset L \subset \mathbb{R}^n$ be a collection of convex bodies that lie in an l-dimensional rational affine subspace L. We define $S_1S_2\dots S_l$ as the l-dimensional *integer mixed volume*, that is, the Minkovskian mixed volume in the affine subspace L normalized in such way that the l-dimensional volume of the minimal parallelepiped with integer vertices equals one.

In this paper we obtain a formula for the zeta-function $\zeta_{F_0,V}(t)$, $V = \{\mathbf{z} \in \mathbb{C}^n \mid F_1(\mathbf{z}) = F_2(\mathbf{z}) = \dots = F_k(\mathbf{z}) = 0\}$ for a generic set of polynomials $F_0, F_1, \dots, F_k$ in terms of the integer mixed volumes of the faces of their Newton polyhedra $\Delta_0, \Delta_1, \dots, \Delta_k$.

§2. Zeta-function of a polynomial deformation

In this section we study the case where $F_0(\mathbf{z}) = z_n$. Consider the set of deformations $f_{i,\sigma}(z_1, \dots, z_{n-1}) := F_i(z_1, \dots, z_{n-1}, \sigma)$ of the functions $f_i := f_{i,0}$ on the set $\mathbb{C}^{n-1}$, $i = 1, 2, \dots, k$, where $\sigma \in \mathbb{C}$ is the deformation parameter. The fiber over the point c of the fibration provided by the function F_0 on the set $\{F_1 = F_2 = \dots = F_k = 0\}$ is $\{f_{1,c} = f_{2,c} = \dots = f_{k,c} = 0\} \times \{c\}$. This fact motivates the following definition.

Definition 2. Consider $V = \{\mathbf{z} \in \mathbb{C}^n \mid F_1(\mathbf{z}) = F_2(\mathbf{z}) = \dots = F_k(\mathbf{z}) = 0\}$. The zeta-function $\zeta_{z_n,V}(t)$ ($\zeta^{\infty}_{z_n,V}(t)$) will be called *the monodromy zeta-function* (*at infinity*) *of the deformation* $\{f_{i,\sigma} \mid i = 1, 2, \dots, k\}$.

2.1. Formulas for the zeta-function of a deformation. Consider the representation $F_i = \sum_{\mathbf{k}\in\mathbb{Z}^n} F_{i,\mathbf{k}} \mathbf{z}^{\mathbf{k}}$, where the $F_{i,\mathbf{k}} \in \mathbb{C}$, $\mathbf{k} \in \mathbb{Z}^n$, are the coefficients of the polynomial F_i and $\mathbf{k} = (k_1, k_2, \dots, k_n)$ are the coordinates in the space $\mathbb{R}^n$ that correspond to the variables $(z_1, z_2, \dots, z_n)$. Let $\Delta_i = \Delta(F_i)$ denote the *Newton polyhedron* of the polynomial F_i, $i = 1, 2, \dots, k$, i.e., the convex hull of the set $\{\mathbf{k} \in \mathbb{Z}^n \mid F_{i,\mathbf{k}} \neq 0\}$. A subset I of the set $\{1, 2, \dots, n\}$ will be called an *index set*. Denote $\mathbb{R}^I = \{\mathbf{k} \in \mathbb{R}^n \mid k_i = 0,\ i \notin I\}$. Let $j_1^I < j_2^I < \dots < j_{k(I)}^I$ be the elements of the set $\{j \in \{1, 2, \dots, k\} \mid \Delta_j \cap \mathbb{R}^I \neq \varnothing\}$. We put $\Delta_i^I = \Delta_{j_i^I} \cap \mathbb{R}^I$, $i = 1, 2, \dots, k(I)$ and $F_i^I = \sum_{\mathbf{k}\in\Delta_i^I} F_{j_i^I,\mathbf{k}} \mathbf{z}^{\mathbf{k}}$.

An integer covector is said to be *primitive* if it is not a multiple of another integer covector. We denote by $\mathcal{Z}^I$ the set of primitive covectors in the dual space $(\mathbb{R}^I)^*$. For a convex set $S \subset \mathbb{R}^I$ and a covector $\alpha \in \mathcal{Z}^I$, let S^α be the subset of S formed by the points where the function $\alpha|_S$ attains its minimal value: $S^\alpha = \{\mathbf{x} \in S \mid \alpha(\mathbf{x}) = \min(\alpha|_S)\}$. For an arbitrary polynomial $P = \sum_{\mathbf{k}\in\Delta} P_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$ with the Newton polyhedron $\Delta \subset \mathbb{R}^I$ and a covector $\alpha \in \mathcal{Z}^I$, we denote by P^α the polynomial $\sum_{\mathbf{k}\in\Delta^\alpha} P_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$. For an index set I containing n, let $\mathcal{Z}_+^I \subset \mathcal{Z}^I$ ($\mathcal{Z}_-^I \subset \mathcal{Z}^I$) be the subset of covectors $\alpha = \dots + \alpha_n\, dk_n$ with strictly positive last component: $\alpha_n > 0$ (strictly negative last component: $\alpha_n < 0$).

Definition 3. Consider a covector $\alpha \in \mathcal{Z}^{\{1,2,\dots,n\}}$. We say that a system of polynomials $F_1, F_2, \dots, F_k$ is *α-nondegenerate* with respect to its Newton polyhedra $\Delta_1, \Delta_2, \dots, \Delta_k$ if the 1-forms dF_i^α, $i = 1, 2, \dots, k$, are linearly independent at all the points of the set $\{\mathbf{z} \in (\mathbb{C}^*)^n \mid F_1^\alpha(\mathbf{z}) = F_2^\alpha(\mathbf{z}) = \dots = F_k^\alpha(\mathbf{z}) = 0\}$.

We say that a system of polynomials $F_1, F_2, \dots, F_k$ is *σ-nondegenerate* (*at infinity*) with respect to its Newton polyhedra if for each index set I containing n and each covector $\alpha \in \mathcal{Z}_+^I$ ($\alpha \in \mathcal{Z}_-^I$), the system of polynomials $F_1^I, F_2^I, \dots, F_{k(I)}^I$ is α-nondegenerate with respect to its Newton polyhedra.

Finally, a system of polynomials $F_1, F_2, \dots, F_k$ is said to be *nondegenerate* with respect to its Newton polyhedra if for each index set I and each $\alpha \in \mathcal{Z}^I$ the system of polynomials $F_1^I, F_2^I, \dots, F_{k(I)}^I$ is α-nondegenerate.

For each index set $I \subset \{1, 2, \dots, n\}$ containing n, we define the following rational functions:

$$\zeta^I_{\Delta_1,\Delta_2,\dots,\Delta_k}(t) = \prod_{\alpha\in\mathcal{Z}_+^I} \left(1 - t^{\alpha(\frac{\partial}{\partial k_n})}\right)^{l!\, Q^l_{k(I)}\left(\Delta_1^{I,\alpha},\Delta_2^{I,\alpha},\dots,\Delta_{k(I)}^{I,\alpha}\right)},$$

$$\zeta^{I,\infty}_{\Delta_1,\Delta_2,\dots,\Delta_k}(t) = \prod_{\alpha\in\mathcal{Z}_-^I} \left(1 - t^{-\alpha(\frac{\partial}{\partial k_n})}\right)^{l!\, Q^l_{k(I)}\left(\Delta_1^{I,\alpha},\Delta_2^{I,\alpha},\dots,\Delta_{k(I)}^{I,\alpha}\right)},$$

where $l = |I| - 1$, $\frac{\partial}{\partial k_n}$ is the vector in $\mathbb{R}^I$ whose only nonzero coordinate is $k_n = 1$, and $Q_k^l(x_1, x_2, \dots, x_k) := \left[\prod_{i=1}^k \frac{x_i}{1+x_i}\right]_l$, where $[\cdot]_l$ denotes the degree l homogeneous part of the power series under consideration. In particular, $Q_0^l \equiv 0$ for $l > 0$ and $Q_0^0 \equiv 1$.

Theorem 1. *Suppose a system of polynomials $F_1, F_2, \dots, F_k$ is σ-nondegenerate with respect to its Newton polyhedra $\Delta_1, \Delta_2, \dots, \Delta_k$. Then*

$$\zeta_{z_n,V\cap(\mathbb{C}^*)^n}(t) = \zeta^{\{1,2,\dots,n\}}_{\Delta_1,\Delta_2,\dots,\Delta_k}(t), \tag{1}$$

$$\zeta_{z_n,V}(t) = \prod_{I\,:\,n\in I\subset\{1,2,\dots,n\}} \zeta^I_{\Delta_1,\Delta_2,\dots,\Delta_k}(t), \tag{2}$$

where $V = \{\mathbf{z} \in \mathbb{C}^n \mid F_1(\mathbf{z}) = F_2(\mathbf{z}) = \dots = F_k(\mathbf{z}) = 0\}$.

Theorem 2. *Suppose a system of polynomials $F_1, F_2, \dots, F_k$ is σ-nondegenerate at infinity with respect to its Newton polyhedra $\Delta_1, \Delta_2, \dots, \Delta_k$. Then*

$$\zeta^\infty_{z_n,V\cap(\mathbb{C}^*)^n}(t) = \zeta^{\{1,2,\dots,n\},\infty}_{\Delta_1,\Delta_2,\dots,\Delta_k}(t), \tag{3}$$

$$\zeta^\infty_{z_n,V}(t) = \prod_{I\,:\,n\in I\subset\{1,2,\dots,n\}} \zeta^{I,\infty}_{\Delta_1,\Delta_2,\dots,\Delta_k}(t), \tag{4}$$

where $V = \{\mathbf{z} \in \mathbb{C}^n \mid F_1(\mathbf{z}) = F_2(\mathbf{z}) = \dots = F_k(\mathbf{z}) = 0\}$.

Remark 1. For $k = 1$, equation (1) implies

$$\zeta_{z_n,V\cap(\mathbb{C}^*)^n}(t) = \prod_{\alpha\in\mathcal{Z}_+^{I_0}} \left(1 - t^{\alpha(\frac{\partial}{\partial k_n})}\right)^{(-1)^n (n-1)!\,\mathrm{Vol}_{n-1}(\Delta_1^\alpha)}, \tag{5}$$

where $\mathrm{Vol}_l(\cdot)$ denotes the l-dimensional integer volume, $I_0 = \{1, 2, \dots, n\}$. This relation is similar to formula (1) in [3, Theorem 2.2] for the zeta-function of a singularity deformation. In fact, let f_σ denote the germ at the origin of the deformation defined by $f_\sigma(z_1, \dots, z_{n-1}) = F_1(z_1, \dots, z_{n-1}, \sigma)$. Using the equation in [3], we obtain $\zeta_{f_\sigma|_{(\mathbb{C}^*)^{n-1}}}(t) = \prod_{\alpha\in\mathcal{Z}_{++}^{I_0}} (1 - t^{\alpha(\frac{\partial}{\partial k_n})})^{(-1)^n (n-1)!\,\mathrm{Vol}_{n-1}(\Delta_1^{I_0,\alpha})}$, where $\mathcal{Z}_{++}^{I_0}$ is the subset of covectors in $\mathcal{Z}_+^{I_0}$ whose components are all strictly positive. Hence, the local zeta-function

$\zeta_{f_\sigma|_{(\mathbb{C}^*)^{n-1}}}(t)$ is a "natural" factor of the global one $\zeta_{z_n,V\cap(\mathbb{C}^*)^n}(t)$. The same observation follows from the localization principle (see below and [2]).

Example 1. Assume that $n = 2$ and $k = 1$. Consider the polynomial $F_1(z_1, z_2) = z_1 + z_2(1 + z_1^2)$. Identities (1), (2) and the corresponding relations in [3, Theorem 2.2] imply that $\zeta_{f_\sigma|_{\mathbb{C}}}(t) = \zeta_{f_\sigma|_{(\mathbb{C}^*)}}(t) = (1-t)$, $\zeta_{z_2,V}(t) = \zeta_{z_2,V\cap(\mathbb{C}^*)^2}(t) = (1-t)^2$. The same results can be obtained by the following arguments. The global fiber is $V_\sigma = \{z_1 \mid F_1(z_1,\sigma) = 0\} = \{\frac{-1+\sqrt{1-4\sigma^2}}{2\sigma}\} = \{x_1(\sigma), x_2(\sigma)\}$, where $x_1(\sigma) \approx -\sigma,\ x_2(\sigma) \approx -\sigma^{-1}$ for $|\sigma| \ll 1$. Thus, it consists of two points, one of them close to the origin and the other close to infinity. The monodromy transformation is the identical map of the fiber itself, whence $\zeta_{z_2,V}(t) = \det((1-t)\,\mathrm{Id}) = (1-t)^2$. Since the local fiber $\{f_\sigma(z_1) = 0\} = \{x_1(\sigma)\}$ consists of one point, we have $\zeta_{f_\sigma|_{\mathbb{C}}}(t) = (1-t)$.

2.2. Proofs of the theorems. We reduce the calculation of the zeta-function to integration with respect to the Euler characteristic (see., e.g., [7]), using the following *localization principle.*

We recall the notion of the zeta-function as applied to a family of sections of a line bundle over a variety, introduced by S. M. Gusein-Zade and D. Siersma in [2]. Let W be a compact complex analytic variety, and let W_1 be the complement to a compact subvariety of W. Let L be a line bundle over W, and let q_σ be a family of sections of L analytic in $\sigma \in \mathbb{C}_\sigma$. Let U be the subset of $W_1 \times \mathbb{C}_\sigma$ given by $q_\sigma(x) = 0$. The restriction to U of the projection $W_1 \times \mathbb{C}_\sigma \to \mathbb{C}_\sigma$ is a fibration over the punctured disk $\mathbb{D}_r^* \subset \mathbb{C}_\sigma$ for $|r| \ll 1$. The *zeta-function of a family of sections q_σ restricted to the set W_1* is the zeta-function of the monodromy transformation of the above fibration. We denote it by $\zeta_{q_\sigma|_{W_1}}(t)$.

The fibration L is trivial over a neighborhood of a point $x \in W$. Therefore, using a fixed coordinate system, we can view the family of germs at the point x of sections q_σ as a deformation in the parameter σ of a function germ. We denote by $\zeta_{q_\sigma|_{W_1},x}(t)$ the zeta-function of the germ at the point $\sigma = 0$ of the above deformation restricted to the set W_1 (see, e.g., [3]).

Theorem 3 ([2], "localization principle"). *We have*

$$\zeta_{q_\sigma|_{W_1}}(t) = \int_W \zeta_{q_\sigma|_{W_1},x}(t)\, d\chi.$$

Using the Newton polyhedra $\Delta_1, \Delta_2, \dots, \Delta_k$ of the polynomials $F_1, F_2, \dots, F_k$, we construct a unimodular simplicial partition Λ of the dual space $(\mathbb{R}^n)^*$; we assume that this partition is sufficiently fine for the system $\{\Delta_i\}$ in the sense of [9]. Consider the toroidal compactification X_Λ of the torus $(\mathbb{C}^*)^n$ that corresponds to the partition Λ. Recall that the standard action of the torus $(\mathbb{C}^*)^n$ on itself uniquely extends to an action of the torus on the variety X_Λ. The cones $\lambda \in \Lambda$ of the partition are in one-to-one correspondence with the orbits $T_\lambda \subset X_\Lambda$ of this action and the orbit T_λ is isomorphic to $(\mathbb{C}^*)^{n-\dim\lambda}$. Denote by X'_Λ the complement in X_Λ to the torus $T_{\{0\}} \cong (\mathbb{C}^*)^n$. Let $\bar{V}$ be the closure of the set $V \cap T_{\{0\}} \subset X_\Lambda$, and let $V' = \bar{V} \cap X'_\Lambda$. We prove the following statement.

Lemma 1. *For a sufficiently fine partition Λ, we have*

$$\begin{aligned} \zeta_{z_n,V\cap(\mathbb{C}^*)^n}(t) &= \int_{V'} \zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x}(t)\, d\chi, \\ \zeta^\infty_{z_n,V\cap(\mathbb{C}^*)^n}(t) &= \int_{V'} \zeta^\infty_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x}(t)\, d\chi, \end{aligned} \tag{6}$$

where, for a germ at $x \in V'$ of a meromorphic function f on the set $\bar{V}$ and for an open subset $A \subset \bar{V}$, the expression $\zeta_{f|_A,x}(t)$ ($\zeta^{\infty}_{f|_A,x}(t)$) denotes the local zeta-function (at infinity) of the germ at x of the function f restricted to A.

Proof. We may assume the partition Λ to be a subdivision of the standard partition Π of the space $(\mathbb{R}^n)^*$ corresponding to the n-dimensional projective space: $X_\Pi = \mathbb{CP}^n \supset (\mathbb{C}^*)^n$. Let $p\colon X_\Lambda \to \mathbb{CP}^n$ be the map of the toric varieties induced by the refinement $\Lambda \prec \Pi$. Consider the family of global sections s_σ, $\sigma \in \mathbb{C}$, of the fibration $\mathcal{O}(1)$ over $\mathbb{CP}^n$ that is defined by the condition $s_\sigma|_{(\mathbb{C}^*)^n} = z_n - \sigma$. Denote $\pi = p \circ inj$, where $inj : \bar{V} \hookrightarrow X_\Lambda$ is the inclusion map. Let $S_\sigma = \pi^*(s_\sigma)$ be the family of sections of the bundle $\pi^*(\mathcal{O}(1))$ that is the pull-back of s_σ. In a similar way, consider a family of sections s'_σ, $\sigma \in \mathbb{C}$, of the fibration $\mathcal{O}(1)$ that is defined by the condition $s'_\sigma|_{(\mathbb{C}^*)^n} = 1 - \sigma z_n$, and consider the pull-back $S'_\sigma = \pi^*(s'_\sigma)$.

By simple reformulations, we can easily show that

$$\zeta_{z_n,V\cap(\mathbb{C}^*)^n}(t) = \zeta_{S_\sigma|_{V\cap(\mathbb{C}^*)^n}}(t), \quad \zeta^{\infty}_{z_n,V\cap(\mathbb{C}^*)^n}(t) = \zeta_{S'_\sigma|_{V\cap(\mathbb{C}^*)^n}}(t),$$
$$\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t) = \zeta_{S_\sigma|_{V\cap(\mathbb{C}^*)^n},x}(t), \quad \zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t) = \zeta_{S'_\sigma|_{V\cap(\mathbb{C}^*)^n},x}(t).$$

Applying Theorem 3 to the families S_c and S'_c, we obtain

$$\zeta_{S_\sigma|_{V\cap(\mathbb{C}^*)^n}}(t) = \int_{\bar{V}} \zeta_{S_\sigma|_{V\cap(\mathbb{C}^*)^n},x}(t)\, d\chi = \int_{\bar{V}} \zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t)\, d\chi,$$
$$\zeta_{S'_\sigma|_{V\cap(\mathbb{C}^*)^n}}(t) = \int_{\bar{V}} \zeta_{S'_\sigma|_{V\cap(\mathbb{C}^*)^n},x}(t)\, d\chi = \int_{\bar{V}} \zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t)\, d\chi.$$

Moreover, it is easily seen that $\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t) = \zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t) = 1$ for $x \notin V'$. Therefore, using the multiplicative property of the integration, we get

$$\int_{\bar{V}} \zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t)\, d\chi = \int_{V'} \zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x}(t)\, d\chi,$$
$$\int_{\bar{V}} \zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},x}(t)\, d\chi = \int_{V'} \zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x}(t)\, d\chi. \qquad \square$$

Let $\Lambda_+ \subset \Lambda$ ($\Lambda_- \subset \Lambda$) be the subset of cones $\lambda \in \Lambda$ generated by a set of primitive covectors $\alpha_1, \alpha_2, \dots, \alpha_l$ lying in $\mathcal{Z}^{\{1,2,\dots,n\}} \setminus \mathcal{Z}^{\{1,2,\dots,n\}}_-$ (in $\mathcal{Z}^{\{1,2,\dots,n\}} \setminus \mathcal{Z}^{\{1,2,\dots,n\}}_+$, respectively). We may assume that Λ is so fine that $\Lambda_- \cup \Lambda_+ = \Lambda$.

Consider an arbitrary point $x_0 \in V'$. It is contained in the torus T_λ that corresponds to some l-dimensional cone $\lambda \in \Lambda$, $l < n$. This cone lies on the border of an n-dimensional cone $\lambda' \in \Lambda$. Denote by $\alpha_1, \alpha_2, \dots, \alpha_l$ the primitive integer covectors that generate the cone λ. The cone λ' is generated by the covectors $\alpha_1, \alpha_2, \dots, \alpha_l$ and some covectors $\alpha_{l+1}, \alpha_{l+2}, \dots, \alpha_n$. Consider the coordinate system $\mathbf{u} = (u_1, u_2, \dots, u_n)$ corresponding to the set of covectors $(\alpha_1, \alpha_2, \dots, \alpha_n)$. We have $u_i(x_0) = 0$, $i \le l$, $u_i(x_0) \ne 0, i > l$. We express the monomial z_n as a function F of the variables $\mathbf{u}$:

$$F(\mathbf{u}) = b \cdot u_1^{\alpha_1(\partial/\partial k_n)} u_2^{\alpha_2(\partial/\partial k_n)} \cdots u_l^{\alpha_l(\partial/\partial k_n)},$$

where $b(\mathbf{u}) = \prod_{j=l+1}^n u_j^{\alpha_j(\partial/\partial k_n)}$, $b(x_0) \in \mathbb{C}^*$. Now we are ready to calculate the values of the integrands $\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x_0}(t)$ and $\zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x_0}(t)$ in the following two cases.

1. If $\lambda \in \Lambda_-$, then the value of the function F at the point x is not zero, so that $\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x_0}(t) = 1$. Accordingly, assume that $\lambda \in \Lambda_+$. Then the point x_0 is not a pole of the function F and therefore $\zeta^{\infty}_{z_n|_{V\cap(\mathbb{C}^*)^n},\,x_0}(t) = 1$.
2. Assume that $\lambda \in \Lambda \setminus \Lambda_-$ ($\lambda \in \Lambda \setminus \Lambda_+$). The system of polynomials $F_1, F_2, \dots, F_k$ is σ-nondegenerate (at infinity) with respect to its polyhedra $\Delta_1, \Delta_2, \dots, \Delta_k$.

Therefore, $l+k \le n$, and there is a coordinate system $(u_1, \dots, u_l, w_{l+1}, \dots, w_n)$ in a neighborhood of x_0 such that $w_i(x_0) = 0$, $i = l+1, \dots, n$, and

$$F_i = a_i\, u_1^{m_{i,1}} u_2^{m_{i,2}} \cdots u_l^{m_{i,l}} \cdot w_{n-i+1}, \quad i = 1, 2, \dots, k, \tag{7}$$

where $m_{i,j} = \min(\alpha_j|_{\Delta_i})$ and a_i is a germ of an analytic function such that $a_i(x_0) \ne 0$. Denote $V_{x_0} = V \cap (C^*)^n \cap U$. By (7), we have

$$V_{x_0} = \{u_i \ne 0,\ i \le l;\ w_i = 0,\ i > n-k\} \subset U.$$

Hence,

$$\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x_0}(t) = \zeta_{g|_{\{u_i\ne 0,\, i\le l\}},\, 0}(t) \quad \big(\zeta^\infty_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x_0}(t) = \zeta^\infty_{g|_{\{u_i\ne 0,\, i\le l\}},\, 0}(t)\big), \tag{8}$$

where g is the germ of the function in the variables $(u_1, \dots, u_l, w_{l+1}, \dots, w_{n-k})$ that is given by the relation

$$g = \prod_{j=1}^{l} u_j^{\alpha_j(\partial/\partial k_n)} \cdot b(u_1, \dots, u_l, w_{l+1}, \dots, w_{n-k}, 0, \dots, 0).$$

Using the Varchenko-type formula for meromorphic functions (see [1]), we calculate the right-hand side of (8). For $l = 1$, we obtain

$$\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x_0}(t) = 1 - t^{\alpha_1(\partial/\partial k_n)} \quad \big(\zeta^\infty_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x_0}(t) = 1 - t^{-\alpha_1(\partial/\partial k_n)}\big). \tag{9}$$

Finally, the two zeta-functions in question are trivial if $l > 1$.

Now we specify the only case where the function $\zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x_0}(t)$ $(\zeta^\infty_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x_0}(t))$ is not trivial. Namely, we assume that $x_0 \in T_\lambda$, $\lambda \in \Lambda \setminus \Lambda_-$ $(\lambda \in \Lambda \setminus \Lambda_+)$ and $\dim \lambda = 1$. Denote $\alpha = \alpha_1$. The set $T_\lambda \cap V'$ can be defined in the coordinates $(u_2, \dots, u_{n+1})$ on the torus $T_\lambda = \{u_1 = 0\}$ by the system of equations $\{Q_1^\alpha = Q_2^\alpha = \dots = Q_k^\alpha = 0\}$, where

$$Q_i^\alpha = \sum_{\mathbf{k}\in\Delta_i^\alpha} F_{i,\mathbf{k}}\, u_2^{\alpha_2(\mathbf{k})} u_3^{\alpha_3(\mathbf{k})} \cdots u_n^{\alpha_n(\mathbf{k})}.$$

Using the main results of [9] and [10], we obtain

$$\chi(T_\lambda \cap V') = (n-1)!\, Q_k^{n-1}(\Delta(Q_1^\alpha), \Delta(Q_2^\alpha), \dots, \Delta(Q_k^\alpha)), \tag{10}$$

where $\Delta(\cdot)$ denotes the Newton polyhedron of the Laurent polynomial under consideration. The covectors $\alpha_2, \alpha_3, \dots, \alpha_n$ determine an isomorphism of the integer lattices of the hyperplane $\{\alpha = 0\} \subset \mathbb{R}^n_{\mathbf{k}}$ and the space $\mathbb{R}^{n-1}$, which contains the polyhedra $\Delta(Q_i^\alpha)$. Under this isomorphism, the polyhedra $\Delta(Q_i^\alpha)$ correspond to parallel shifts of the polyhedra Δ_i. Therefore, the corresponding mixed integer volumes coincide and

$$Q_k^{n-1}(\Delta(Q_1^\alpha), \Delta(Q_2^\alpha), \dots, \Delta(Q_k^\alpha)) = Q_k^{n-1}(\Delta_1^\alpha, \Delta_2^\alpha, \dots, \Delta_k^\alpha). \tag{11}$$

Relations (9), (10), (11) imply the following answers:

$$\begin{aligned} &\int_{T_\lambda\cap V'} \zeta_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x}(t)\, d\chi = \big(1 - t^{\alpha(\frac{\partial}{\partial k_n})}\big)^{(n-1)!\, Q_k^{n-1}(\Delta_1^\alpha, \Delta_2^\alpha, \dots, \Delta_k^\alpha)}, \\ &\int_{T_\lambda\cap V'} \zeta^\infty_{z_n|_{V\cap(\mathbb{C}^*)^n},\, x}(t)\, d\chi = \big(1 - t^{-\alpha(\frac{\partial}{\partial k_n})}\big)^{(n-1)!\, Q_k^{n-1}(\Delta_1^\alpha, \Delta_2^\alpha, \dots, \Delta_k^\alpha)}. \end{aligned} \tag{12}$$

We can multiply identities (12) over all strata $T_\lambda \subset X'_\Lambda$ of dimension $(n-1)$ corresponding to the tori $\lambda \in \Lambda \setminus \Lambda_-$ $(\lambda \in \Lambda \setminus \Lambda_+)$, and apply (6), obtaining the required formulas (1) and (3). Formulas (2) and (4) follow from (1) and (3) (respectively) by the multiplicative property of zeta-functions.

§3. Zeta-function of a polynomial on a complete intersection

In this section we obtain the general formula for the zeta-function at the origin of a polynomial $F_0 = \sum_{\mathbf{k}\in\mathbb{Z}^n} F_{0,\mathbf{k}}\mathbf{z}^{\mathbf{k}}$ on the set of common zeros of a set of polynomials $F_1, F_2, \dots, F_k$. We use the notation and definitions introduced in §2. Let Δ_0 be the Newton polyhedron of F_0. For an index set I, we denote $\Delta_0^I = \Delta_0 \cap \mathbb{R}^I$, $F_0^I = \sum_{\mathbf{k}\in\Delta_0^I} F_{0,\mathbf{k}}\mathbf{z}^{\mathbf{k}}$.

For each index set $I \subset \{1, 2, \dots, n\}$, consider the following rational function:

$$\widetilde{\zeta}^I_{\Delta_0;\Delta_1,\dots,\Delta_k}(t) := \prod_{\alpha\in\mathcal{Z}^I_{\Delta_0}} \left(1 - t^{m_{\Delta_0^I}(\alpha)}\right)^{l!\,\widetilde{Q}^l_{k(I)+1}\left(\Delta_0^{I,\alpha},\Delta_1^{I,\alpha},\dots,\Delta_{k(I)}^{I,\alpha}\right)}, \tag{13}$$

where $m_{\Delta_0^I}(\alpha) = \min(\alpha|_{\Delta_0^I})$ is the minimal value of the covector α on the set Δ_0^I, the symbol $\mathcal{Z}^I_{\Delta_0}$ stands for the set of covectors $\alpha \in \mathcal{Z}^I$ such that $\min(\alpha|_{\Delta_0^I}) > 0$ (for $\Delta_0^I = \varnothing$, we put $\mathcal{Z}^I_{\Delta_0} = \varnothing$), and

$$\widetilde{Q}^l_{k+1}(x_0, x_1, \dots, x_k) = Q^l_k(x_1, x_2, \dots, x_k) - Q^l_{k+1}(x_0, x_1, \dots, x_k) \tag{14}$$

is a homogeneous polynomial of degree $l := |I| - 1$. The following statement is a consequence of Theorem 1 and some observations concerning the formula for the Euler characteristic of a nondegenerate complete intersection that was obtained in [10].

Theorem 4. *Suppose that systems of polynomials $F_0, F_1, \dots, F_k$ and $F_1, F_2 \dots, F_k$ are nondegenerate with respect to their Newton polyhedra $\Delta_0, \Delta_1, \dots, \Delta_k$ and $\Delta_1, \Delta_2, \dots, \Delta_k$, respectively. Then*

$$\zeta_{F_0,V\cap(\mathbb{C}^*)^n}(t) = \widetilde{\zeta}^{\{1,\dots,n\}}_{\Delta_0;\Delta_1,\dots,\Delta_k}(t), \tag{15}$$

$$\zeta_{F_0,V}(t) = \prod_{I\subset\{1,\dots,n\}\,:\,I\neq\varnothing} \widetilde{\zeta}^I_{\Delta_0;\Delta_1,\dots,\Delta_k}(t), \tag{16}$$

where $V = \{\mathbf{z} \in \mathbb{C}^n \mid F_1(\mathbf{z}) = F_2(\mathbf{z}) = \dots = F_k(\mathbf{z}) = 0\}$ is the set of common zeros of the system $F_1, F_2, \dots, F_n$.

Remark 2. Consider the case where $k = 0$. Using (16) and (13), we can obtain the relation

$$\zeta_{F_0,\mathbb{C}^n}(t) = \prod_{I\neq\varnothing}\prod_{\alpha\in\mathcal{Z}^I_{\Delta_0}} \left(1 - t^{m_{\Delta_0}(\alpha)}\right)^{(-1)^l l!\,\mathrm{Vol}_l(\Delta_0^{I,\alpha})}$$

(here we put $\mathrm{Vol}_0(\mathrm{pt}) = 1$). This is an analog of the Libgober–Sperber theorem (see [4]) and (in a slightly different form) was obtained by Y. Matsui and K. Takeuchi ([5, §4]).

Proof of the theorem. Note that formula (16) follows from (15) by the multiplicative property of zeta-functions. We prove (15).

Consider the system of polynomials $G_1, G_2, \dots, G_{k+1}$ in $n + 1$ variables $(\mathbf{z}, z_{n+1}) = (z_1, z_2, \dots, z_{n+1})$ given by

$$\begin{aligned} G_i(z_1, z_2 \dots, z_{n+1}) &= F_i(z_1, z_2 \dots, z_n), \quad i = 1, 2, \dots, k,\\ G_{k+1}(z_1, z_2 \dots, z_{n+1}) &= F_0(z_1, z_2 \dots, z_n) - z_{n+1}. \end{aligned} \tag{17}$$

Consider the set $W = \{(\mathbf{z}, z_{n+1}) \in \mathbb{C}^{n+1} \mid G_1(\mathbf{z}) = G_2(\mathbf{z}) = \dots = G_{k+1}(\mathbf{z}) = 0\}$. Since, obviously, the fibrations defined by the maps

$$V \cap (\mathbb{C}^*)^n \cap F_0^{-1}(\mathbb{D}^*_\delta) \xrightarrow{F_0} \mathbb{D}^*_\delta \text{ and } W \cap (\mathbb{C}^*)^{n+1} \cap \{0 < |z_{n+1}| \le \delta\} \xrightarrow{z_{n+1}} \mathbb{D}^*_\delta$$

are isomorphic, we have

$$\zeta_{F_0,V\cap(\mathbb{C}^*)^n}(t) = \zeta_{z_{n+1},W\cap(\mathbb{C}^*)^{n+1}}(t). \tag{18}$$

The space $\mathbb{R}^n$ with the coordinates $(k_1, k_2, \dots, k_n)$ is enclosed in a standard manner in the space $\mathbb{R}^{n+1}$ with the additional coordinate k_{n+1} that corresponds to the variable z_{n+1}. For $i \le k$, the Newton polyhedra of the polynomials F_i and G_i coincide: $\Delta(G_i) = \Delta_i$. The Newton polyhedron of the polynomial G_{k+1} is a cone of integer height 1 over the Newton polyhedron of the polynomial F_0, $\Delta(G_{k+1}) = C\Delta_0$.

Proposition 1. *For a system of polynomials $F_0, F_1, \dots, F_k$ such that both this system itself and the system $F_1, F_2, \dots, F_k$ are nondegenerate with respect to their Newton polyhedra, the system of polynomials $G_1, G_2, \dots, G_{k+1}$ is also nondegenerate with respect to its Newton polyhedra.*

Proof. Consider an arbitrary subset $I \subset \{1, 2, \dots, (n+1)\}$ and an arbitrary covector $\alpha \in \mathcal{Z}^I$. For $n+1 \notin I$, obviously, the conditions of α-nondegeneracy as applied to the system $\{G_i^I\}$ and to the system $\{F_i^I\}$ are equivalent. Assume that $n+1 \in I$. Denote $I' = I \setminus \{n+1\}$, $\alpha' = \alpha|_{\mathbb{R}^{I'}}$. Extending the notation of Subsection 2.1 to the system of polynomials $G_1, G_2, \dots, G_{k+1}$, we see that $k(I) = k(I') + 1$, $G_i^{I,\alpha}(\mathbf{z}, z_{n+1}) = F_i^{I',\alpha'}(\mathbf{z})$ for $i \le k(I')$. Three cases are possible.

1. $\alpha\bigl(\frac{\partial}{\partial k_{n+1}}\bigr) > \min(\alpha'|_{\Delta_0^{I'}})$. Then $(C\Delta_0 \cap \mathbb{R}^I)^\alpha = \Delta_0^{I',\alpha'}$, $G_{k(I)}^{I,\alpha}(\mathbf{z}, z_{n+1}) = F_0^{I',\alpha'}(\mathbf{z})$.
2. $\alpha\bigl(\frac{\partial}{\partial k_{n+1}}\bigr) < \min(\alpha'|_{\Delta_0^{I'}})$. Then $(C\Delta_0 \cap \mathbb{R}^I)^\alpha = \{\frac{\partial}{\partial k_{n+1}}\}$, $G_{k(I)}^{I,\alpha} = -z_{n+1}$.
3. $\alpha\bigl(\frac{\partial}{\partial k_{n+1}}\bigr) = \min(\alpha'|_{\Delta_0^{I'}})$. Then $(C\Delta_0 \cap \mathbb{R}^I)^\alpha = C(\Delta_0^{I',\alpha'})$ is a cone of integer height 1 over $\Delta_0^{I',\alpha'}$, $G_{k(I)}^{I,\alpha}(\mathbf{z}, z_{n+1}) = F_0^{I',\alpha'}(\mathbf{z}) - z_{n+1}$.

Using the α'-nondegeneracy of the systems $F_0, F_1, \dots, F_k$ and $F_1, F_2, \dots, F_k$, we verify easily that the 1-forms $dG_i^{I,\alpha}$, $i = 1, 2, \dots, k(I)$, are linearly independent at the points of the set

$$\bigl\{(\mathbf{z}, z_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid G_1(\mathbf{z}, z_{n+1}) = G_2(\mathbf{z}, z_{n+1}) = \dots = G_{k+1}(\mathbf{z}, z_{n+1}) = 0\bigr\}. \qquad \square$$

Proposition 1 shows that Theorem 1 is applicable to the polynomials $G_1, G_2, \dots, G_{k+1}$:

$$\zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t) = \prod_{\alpha \in \mathcal{Z}_+^{I_0}} \Bigl(1 - t^{\alpha\left(\frac{\partial}{\partial k_{n+1}}\right)}\Bigr)^{n!\, Q_{k+1}^n((C\Delta_0)^\alpha, \Delta_1^\alpha, \dots, \Delta_k^\alpha)}, \tag{19}$$

where $I_0 = \{1, 2, \dots, n+1\}$. It is easily seen that in the above cases 1 and 2 the exponent $Q_{k+1}^n((C\Delta_0)^\alpha, \Delta_1^\alpha, \dots, \Delta_k^\alpha)$ equals 0. Therefore,

$$\zeta_{z_{n+1}, W \cap (\mathbb{C}^*)^{n+1}}(t) = \prod_{\alpha \in \mathcal{Z}_{\Delta_0}^{I_0'}} \bigl(1 - t^{m_{\Delta_0}(\alpha)}\bigr)^{n!\, Q_{k+1}^n(C(\Delta_0^\alpha), \Delta_1^\alpha, \dots, \Delta_k^\alpha)}, \tag{20}$$

where $I_0' = \{1, 2, \dots, n\}$. Now, formula (15) follows from (18), (20), and the identity

$$n!\, Q_{k+1}^n((C(\Delta_0^\alpha), \Delta_1^\alpha, \dots, \Delta_k^\alpha) = (n-1)!\, \widetilde{Q}_{k+1}^{n-1}(\Delta_0^\alpha, \Delta_1^\alpha, \dots, \Delta_k^\alpha),$$

which is a consequence of the following statement. $\square$

Proposition 2. *Let $\Delta_0, \Delta_1, \dots, \Delta_k$ be a set of integer polyhedra lying in a rational affine hyperplane $L \subset \mathbb{R}^{n+1}$. Let $C\Delta_0$ be the cone over Δ_0 with vertex at some point $v \in \mathbb{R}^{n+1}$ that lies at the integer distance 1 from the hyperplane L. Then*

$$(n+1)!\, Q_{k+1}^{n+1}(C\Delta_0, \Delta_1, \dots, \Delta_k) = n!\, \widetilde{Q}_{k+1}^n(\Delta_0, \Delta_1, \dots, \Delta_k). \tag{21}$$

Proof. We choose an affine integer coordinate system $\mathbf{k} = (k_1, k_2, \dots, k_{n+1})$ in the space $\mathbb{R}^{n+1}$ in such a way that $L = \{\mathbf{k} \in \mathbb{R}^n \mid k_{n+1} = 0\}$ and $v = (0, 0, \dots, 1)$. Choose Laurent polynomials $F_0, F_1, \dots, F_k$ in the variables $\mathbf{z} = (z_1, z_2, \dots, z_n)$ with fixed Newton polyhedra $\Delta_0, \Delta_1, \dots, \Delta_k$ in such a way that the systems $F_0, F_1, \dots, F_k$ and $F_1, F_2, \dots, F_k$

are nondegenerate in the sense of [9] with respect to their Newton polyhedra. One can easily show (see Proposition 1) that the system of Laurent polynomials $G_1, G_2, \dots, G_{k+1}$ in $n+1$ variables defined in terms of the polynomials $\{F_i\}$ by formulas (17) is also nondegenerate in the sense of [9] with respect to its Newton polyhedra $\Delta_1, \dots, \Delta_k, C(\Delta_0)$. We put

$$V = \{F_0 = F_1 = \dots = F_k = 0\} \subset (\mathbb{C}^*)^n,$$
$$V_1 = \{F_1 = F_2 = \dots = F_k = 0\} \subset (\mathbb{C}^*)^n,$$
$$W = \{G_1 = G_2 = \dots = G_{k+1} = 0\} \subset (\mathbb{C}^*)^{n+1}.$$

Applying the results of [10], we find the Euler characteristics of the sets V, V_1, W:

$$\chi(V) = n!\, Q^n_{k+1}(\Delta_0, \Delta_1, \dots, \Delta_k), \quad \chi(V_1) = n!\, Q^n_k(\Delta_1, \Delta_2, \dots, \Delta_k), \tag{22}$$

$$\chi(W) = (n+1)!\, Q^{n+1}_{k+1}(C\Delta_0, \Delta_1, \dots, \Delta_k). \tag{23}$$

Consider the projection $p\colon (\mathbb{C}^*)^{n+1} \to (\mathbb{C}^*)^n$ to the coordinate hyperplane with the coordinates $(z_1, z_2, \dots, z_n)$. Its restriction $p|_W$ provides an isomorphism between W and $V_1 \setminus V$. Therefore,

$$\chi(W) = \chi(V_1) - \chi(V).$$

Applying this relation and using (22), (23), and (14), we get (21). □

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