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CONWAY POLYNOMIAL AND MAGNUS EXPANSION

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Abstract. The Magnus expansion is a universal finite type invariant of pure braids with values in the space of horizontal chord diagrams. The Conway polynomial composed with the short-circuit map from braids to knots gives rise to a series of finite type invariants of pure braids and thus factors through the Magnus map. In the paper, the resulting mapping from horizontal chord diagrams on 3 strands to univariate polynomials is described explicitly and evaluated on the Drinfeld associator, which leads to a beautiful generating function whose coefficients are, conjecturally, alternating sums of multiple zeta values.

§1. Introduction

We assume that the reader is familiar with the fundamentals of knot theory, braid groups, and finite type (Vassiliev) invariants. All these preliminaries can be found, for instance, in [15].

The short-circuit closure of pure braids [10] induces a map from pure braids onto the set of (topological types of) oriented knots. Thus, any Vassiliev invariant of knots becomes a finite type invariant of pure braids. There is a universal finite type invariant of pure braids given by the Magnus expansion. In this paper, we explicitly describe the map from horizontal chord diagrams on 3 strands that is obtained by factoring the Conway polynomial (pulled back to pure 3-braids) through the Magnus expansion. The result is described by a peculiar combinatorial map from ordered partitions of an integer into nonordered partitions of the same integer.

In §2 we say introductory words about the group of pure braids and introduce the Magnus expansion. §3 is devoted to the construction of the short-circuit closure, relating braids to knots. In §4 we talk about the Conway polynomial of braids transferred from knots via short-circuit closure and state the main theorem, whose proof is given in §5. In §6, an attempt of evaluation at the Drinfeld associator is undertaken for the mapping obtained in the main theorem; also, we state the results of our computer calculations and the corresponding conjecture. Finally, in §7 we list some open problems related to the material of the paper.

I am indebted to Jacob Mostovoy who read the first version of the paper and made numerous useful remarks.

§2. Pure braids and Magnus expansion

Let P_m be the group of pure braids on m enumerated vertical strands with multiplication defined as the concatenation from top to bottom. This group is generated by the elements x_{ij}, $1 \le i < j \le m$, representing one full positive twist between the ith and the jth strands with all the remaining strands of the braid being strictly vertical and placed

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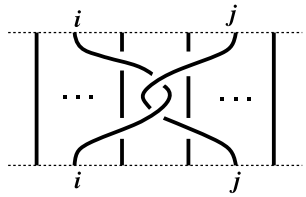
behind these two:

$x_{ij} =$ 

The defining relation between these generators can be found, for instance, in [2, 4]; we do not need them here.

There is a semidirect decomposition

$$P_m \cong F_{m-1} \ltimes \cdots \ltimes F_2 \ltimes F_1$$

(see [2]), where F_k is a free group on k generators, implemented in our case as the subgroup of P_m generated by the set $x_{1,k+1}, x_{2,k+1}, \dots, x_{k,k+1}$. This decomposition assures that any pure braid can uniquely be written in the *combed form* $\prod_s x_{i_s j_s}^{a_s}$ with $j_1 \geq j_2 \geq \cdots$, where the a_s are arbitrary nonzero integers and, in this product, no two identical generators follow each other (that is, the word is reduced).

The *Magnus expansion* is a map from P_m into the $\mathbb{Z}$-algebra of formal power series in $\binom{m}{2}$ noncommuting variables t_{ij}, $1 \leq i < j \leq m$, defined by

$$\mu_m(\beta) = \prod_s (1 + t_{i_s j_s})^{a_s}$$

if $\prod_s x_{i_s j_s}^{a_s}$ is the combed form of the braid β. Here the negative powers are understood as usual, in accordance with the rule $(1+t)^{-1} = 1 - t + t^2 - t^3 + \cdots$; this is why we need power series rather than merely polynomials in the construction of μ_m.

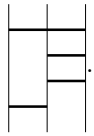
Example. To compute the value $\mu_3(x_{12}x_{23})$, first we find the combed form of this braid,

$$x_{12}x_{23} = x_{13}x_{23}x_{13}^{-1}x_{12},$$

and then write

$$\begin{aligned}\mu_3(x_{12}x_{23}) &= (1 + t_{13})(1 + t_{23})(1 - t_{13} + t_{13}^2 - \cdots)(1 + t_{12}) \\ &= 1 + t_{12} + t_{23} + t_{13}t_{23} - t_{23}t_{13} + t_{23}t_{12} + \cdots .\end{aligned}$$

From a broader perspective, it makes sense to view the codomain of the mapping μ_m as the completed quotient of the algebra $\mathbb{Z}[\{t_{ij}\}_{1 \leq i < j \leq m}]$ over the ideal generated by the elements $[t_{ij}, t_{kl}]$ and $[t_{ij}, t_{ik} + t_{jk}]$, where all indices are assumed to be distinct and t_{pq} is understood as t_{qp} if $p > q$. We denote this algebra by $\mathcal{A}^h(m)$ and view its generating monomials as horizontal chord diagrams on m vertical strands (each variable t_{ij} represents a chord connecting the ith and jth strands; the product of variables is understood as vertical concatenation from top to bottom). Example:

$t_{13}t_{23}^2t_{12} =$  .

We will denote the completion of $\mathcal{A}^h(m)$, i.e., the corresponding algebra of formal series, by $\widehat{\mathcal{A}}^h(m)$.

We say that a horizontal chord diagram is *descending* if it is represented by a monomial $\prod_s t_{i_s j_s}^{a_s}$ satisfying $j_1 \geq j_2 \geq \cdots$. By definition, the set of descending diagrams is in one-to-one correspondence with the set of positive combed braids P_m^+ (the braids whose combed form involves only positive powers of the generators x_{ij}). The set of descending

chord diagrams forms a basis of the free Abelian group $\mathcal{A}^h(m)$ (see Section 3–2 in [11])[1], so that we have a module isomorphism $\mathbb{Z}P_m^+ \cong \mathcal{A}^h(m)$.

By an *invariant of braids* we understand any mapping from the braid group P_m into an arbitrary set; we are interested only in its invariance under the braid isotopy, tacitly assumed in the definition of P_m, and not in the invariance under the renumbering of strands etc. For pure braids, as in the classical case of knots, the notion of finite type (Vassiliev) invariants, can be defined; see [1, 11, 3]. It turns out that the Magnus expansion truncated to any degree n is a Vassiliev invariant of order n. Moreover, the following theorem holds true.

Theorem ([11, 13, 3])**.** *The mapping $\mu_m : P_m \to \widehat{\mathcal{A}}^h(m)$ is a universal finite type invariant of pure braids in the sense that, for any degree n invariant $f : P_m \to \mathbb{Z}$, there exists a map $g : \widehat{\mathcal{A}}^h(m) \to \mathbb{Z}$ vanishing at all monomials of degree greater than n and such that $f = g \circ \mu_m$.*

Remark. In fact, a universal finite type invariant of pure braids can be defined, by sending each x_{ij} in the combed form into $1 + c_{ij}t_{ij} + T_{ij}$, where the c_{ij} are any nonzero constants and T_{ij} are arbitrary series starting with degree greater than one. A remarkable instance of this construction (with values in $\widehat{\mathcal{A}}^h(m) \otimes \mathbb{C}$) is provided by the Kontsevich integral ([1, 3]). Its advantage over the usual Magnus expansion consists in multiplicativity; however, the definition of the Kontsevich integral is much more involved and its computation is much more difficult; moreover, its value depends on the placement of the endpoints of the braid. For example, the Kontsevich integrals of the generating braids of the group P_3, where the strand endpoints are collinear and equidistant, are infinite series with the following terms up to degree 2:

$$\begin{aligned} I(x_{12}) &= 1 - A + \frac{1}{2}A^2 - \frac{i\ln 2}{2\pi}[B, C] + \cdots, \\ I(x_{13}) &= 1 - C + \frac{1}{2}C^2 + \frac{1}{2}[A, B] + \cdots, \\ I(x_{23}) &= 1 - B + \frac{1}{2}B^2 + \frac{i\ln 2}{2\pi}[C, A] + \cdots, \end{aligned}$$

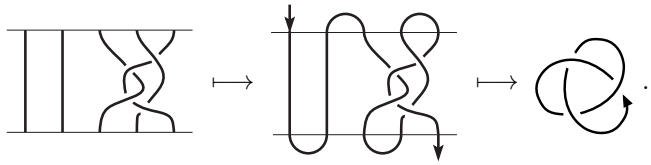
where $A = t_{12}, B = t_{23}, C = t_{13}$, and we recall that $[A, B] = [B, C] = [C, A]$ in accordance with the definition of $\widehat{\mathcal{A}}^h(3)$. The reader may wish, by way of exercise, to check that these relations agree with the commutation relations in the group P_3 (all they are expressed by the condition that the element $x_{12}x_{13}x_{23}$ is central, see [4]).

§3. Short-circuit closure

Alongside with the usual (Artin) closure that turns braids into links, there is another operation of closing pure braids into oriented knots, called *short-circuit closure*; see [10, 3]. It is defined by connecting pairwise by short arcs the upper endpoints number $2i$ and $2i + 1$ and the lower endpoints number $2i - 1$ and $2i$, which results in a long knot with two loose endpoints. Attaching an additional arc, one obtains a conventional compact knot. Orientation is chosen so that the leftmost strand of the braid is oriented

[1]In a more general setting, this fact easily follows from Theorem 3.1 in [13]. This can also be proved by using noncommutative Gröbner bases (I thank A. Khoroshkin for teaching me the idea of this proof). Closely related formulations are scattered in the work by T. Kohno, V. Drinfeld, An. Kirillov, S. Yuzvinsky, etc.

downwards. For example:



(Earlier, the nonoriented version of this operation, called *plat closure*, was introduced and studied by J. Birman [2].)

It is easily seen that the short-circuit closures of braids with different numbers of strands are consistent with the inclusions $P_m \to P_{m+1}$ (adding a vertical strand on the right), so that we obtain a well-defined map $\varkappa$ from the group $P_\infty := \bigcup_{m\geq 1} P_m$ to the set $\mathcal{K}$ of oriented knots. A theorem of Mostovoy and Stanford asserts that this map is onto and that it identifies $\mathcal{K}$ with double cosets of the group P_∞ over two special subgroups; see [10].

In the particular case where $m = 3$, the image of $\varkappa_3 = \varkappa\,|_{P_3}$ coincides with the set of all 2-bridge (rational) knots (see [12] for an introduction to rational knots). Indeed, the short-circuit closure of the braid $x_{13}^{a_1}x_{23}^{b_1}\dots x_{13}^{a_k}x_{23}^{b_k}$ (respectively, $x_{13}^{a_1}x_{23}^{b_1}\dots x_{13}^{a_k}x_{23}^{b_k}x_{13}^{a_{k+1}}$), where all a_i, b_i are nonzero integers, is the rational knot corresponding to the continued fraction with the denominators

$$(2a_1, -2b_1, \dots, 2a_k, -2b_k + 1)$$

(respectively,

$$(2a_1, -2b_1, \dots, 2a_k, -2b_k, 2a_k + 1)).$$

Now, a simple number-theoretic argument shows that any rational number with odd numerator and denominator has a continued fraction of this kind (the last number is odd, while all the previous ones are even). Rational knots, in distinction to links, correspond to rational fractions with odd denominators. If the numerator of the number p/q happens to be even, then the relevant knot is equivalent to the knot $(p \pm q)/q$ (see [12]); thus, our assertion is proved.

§4. Symbol of the Conway polynomial

By taking composition with the short-circuit closure, any invariant of knots can be converted into an invariant of braids. For example, the Conway polynomial of knots $\nabla : \mathcal{K} \to \mathbb{Z}[t]$ induces an invariant of pure braids $\nabla \circ \varkappa_m : P_m \to \mathbb{Z}[t]$ (the Conway polynomial of braids). For every n, the coefficient of t^{2n} in this polynomial is a Vassiliev invariant of degree $2n$. By the universality of the Magnus expansion, there is a mapping (the "symbol" of the Conway polynomial) $\chi_m : \widehat{\mathcal{A}}^h(m) \to \mathbb{Z}[t]$ such that $\nabla \circ \varkappa_m = \chi_m \circ \mu_m$. I succeeded in finding an explicit description of the symbol only for pure braids on 3 strands.

The following theorem describes the values of the mapping $\chi = \chi_3$ on descending chord diagrams (which form a basis of the free Abelian group $\mathcal{A}^h(3)$). Let $A = t_{12}$, $B = t_{23}$, $C = t_{13}$.

Theorem. *Any descending chord diagram on three strands is a* (*positive*) *word in the letters* A, B, C *where all* A*'s come at the end. We claim that, for any words* w, w_1, w_2,

(1) $\chi(wA) = 0$;
(2) $\chi(Bw) = 0$;
(3) $\chi(w_1B^2w_2) = 0$.

Assertions (1), (2), (3) *leave us with only two kinds of words*:

$$C^{c_1}B\dots C^{c_{k-1}}BC^{c_k}$$

and

$$C^{c_1}B\dots C^{c_{k-1}}BC^{c_k}B,$$

which we encode respectively by $[c_1,\dots,c_k]$ *and* $[c_1,\dots,c_k]'$.

(4) *The values of* χ *on the elements of the second kind are reduced to its values on the elements of the first kind*:

$$\chi([c_1,\dots,c_k]') = t^{-2}\chi([c_1,\dots,c_k,1]).$$

Thus, it remains to determine χ *on the elements* $[c_1,\dots,c_k]$.

(5) *We have*

$$\chi([c_1,\dots,c_k]) = (-1)^{k-1}\Big(\prod_{i=1}^{k-1} p_1 p_{c_i-1}\Big)\cdot p_{c_k},$$

where $p_s = \chi([s])$ *is a sequence of polynomials in* t *that can be defined recursively by* $p_0 = 1$, $p_1 = t^2$, *and* $p_{s+2} = t^2(p_s + p_{s+1})$ *for* $s \geq 0$. *In particular, the value of* χ *on the empty chord diagram* (*the unit of the algebra* $\widehat{\mathcal{A}}^h(3)$) *is* 1.

Remark 1. Note that the polynomial $p_k = \chi([k])$ is equal to $t^k\nabla(T_{k+1,2})$, where the letter T denotes the torus link with given parameters (in the case where this is a 2-component link, correct orientations of the components must be chosen), and can be written explicitly as

$$p_k = \sum_{k/2\leq j\leq k}\binom{j}{2j-k}t^{2j}.$$

Remark 2. The image of χ belongs to the commutative algebra generated by the polynomials p_1, p_2 etc., whose additive basis can be identified with (unordered) partitions. In this setting, the map χ is defined by the transformation of ordered partitions into unordered partitions by the rule

$$[c_1,\dots c_k] \mapsto (1^{k-1}, c_1-1,\dots,c_{k-1}-1,c_k).$$

Examples.

$$\begin{aligned}
\chi(1) &= 1,\\
\chi(B) &= 0,\\
\chi(C) &= t^2,\\
\chi(CB) &= -t^2,\\
\chi(BC) &= 0,\\
\chi(C^3BC^3) &= -p_1p_2p_3 = -t^2(t^4+t^2)(t^6+2t^4).
\end{aligned}$$

§5. Proof of the theorem

We must find a map χ rendering the following diagram commutative:

$$\begin{tikzcd}
P_3 \arrow[r, "\mu_3"] \arrow[d, "\varkappa"] & \widehat{\mathcal{A}}^h(3) \arrow[d, "\chi"] \\
\mathcal{K} \arrow[r, "\nabla"] & \mathbb{Z}[[t]]
\end{tikzcd}$$

We extend the Magnus expansion linearly to the map $\widehat{\mathbb{Z}P_3} \to \widehat{\mathcal{A}}^h(3)$ denoted by the same letter μ_3. We shall prove the theorem by finding a left inverse of μ_3, that is, a mapping $\nu_3 : \widehat{\mathcal{A}}^h(3) \to \widehat{\mathbb{Z}P_3}$ such that $\nu_3\circ\mu_3 = \mathrm{id}$. Indeed, the set of decreasing chord diagrams on 3 strands is in one-to-one correspondence with the set of positive braids P_3^+.

The correspondence is defined simply by setting $x_{ij} \leftrightarrow t_{ij}$. Identifying a word w in x_{ij} with the corresponding word in t_{ij}, we see that

$$\mu_3(w) = \sum_{w' \subseteq w} w'$$

for any positive word w. It is easy to check that the inverse of this map $\widehat{\mathbb{Z}P_3^+} \to \widehat{\mathcal{A}}^h(3)$ is given by the formula

$$\nu_3(w) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|} w',$$

where the absolute value of a word denotes its length (or total exponent). The diagram

$$\begin{tikzcd}
\widehat{\mathbb{Z}P_3} \arrow[r, shift left, "\mu_3"] \arrow[d, "\varkappa"] & \widehat{\mathcal{A}}^h(3) \arrow[l, shift left, "\nu_3"] \arrow[d, "\chi"] \\
\widehat{\mathbb{Z}\mathcal{K}} \arrow[r, "\nabla"] & \mathbb{Z}[[t]]
\end{tikzcd}$$

shows that we have $\chi = \nabla \circ \varkappa \circ \nu_3$ and, consequently,

$$\chi(w) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|} \nabla(\varkappa(w')),$$

where w' (a word in the letters t_{ij}) is understood, via the mentioned identification, as a word in the letters x_{ij}, that is, as a positive pure braid.

We will consecutively prove the five assertions of the theorem by applying this expression for χ and splitting the sum over all 2^n subwords $w' \subseteq w$ into appropriate subsums of $2, 4, \dots, 2^k$ terms.

(1) Split the sum into pairs $\pm\big(\nabla(\varkappa(w'A)) - \nabla(\varkappa(w'))\big)$ and notice that the knots $\varkappa(w'A)$ and $\varkappa(w')$ are isotopic.

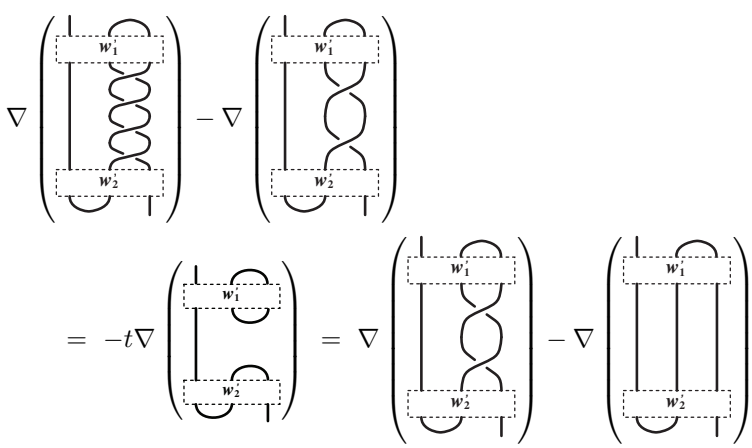
(2) The same argument for the pairs of knots $\varkappa(Bw')$ and $\varkappa(w')$.

(3) The sum giving $\chi(w_1 B^2 w_2)$ consists of quadruples defined by the choice of subwords $w_1' \subseteq w_1$, $w_2' \subseteq w_2$:

$$\pm\big(\nabla(\varkappa(w_1' B^2 w_2')) - 2\nabla(\varkappa(w_1' B w_2')) + \nabla(\varkappa(w_1' w_2'))\big).$$

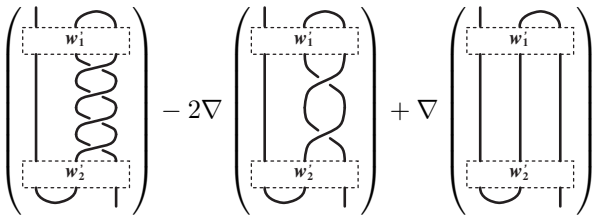
We prove that any such quadruple sums to zero.

Indeed, the defining skein relation for the Conway polynomial implies that



where the braids corresponding to the words w_1' and w_2' are depicted as rectangular boxes.

Therefore,



as required.

(4) We prove that for any word w we have $\chi(wBC) = t^2\chi(wB)$. Indeed,

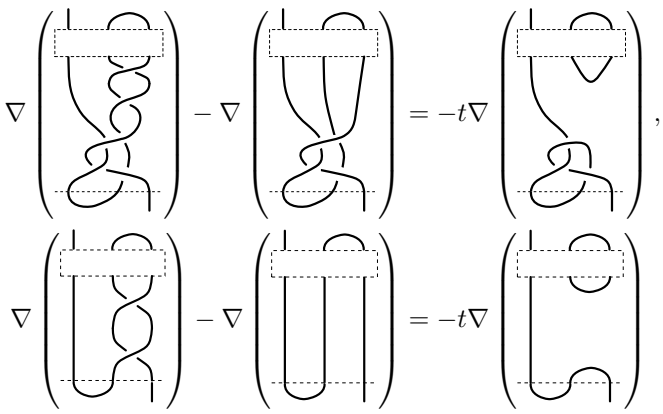
$$\chi(wB) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|}\big(\nabla(w'B) - \nabla(w')\big),$$

$$\chi(wBC) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|}\big(\nabla(w'BC) - \nabla(w'B) - \nabla(w'C) + \nabla(w')\big).$$

Now, the claim follows from the identity

$$\nabla(w'BC) - \nabla(w'C) = (t^2+1)(\nabla(w'B) - \nabla(w')),$$

which is proved by using the Conway skein relation:



combined with the observation that adding a trefoil knot to a link (which occurs in the pictures on the right) leads to multiplication of the corresponding polynomial by t^2+1.

(5) Here we must show that $\chi(C^nBw) = -p_1p_{n-1}\chi(w)$, where w is an arbitrary word in C and B. Indeed, let us split the alternating sum for $\chi(C^nBw)$ into the parts corresponding to a fixed subword $w' \subseteq w$:

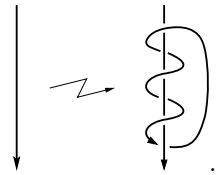
$$\chi(C^nBw) = \sum_{w' \subseteq w} (-1)^{|w|-|w'|} \sum_{l=1}^{n} (-1)^{n-l} \binom{n}{l} \big(\nabla\varkappa(C^lBw') - \nabla\varkappa(C^lw')\big).$$

Using the Conway skein relations on a proper crossing, we get:

$$\nabla\varkappa(C^lBw') - \nabla\varkappa(C^lw') \;=\; -t\nabla(K_{(l)}),$$

where $K = \varkappa(w')$, and $K_{(l)}$ denotes the oriented 2-component link obtained from the oriented knot K by adding a trivial l-linked component to K in accordance with the

picture (drawn for $l = 3$):



Lemma. *For any knot K and any natural number l, we have*

$$\nabla(K_{(l)}) = t(q_0 + q_1 + \cdots + q_{l-1})\nabla(K),$$

where q_s is the Conway polynomial of the torus knot of type $(2, 2s+1)$ given explicitly by

$$q_s = \sum_{j=0}^{s} \binom{s+j}{s-j} t^{2j}.$$

This identity is proved, as usual, by recursively applying the Conway skein relation. Substituting it in the previous formula for $\chi(C^n Bw)$, we get

$$-t^2 \cdot \sum_{l=1}^{n} (-1)^{n-l} \binom{n}{l} \sum_{s=0}^{l-1} q_s \cdot \sum_{w' \subseteq w} (-1)^{|w|-|w'|} \nabla(\varkappa(w')),$$

and it remains to show that the middle term of this product is equal to p_{n-1}. Indeed, this term is easily transformed to

$$\sum_{s=0}^{n-1} (-1)^{n-1-s} \binom{n-1}{s} q_s$$

or, recalling the expression for q_s, to

$$\sum_{s=0}^{n-1} \sum_{j=0}^{s} (-1)^{n-1-s} \binom{n-1}{s} \binom{s+j}{s-j} t^{2j}.$$

Changing the order of summation, we can rewrite this as

$$(-1)^{n-1} \sum_{j=0}^{n-1} \Big[\sum_{s=j}^{n-1} (-1)^s \binom{n-1}{s} \binom{s+j}{2j} \Big] t^{2j}.$$

An application of the product summation formula from [7, (5.24)] to the sum over s inside the brackets gives $(-1)^{n-1} \binom{j}{2j-n+1}$, thus proving the required assertion.

Remark. The coefficients of the polynomials p_n and q_n can be read off the Pascal triangle as follows:

	q_0	q_1	q_2	q_3	q_4	q_5	q_6
p_0	**1**						
p_1	**1**	**1**					
p_2	**1**	**2**	**1**				
p_3	**1**	**3**	**3**	**1**			
p_4	**1**	**4**	**6**	**4**	**1**		
p_5	**1**	**5**	**10**	**10**	**5**	**1**	
p_6	**1**	**6**	**15**	**20**	**15**	**6**	**1**

§6. Evaluation at the associator

The Drinfeld associator [4, 9], a remarkable element of the algebra $\widehat{\mathcal{A}}^h(3)\otimes\mathbb{C}$, is given by an infinite series in the (noncommuting) variables $a = A/(2\pi i)$, $b = B/(2\pi i)$ with coefficients in the algebra of multiple zeta values (MZV; see [8]):

$$\begin{aligned}\Phi = 1 &- \zeta_2[a,b] - \zeta_3([a,[a,b]] + [b,[a,b]]) \\ &- \zeta_4[a,[a,[a,b]]] - \zeta_{3,1}[b,[a,[a,b]]] - \zeta_{2,1,1}[b,[b,[a,b]]] + \frac{1}{2}\zeta_2^2[a,b]^2\cdots\end{aligned}$$

(see [5] for an explicit expansion of Φ up to degree 12).

Taking the value of the complexification mapping $\chi_{\mathbb{C}} : \widehat{\mathcal{A}}^h(3)\otimes\mathbb{C} \to \mathbb{C}[t]$ at this element Φ, we obtain the following result.

Conjecture.

$$\begin{aligned}\chi_{\mathbb{C}}(\Phi) &= -\zeta_2 T^2 + (-\zeta_3 + \zeta_{2,2})T^4 + (-\zeta_4 + \zeta_{2,3} + \zeta_{3,2} - \zeta_{2,2,2})T^6 + \cdots \\ &= \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(-1)^k\zeta_{n+k}^{(k)}\Big)T^{2n},\end{aligned}$$

where $T = t/(2\pi i)$, the numbers $\zeta_{l_1,\dots,l_k} = \zeta(l_1,\dots,l_k)$ are multiple zeta values, and $\zeta_m^{(k)}$ is the short-hand notation for the sum of all $\zeta_{l_1,\dots,l_k}$, where $l_i \geq 2$ for $i = 1,\dots,k$ and $l_1 + l_2 + \cdots + l_k = m$.

We have checked this formula by computer up to T^{10} (see [5]) using the table of relations between MZV's provided in [14]. It is an interesting remark that, when evaluated numerically, the coefficients of this polynomial

$$\begin{aligned}&- 1.644934\,T^2 - 0.390314\,T^4 - 0.332698\,T^6 - 0.312405\,T^8 \\ &- 0.303958\,T^{10} - 0.300153\,T^{12} - 0.298365\,T^{14} - 0.297505\,T^{16} + \cdots\end{aligned}$$

seem to tend to a limit whose nature remains unclear.

§7. Open problems

(1) For what triples of reduced rational fractions p_1/q_1, p_2/q_2, p_3/q_3, where both numerators and denominators form arithmetical progressions, do the values of the Conway polynomial at the corresponding rational knots also form an arithmetical progression? Our proof of assertion 3 of the main theorem gives an abnormally big number of such triples, for instance, $(\frac{3}{11}, \frac{13}{31}, \frac{23}{51})$, $(\frac{5}{11}, \frac{9}{19}, \frac{13}{27})$, $(\frac{5}{17}, \frac{11}{41}, \frac{17}{65})$, $(\frac{13}{75}, \frac{19}{111}, \frac{25}{147})$, yet the claim about arithmetical progressions is not true in general.

(2) Generalize the main theorem (§4) in two directions: (A) to pure braids with an arbitrary number of strands, (B) to the HOMFLY polynomial, which is a generalization of the Conway polynomial.

(3) This is related to the remark at the end of the Introduction. Describe all (or some) triples of formal series P, Q, R in $\widehat{\mathcal{A}}^h(3)\otimes\mathbb{C}$ starting with terms of degree higher than one, such that the correspondence $x_{12} \mapsto 1+t_{12}+P$, $x_{23} \mapsto 1+t_{23}+Q$, $x_{13} \mapsto 1 + t_{13} + R$ defines a group homomorphism.

(4) Prove the conjecture in §6. Find similar facts for other horizontal associators and relate them to the action of the Grothendieck–Teichmüller group (see [4, 6]).

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