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WIDTH OF GROUPS OF TYPE E_6 WITH RESPECT TO ROOT ELEMENTS. I

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Abstract. Simply connected and adjoint groups of type E_6 over fields are considered. Let K be a field such that every polynomial of degree not exceeding six has a root. It is shown that any element of the adjoint group of type E_6 over K can be expressed as a product of at most eight root elements.

Introduction

This work is a sequel to the paper [22]. Here is the main result of the present paper.

Theorem. *Suppose that every nonconstant polynomial of degree not exceeding* 6 *over* K *has a root in* K. *Then any element of the group* $G_{\mathrm{ad}}(\mathrm{E}_6, K)$ *can be expressed as a product of at most* 8 *root elements.*

Recall that the width of a group G with respect to a generating set S is either the minimal natural number n such that any element of G is a product of at most n elements of S, or ∞ if there is no such n. This definition can be extended to the notion of the width of a group G with respect to an arbitrary set $S \subset G$. In that case, the width of $\langle S\rangle$ with respect to S is called the width of G with respect to S. This definition can be useful if our choice of the set S is closely related to the group G and has little relation to $\langle S\rangle$. Probably the most well-known set S in this context is the set of commutators. The width of a group G with respect to commutators is usually denoted by $c(G)$.

1. Width of a group with respect to commutators. For the first time, the commutator width of groups was considered by Shoda; in 1936, he proved that $c(\mathrm{GL}(n, F)) = 1$ for any algebraically closed field F. Later, in 1951, he also proved that $c(\mathrm{GL}(n, F)) \leq n$ for any infinite field F. Tôyama and Goto showed that $c(G) = 1$ for any connected compact semisimple Lie group G. In 1951, Ore [89] proved that the commutator width of any symmetric group (finite or infinite) is at most 1. He also proved that $c(\mathrm{A}_n) = 1$ for $n \geq 5$ and conjectured that $c(G) \leq 1$ for any finite simple group G ("Ore conjecture").

In 1954–1955, Griffiths studied commutators in the free product $G = G_1 * G_2 * \cdots * G_n$ of finitely presented groups G_i; he showed that $c(G) \geq n$ if $[G_i, G_i]$ is nontrivial for every i. Later, Goldstein and Turner [62] proved that in fact $c(G) \geq \sum c(G_i)$. In 1963, Macdonald began studying the groups G with cyclic commutator subgroup $[G, G]$. In [80], he proved that $c(G) \leq m/2$ if G is nilpotent and $[G, G]$ is cyclic of order m. He also showed that finite groups G with cyclic commutator subgroup can have arbitrarily large width. The investigation of groups with cyclic commutator subgroups continued in publications of Rodney [92], Liebeck [79], Isaacs [73], Guralnick [66, 67, 68], and many others. The width of operator groups was studied by de la Harpe and Skandalis

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[70, 71, 72], and by Brown and Pearcy [36]; commutator subgroups of diffeomorphism groups were considered by McDuff [87], Mather [82]–[85], and Epstein [59].

In [112], Wood showed that $c(G) = \infty$ if G is the universal covering group of $\mathrm{SL}(n, \mathbb{R})$. The group $\mathrm{SL}(n, A)$ for a ring A of continuous functions on a topological space was investigated in the paper by Thurston and Vaserstein [99], and in Vaserstein's own papers [101, 102]. Later, the group $\mathrm{SL}(n, A)$ for arbitrary principal ideal domains was studied by Newman [88], Dennis and Vaserstein [46], and Vaserstein and Wheland [103]. Newman [88] proved that $c(\mathrm{SL}(n, A)) \leq (2 \log n)/\log \frac{3}{2} + c(\mathrm{SL}(3, A))$ for any principal ideal domain A. He conjectured that $c(\mathrm{SL}(3, A))$ is always finite; Dennis and Vaserstein disproved this conjecture, showing that $A = \mathbb{C}[x]$ is a counterexample. Notice that, as was showed by Sivatski and Stepanov in [94], this result in fact follows from the paper [77], published in 1982. There van der Kallen proved that the width of $E(n, \mathbb{C}[x])$ with respect to the set of all elementary transvections is infinite, while Sivatski and Stepanov showed in 1999 that the set of all commutators in $E(n, R)$ has finite width with respect to the set of all elementary transvections for any finite-dimensional commutative ring R. Moreover, Dennis and Vaserstein proved in [46] that $c(\mathrm{SL}(n, A)) \leq 5 + c(\mathrm{SL}(3, A))$, and that $c(\mathrm{SL}(n, A)) < \infty$ implies $c(\mathrm{SL}(n, A)) \leq 6$ for sufficiently large n. They obtained similar results for arbitrary associative rings of finite stable rank. This boundary was improved from 6 to 4 by Vaserstein and Wheland.

2. Width of a group with respect to involutions. Another interesting generating set is the set of all involutions. It is well known (see, for example, [111]) that any element of the group G of isometries of a quadratic form over a field is a product of two involutions in G. In 1976, Gustafson, Halmos, and Radjavi proved in [69] that any matrix in $\mathrm{GL}(n, K)$ with determinant ± 1 is a product of four involutory matrices (see also [35]). In [78] Knüppel and Nielsen proved that any element of $\mathrm{SL}(n, K)$ is a product of four involutory matrices in $\mathrm{SL}(n, K)$. In 1999, Austin proved (see [34]) that every element of a Chevalley group $G = G(\mathrm{F}_4, K)$ is a product of four involutions in G if $|K| \geq 25$. In 2000, Ellers [54] proved that if G is a Chevalley group over a field K containing sufficiently many elements, then any noncentral element of G is a product of four involutions in G, while any central element is a product of five involutions in G. Involutions in Chevalley groups over rings are also considered sometimes; for instance, Röpcke [93] proved that if R is a local ring with $2 \in R^*$, V is a free R-module of rank n, f is a regular symmetric bilinear form, and G is a group of isometries of f, then every element of G is a product of four involutions in G.

The width of a group with respect to involutions is closely related to the problem of determining the minimal number of involutions that generate a group in question. In 1978, Wagner [109] proved that four involutions are needed to generate $\mathrm{U}(3, 3)$. In [61], Gillio and Tamburini proved that the alternating groups, the linear group, the symplectic groups of dimension at least six, and the Suzuki groups are generated by three involutions. It is proved that the groups $G = \mathrm{PSp}(4, K)$ for any finite field K and $\mathrm{U}(3, q)$ for $q \neq 3$ can be generated by three involutions; see [44] and [109].

Dalla Volta and Tamburini [45, 42, 43] proved a similar proposition for orthogonal groups. For simple Chevalley groups over finite fields of characteristic 2, with the exception of A_2, ${}^2\mathrm{A}_2$, A_3, and ${}^2\mathrm{A}_3$, the same estimate was obtained by Nuzhin in [16]; for the sporadic groups this was done by Dalla Volta in [41], and for the most of exceptional groups by Weigel in [110]. Finally, in 1994, Malle, Saxl, and Weigel [81] proved that *every* finite simple non-Abelian group G distinct from $\mathrm{U}(3, 3)$ is generated by three involutions (note that two involutions can generate a dihedral group only).

In [12], Mazurov posed the question of describing the finite simple groups generated by three involutions such that two of them commute. This question turned out to be closely related to the problem of finding a Hamiltonian cycle in a Cayley graph of the group under study. Nuzhin answered this question in [16]–[19] for simple alternating groups and for simple groups of Lie type. Later, Timofeenko, Nuzhin, and other authors investigated sporadic groups with the help of computers. It turned out that a sporadic group G is not generated by three involutions two of which commute if and only if $G = M_{11}, M_{22}, M_{23}$, or McL. In 2003, Mazurov [15] used character tables to give a unified proof for all sporadic groups. Tamburini and Zucca [98] obtained some very interesting results concerning matrix groups over arbitrary finitely generated commutative rings.

3. Width of Chevalley groups with respect to root elements. Another natural set of generators for Chevalley groups is the set of root elements. Since the root elements in $G_P(\phi, R)$ generate the elementary subgroup $E_P(\Phi, R)$, which, in general, does not coincide with $G_P(\Phi, R)$, the width of a Chevalley group with respect to root elements is defined as the width of its elementary subgroup. We already mentioned a similar situation in the case of commutators.

Dieudonné computed the width of classical groups with respect to root elements in his remarkable paper [49]. Later, Dieudonné's results were extended to groups that preserve quadratic forms with nontrivial radical (see, e.g., [95]). Dieudonné also studied the width of classical groups with respect to reflection symmetries, i.e., involutions that fix some hyperplane. These symmetries are closely related to both involutions and root elements; after Dieudonné they were considered by Götzky, Ishibashi, Ellers, and many others (see [50]–[53], [55]–[58], [64, 65, 74, 75, 76, 97, 113], and also [20, 21]).

Unlike the classical case, the width of *exceptional* groups with respect to root elements was poorly investigated; the only exceptions are the natural lower bounds. We explain the origin of these lower bounds.

Recall that the rank of the matrix $A - E$ is called the residue of A. It is easily seen that the residue is invariant under conjugation, and that the residue of a product of matrices is not greater than the sum of their residues. These properties easily imply the lower bounds for the width of Chevalley groups. For instance, we are interested in the case of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ in the minimal, 27-dimensional representation. The residue of a root element is equal to 6, while $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ contains elements of residue 27; hence, the width of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is at least 5. Curiously, for classical groups similar bounds coincide with the answer found in [49], so it is natural to believe that this is the case also for exceptional groups (however, it seems that the width of groups of type E_6 is equal to 6, not 5). Recently, Cohen, Steinbach, Ushirobira, and Wales proved that $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is generated by five root subgroups. This result implies that the width of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is at most 10. No other known bounds for the width of exceptional groups have been known until the present time.

We say that a field K is *k-closed* if every polynomial of degree not exceeding k over K has a root in K (unfortunately, we could not find a generally accepted name for such fields). Let $G = G_{\mathrm{ad}}(\mathrm{E}_6, K)$, where K is a 6-closed field. We have already mentioned that our main goal is to prove that the width of G is at most 8. However, this bound is not sharp, and we plan to improve it to 7 (and, most likely, to 6). We also plan to prove a similar result for the simply connected $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ and to generalize these results to an arbitrary field. The natural course of transferring our proof to the case of 2-closed field presents some difficulties in the second part of §4; presumably, the Frobenius normal form should be used instead of the Jordan normal form. The case of an arbitrary field is even harder. That is why in the present paper we do not struggle to lift the restriction on the field K.

The paper is organized as follows. In §1 we introduce the basic notation and describe the necessary results of [22]. In §2 we consider some subgroups of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$. The first part of §2 is concerned with diagonal matrices, in the second part we deal with matrices from the group $D_\alpha = \langle X_\beta; \beta \perp \alpha \rangle$, and in the third part with matrices from the parabolic subgroup. In the fourth part of §2 we consider matrices from the Levi subgroup; we describe them in Theorem 1. We start the investigation of the unipotent radical in the fifth part of §2. In §3 we continue with the unipotent radical and prove Theorem 2, which asserts that every unipotent is a product of at most three root elements. Products of matrices are considered in §4. Let $\overline{g} = \{g_{ij}\}$, $1 \le i \le 6$, $22 \le j \le 27$, be the (6×6)-matrix located at the top right corner of a matrix $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$. We prove in §4 that if K is a 6-closed field and $A \in \mathrm{GL}(6, K)$, then there exists a matrix $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $\overline{g} = A$ and g is a product of at most 4 root elements (the corollary to Theorem 5). In §5 we prove Theorem 6: for any noncentral $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$, there exists $h \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that the submatrix $\overline{hgh^{-1}}$ is invertible. Finally, in §6 we prove our main theorem: for a 6-closed field K, any element of $G_{\mathrm{ad}}(\mathrm{E}_6, K)$ is a product of at most 8 root elements.

Notice that most of the facts of the first few sections are either well known to specialists, or follow easily from classical and/or well-known facts, or, at the worst, are perfectly natural and even intuitively obvious, although their proof can take some time. Many of these facts were more than once mentioned by different authors. Nevertheless, it is quite hard to find the one who was first to explicitly state such an easy fact, and virtually impossible to pinpoint the one who discovered it. Next, it is often easier to prove some claim from scratch than to show that it is equivalent to a known fact. Because of that, we have taken the liberty to omit the authors of some easy facts that we prove.

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§1. Basic notions

The present paper is a sequel to [22], and all our notation and terminology pertaining to Chevalley groups, Weyl modules, and trilinear forms, is preserved. We have taken the liberty not to reiterate §1 of [22], which describes these things in detail. We need to recall some general facts not included in [22]. Let $\Phi = \mathrm{E}_6$. There are two possibilities for $G = G_P(\Phi, R)$, namely, the adjoint group $G_{\mathrm{ad}}(\mathrm{E}_6, R) = G_{P(\Phi)}(\mathrm{E}_6, R)$ and the simply connected group $G_{\mathrm{sc}}(\mathrm{E}_6, R) = G_{Q(\Phi)}(\mathrm{E}_6, R)$. The adjoint group is the scheme-theoretic quotient of the simply connected one by the constant scheme μ_3. Therefore, if every polynomial of degree at most 3 over K has a root in K, then the adjoint group is the quotient of the simply connected one by its center, which is isomorphic to the group μ_3 of cubic roots of 1: $G_{\mathrm{ad}}(\mathrm{E}_6, K) \cong G_{\mathrm{sc}}(\mathrm{E}_6, K)/\mu_3$. We shall use this fact later.

We make heavy use of the results of [22]. For convenience, here we list the basic results and notation from [22]. Propositions from [22] will be cited in square brackets, e.g., [22, Prop. 6]. From now on, K denotes an arbitrary field if it is not stated otherwise.

Recall that δ denotes the maximal root of E_6, i.e., $\delta = {}^{12321}_{\quad 2}$; $x_\alpha(a)$ denotes the elementary root element corresponding to $\alpha \in \mathrm{E}_6$, $a \in K$, and $X_\alpha = \big\{x_\alpha(a) \mid a \in K\big\}$ denotes the unipotent root subgroup. Let $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$, and let $\widetilde{W}$ denote the extended Weyl group $\langle w_\alpha(1); \alpha \in \Phi \rangle$. It is well known that $w_\alpha(1)x_\beta(a)w_\alpha(1)^{-1} = x_{w_\alpha\beta}(\pm a)$ for all $\alpha, \beta \in \Phi = \mathrm{E}_6$ and $a \in K$. As in [22], we put

$$I_1^\alpha = \{\rho; \rho, \rho - \alpha \in \Lambda\}, \quad I_2^\alpha = \{\rho; \rho \in \Lambda, \rho \pm \alpha \notin \Lambda\}, \quad \text{and} \quad I_3^\alpha = \{\rho; \rho, \rho + \alpha \in \Lambda\}.$$

It is easy to check that conjugation with respect to w_β maps the sets I_1^α, I_2^α, I_3^α to the sets $I_1^{w_\beta(\alpha)}, I_2^{w_\beta(\alpha)}$, $I_3^{w_\beta(\alpha)}$, respectively. Also, we put $I_1 = I_1^\delta, I_2 = I_2^\delta$, $I_3 = I_3^\delta$. In other words, $I_1 = \{i; 1 \le i \le 6\}, I_2 = \{i; 6 < i < 22\}$, and $I_3 = \{i; 22 \le i \le 27\}$.

Definition. Let ρ, σ be two distinct weights. The minimal number of roots with sum equal to $\rho - \sigma$ is called the distance between ρ and σ and is denoted by $d(\rho, \sigma)$. We put $d(\rho, \sigma) = 0$ for $\rho = \sigma$.

It is known (see [22, Prop. 5]) that in our case of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ the distance between weights can be equal to 0, 1, or 2. We say that weights within distance 1 are *adjacent*, and weights within distance 2 are *nonadjacent*. Triples of pairwise nonadjacent weights are called triads. Following Aschbacher, by a 3-form $\mathfrak{F}$ we mean a triple (T, Q, F), where T is a cubic form, Q is its partial polarization, and F is its full polarization.

Definition. A vector v is said to be *singular* (with respect to a 3-form $\mathfrak{F}$) if $Q(x, v) = 0$ for any vector x. A subspace is *singular* if each vector in it is singular. We put the distance $d(u, v)$ between two distinct singular vectors u and v to be 1 if $u - v$ is singular, and 2 otherwise. In the former case we say that these vectors are *adjacent*, whereas in the latter case we say that they are *nonadjacent*. If $u = v$, we say that the distance between them is equal to 0.

Every matrix $A \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ preserves F and, therefore, maps singular vectors to singular vectors, and nonsingular vectors to nonsingular vectors. Hence, A preserves distance between vectors.

We mentioned in [22] that if g is a root element, then $g - E$ lies in the Lie algebra; therefore, it can be represented in the Chevalley basis. We introduce a convenient notation.

Definition. Let g be a root element, and let $\alpha \in \Phi = \mathrm{E}_6$. The coefficient of the vector e_α in the expansion of $g - E$ with respect to the Chevalley basis is called the coefficient of the root element g (or of the matrix g) at the root α and is denoted by $(\alpha)_g$.

Let g be a root element. Denote the six-dimensional subspace $\mathrm{Im}(g - E)$ by V^g. In other words, V^g is the subspace spanned by the columns of the matrix $g - E$. Similarly, the six-dimensional subspace of V^* spanned by the rows of the matrix $g - E$ is denoted by V_g.

It is easy to check that the subspace V^g is singular. The corollary to Theorem 2 shows that the converse is also true; for any six-dimensional singular subspace W there exists a root element g such that $V^g = W$.

One of the most important concepts in the theory of root subgroups is the notion of the angle between root subgroups. Some well-known facts concerning it were listed in [22] with new proofs. In particular, the following proposition relates the notion of the angle between root subgroups to six-dimensional singular subspaces.

Proposition 12 of [22]. *Let g, h be two root elements. Then precisely one of the following cases occurs*:

(1) *if* $V^g = V^h$, *then* $\angle(g, h) = 0$;
(2) *if* $\dim(V^g \cap V^h) = 3$, *then* $\angle(g, h) = \pi/3$;
(3) *if* $\dim(V^g \cap V^h) = 1$, *then* $\angle(g, h) = \pi/2$;
(4) *if* $V^g \cap V^h = 0$ *and there exists a six-dimensional singular subspace W such that* $\dim(V^g \cap W) = \dim(V^h \cap W) = 3$, *then* $\angle(g, h) = 2\pi/3$;
(5) *if* $V^g \cap V^h = 0$ *and for any $v \in V^g$ there exists a unique* (*up to a scalar factor*) *vector $u \in V^h$ such that $v + u$ is singular, then* $\angle(g, h) = \pi$.

We shall use this proposition later.

§2. A few subgroups

Here we consider some subgroups of the group $G = G_{\mathrm{sc}}(\mathrm{E}_6, K)$, which will be needed in the sequel, and establish some general properties of their elements.

1. Subgroup of diagonal matrices. Let $T = \{A \in G_{\mathrm{sc}}(\mathrm{E}_6, K);\ i \neq j \Leftrightarrow A_{ij} = 0\}$ be the subgroup of diagonal matrices.

Lemma 2.1. *Let A be an arbitrary diagonal matrix of size 27×27. Then $A \in T$ if and only if $A_{\rho\rho}A_{\sigma\sigma}A_{\tau\tau} = 1$ for any triad $\{\rho, \sigma, \tau\}$.*

Proof. As was noted in [22], the group $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is the group of isometries of a trilinear form F. In particular, if $A \in T$, then $F(e^\rho, e^\sigma, e^\tau) = F(Ae^\rho, Ae^\sigma, Ae^\tau) = A_{\rho\rho}A_{\sigma\sigma}A_{\tau\tau}F(e^\rho, e^\sigma, e^\tau)$, which proves one implication, because $F(e^\rho, e^\sigma, e^\tau) \neq 0$. Now we need to prove that $F(x, y, z) = F(Ax, Ay, Az)$ for all x, y, z. By linearity, it suffices to show this for the case where x, y, z are basis vectors. By assumption, equality occurs when the weights corresponding to x, y, z form a triad. Otherwise, both sides are zero. Hence, $F(x, y, z) = F(Ax, Ay, Az)$ for all x, y, z. □

Recall that (by [22, Prop. 6]) for any weight in I_2^α there exist exactly two weights I_1^α that are nonadjacent to it. In the present section, we denote these weights by adding lower indices 1 and 2 to the weight from I_2^α; so if $\rho \in I_2^\alpha$, then $\rho_1, \rho_2 \in I_1^\alpha$ and $d(\rho, \rho_1) = d(\rho, \rho_2) = 2$.

Proposition 1. *Let A be an arbitrary invertible diagonal matrix of size 27×27. Then $A \in T$ if and only if there exists $d_A \in K$ such that*

1) $d_A^3 = \prod_{\phi \in I_1} A_{\phi\phi}$,
2) $A_{\phi\phi} = \frac{A_{\phi+\delta,\phi+\delta}}{d_A}$, *where* $\phi \in I_3$,
3) $A_{\phi\phi} = \frac{d_A}{A_{\phi_1,\phi_1}A_{\phi_2,\phi_2}}$, *where* $\phi \in I_2$.

Proof. First, we prove that $A \in T$ if such a scalar d_A exists. By Lemma 2.1, it suffices to show that $A_{\rho\rho}A_{\sigma\sigma}A_{\tau\tau} = 1$ for an arbitrary triad $\{\rho, \sigma, \tau\}$. By [22, Prop. 5], we have either $\rho, \sigma, \tau \in I_2$, or, up to renumbering, $\rho \in I_1, \sigma \in I_2$, and $\tau \in I_3$. In the former case, we can apply [22, Prop. 6] to obtain

$$A_{\rho\rho}A_{\sigma\sigma}A_{\tau\tau} = \frac{d_A}{A_{\rho_1,\rho_1}A_{\rho_2,\rho_2}} \frac{d_A}{A_{\sigma_1,\sigma_1}A_{\sigma_2,\sigma_2}} \frac{d_A}{A_{\tau_1,\tau_1}A_{\tau_2,\tau_2}} = \frac{d_A^3}{\prod_{i \in I_1} A_{ii}} = 1.$$

In order to deal with the latter case ($\rho \in I_1, \sigma \in I_2$, and $\tau \in I_3$), we recall that, by [22, Prop. 6], the weight σ is nonadjacent to both ρ and $\tau + \delta$; therefore, up to reordering, $\rho = \sigma_1$ and $\tau + \delta = \sigma_2$. Hence,

$$A_{\rho\rho}A_{\sigma\sigma}A_{\tau\tau} = A_{\sigma_1,\sigma_1} \frac{d_A}{A_{\sigma_1,\sigma_1}A_{\sigma_2,\sigma_2}} \frac{A_{\sigma_2,\sigma_2}}{d_A} = 1.$$

Now we prove that $A \in T$ implies the existence of such a scalar d_A. Consider the triads $\{\rho, \rho_1, \rho_2 - \delta\}$ and $\{\rho, \rho_2, \rho_1 - \delta\}$, where $\rho \in I_2$. Since $A \in T$, we have

$$A_{\rho\rho}A_{\rho_1\rho_1}A_{\rho_2-\delta,\rho_2-\delta} = A_{\rho\rho}A_{\rho_2\rho_2}A_{\rho_1-\delta,\rho_1-\delta} = 1. \tag{2.1}$$

By [22, Prop. 6], we have $A_{\phi-\delta,\phi-\delta} = kA_{\phi\phi}$, where $k \in K$ is one and the same for all $\phi \in I_1$. Using (2.1), we obtain $A_{\rho\rho} = \frac{1}{kA_{\rho_1,\rho_1}A_{\rho_2,\rho_2}}$ for an arbitrary weight $\rho \in I_2$. Now, let $\{\rho, \sigma, \tau\}$ be a triad such that $\rho, \sigma, \tau \in I_2$. We have

$$A_{\rho\rho}A_{\sigma\sigma}A_{\tau\tau} = \frac{1}{kA_{\rho_1,\rho_1}A_{\rho_2,\rho_2}} \frac{1}{kA_{\sigma_1,\sigma_1}A_{\sigma_2,\sigma_2}} \frac{1}{kA_{\tau_1,\tau_1}A_{\tau_2,\tau_2}}.$$

By [22, Prop. 6], this is equal to $\frac{1}{k^3 \prod_{\phi \in I_1} A_{\phi\phi}}$. Hence, we can take $d_A = \frac{1}{k}$. □

2. The subgroup D_α. Let $\alpha \in E_6$ be an arbitrary root. Let $D_\alpha = \langle X_\beta; \beta \perp \alpha\rangle$. By [22, Prop. 4], our space can be decomposed into a direct sum of three D_α-invariant subspaces: $V_k^\alpha = \langle e^\phi; \phi \in I_k^\alpha\rangle$, where $1 \le k \le 3$. Consider the restriction of D_α to V_k^α. Again by [22, Prop. 4], every X_β becomes a transvection after restricting to V_1^α. Therefore, the restriction of D_α to V_1^α coincides with $\mathrm{SL}(V_1^\alpha, K)$. A similar statement is valid for the restriction to V_3^α.

Proposition 2. *Suppose that $A \in D_\alpha$, $\phi \ne \psi \in I_1^\alpha$. Then $A_{\phi-\alpha,\phi-\alpha} = A_{\phi\phi}$ and $c_{\phi-\alpha,\psi-\alpha}A_{\phi-\alpha,\psi-\alpha} = c_{\phi\psi}A_{\phi\psi}$.*

Proof. By the definition of D_α, the matrix A is the product of n elementary root elements. The proof is by induction on n. If A is the identity, there is nothing to prove. We multiply a matrix A that satisfies the inductive assumption by an elementary root element $x_\beta(a)$ for some $\beta \perp \alpha$. Let $B = x_\beta(a)A$. If $\phi - \beta \notin \Lambda$, the coefficients in the ϕth row do not change. Applying [22, Prop. 4], we see that $\phi - \alpha - \beta \notin \Lambda$. Hence, the coefficients in the $(\phi-\alpha)$th row do not change either. Therefore, we may assume that $\phi-\beta, \phi-\alpha-\beta \in \Lambda$. Hence,

$$B_{\phi-\alpha,\phi-\alpha} = A_{\phi-\alpha,\phi-\alpha} + c_{\phi-\alpha,\phi-\alpha-\beta}aA_{\phi-\alpha-\beta,\phi-\alpha}.$$

By the inductive assumption, this is equal to

$$A_{\phi\phi} + c_{\phi-\alpha,\phi-\alpha-\beta}a(c_{\phi-\alpha-\beta,\phi-\alpha}c_{\phi-\beta,\phi}A_{\phi-\beta,\phi}) = A_{\phi\phi} + c_{\phi,\phi-\beta}aA_{\phi-\beta,\phi} = B_{\phi\phi}.$$

Now assume that $\phi - \beta = \psi$. Then

$$B_{\phi-\alpha,\psi-\alpha} = A_{\phi-\alpha,\psi-\alpha} + c_{\phi-\alpha,\psi-\alpha}aA_{\psi-\alpha,\psi-\alpha};$$

by the inductive assumption, this is equal to

$$c_{\phi-\alpha,\psi-\alpha}c_{\phi\psi}A_{\phi\psi} + c_{\phi-\alpha,\psi-\alpha}aA_{\psi\psi} = c_{\phi-\alpha,\psi-\alpha}c_{\phi\psi}B_{\phi\psi},$$

and the proof is finished.

Finally, assume that $\phi - \beta \ne \psi$. Then

$$B_{\phi-\alpha,\psi-\alpha} = A_{\phi-\alpha,\psi-\alpha} + c_{\phi-\alpha,\phi-\alpha-\beta}aA_{\phi-\alpha-\beta,\psi-\alpha}.$$

By the inductive assumption, this is equal to

$$c_{\phi-\alpha,\psi-\alpha}c_{\phi\psi}A_{\phi\psi} + c_{\phi-\alpha,\phi-\alpha-\beta}a(c_{\phi-\alpha-\beta,\psi-\alpha}c_{\phi-\beta,\psi}A_{\phi-\beta,\psi}).$$

On the other hand, since

$$B_{\phi\psi} = A_{\phi\psi} + c_{\phi,\phi-\beta}aA_{\phi-\beta,\psi},$$

it suffices to prove that

$$c_{\phi-\alpha,\psi-\alpha}c_{\phi\psi}c_{\phi-\alpha,\phi-\alpha-\beta}c_{\phi-\alpha-\beta,\psi-\alpha}c_{\phi-\beta,\psi}c_{\phi,\phi-\beta} = 1. \tag{2.2}$$

Note that the root $\psi-\phi \in E_6$ is orthogonal to α, so that (2.2) follows from [22, Lemma 2.1] with γ equal to $\psi - \phi$ and σ equal to $\phi - \alpha - \beta$. $\square$

Mostly, we shall use the group D_δ; we denote it simply by D. The subspaces V_k^δ will be denoted by V_k. The preceding claim can be simplified for elements of D.

Corollary. *Suppose $A \in D$ and $\phi, \psi \in I_1$. Then $A_{\phi-\delta,\psi-\delta} = A_{\phi\psi}$.*

Proposition 3. *Suppose $A \in D$ and $\rho, \sigma \in I_2$. Moreover, let $A_1 = A|_{V_1}$ and let $A_1^{(\rho,\sigma)}$ be the submatrix of A_1 obtained by omitting the rows ρ_1, ρ_2 and columns σ_1, σ_2. Let $d_A^{(\rho,\sigma)} = \det A_1^{(\rho,\sigma)}$. Then $A_{\rho\sigma} = d_A^{(\rho,\sigma)}$.*

Proof. 1. By the definition of D, the matrix A is the product of n elementary root elements corresponding to *simple and minus simple roots*. The proof is by induction on n. If A is the identity matrix, there is nothing to prove. Suppose the inductive assumption for A is fulfilled. We multiply A by an elementary root element $x_\alpha(a)$, where $\alpha \in \Pi^{\pm}, \alpha \perp \delta$, and put $B = x_\alpha(a)A$. Note that the action structure constants are equal to 1 for simple and minus simple roots. Thus, we have $B_{\rho\sigma} = A_{\rho\sigma} + aA_{\rho-\alpha,\sigma}$ for $\rho - \alpha \in \Lambda$, and $B_{\rho\sigma} = A_{\rho\sigma}$ for $\rho - \alpha \notin \Lambda$

2. Consider the case where $\rho - \alpha \in \Lambda$. By [22, Prop. 6], I_1 contains three weights adjacent to both ρ and $\rho - \alpha$; one weight (say, ρ_1) nonadjacent to both ρ and $\rho - \alpha$; one weight (say, ρ_2) nonadjacent to ρ and adjacent to $\rho - \alpha$; and one weight, $\rho_2 + \alpha$, adjacent to ρ and nonadjacent to $\rho - \alpha$. By [22, Prop. 4], ρ_2 is a unique weight in I_1 such that $\rho_2 + \alpha \in \Lambda$; hence, B_1 can be obtained from A_1 by an elementary row operation, namely, by adding the row ρ_2 multiplied by a to the row $\rho_2 + \alpha$. Therefore, the matrices $B_1^{(\rho,\sigma)}, A_1^{(\rho,\sigma)}$, and $A_1^{(\rho-\alpha,\sigma)}$ have three common rows, while the row $\rho_2 + \alpha$ of $B_1^{(\rho,\sigma)}$ is a sum of the row $\rho_2 + \alpha$ of $A_1^{(\rho,\sigma)}$ and the row ρ_2 of $A_1^{(\rho-\alpha,\sigma)}$ multiplied by a. Since α is a simple root, the numbers of the weights ρ_2 and $\rho_2 + \alpha$ differ by 1, i.e., the rows ρ_2 and $\rho_2 + \alpha$ are adjacent. Therefore, the rows ρ_2 and $\rho_2 + \alpha$ of the matrices $A_1^{(\rho-\alpha,\sigma)}$ and $A_1^{(\rho,\sigma)}$ are at the same position. Using Laplace's formula, we get $d_B^{(\rho,\sigma)} = d_A^{(\rho,\sigma)} + ad_A^{(\rho-\alpha,\sigma)}$, and the proof is finished.

3. Suppose $\rho - \alpha \notin \Lambda$, and let $\phi \in I_1$ be a weight such that $\phi + \alpha \in \Lambda$ (it is unique by [22, Prop. 4]). Then [22, Prop. 6] implies that either both ϕ and $\phi + \alpha$ are nonadjacent to ρ, or ϕ is adjacent to ρ whereas $\phi + \alpha$ is not, or both ϕ and $\phi + \alpha$ are adjacent to ρ. It is easy to show that in the first two cases we have $B_1^{(\rho,\sigma)} = A_1^{(\rho,\sigma)}$, whence $d_B^{(\rho,\sigma)} = d_A^{(\rho,\sigma)}$. In the last case the matrix $B_1^{(\rho,\sigma)}$ can be obtained from $A_1^{(\rho,\sigma)}$ by adding one of its rows multiplied by a to another row; therefore, it has the same determinant and we have $d_B^{(\rho,\sigma)} = d_A^{(\rho,\sigma)}$ as before. □

Corollary. *The map $D \to \mathrm{SL}(6, K)$ taking an element $A \in D$ to $A_1 = A|_{V_1}$ is an isomorphism. Moreover, the map $D_\alpha \to \mathrm{SL}(6, K)$ taking an element $A \in D_\alpha$ to $A|_{V_1^\alpha}$ is also an isomorphism. In particular, $A|_{V_2^\alpha}$ is uniquely determined by the matrix $A|_{V_1^\alpha}$.*

Proof. The first assertion easily follows from Proposition 3 and the corollary to Proposition 2. We mentioned in §1 that $w_\alpha(1)x_\beta(a)w_\alpha(1)^{-1} = x_{w_\alpha\beta}(\pm a)$. Consequently, $w_\alpha(1)D_\beta w_\alpha(1)^{-1} \subseteq D_{w_\alpha\beta}$. Notice that $w_\alpha(1)^4 = E$. Therefore, conjugation by $w_\alpha(1)$ is an isomorphism between D_β and $D_{w_\alpha\beta}$. The second assertion follows now from the first and from the fact that w_α takes I_k^β to $I_k^{w_\alpha\beta}$ for $1 \le k \le 3$. The third assertion follows readily from the second. □

3. Parabolic subgroup. Let $P = \{A \in G_{\mathrm{sc}}(\mathrm{E}_6, K); AV_1 = V_1\}$ be the parabolic subgroup of type P_2. In other words, P consists of matrices having the block form

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix},$$

where A_{11}, A_{13}, A_{33} are blocks of size 6×6, A_{22} is a block of size 15×15, A_{12} and A_{32} are blocks of size 6×15 blocks, and A_{23} is a block of size 15×6. We denote the group $P^T = P_{-2}$ by P^-.

Proposition 4. *Suppose $A \in P$. The subspace AV_2 is contained in $V_1 \oplus V_2$. In other words, the entry A_{32} in the above block matrix is zero.*

Proof. Let Ω be the set of singular vectors $v \in V$ such that there exists a four-dimensional subspace in V_1 consisting of vectors adjacent to v. Let $V' \le V$ be a subspace spanned by Ω. It is easily seen that $V' \supseteq V_1 \oplus V_2$, because e^i lies in Ω for $i \in I_1 \cup I_2$. On the other hand, [22, Lemma 2.2] implies that for $v \notin V_1 \oplus V_2$ there is at most one (up to conjugacy) vector in V_1 adjacent to v. Therefore, $v \notin \Omega$. This shows that $\Omega \subseteq V_1 \oplus V_2$, whence $V' = V_1 \oplus V_2$. Moreover, since $AV_1 = V_1$, we have $A\Omega = \Omega$ and $AV' = V'$. Therefore, $AV_2 \subset V_1 \oplus V_2$, as claimed. □

Remark. There is a statement similar to Proposition 4: suppose that $A \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ and the blocks A_{31}, A_{32} are zero; then A_{21} is zero, i.e., $A \in P$. Moreover, both Proposition 4 and this statement can be transposed to obtain statements about P^-. The proofs of these claims are similar to that of Proposition 4.

4. Levi subgroup. Let L be the Levi subgroup corresponding to the parabolic subgroup P, i.e., $L = P \cap P^-$. By Proposition 4 and its "transpose", L consists of all matrices in $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ that have the block form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \tag{2.3}$$

where A and C are (6×6)-blocks, and B is a (15×15)-block. It is clear that L contains both T and D.

Proposition 5. *$TD = L = DT$.*

First, we need to prove an auxiliary lemma.

Lemma 2.2. *Suppose $C \in L$ and the matrix $C_1 = C|_{V_1}$ is diagonal. Then C is diagonal.*

Proof. Suppose $C|_{V_3}$ is not diagonal. This means that there exist $\rho \ne \sigma \in I_3$ such that $C_{\rho\sigma} \ne 0$. Then σ is adjacent to $\sigma + \delta \in I_1$; in other words, e^σ is adjacent to $e^{\sigma+\delta}$. Put $x = C_{*\sigma} = Ce^\sigma \in V_3$. By assumption, $x_\rho \ne 0$. At the same time x is adjacent to $e^{\sigma+\delta}$, since $e^{\sigma+\delta}$ is proportional to $e^{\sigma+\delta}$. Therefore, the vector $y = x + e^{\sigma+\delta}$ is singular. Furthermore, by [22, Corollary to Prop. 2] the weight ρ is nonadjacent to $\sigma + \delta$. Let $\tau \in I_2$ be a weight nonadjacent to both ρ and $\sigma + \delta$. Consider $Q(e^\tau, y)$. The definition of Q shows that $Q(e^\tau, y) = \sum \pm y_\phi y_\psi$, where the sum is taken over all unordered pairs $\{\phi, \psi\}$ that form a triad with τ. By [22, Prop. 5], either $\phi, \psi \in I_2$, or one of the weights (say, ϕ) lies in I_1 whereas the other lies in I_3. We are not interested in the triads of the former type, because for $\phi \in I_2$ we have $y_\phi = 0$. Moreover, by the definition of y, if $\phi \in I_1$ and $y_\phi \ne 0$, then $\phi = \sigma + \delta$, whence $\psi = \rho$. Therefore, $Q(e^\tau, y) = \pm y_{\sigma+\delta} y_\rho = \pm y_\rho$, and this is not zero by our assumption. On the other hand, since y is a singular vector, we have $Q(e^\tau, y) = 0$. This contradiction proves that the matrix $C|_{V_3}$ is diagonal.

Similarly, we can prove that $C|_{V_2}$ is a diagonal matrix. If not, there exist $\rho \ne \sigma \in I_2$ such that $C_{\rho\sigma} \ne 0$. Take $x = C_{*\sigma} = Ce^\sigma \in V_2$. By assumption, $x_\rho \ne 0$. By [22, Prop. 6], there exists a weight $\rho_1 \in I_1$ that is adjacent to σ and nonadjacent to ρ. Let $\rho_2 - \delta \in I_3$ be a weight nonadjacent to both ρ and ρ_1. Take $y = x + e^{\rho_1}$ and consider $Q(e^{\rho_2-\delta}, y)$. As in the previous paragraph, $Q(e^{\rho_2-\delta}, y) = \sum \pm y_\phi y_\psi$, where the sum is taken over all unordered pairs $\{\phi, \psi\}$ that form a triad with $\rho_2 - \delta$. By [22, Prop. 6], one of these weights, say, ϕ, lies in I_1, whereas the other lies in I_2. Moreover, by the definition of y, if $\phi \in I_1$ and $y_\phi \ne 0$, then $\phi = \rho_1$, whence $\psi = \rho$. Therefore, $Q(e^{\rho_2-\delta}, y) = \pm y_{\rho_1} y_\rho = \pm y_\rho$, and by our assumption this is not zero. On the other hand, since y is a singular vector, we have $Q(e^{\rho_2-\delta}, y) = 0$. This contradiction shows that $C|_{V_2}$ is a diagonal matrix, and the proof is finished. □

Proof of Proposition 5. Suppose $A \in L$, $A_1 = A|_{V_1}$. Consider the matrix $B_1 \in \mathrm{SL}(V_1, K)$ obtained from A_1 by dividing the last row by $\det A_1$. The matrix $C_1 = A_1(B_1)^{-1}$ is diagonal. By the corollary to Proposition 3, there exists $B \in D$ such that $B|_{V_1} = B_1$. Let $C = AB^{-1}$; we have $C|_{V_1} = C_1$. By Lemma 2.2, C is a diagonal matrix; hence, $A = CB$ for $B \in D$, $C \in T$. The second part is obtained from the first by transposition. $\square$

Theorem 1. *Suppose A is a matrix in* $\mathrm{GL}(V, K)$ *such that the subspaces V_1, V_2, V_3 are A-invariant. Let $A_1 = A|_{V_1}$, and, for $\rho, \sigma \in I_2$, let $A_1^{(\rho,\sigma)}$ be the (4×4)-matrix obtained by removing the rows ρ_1, ρ_2 and columns σ_1, σ_2 from A_1. We have $d_A^{(\rho,\sigma)} = \det A_1^{(\rho,\sigma)}$. Then $A \in L$ if and only if there exists $d_A = \sqrt[3]{\det A_1}$ such that $A_{\rho\sigma} = \frac{A_{\rho+\delta,\sigma+\delta}}{d_A}$ for $\rho, \sigma \in I_3$ and $A_{\rho\sigma} = \frac{d_A^{(\rho,\sigma)}}{d_A^2}$ for $\rho, \sigma \in I_2$.*

Proof. 1. First, we prove that if $A \in L$, then such a scalar d_A exists. By Proposition 5, we have $A = BC$ for some $B \in D$, $C \in T$. Put $B_1 = B|_{V_1}$, $C_1 = C|_{V_1}$. Take d_A to be d_C from Proposition 1 applied to C. By the definition of C, we have $d_A = \sqrt[3]{\det A_1}$. We check that d_A satisfies other conditions. Take $\rho, \sigma \in I_3$. We have $A_{\rho\sigma} = B_{\rho\sigma} C_{\sigma\sigma} = \frac{B_{\rho+\delta,\sigma+\delta} C_{\sigma+\delta,\sigma+\delta}}{d_A}$ by Proposition 1 and the corollary to Proposition 2. It remains to note that this is equal to $\frac{A_{\rho+\delta,\sigma+\delta}}{d_A}$.

Now we take $\rho, \sigma \in I_2$. We have $A_{\rho\sigma} = B_{\rho\sigma} C_{\sigma\sigma} = \frac{d_B^{(\rho,\sigma)} d_A}{C_{\sigma_1,\sigma_1} C_{\sigma_2,\sigma_2}}$ by Proposition 1 and Proposition 3. It is easy to check that $d_A^{(\rho,\sigma)} = d_B^{(\rho,\sigma)} \prod C_{\tau\tau}$, where the product is taken over all $\tau \in I_1$ adjacent to σ. Therefore,

$$\frac{d_B^{(\rho,\sigma)} d_A}{C_{\sigma_1,\sigma_1} C_{\sigma_2,\sigma_2}} = \frac{d_A^{(\rho,\sigma)} d_A}{\prod_{\tau \in I_1} C_{\tau\tau}} = \frac{d_A^{(\rho,\sigma)}}{d_A^2},$$

as required.

2. Suppose such a scalar d_A exists. We need to prove that $A \in L$. Let $C_1 \in \mathrm{GL}(V_1, K)$ be a diagonal matrix such that $B_1 = A_1 C_1 \in \mathrm{SL}(V_1, K)$. It is easily seen that we can take d_A^{-1} to be d_C of Proposition 1. Now, let $C \in T$ be the matrix constructed in Proposition 1 applied to C_1 and d_C. Let $B = AC$; we have $B|_{V_1} = B_1$. We need to show that $B \in D$. By Proposition 3 and the corollaries to Propositions 2 and 3, it remains to prove that $B|_{V_3} = B|_{V_1}$ and that $B_{\rho\sigma} = d_B^{(\rho,\sigma)}$ for $\rho, \sigma \in I_2$.

We prove the first relation. Take $\rho, \sigma \in I_3$. We have

$$B_{\rho\sigma} = A_{\rho\sigma} C_{\sigma\sigma} = \frac{A_{\rho+\delta,\sigma+\delta}}{d_A} \cdot \frac{C_{\sigma+\delta,\sigma+\delta}}{d_C} = A_{\rho+\delta,\sigma+\delta} C_{\sigma+\delta,\sigma+\delta} = B_{\rho+\delta,\sigma+\delta}.$$

To prove the second relation, we take $\rho, \sigma \in I_2$. We have

$$B_{\rho\sigma} = A_{\rho\sigma} C_{\sigma\sigma} = \frac{d_A^{(\rho,\sigma)}}{d_A^2} \cdot \frac{d_C}{C_{\sigma_1,\sigma_1} C_{\sigma_2,\sigma_2}}.$$

It is easy to show that $d_B^{(\rho,\sigma)} = d_A^{(\rho,\sigma)} \prod C_{\tau\tau}$, where the product is taken over all $\tau \in I_1$ adjacent to σ. Therefore,

$$\frac{d_A^{(\rho,\sigma)}}{d_A^2} \cdot \frac{d_C}{C_{\sigma_1,\sigma_1} C_{\sigma_2,\sigma_2}} = \frac{d_B^{(\rho,\sigma)} d_C^3}{\prod_{\tau \in I_1} C_{\tau\tau}} = d_B^{(\rho,\sigma)},$$

as claimed. $\square$

5. Unipotent subgroup. Let $U_2 = U_2^+$ be a unipotent radical of the parabolic subgroup P_2, i.e., $U_2 = \langle X_\alpha; \angle(\alpha, \delta) \le \pi/3\rangle$. In other words, all matrices in U_2 have the block form

$$\begin{pmatrix} E_6 & A & B \\ 0 & E_{15} & C \\ 0 & 0 & E_6 \end{pmatrix}, \tag{2.4}$$

where E_n denotes the identity matrix of size $n \times n$. Denote by U_2' the group of all matrices in $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ that have the block form described above. It is well known that $U_2 = U_2'$, but, as has been mentioned, we prefer to lean upon outside facts as little as possible. In §4 we shall prove that $U_2 = U_2'$, but for the purposes of the present paper it suffices to know that $U_2 \le U_2'$, which is obvious. Similarly, $U_2^- = U_2^T = \langle X_\alpha; \angle(\alpha, -\delta) \le \pi/3\rangle$.

It is easy to check that for any two root subgroups in U_2 their commutator is contained in X_δ, and that X_δ commutes with the entire group U_2. Therefore, if $\angle(\alpha, \delta) = \pi/3$, then the coordinate x_α in the decomposition of the unipotent g into the product of elementary root unipotents is determined uniquely. We denote this coordinate by $(\alpha)_g$. Note that if two unipotents are multiplied, then their blocks denoted by A (respectively, C) in (2.4) are added. This means that for an arbitrary unipotent g its coefficients in the block A (respectively, C) satisfy the conditions similar to those in [22, Lemma 3.1]. More precisely, we have proved the following lemma.

Lemma 2.3. *Suppose $g \in U_2$ is a unipotent and either $\phi \in I_1$, $\psi \in I_2$ or $\phi \in I_2$, $\psi \in I_3$. Then*

(1) *$g_{\phi\psi} = 0$ if $d(\phi, \psi) = 2$;*
(2) *$g_{\phi\psi} = c_{\phi\psi}(\psi - \phi)_g$ if $d(\phi, \psi) = 1$.*

In particular, our old definition of $(\alpha)_g$, given in [22], is compatible with the new one: when they are both meaningful (i.e., for root elements that are unipotent and for roots α such that $\angle(\alpha, \delta) = \pi/3$), their values coincide.

§3. Unipotent elements

Suppose $g \in U_2$ is a unipotent, and $x_\alpha(a)$, $\alpha \perp \delta$, is an elementary root element. Lemma 2.3 implies that

$$g = E + \sum_{\beta; \angle(\beta,\delta)=\pi/3} (\beta)_g e_\beta + \sum_{i,j; i\in I_1, j\in I_3} g_{ij}.$$

Conjugating this by $x_\alpha(a)$, we obtain

$$x_\alpha(a) g x_\alpha(-a) = E + \sum_{\beta; \angle(\beta,\delta)=\pi/3} (\beta)_g x_\alpha(a) e_\beta x_\alpha(-a) + \sum_{i\in I_1, j\in I_3} x_\alpha(a) g_{ij} x_\alpha(-a).$$

Denote $x_\alpha(a) g x_\alpha(-a)$ by g'. The second sum is equal to $\sum_{i,j;i\in I_1,j\in I_3} g'_{ij}$ by [22, Prop. 4]. Consider the first sum. The angle between α and β can be equal to $\pi/3, \pi/2$, or $2\pi/3$. If it is equal to $\pi/3$ or $\pi/2$, then $x_\alpha(a)$ commutes with $x_\beta(b)$, so that $x_\alpha(a) e_\beta x_\alpha(-a) = e_\beta$. On the other hand, if $\angle(\alpha, \beta) = 2\pi/3$, then $x_\alpha(a) e_\beta x_\alpha(-a) = e_\beta + N_{\alpha\beta} e_{\alpha+\beta}$. We have proved the following statement.

Lemma 3.1. *Suppose that $g \in U_2$ is a unipotent, and that $x_\alpha(a)$, $\alpha \perp \delta$, is an elementary root element, $g' = x_\alpha(a) g x_\alpha(-a)$. If $\angle(\alpha, \beta) = \pi/3$, then $(\beta)_{g'} = (\beta)_g + N_{\alpha\beta} a (\beta - \alpha)_g$; otherwise $(\beta)_{g'} = (\beta)_g$.*

Lemma 3.2. *Suppose $g \in U_2$ is a unipotent, and that there exists a root β such that $\angle(\beta, \delta) = \pi/3$ and $(\beta)_g \ne 0$. Conjugating g, if necessary, by an element of D, we may assume that $(\alpha)_g \ne 0$ for $\alpha = {}^{12321}_{\ \ 1}$.*

Proof. It is clear that the roots α and β can be connected by a sequence of roots $\beta = \beta_0, \beta_1, \dots, \beta_n = \alpha$ such that $\angle(\delta, \beta_i) = \angle(\beta_{i-1}, \beta_i) = \pi/3$ for all $1 \le i \le n$. It remains to show that if $(\beta_{i-1})_g \neq 0$, then, conjugating g by an element of D, we may assume that $(\beta_i)_g \neq 0$. This follows easily from Lemma 3.1 for $\alpha = \beta_i - \beta_{i-1}$. $\square$

Lemma 3.3. *Suppose $g \in U_2$ is a unipotent, and that $(\alpha)_g \neq 0$ for $\alpha = {}^{12321}_{\ \ 1}$. Conjugating g, if necessary, by an element of D, we may assume that $(\alpha')_g = 0$ for all roots α' such that $\angle(\alpha', \delta) = \angle(\alpha', \alpha) = \pi/3$.*

Proof. Let β be a root such that $\angle(\beta, \delta) = \angle(\beta, \alpha) = \pi/3$. Assume that $(\beta)_g \neq 0$. Conjugate g by $x_{\beta-\alpha}(a)$. By Lemma 3.1 such conjugation affects only the coefficients for the roots forming an angle of $\pi/3$ with $\beta - \alpha$. We have $\angle(\alpha, \beta - \alpha) = 2\pi/3$; therefore, if α' is a root such that $\angle(\alpha', \alpha) = \pi/3$, then all the coefficients except for the root β stay the same. Substituting $\frac{-N_{\alpha\beta}(\beta)_g}{(\alpha)_g}$ for a, we obtain $(\beta)_g = 0$. Repeating this for all β such that $\angle(\beta, \delta) = \angle(\beta, \alpha) = \pi/3$, we finish the proof. $\square$

Consider the roots forming an angle of $\pi/3$ with δ, and their expansions into simple roots. In this expansion, the coefficient of α_2 is equal to 1. Note that there is exactly one root with coefficient 3 of α_4, namely, $\alpha = {}^{12321}_{\ \ 1}$, and exactly one root with coefficient 0, namely, $\delta - \alpha = {}^{00000}_{\ \ 1}$. The other roots have coefficient 1 or 2, and there are nine roots of each type. Finally, α forms an angle of $\pi/3$ with the roots with coefficient 2 of α_4, and α is orthogonal to the roots with coefficient 1 of α_4; while for $\delta - \alpha$ we have the reverse situation.

Lemma 3.4. *Suppose $g \in U_2$ is a unipotent, $(\alpha)_g \neq 0$ for $\alpha = {}^{12321}_{\ \ 1}$. Moreover, suppose $(\alpha')_g = 0$ for all roots α' such that $\angle(\alpha', \delta) = \angle(\alpha', \alpha) = \pi/3$. Suppose there exists a root $\gamma \perp \alpha$ such that $\angle(\gamma, \delta) = \pi/3$ and $(\gamma)_g \neq 0$. Conjugating g, if necessary, by an element of D, we may assume that $(\beta)_g \neq 0$ for $\beta = {}^{11100}_{\ \ 1}$.*

Proof. It is easy to show that the angle between the roots β and γ is either $\pi/3$ or $\pi/2$. In the latter case there exists a root γ' such that $\angle(\gamma', \gamma) = \angle(\gamma', \beta) = \angle(\gamma', \delta) = \pi/3$. We prove that if there are two weights γ_1, γ_2 such that $\angle(\gamma_1, \gamma_2) = \angle(\gamma_i, \delta) = \pi/3$, $\angle(\gamma_i, \alpha) = \pi/2$ for $i = 1, 2$, and $(\gamma_1)_g \neq 0$, then, conjugating g by an element of D, we may assume that $(\gamma_2)_g \neq 0$. By Lemma 3.1, it suffices to show that after conjugating g by the root element $x_{\gamma_2 - \gamma_1}(a)$ we still have $(\alpha)_g \neq 0$ and $(\alpha')_g = 0$. Note that $\gamma_2 - \gamma_1$ has coefficient 0 of α_4 in its expansion into simple roots. The rest follows from Lemma 3.1. $\square$

Lemma 3.5. *Suppose $g \in U_2$ is a unipotent and $(\alpha)_g \neq 0$ for $\alpha = {}^{12321}_{\ \ 1}$. Moreover, suppose $(\alpha')_g = 0$ for all roots α' such that $\angle(\alpha', \delta) = \angle(\alpha', \alpha) = \pi/3$. Suppose $(\beta)_g \neq 0$ for $\beta = {}^{11100}_{\ \ 1}$. Conjugating g, if necessary, by an element of D, we may assume that $(\beta')_g = 0$ for all roots β' such that $\angle(\beta', \delta) = \angle(\beta', \beta) = \pi/3$.*

Proof. The proof is similar to that of Lemma 3.3. Let γ be a root such that $\angle(\gamma, \delta) = \angle(\gamma, \beta) = \pi/3$. Suppose $(\gamma)_g \neq 0$. Conjugate g by $x_{\gamma-\beta}(a)$. As was shown in the proof of Lemma 3.3, this conjugation does not affect the coefficients of the roots $\beta' \neq \gamma$ with $\angle(\beta', \delta) = \angle(\beta', \beta) = \pi/3$. Note that $\gamma - \beta$ has coefficient 0 of α_4 in its expansion into simple roots. By Lemma 3.1, the coefficients of α and α' are not altered either. Choosing a scalar a such that $(\gamma)_g$ becomes 0, and repeating this procedure for all γ such that $\angle(\gamma, \delta) = \angle(\gamma, \beta) = \pi/3$ and $(\gamma)_g \neq 0$, we finish the proof. $\square$

Lemma 3.6. *Suppose $g \in U_2$ is a unipotent. Conjugating g, if necessary, by an element of D, we may assume that $(\gamma)_g = 0$ for all $\gamma \neq {}^{12321}_{\ \ 1}, {}^{11100}_{\ \ 1}, {}^{01110}_{\ \ 1}, {}^{00111}_{\ \ 1}, {}^{00000}_{\ \ 1}$ such that $\angle(\gamma, \delta) = \pi/3$.*

Proof. We apply Lemmas 3.2–3.5. It remains to conjugate g by an element of D in order to get $\left(\substack{01111\\1}\right)_g = \left(\substack{00110\\1}\right)_g = 0$ (without changing the other coefficients). Suppose that one of $\left(\substack{01111\\1}\right)_g$ and $\left(\substack{00110\\1}\right)_g$ is not 0. Conjugating g, if necessary, by root elements $x_{\gamma'}(a)$ for $\gamma' = -\substack{00001\\0}$ or $\substack{01000\\0}$, we may assume that $\left(\substack{01110\\1}\right)_g \neq 0$. Then, conjugating g by root elements $x_{\gamma'}(a)$ for $\gamma' = \substack{00001\\0}$ and $-\substack{01000\\0}$, we may assume that $\left(\substack{01111\\1}\right)_g = \left(\substack{00110\\1}\right)_g = 0$, as required. $\square$

Theorem 2. *Every unipotent $g \in U_2$ can be expressed as a product of at most three root elements.*

Proof. By Lemma 3.6, we may assume that $(\gamma)_g = 0$ for all roots $\gamma \neq \substack{12321\\1}, \substack{11100\\1}, \substack{01110\\1}, \substack{00111\\1}, \substack{00000\\1}$ such that $\angle(\gamma, \delta) = \pi/3$. Suppose $\left(\substack{00000\\1}\right)_g \neq 0$. Then, conjugating g by $x_{\gamma'}(a)$, where $\gamma' = \substack{00111\\0}$, we may assume that $\left(\substack{00111\\1}\right)_g = 0$. However, after this conjugation we can have nonzero coefficients of $\substack{11211\\1}$ and $\substack{01221\\1}$. In other words, the unipotent g is expressed as a product of elementary root elements corresponding to the roots $\substack{00000\\1}, \substack{01110\\1}, \substack{11100\\1}, \substack{11211\\1}, \substack{01221\\1}, \substack{12321\\1}$, and maybe $\substack{12321\\2}$. Notice that a product of two root elements forming an angle of $\pi/3$ is also a root element. Indeed, [22, Prop. 1] allows us to assume that those root elements are elementary, and the claim becomes obvious. Thus, the products of the first and second, of the third and fourth, and of the remaining three root elements are in fact three root elements, and their total product is g.

The only remaining case is $\left(\substack{00000\\1}\right)_g = 0$. If either one of the remaining coefficients is also 0, the theorem is obvious. Otherwise, suppose that the remaining four coefficients are nonzero. Conjugate g by the root element $x_{\gamma'}(1)$, where $\gamma' = -\substack{11100\\0}$. The coefficients of $\substack{00000\\1}$ and $\substack{01221\\1}$ become nonzero. Now, conjugating g by $x_{\gamma'}(a)$ for $\gamma' = \substack{00111\\0}$, we may assume that $\left(\substack{00111\\1}\right)_g = 0$. Thus, the coefficient of $\substack{11211\\1}$ becomes nonzero. In other words, as in the first case, g can be expressed as a product of elementary root elements corresponding to the roots $\substack{00000\\1}, \substack{01110\\1}, \substack{11100\\1}, \substack{11211\\1}, \substack{01221\\1}, \substack{12321\\1}$, and maybe $\substack{12321\\2}$. Therefore, g is again a product of three root elements. $\square$

Remark. It is easily seen that the roots $\substack{12321\\1}, \substack{11100\\1}, \substack{01110\\1}, \substack{00111\\1}$, and $\substack{00000\\1}$ from Lemmas 3.2–3.6, and also the maximal root $\delta = \substack{12321\\2}$, lie in a root subsystem of type D_4. This means that the proof of Theorem 2 in fact happens inside D_4.

We shall often consider the (6×6)-submatrix of g situated at the top right corner. This matrix will be denoted by $\bar{g} = \{g_{ij}\}_{i\in I_1, j\in I_3}$. It takes the subspace V_3 to V_1. However, it is clear that there is a standard isomorphism between V_1 and V_3 taking e^i to $e^{i-\delta}$ for $i \in I_1$. In order to simplify the notation, we shall implicitly make use of this isomorphism and talk about conjugation of $\bar{g}$, its Jordan normal form, etc.

The following definition was mentioned in the Introduction.

Definition. A field K is *n-closed* if every nonconstant polynomial over K of degree at most n has a root.

Assertion 1.

(1) *Suppose $g \in U_2$ is a unipotent. Then $\bar{g}$ is conjugated to*

$$\begin{pmatrix} a & 0 & 0 & kx & 0 & 0 \\ 0 & a & 0 & 0 & ky & 0 \\ 0 & 0 & a & 0 & 0 & kz \\ yz & 0 & 0 & b & 0 & 0 \\ 0 & xz & 0 & 0 & b & 0 \\ 0 & 0 & xy & 0 & 0 & b \end{pmatrix}$$

for some $a, b, k, x, y, z \in K$.

(2) *Suppose K is 2-closed and $g \in U_2$ is a unipotent. Then the Jordan normal form of $\overline{g}$ looks like one of the following matrices*:

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \quad \begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix}, \quad \begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 1 \\ 0 & 0 & 0 & 0 & 0 & a \end{pmatrix},$$

or

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}$$

for some $a \neq b \in K$.

(3) *Suppose K is 2-closed and $A \in \mathrm{M}(6, K)$ is a matrix such that its Jordan form is one of the matrices described above. Then there exists a unipotent $g \in U_2$ such that $\overline{g} = A$.*

We shall say that a matrix $A \in \mathrm{M}(6, K)$ is of *type* † if it is conjugate to a matrix as in (1). Note that A may fail to be invertible.

Proof. It is easy to check that if we conjugate g by $f \in D$, then the submatrix $\overline{g}$ is multiplied by $f|_{V_1}$ on the left and by $f^{-1}|_{V_3}$ on the right. Since $f^{-1} \in D$, we have $f^{-1}|_{V_3} = f^{-1}|_{V_1} = (f|_{V_1})^{-1}$. Therefore, if we conjugate g by f, the submatrix $\overline{g}$ is conjugated by $f|_{V_1}$. We apply Lemma 3.6. In order to prove (1), it remains to show that if g is as in Lemma 3.6, then $\overline{g}$ has the form we need. Take $i \in I_1, j \in I_3$ such that $i \neq j + \delta$. Since g_{*j} is a singular vector, $Q(e^k, g_{*j})$ equals 0 for all $k \in I_2$ nonadjacent to both i and j. On the other hand, the description of Q shows that $Q(e^k, g_{*j}) = \pm g_{ij} g_{jj} \pm g_{i-\delta,j} g_{j+\delta,j} + \sum \pm g_{lj} g_{mj}$. By [22, Prop. 6], the last sum consists of three terms and $l, m \in I_2$; at the same time, $g_{jj} = 1$ and $g_{i-\delta,j} = 0$. Now, direct computation shows that g_{ij} has the value we need for $i \neq j + \delta$. The required relations for the diagonal elements of $\overline{g}$, say, for $g_{i,i-\delta}$ and $g_{j,j-\delta}$, can be deduced by a similar computation from the fact that $Q(e^k, g_{*,i-\delta} + g_{*,j-\delta}) = 0$ for $k \in I_2$ and $d(k, i) = d(k, j) = 2$.

The second part follows directly from the first. Now we prove the third part. It is easily seen that there exist unipotents g such that $\overline{g}$ has the same Jordan normal form as A. Moreover, as was mentioned in the previous paragraph, if we conjugate g by $f \in D$, the submatrix $\overline{g}$ is conjugated by the matrix $f|_{V_1} \in \mathrm{SL}(6, K)$. Conversely, by the corollary to Proposition 3, for any $B \in \mathrm{SL}(6, K)$ there exists a matrix $f \in D$ such that $f|_{V_1} = B$. Thus, it remains to show that A is conjugated to some matrix as in (2) by a matrix from $\mathrm{SL}(6, K)$. This follows from the fact that K is a 2-closed field and that the smallest size of a Jordan block in the matrices in (2) is 1 or 2. □

§4. Products

1. Reducing products of matrices in $G_{\mathrm{sc}}(\mathrm{E}_6, 0)K$ to products of matrices in $\mathrm{GL}(6, K)$ and to unipotent elements. Suppose $g \in U_2$ is a unipotent. The last six columns of g span a six-dimensional singular subspace W. By [22, Theorem 2], there exists a root element h such that $V^h = W$. Moreover, by [22, Lemma 3.5] we have $(-\delta)_h \neq 0$. Therefore, we have a map θ from U_2 to the set of root elements $\mathrm{RootEl}_{-\delta} = \{h; h \text{ is a root element}\}, (-\delta)_h = 1\}$. Notice that by Lemma 2.3, the unipotent g is uniquely determined by the six-dimensional singular subspace W. Thus, θ is an injection. Also, if $g \in U_2$ and $h = \theta(g) \in \mathrm{RootEl}_{-\delta}$, then the first six columns of $h - e$ coincide with the last six columns of g. In other words, $h_{\rho\sigma} - \delta_{\rho,\sigma} = g_{\rho\sigma-\delta}$ for $\rho \in \Lambda$ and $\sigma \in I_1$. In particular, $(\gamma)_g = \pm(\gamma - \delta)_h$ for every root γ such that $\angle(\gamma, \delta) = \pi/3$.

We apply [22, Theorem 1] to the case we are interested in: $\alpha = -\delta$. In that theorem, it was proved that every element of $\mathrm{RootEl}_{-\delta}$ is obtained by conjugation of an elementary root element $x_{-\delta}(1)$ by the root elements $x_{\beta+\delta}(\cdot)$, where $\beta, \beta + \delta \in \mathrm{E}_6$, and by $x_\delta(\cdot)$. In other words, for every $h \in \mathrm{RootEl}_{-\delta}$ we have $h = gx_{-\delta}(1)g^{-1}$ for some $g \in U_2$. Thus, we have a map η: $\mathrm{RootEl}_{-\delta} \to U_2$ that takes h to g. Comparing he^λ with $gx_{-\delta}(1)g^{-1}e^\lambda$, we see that η is inverse to θ; hence θ is a bijection. Furthermore, both η and θ are invariant under the conjugation of U_2 and $\mathrm{RootEl}_{-\delta}$ by elements of D. It is clear that there are three more similar bijections besides θ. The first is the bijection between U_2 and $\mathrm{RootEl}_{-\delta}$ that takes a unipotent g to a root element h such that $V_h = \langle g_{i*}; i \in I_1\rangle$. The second is the bijection between U_2^- and RootEl_δ that takes a unipotent g to a root element h such that $V^h = \langle g_{*i}; i \in I_1\rangle$. The third is the bijection between U_2^- and RootEl_δ that takes a unipotent g to a root element h such that $V_h = \langle g_{i*}; i \in I_3\rangle$. We have proved the following assertion.

Assertion 2.

(1) *For every root element h with $(-\delta)_h \neq 0$, there exists a unipotent $g \in U_2$ such that $V^h = \langle g_{*i}; i \in I_3\rangle$.*

(2) *For every root element h with $(-\delta)_h \neq 0$, there exists a unipotent $g \in U_2$ such that $V_h = \langle g_{i*}; i \in I_1\rangle$.*

(3) *For every root element h with $(\delta)_h \neq 0$, there exists a unipotent $g \in U_2^-$ such that $V^h = \langle g_{*i}; i \in I_1\rangle$.*

(4) *For every root element h with $(\delta)_h \neq 0$, there exists a unipotent $g \in U_2^-$ such that $V_h = \langle g_{i*}; i \in I_3\rangle$.*

Corollary.

(1) *Suppose g is a root element and either $(\delta)_g$ or $(-\delta)_g$ is nonzero. Then the matrices $\{g_{ij}\}_{i,j=1}^6$ and $\{g_{ij}\}_{i,j=22}^{27}$ are of type †.*

(2) *Suppose K is a 2-closed field. Then for every matrix A of type † there exist root elements g, h such that $(\delta)_g \neq 0$, $(\delta)_h \neq 0$, $\{g_{ij}\}_{i,j=1}^6 = A$, and $\{h_{ij}\}_{i,j=22}^{27} = A$.*

Proof. Both parts follow from Assertions 1 and 2. □

Remark. It is clear that the conditions $(\delta)_g \neq 0$ and $(\delta)_h \neq 0$ in part (2) can be replaced with $(-\delta)_g \neq 0$ and $(-\delta)_h \neq 0$.

We studied the groups U_2 and U_2' in §2. There we noticed that $U_2 \leq U_2'$. Now we can easily prove that these groups are in fact equal.

Assertion 3. *$U_2' = U_2$.*

Proof. It remains to show that $U_2' \leq U_2$. Suppose $g' \in U_2'$. The last six columns of g' span a six-dimensional singular subspace V'. By [22, Corollary to Theorem 2], there exists a root element $h \in \mathrm{RootEl}_{-\delta}$ such that $V^h = V'$. By Assertion 2, there exists $g \in U_2$ such that the subspace spanned by the last six rows of g coincides with V'. This means that $gV_1 = g'V_1 = V'$. Thus, $g^{-1}g'$ stabilizes V_1, i.e., $g^{-1}g' \in P^-$. Now Proposition 4 (more precisely, its transposed version, cf. remark after Proposition 4), combined with the definition of U_2', implies that $g^{-1}g' = E$. This concludes the proof. □

Suppose $J \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is a matrix equal to $w_\delta(1) = x_\delta(1)x_{-\delta}(-1)x_\delta(1)$. Consider the set L' obtained by multiplication of the Levi subgroup L by J from the right: $L' = \{AJ; A \in L\}$.

Remark. We use this new notation for $w_\delta(1)$ in order to distinguish between multiplication and conjugation by it. As before, by the "action of $w_\delta(1)$", we mean conjugation by this element.

Theorem 3. *Suppose $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is a matrix with $\det \overline{g} \neq 0$. Then $g = vdw$ for some $v, w \in U_2^-$ and $d \in L'$.*

Proof. Consider the six-dimensional subspace W spanned by the last six columns of g. Let h be a root element such that $V^h = W$. By assumption, we have $\det \overline{g} \neq 0$, and then [22, Lemma 3.5] yields $(\delta)_h \neq 0$. Therefore, $\frac{1}{(\delta)_h}(h - E) + E \in \mathrm{RootEl}_\delta$. This implies that there exists a unipotent $v \in U_2^-$ such that the subspace spanned by the first six columns of v coincides with $V^{\frac{1}{(\delta)_h}(h-E)+E} = V^h = W$. Furthermore, consider the six-dimensional subspace $W' < V^*$ spanned by the first six rows of g. Transposing the previous argument, we obtain the existence of a unipotent $w \in U_2^-$ such that the subspace spanned by the last six rows of w coincides with W'.

Now we put $d = v^{-1}gw^{-1}$. Consider the subspace V' generated by the last six rows of d. At the same time $w^{-1}V_3 = V_3$, so that $V' = v^{-1}gw^{-1}V_3 = v^{-1}V$. From our choice of v it follows that $V = vV_1$, whence $V' = V_1$. In other words, $d_{ij} = 0$ for $7 \leq i \leq 27, 22 \leq j \leq 27$. Similarly, transposing the argument above, we obtain $d_{ij} = 0$ for $1 \leq i \leq 6, 1 \leq j \leq 21$. Finally, consider dJ^{-1}. It is easy to check that this matrix belongs to the parabolic subgroups P and P^-. Therefore, $dJ^{-1} \in L$ and $d \in L'$. □

Proposition 6.

(1) *Suppose g_1, g_2 are two root elements such that $(\delta)_{g_1}$ and $(\delta)_{g_2}$ are both nonzero. Then $\overline{g_1g_2}$ is of type †.*
(2) *Suppose K is a 2-closed field. Let g_1 be a root element such that $(\delta)_{g_1} \neq 0$. Then for every matrix A of type † there exists a root element g_2 such that $\overline{g_1g_2} = A$.*

Proof. We prove (1). By [22, Theorem 1], we have $g_1 = fx_\delta(a)f^{-1}$ for some $f \in U_2^-$. It is easy to show that for every $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ and every unipotent $u \in U_2^-$ the matrix $\overline{g}$ is equal both to $\overline{ug}$ and to $\overline{gu}$. In particular, $\overline{g_1g_2} = \overline{f^{-1}g_1g_2f} = \overline{x_\delta(a)(f^{-1}g_2f)}$. Furthermore, $\overline{x_\delta(a)(f^{-1}g_2f)} = \overline{f^{-1}g_2f} + a\{(f^{-1}g_2f)_{ij}\}_{i,j \in I_3}$. Since g_2 is a root element, the matrix $\overline{f^{-1}g_2f} = \overline{g}_2$ is scalar. By the corollary to Assertion 2, the matrix $\{(f^{-1}g_2f)_{ij}\}_{i,j \in I_3}$ is of type †; therefore, $\overline{g_1g_2}$ is also of type †.

Now we prove (2). As before, $\overline{g_1g_2} = \overline{x_\delta(a)(f^{-1}g_2f)} = \overline{f^{-1}g_2f} + a\{(f^{-1}g_2f)_{ij}\}_{i,j \in I_3}$, where f and a depend on g_1 only. For $h = f^{-1}g_2f$ we have $\overline{g_1g_2} = \overline{h} + a\{h_{ij}\}_{i,j \in I_3}$. Here h can be an arbitrary root element. Since $(\delta)_h = (\delta)_{g_2}$, the condition $(\delta)_{g_2} \neq 0$ becomes $(\delta)_h \neq 0$. Now the corollary to Assertion 2 concludes the proof. □

Theorem 4.

(1) *Suppose g_1 and g_2 are two elements of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that both $\det \overline{g_1}$ and $\det \overline{g_2}$ are nonzero. Then $\overline{g_1g_2} = \overline{g_1}A\overline{g_2}$, where A is a matrix of type †. In particular, if g_2 is a root element, then $\overline{g_1g_2} = \overline{g_1}A$.*
(2) *Suppose K is a 2-closed field. Take $g_1 \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ with $\det \overline{g_1} \neq 0$. Then for any matrix A of type † there exists a root element g_2 such that $\overline{g_1g_2} = \overline{g_1}A$.*

Proof. Let f_1 be a root element such that $\langle \{f_1\}_{i*}; i \in I_1 \rangle = \langle \{g_1\}_{i*}; i \in I_1 \rangle$ and $(\delta)_{f_1} = 1$. In other words, if $W = \langle \{g_1\}_{i*}; i \in I_1 \rangle$ and h_1 is a root element such that $V^{h_1} = W$ and $(\delta)_{h_1} = 1$, then $f_1 = x_{-\delta}(1)h_1x_{-\delta}(-1)$. Similarly, let f_2 be a root element such that $\langle \{f_2\}_{*i}; i \in I_3 \rangle = \langle \{g_2\}_{*i}; i \in I_3 \rangle$ and $(\delta)_{f_2} = 1$. Then for $i \in I_1, j \in I_3$, and $k \in \Lambda$ we have $\{g_1\}_{ik} = \sum_{l \in I_1} \{g_1\}_{i,l-\delta}\{f_1\}_{lk}$, because the ith row of g_1 is the sum over l of the rows of f_1 with number l multiplied by $\{g_1\}_{i,l-\delta}$; similarly $\{g_2\}_{kj} = \sum_{m\in I_3}\{g_2\}_{m+\delta,j}\{f_2\}_{km}$. Therefore,

$$\begin{aligned}\{g_1g_2\}_{ij} = \sum_{k\in\Lambda}\{g_1\}_{ik}\{g_2\}_{kj} &= \sum_{k\in\Lambda}\Big(\sum_{l\in I_1}\{g_1\}_{i,l-\delta}\{f_1\}_{lk}\Big)\Big(\sum_{m\in I_3}\{g_2\}_{m+\delta,j}\{f_2\}_{km}\Big)\\ &= \sum_{l\in I_1}\sum_{m\in I_3}\Big(\{g_1\}_{i,l-\delta}\{g_2\}_{m+\delta,j}\Big(\sum_{k\in\Lambda}\{f_1\}_{lk}\{f_2\}_{km}\Big)\Big)\\ &= \sum_{l\in I_1}\sum_{m\in I_3}(\{g_1\}_{i,l-\delta}\{f_1f_2\}_{lm}\{g_2\}_{m+\delta,j}).\end{aligned}$$

This expression is precisely what we need in (1), if we recall the standard implicit isomorphism $V_3 \to V_1$. Now, part (2) follows from this argument and Proposition 6. $\square$

2. Products of matrices in $\mathrm{GL}(6, K)$**.** Here we work with Jordan normal forms of (6×6)-matrices. For brevity, we denote the block diagonal matrices with the blocks $B_1, \ldots, B_k$ by $D(B_1, \ldots, B_k)$. Denote by $(a, \ldots, a)$ the Jordan block corresponding to the eigenvalue a, where the number of a's is equal to the size of the block. We denote different eigenvalues by different letters. For example, $D(a, a, (b, b), b)$ denotes the matrix

$$\begin{pmatrix} a & 0 & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0\\ 0 & 0 & b & 1 & 0\\ 0 & 0 & 0 & b & 0\\ 0 & 0 & 0 & 0 & b\end{pmatrix}.$$

Now we need a more precise version of the term "matrices of type †". We say that a matrix $A \in \mathrm{GL}(6, K)$ is a matrix of type ‡ if it is conjugate to a matrix $D(a, a, a, b, b, b)$ for some $a, b \neq 0$. In particular, every matrix of type ‡ is of type †.

Lemma 4.1. *Suppose K is a 2-closed field.*

(1) *Let $A = \left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} c & 0\\ 0 & d\end{smallmatrix}\right)$ be two arbitrary nonscalar matrices in* $\mathrm{GL}(2, K)$. *Then every nonscalar matrix in* $\mathrm{GL}(2, K)$ *with the determinant $abcd$ can be expressed as a product of matrices A' and B', where A' is conjugate to A, and B' is conjugate to B.*
(2) *Suppose $A \in \mathrm{GL}(6, K)$ is a block diagonal matrix with three nonscalar blocks of size 2×2 with the same determinant. Then it is a product of two matrices of type ‡.*
(3) *Suppose a matrix $B \in \mathrm{GL}(6, K)$ has the eigenvalues $x, \frac{1}{x}, y, \frac{1}{y}, z, \frac{1}{z}$ in K and that they are pairwise distinct. Then B is a product of two matrices of type ‡.*

Proof. It is easily seen that in order to prove (1) it suffices to show that the product $A'B'$ can have arbitrary trace. Put $A' = A$ and $B' = \left(\begin{smallmatrix} x & y\\ z & t\end{smallmatrix}\right)$. Note that B' is conjugate to B if and only if $x + t = c + d$ and $xt - yz = cd$. Furthermore, the trace of $A'B'$ equals $ax + bt$. Since $a \neq b$, the linear system

$$\begin{cases} x + t = c + d,\\ ax + bt = k,\\ xt - yz = cd\end{cases}$$

has a solution for every k. This proves (1). Now (2) directly follows from (1), and (3) from (2). $\square$

Lemma 4.2. *Suppose K is a 2-closed field and $A \in \mathrm{SL}(6, K)$ is a block upper-triangular matrix with three diagonal (2×2)-blocks. Moreover, suppose that either*

(1) *all diagonal blocks are nonscalar, or*
(2) *exactly one of the three diagonal blocks is scalar.*

Then A is a product of three matrices of type ‡.

Proof. 1. Denote by a, b, c the determinants of the first, the second, and the third block, respectively. We have $abc = 1$. Moreover, it is easily seen that every 2-closed field is infinite. Consider any nonscalar matrix $P = D(p, \frac{1}{p}, p, \frac{1}{p}, p, \frac{1}{p})$, i.e., $p \neq \pm 1$. Then, by Lemma 4.1 (1), we can conjugate P by a block diagonal matrix $Q \in \mathrm{GL}(6, K)$ with (2×2)-blocks such that $AQPQ^{-1}$ has the eigenvalues $\{x, \frac{a}{x}\}, \{y, \frac{b}{y}\}$, and $\{z, \frac{c}{z}\}$ in the corresponding diagonal blocks for some $x, y, z \in K^*$ with $x \neq \frac{a}{x}, y \neq \frac{b}{y}$, $z \neq \frac{c}{z}$. Now we need to pick x, y, z such that $AQPQ^{-1}$ satisfies the assumption of Lemma 4.1 (3).

2. Suppose neither of a, b, c is equal to 1. Take $y = \frac{x}{a}$ and $z = \frac{y}{b} = cx$. Now $AQPQ^{-1}$ has the eigenvalues $x, \frac{1}{y}, y, \frac{1}{z}, z, \frac{1}{x}$. For $AQPQ^{-1}$ to satisfy the assumption of Lemma 4.1 (3), we need to choose $x \in K^*$ such that all eigenvalues of this matrix are pairwise distinct. This condition yields a system of 15 inequalities on x. It is easy to check that all these inequalities are polynomial (or can be made polynomial after multiplication by x). Since K is infinite and we have finitely many inequalities, in order to pick x we only need to show that there are no inequalities of degree $-\infty$, i.e., of the form $0 \neq 0$. Note that we have exactly 6 inequalities of degree at most 0. At the same time, the inequalities expressing the fact that x, $y = \frac{x}{a}$, and $z = cx$ are pairwise distinct, follow from the fact that $a, b, c \neq 1$ (together with the fact that $x \neq 0$); similar inequalities for $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ also follow. Therefore, all inequalities have degree at least 0, and we can pick $x \in K^*$ we need. With this x, the matrix $AQPQ^{-1}$ satisfies the assumption of Lemma 4.1 (3); therefore, it is the product of two matrices of type ‡. Hence, A is the product of three matrices of type ‡.

3. We need to consider two cases: $a = 1$ with $b, c \neq 1$ and $a = b = c = 1$. In the latter case, the matrix $AQPQ^{-1}$ has the eigenvalues $x, \frac{1}{x}, y, \frac{1}{y}, z, \frac{1}{z}$. In the former case, substituting $z = \frac{y}{b}$ we obtain the same matrix. For this matrix to satisfy the assumption of Lemma 4.1 (3), it must satisfy the same 15 inequalities in two (in the former case) or three (in the latter case) variables. It is easy to show that, as before, this system does not contain inequalities of degree $-\infty$, hence it has a solution. As in the previous case, we conclude that A is the product of three matrices of type ‡, which proves the first part of the lemma.

4. Now we prove the second part. Assume that the first block is the scalar block $D(d, d)$ with $d^2 = a$. It is easily seen that, as in the first part, to obtain $AQPQ^{-1}$ we need to take not an arbitrary P but the one with $p = \frac{x}{d}$ (since x, y, z do not depend on p, we can pick p after the choice of x, y, z). In this case, since $p \neq \pm 1$, we have two more inequalities on x, namely, $x \neq \pm d$. Since these inequalities have degree 1, the proof of the first part applies. $\square$

Theorem 5. *Suppose K is a 6-closed field. Then every matrix $A \in \mathrm{GL}(6, K)$ is a product of at most 4 matrices of type †.*

Proof. Since K is 6-closed, all the eigenvalues of A are in K. Therefore, the Jordan form of A is in $\mathrm{GL}(6, K)$, and A is conjugate to it. Moreover, since matrices of type † are defined up to conjugation, we may assume that A coincides with its Jordan form. We list all Jordan forms of matrices in $\mathrm{GL}(6, K)$. We partition them in accordance with eigenvalues.

I	$D(a,b,c,d,e,f)$		
II	a) $D(a,b,c,d,e,e)$,	b) $D(a,b,c,d,(e,e))$;	
III	a) $D(a,b,c,d,d,d)$,	b) $D(a,b,c,d,(d,d))$,	c) $D(a,b,c,(d,d,d))$;
IV	a) $D(a,b,c,c,d,d)$,	b) $D(a,b,c,c,(d,d))$,	c) $D(a,b,(c,c),(d,d))$;
V	a) $D(a,b,c,c,c,c)$,	b) $D(a,b,c,c,(c,c))$,	c) $D(a,b,c,(c,c,c))$,
	d) $D(a,b,(c,c),(c,c))$,	e) $D(a,b,(c,c,c,c))$;	
VI	a) $D(a,b,b,c,c,c)$,	b) $D(a,b,b,c,(c,c))$,	c) $D(a,b,b,(c,c,c))$,
	d) $D(a,(b,b),c,c,c)$,	e) $D(a,(b,b),c,(c,c))$,	f) $D(a,(b,b),(c,c,c))$;
VII	a) $D(a,a,b,b,c,c)$,	b) $D(a,a,b,b,(c,c))$,	c) $D(a,a,(b,b),(c,c))$,
	d) $D((a,a),(b,b),(c,c))$;		
VIII	a) $D(a,b,b,b,b,b)$,	b) $D(a,b,b,b,(b,b))$,	c) $D(a,b,b,(b,b,b))$,
	d) $D(a,b,(b,b),(b,b))$,	e) $D(a,b,(b,b,b,b))$,	f) $D(a,(b,b),(b,b,b))$,
	g) $D(a,(b,b,b,b,b))$;		
IX	a) $D(a,a,b,b,b,b)$,	b) $D(a,a,b,b,(b,b))$,	c) $D(a,a,b,(b,b,b))$,
	d) $D(a,a,(b,b),(b,b))$,	e) $D(a,a,(b,b,b,b))$,	f) $D((a,a),b,b,b,b)$,
	g) $D((a,a),b,b,(b,b))$,	h) $D((a,a),b,(b,b,b))$,	i) $D((a,a),(b,b),(b,b))$,
	j) $D((a,a),(b,b,b,b))$;		
X	a) $D(a,a,a,b,b,b)$,	b) $D(a,a,a,b,(b,b))$,	c) $D(a,a,a,(b,b,b))$,
	d) $D(a,(a,a),b,(b,b))$,	e) $D(a,(a,a),(b,b,b))$,	f) $D((a,a,a),(b,b,b))$;
XI	a) $D(a,a,a,a,a,a)$,	b) $D(a,a,a,a,(a,a))$,	c) $D(a,a,a,(a,a,a))$,
	d) $D(a,a,(a,a),(a,a))$,	e) $D(a,a,(a,a,a,a))$,	f) $D(a,(a,a),(a,a,a))$,
	g) $D(a,(a,a,a,a,a))$,	h) $D((a,a),(a,a),(a,a))$,	i) $D((a,a),(a,a,a,a))$,
	j) $D((a,a,a),(a,a,a))$,	k) $D((a,a,a,a,a,a))$.	

Since the matrices of type † can be multiplied by a scalar, we may assume that $\det A = 1$; moreover, in Case XI we assume that $a = 1$.

Notice that Cases I, II, III, IV, V without a), VI, VII, VIII without a), b), and c), IX without a) and f), X, and Cases h), i), k) in XI satisfy the assumption of Lemma 4.2 (1). Therefore, the theorem is proved for those cases. Furthermore, Cases V a), VIII b), c), IX a), f), XI d)–g), j) satisfy the assumption of Lemma 4.2 (2). Therefore, it remains to consider Cases VIII a) and XI a)–c). It is easy to check that we can take two, one, or zero matrices with Jordan normal form $D(1,1,1,1,(1,1))$ in Case XI c), b), a) respectively. Clearly, in the remaining Case VIII a) we can take four matrices of type †. $\square$

Corollary. *Suppose K is a 6-closed field and $A \in \mathrm{GL}(6, K)$. Then there exists $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $\overline{g} = A$ and g is a product of five root elements.*

Proof. By the preceding theorem, A is a product of four matrices of type †; we denote them by A_1, A_2, A_3, A_4. Observe that all these matrices are invertible. Take a root element $g_0 = x_\delta(1)$. By Theorem 4 (2) there exists a root element g_1 such that $\overline{g_0 g_1} = \overline{g_0} A_1 = A_1$. Applying Theorem 4 (2) three more times, we get $\overline{g_0 g_1 g_2 g_3 g_4} = A_1 A_2 A_3 A_4 = A$, as claimed. $\square$

§5. Nondegeneracy theorem

Our goal in this section is to prove the following theorem.

Theorem 6. *For every noncentral element $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ there exists $h \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that the submatrix $\{hgh^{-1}\}_{ij}$, where $i \in I_1$ and $j \in I_3$, is invertible.*

Recall that we denote the submatrix $\{g_{ij}\}_{i\in I_1, j\in I_3}$ of $g\in G_{\mathrm{sc}}(\mathrm{E}_6,K)$ by $\bar{g}$.

We need some auxiliary results.

Lemma 5.1. *Let u and v be two singular vectors.*

1. *If $v=au$, then there exists a matrix $h\in G_{\mathrm{sc}}(\mathrm{E}_6,K)$ such that $hu=e^1$ and $hv=ae^1$.*
2. *If $\langle u,v\rangle$ is a two-dimensional singular subspace, then there exists a matrix $h\in G_{\mathrm{sc}}(\mathrm{E}_6,K)$ such that $hu=e^1$ and $hv=e^{22}$.*
3. *If the subspace $\langle u,v\rangle$ is not singular, then there exists a matrix $h\in G_{\mathrm{sc}}(\mathrm{E}_6,K)$ such that $hu=e^1$ and $hv=e^{23}$.*

Proof. The first two parts follow immediately from [22, Theorem 2]. We prove the third part. By [22, Theorem 2], we may assume that $u=e^1$. Since the subspace $\langle u,v\rangle$ is not singular, by [22, Prop. 7] there exists a weight k such that it is nonadjacent to the first weight and $v_k\neq 0$.

Furthermore, let l be a weight adjacent to k such that $v_l\neq 0$. Multiply v by the root element $h_1=x_{l-k}\big(-c_{lk}\frac{v_l}{v_k}\big)$. Now, it is easy to check that $v_l=0$. By [22, Corollary to Prop. 2], the other coordinates corresponding to the weights adjacent to k do not change. The coordinate corresponding to k does not change either. Finally, observe that $h_1e^1\neq e^1$ if and only if the coordinate corresponding to α_1 in the decomposition of $l-k$ into simple roots is equal to -1. Clearly, this coordinate can be either 0 or 1, so that $u=e^1$ is invariant under h_1. Repeating this argument for every l, we can show that the coordinates of v corresponding to all weights adjacent to k are zero. At the same time, we still have $v_k\neq 0$ and $u=e^1$. Now [22, Prop. 13] yields $v=v_ke^k$.

It remains to show that we can obtain the pair (e^1,e^{23}) from the pair $(u,v)=(e^1,ae^k)$ for $d(1,k)=2$. It suffices to show that we can obtain the pair (e^1,e^l) for any weight l adjacent to k and nonadjacent to 1 from $(u,v)=(e^1,ae^k)$. This can be achieved by transforming $(u,v)=(e^1,ae^k)$ into (e^1,ae^k+be^l), where l is as before, and $b\in K$ is arbitrary. It is clear that we can do that by multiplying u and v by $x_{l-k}(c_{lk}\frac{b}{a})$. ☐

Lemma 5.2. *Suppose $A\in \mathrm{M}(6,K)$.*

1. *If $\operatorname{rk}A=1$, then there exists $B\in \mathrm{SL}(6,K)$ such that $\{BAB^{-1}\}_{ij}=0$ for $1\le i\le 6$, $1<j\le 6$;*
2. *If $\operatorname{rk}A=3$, then there exists $B\in\mathrm{SL}(6,K)$ such that $\{BAB^{-1}\}_{ij}=0$ for $1\le i\le 6$, $4\le j\le 6$, and the submatrix $\{BAB^{-1}\}_{ij}$, where $4\le i\le 6$, $1\le j\le 3$, is invertible.*

Proof. This is obvious. ☐

Assertion 4.

1. *Suppose v is a singular vector with $v_{22}\neq 0$ and $v_i=0$ for $22<i\le 27$. Then v is adjacent to e^1.*
2. *Suppose u,v are two singular vectors, $u\in V_1$, $v\notin V_1\oplus V_2$. Assume that $u_iv_{j-\delta}=u_jv_{i-\delta}$ for every $i,j\in I_1$. Then u is adjacent to v.*

Proof. First, we prove that for every $k\in I_2$ nonadjacent to the 22nd weight we have $v_k=0$. Suppose the contrary. Then there exists a weight $k\in I_2$ nonadjacent to both k and the 22nd weight with $v_k\neq 0$. Let $j\in I_1$ be a weight that is nonadjacent to both k and the 22nd weight. By definition, $Q(e^j,v)=\sum \pm v_lv_m$, where $d(j,l)=d(j,m)=d(l,m)=2$ and the pair $\{l,m\}$ is unordered. By [22, Prop. 5], one of the weights in each of the pairs $\{l,m\}$ belongs to I_3. Therefore, all the terms in that sum except for $\pm v_kv_{22}$ are zero. From the singularity of v it follows that $Q(e^j,v)=0$, whence $v_k=0$, a contradiction. Furthermore, by [22, Corollary to Prop. 2] and [22, Prop. 6] the weights nonadjacent to the first weight are precisely the weights j with $22 < j \le 27$ and the weights $k \in I_2$ nonadjacent to the 22nd weight. Therefore, the corresponding coordinates of v are equal to zero. Now the explicit description of F shows that $F(v, e^1, x) = 0$ for every x; therefore, v is adjacent to e_1.

Now we prove the second part. Let A_1 be a matrix in $\mathrm{SL}(V_1, K)$ taking u to e^1. As before, there exists a matrix $A \in D$ such that $A|_{V_1} = A_1$. Put $v' = Av$. Since $A_{V_1} = A_{V_3}$ and $u_i v_{j-\delta} = u_j v_{i-\delta}$ for all $i, j \in I_1$, we have $v'_i = 0$ for $22 < i \le 27$. Since $v \notin V_1 \oplus V_2$, we have $v' \notin V_1 \oplus V_2$, whence $v'_{22} \neq 0$. It remains to apply the first part, and we are done. □

Lemma 5.3. *Suppose $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ and $g_{ij} = 0$ for $1 \le i \le 6$ and $22 < j \le 27$. Suppose also that there exists i such that $g_{i,22} \neq 0$. Then there exists $h \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $(hgh^{-1})_{i*} = e_{22}$.*

Proof. Assume that $i = 1$. Transposing Assertion 4 (1) and applying it to $v = g_{i*}$, we see that the covector g_{i*} is adjacent to e_1. This means that we can apply the transpose of Lemma 5.1 (2) to obtain a matrix $f \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $e_1 f = e_1$ and $g_{1*} f = e_{22}$. Hence, $e_1 f^{-1} g f = e_{22}$. Therefore, we can take h to be f^{-1}.

Assume that $i \neq 1$. Applying the transpose of [22, Lemma 2.2] to $v = g_{i*}$ and $u = e_i$, we see that the covector g_{i*} is adjacent to e_i. In this case we can apply the transpose of Lemma 5.1 (3). This means that there exists a matrix $f \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $e_i f = e_i$ and $g_{i*} f = e_{22}$. Hence, $e_i f^{-1} g f = e_{22}$. Therefore, we can take h to be f^{-1}. □

Lemma 5.4. *Suppose* $\operatorname{rk} \bar{g} = 1$, *$\{\bar{g}\}_{ij} = 0$ for all $1 \le i \le 6$, $22 < j \le 27$, and there exists $1 \le i \le 6$ such that $\bar{g}_{i,22} \neq 0$. Then there exist weights $6 < k < 22$, $22 < j \le 27$ such that $d(i,k) = 1$, $j \neq i - \delta$, and $g_{kj} \neq 0$.*

Proof. Suppose the contrary. Take any weight $22 < j \le 27$ not equal to $i - \delta$. By assumption, $g_{kj} = 0$ for every $k \in I_2$ such that $d(i,k) = 1$. Suppose $l \in I_2$ and $d(i,l) = 2$. Furthermore, let $m \in I_3$ be a weight such that $d(i,m) = d(l,m) = 2$. Consider the pair of singular vectors ge^{22}, ge^j. They should span a two-dimensional singular subspace of V, so that $F(ge^{22}, ge^j, e^m) = 0$. By the definition of F, the left-hand side is equal to $\sum \pm (ge^{22})_s (ge^j)_t$, where the sum is taken over all ordered pairs of nonadjacent weights s, t nonadjacent to m. Note that any weight t nonadjacent to m is either nonadjacent to i and, by [22, Prop. 6], equals l, or is adjacent to i, or equals i. However, by assumption, $g_{tj} = 0$ for every $t \neq i - \delta$ such that $d(i,t) \le 1$. Since $i - \delta$ is adjacent to m, all the terms in that sum except for $\pm g_{i,22} g_{lj}$ are zero. Therefore, $g_{lj} = 0$ for every $l < 22$.

Now we have $ge^j \in V_3 = \langle e^{22}, e^{23}, e^{24}, e^{25}, e^{26}, e^{27} \rangle$ for every j such that $22 < j \le 27$, $j \neq i - \delta$. Moreover, $ge^j \in gV_3$. By [22, Prop. 12], if the intersection of two six-dimensional singular subspaces contains a four-dimensional subspace, they coincide. Therefore, $V_3 = gV_3$ and $ge^{22} \in V_3$. In particular, $g_{i,22} = 0$, which contradicts our assumption. □

Lemma 5.5. *Suppose $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$, $i_1, i_2, i_3 \in I_1$, and $g_{ij} = 0$ for $1 \le i \le 6$, $25 \le j \le 27$. Moreover, suppose that* $\det\{g_{ij}\} \neq 0$ *for $i = i_1, i_2, i_3$, and $22 \le j \le 24$. Then there exists $h \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $(hgh^{-1})_{ij} = 0$ for $i = i_1, i_2, i_3$, and $j \neq 22, 23, 24$.*

Proof. Consider the three-dimensional subspace

$$W = \langle g_{i*}; i = i_1, i_2, i_3 \rangle < V^*.$$

By assumption, it has a basis l^1, l^2, l^3 such that $l^t_s = \delta_{t,s+\delta}$ for $s \in I_3$. Since W is singular, we may apply the transpose of [22, Lemma 4.2] to obtain a root element f such that $f_{i*} = l^i$ for $1 \le i \le 3$. At the same time we have $(\delta)_f = 1$. Furthermore, put $f' = x_{-\delta}(-1) f x_{-\delta}(1)$. It follows that $V^{f'} = \langle f_{i*}; i \in I_1 \rangle$ and $f'_{ij} = f_{ij} + \delta_{i,j}$

for every $i \in I_1$. Assertion 2 (4) shows that there exists a unipotent $h \in U_2^-$ such that $V^{f'} = \langle h_{j*}; j \in I_3 \rangle$. As before, this means that $f_{ij} = f'_{ij} - \delta_{i,j} = h_{i-\delta,j}$ for every $i \in I_1$, whence $W = \langle h_{i-\delta,*}; 1 \le i \le 3 \rangle$. In other words, h takes the subspace $\langle e_{22}, e_{23}, e_{24} \rangle < V^*$ to W, while fixes the subspace $\langle e_i; i = i_1, i_2, i_3 \rangle < V^*$ (because $h \in U_2^-$). This implies that $e_i hgh^{-1} = e_i gh^{-1} = g_{i*}h^{-1} \in \langle e_{22}, e_{23}, e_{24} \rangle$ for $i = i_1, i_2, i_3$, completing the proof. □

Proof of Theorem 6. 1. By [22, Prop. 1], we can use conjugation to map any pair of root subgroups to any other pair with the same angle between them. Notice that the conjugation of a root element X by some $f \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ is the same as the action of f on the subspace V^X. In other words, $fV^X = V^{fXf^{-1}}$. By [22, Corollary to Theorem 2], any pair of six-dimensional singular subspaces can be mapped to any other pair with the same angle between them (= angle between the corresponding root subgroups) by multiplication (= by a change of basis). This means that the relative position of two six-dimensional singular subspaces is determined by the angle between them.

Consider the six-dimensional singular subspaces V_3 and gV_3. The argument above shows that there exists an element $f \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ such that $fV_3 = V_3$ and fgV_3 is equal to one of the following: $V^{X_{-\delta}} = V_3$, $V^{X_{-\alpha_2}}$, $V^{X_{\alpha_1}}$, $V^{X_{\alpha_2}}$, $V^{X_\delta} = V_1$. Since $f(gV_3) = (fgf^{-1})(fV_3)$, we can conjugate g if necessary and assume that gV_3 is one of the five subspaces described in the previous sentence. Moreover, $fV_3 = V_3$ implies $f \in P^-$. Thus, $\operatorname{rk} \overline{g} = \operatorname{rk} \overline{fgf^{-1}}$. It remains to show that this rank is determined by the angle between the subspaces V_3 and gV_3: if $gV_3 = V_3$ or $V^{X_{-\alpha_2}}$, i.e., $\angle(V_3, gV_3) = 0$ or $\pi/3$, then the matrix $\overline{g}$ is zero; if $gV_3 = V^{X_{\alpha_1}}$, i.e., $\angle(V_3, gV_3) = \pi/2$, then $\operatorname{rk} \overline{g} = 1$; if $gV_3 = V^{X_{\alpha_2}}$, i.e., $\angle(V_3, gV_3) = 2\pi/3$, then $\operatorname{rk} \overline{g} = 3$; finally, if $gV_3 = V_1$, i.e., $\angle(V_3, gV_3) = \pi$, then $\operatorname{rk} \overline{g} = 6$. Suppose W is a six-dimensional singular subspace such that $\angle(W, gW) = \pi$ and $fW = V_3$ for some $f \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$. Then the angle between W and gW is the same as the angle between $V_3 = fW$ and $f(gW) = fgf^{-1}V_3$. Therefore, the claim of the theorem is equivalent to the existence of a six-dimensional subspace W forming the angle π with gW.

Since g is noncentral, there is a singular vector not proportional to its image. By Lemma 5.1, we can map this vector to either e^{22} or e^{23}, while its image maps to e^1. This means that the angle between V_3 and gV_3 is at least $\pi/2$. We need to conjugate g in order to make this angle equal to π. We prove that if this angle is equal to $\pi/2$ or $2\pi/3$, we can increase it. As was mentioned in §3, the conjugation of g by a matrix $A \in D$ corresponds to the conjugation of $\overline{g}$ by $A_1 = A|_{V_1}$. Since we can take A_1 to be an arbitrary matrix in $\mathrm{SL}(6, K)$, we can apply Lemma 5.2 in both cases.

2. Assume that the angle between V_3 and gV_3 equals $\pi/2$. We write the matrix $\overline{g}$ in the form described in Lemma 5.2. Since $\operatorname{rk} \overline{g} = 1$, it is clear that there exists $i \in I_1$ such that $g_{i,22} \ne 0$. By Lemma 5.3, we may assume that $g_{it} = 0$ for $t \ne 22$. If g_{kj} is still nonzero for some $1 \le k \le 6, 22 < j \le 27$, then $\operatorname{rk} \overline{g} > 1$, and this is exactly what we need (as was mentioned above, if $\operatorname{rk} \overline{g} > 1$, then the angle between V_3 and gV_3 is greater than $\pi/2$). Otherwise, we can pick k and j in accordance with Lemma 5.4. Let $i_1 = i, i_2, i_3$ be three of the first six weights adjacent to k and not equal to $j + \delta$ (if $d(k, j) = 1$, there are exactly three such weights; otherwise there are four and we pick any three of them). Let i_4, i_5, i_6 are the remaining weights, and assume that $d(k, i_4) = 1$.

Let $\alpha = i_4 - k \in \mathrm{E}_6$. By [22, Prop. 4], $\angle(\alpha, \delta) = \pi/3$. We conjugate g by a root element $x_\alpha(a)$. Notice that after multiplication of g by $x_\alpha(a)$ on the left, i.e., after replacing g with $x_\alpha(a)g$, the rows $\rho_l - \alpha$ multiplied by $\pm a$ add to the rows ρ_l if $\rho_l - \alpha \in \Lambda$. In particular, the row i_4 is replaced by the sum of that row and the row k multiplied by $\pm a$. By [22, Corollary of Prop. 2], all the weights ρ_l except for i_4 are nonadjacent to k. By [22, Prop. 4], there are exactly three weights ρ_l in I_1; hence, these are the weights i_4, i_5, and i_6. Furthermore, multiplying g by $x_\alpha(-a)$ on the right, i.e., replacing g with $gx_\alpha(-a)$, we see that the columns σ_l for $\sigma_l + \alpha \in \Lambda$ are replaced with the sums of those columns and the columns $\sigma_l + \alpha$ multiplied by $\pm a$. In particular, the column k is replaced by the sum of that column and the column i_4 multiplied by $\pm a$. All the weights σ_l except for k are adjacent to k and nonadjacent to i_4 (by [22, Corollary to Prop. 2]). By [22, Prop. 4], there are exactly three weights σ_l in I_3, and [22, Prop. 6] shows that these are the weights $i_1 - \delta, i_2 - \delta$, and $i_3 - \delta$.

Consider the impact of the conjugation of g by $x_\alpha(a)$ on $\bar{g}$. First, something is added to the rows i_4, i_5, i_6. Note that the row k multiplied by $\pm a$ gets added to the row i_4. The result is that $g_{i_4,j}$ equals $\pm a g_{kj} \neq 0$ (by the choice of k and j), while still $g_{it} = 0$ for $t \neq 22$. Next, something gets added to the columns $i_1 - \delta, i_2 - \delta$, $i_3 - \delta$. Thus, $g_{i_4,j} \neq 0$ (since $j \neq i_1 - \delta, i_2 - \delta, i_3 - \delta$), while still $g_{it} = 0$ for $t \neq 22$. It remains to observe that the rank of the result is greater than 1.

3. Assume that the angle between V_3 and gV_3 equals $2\pi/3$. We write the matrix $\bar{g}$ in the form described in Lemma 5.2. Suppose that the submatrix $\{g_{ij}\}$, $1 \leq i \leq 3$, $22 \leq j \leq 24$, is invertible. Then, by Lemma 5.5, we can conjugate g and assume that $g_{ij} = 0$ for every $1 \leq i \leq 3$ and $j \neq 22, 23, 24$. If there exists $i \in I_1$ and $25 \leq j \leq 27$ such that $g_{ij} \neq 0$, then $\operatorname{rk} \bar{g} > 3$, and we are done. Otherwise, since g is invertible, there are covectors x_1, x_2, x_3 such that $x_i g = e_{i+24}$ for $1 \leq i \leq 3$. They are singular, because their images are singular. Together with e_1, e_2, e_3 they span a six-dimensional singular subspace in V^*.

Applying the transpose of [22, Prop. 7] several times, we see that the covectors x_1, x_2, x_3 may have nonzero coefficients only at the positions 1–8 and 12. By the definition of x_i, since $g_{ij} = 0$ for $1 \leq i \leq 6$, $25 \leq j \leq 27$, the (3×3)-submatrix on the intersection of the rows 7, 8, 12 with the last three columns is invertible.

Conjugate g by the elementary root unipotent $x_{\alpha_2}(1)$. Multiplying g by $x_{\alpha_2}(1)$ from the left, i.e., replacing g with $x_{\alpha_2}(1)g$, we add the rows 7, 8, 12 to the rows 4, 5, 6, respectively (here we list only the changes in the submatrix $\bar{g}$). Therefore, after this multiplication we still have $\det\{g_{ij}\} \neq 0$, $1 \leq i \leq 3$, $22 \leq j \leq 24$, and $g_{ij} = 0$, $1 \leq i \leq 3$, $25 \leq j \leq 27$. At the same time, $\det\{g_{ij}\} \neq 0$, $4 \leq i \leq 6$, $25 \leq j \leq 27$. Notice that, after multiplication of $x_{\alpha_2}(1)g$ by $x_{\alpha_2}(-1)$ from the right, the columns 25, 26, 27 stay the same, and the columns 22, 23, 24 are replaced by the sum of these columns and the columns 19, 20, 21, respectively. Observe that we still have $\det\{g_{ij}\} \neq 0$, $1 \leq i \leq 3$, $22 \leq j \leq 24$, $g_{ij} = 0$ for $1 \leq i \leq 3$, $25 \leq j \leq 27$, and $\det\{g_{ij}\} \neq 0$ for $4 \leq i \leq 6$, $25 \leq j \leq 27$. Clearly, the rank of this matrix is 6. This completes the proof.

4. It remains to consider the case where the angle between V_3 and gV_3 equals $2\pi/3$ and the matrix $\{g_{ij}\}$, $1 \leq i \leq 3$, $22 \leq j \leq 24$, is not invertible. Recall that the matrix $\{g_{ij}\}$, $4 \leq i \leq 6$, $22 \leq j \leq 24$, is invertible by Lemma 5.2. By Lemma 5.5, we may assume that $g_{ij} = 0$, $4 \leq i \leq 6$, $j \neq 22, 23, 24$. Consider the subgroup $D' = \langle X_\alpha; \alpha \perp \delta, \alpha_2 \rangle$. It is clear that $D' < D$ and $D' \cong \mathrm{SL}(3, K) \times \mathrm{SL}(3, K)$, because $\mathrm{SL}(3, K) = G_{\mathrm{sc}}(\mathrm{A}_2, K)$ coincides with $E_{sc}(\mathrm{A}_2, K)$ (we mentioned this in the Introduction), i.e., it is generated by transvections. Suppose $A \in D'$ and $A_1 = A|_{V_1}$. Then $A_1 = \left(\begin{smallmatrix} A_1^1 & 0 \\ 0 & A_1^2 \end{smallmatrix}\right)$, where A_1^1 and A_1^2 belong to $\mathrm{SL}(3, K)$. Suppose $\bar{g} = \left(\begin{smallmatrix} X^1 & X^3 \\ X^2 & X^4 \end{smallmatrix}\right)$, where the X^i are (3×3)-matrices. If we conjugate g by A, the submatrix X^1 gets conjugated by the submatrix A_1^1, while X^4 gets conjugated by A_1^2. At the same time, the submatrix X^2 is multiplied by A^2 from the left and by $(A_1^1)^{-1}$ from the right. Finally, the submatrix X^3 is multiplied by A_1^1 from the left and by $(A_1^2)^{-1}$ from the right. In our case $X^3 = X^4 = 0$ and $\det X^2 \neq 0$, $\det X^1 = 0$. It is well known that there exists a matrix $A_1^1 \in \mathrm{SL}(3, K)$ such that the first row of X^1 is zero. Consider the first six basis covectors. Since they span a six-dimensional singular subspace in V^*, their images also span a six-dimensional singular subspace. Applying

[22, Prop. 7] several times, we see that if a covector is adjacent to e_{22}, e_{23}, and e_{24}, then all of its nonzero coefficients are at the positions 7, 8, 12, and at the last six positions. In particular, $g_{1j} = 0$ for $j \neq 7, 8, 12$. Furthermore, denote by X^5 the submatrix $\{g_{ij}\}$, $1 \leq i \leq 3, j = 7, 8, 12$. The conjugation of g by the same matrix A corresponds to the multiplication of X^5 by A_1^1 from the left and by $(A_1^2)^{-1}$ from the right. It is easily seen that we can choose A_1^1 and A_1^2 such that

(1) the first row of X^1 is zero;
(2) X^2 is a diagonal matrix;
(3) $g_{1,7}$ is not equal to 0.

Consider the six-dimensional singular subspace $W = \langle e_1, e_5, e_6, e_{16}, e_{17}, e_{19} \rangle < V^*$ and its image. Suppose $e_1 g = ae_7 + be_8 + ce_{12}$; it is clear that $e_5 g = ke_{23}$ and $e_6 g = le_{24}$. As above, $e_{19} g$ can have nonzero coordinates 7, 8, 12 and the last six. Since g is invertible, at least one of the three last coordinates is nonzero. Since the first weight is adjacent to the 19th weight, $e_{19} g$ is adjacent to $e_1 g = ae_7 + be_8 + ce_{12}$. The explicit description of F shows that

$$e_{19} g = m(ae_7 + be_8 + ce_{12}) + n(ae_{25} + be_{26} + ce_{27}) + g_{19,22} e_{22} + g_{19,23} e_{23} + g_{19,24} e_{24}.$$

Moreover, $n \neq 0$. By [22, Prop. 7], there exists a unique (up to a scalar multiplication) covector in W adjacent to e_{23}, and a similar statement holds true for e_{24}. Furthermore, it is easily seen that this statement is true also for $ae_7 + be_8 + ce_{12}$ and for $e_{19}^T g$. However, in the latter case it is simpler to check this for the covector $n(ae_{25} + be_{26} + ce_{27}) + g_{19,22} e_{22}$ in Wg. Now, by the transpose of [22, Prop. 12], the subspaces W and Wg are opposite; in other words, the angle between them equals π. Now we can transpose the argument used in (1) to finish the proof. □

§6. Main theorem

In this section we prove the main theorem of the present paper.

Main theorem. *Suppose that K is a 6-closed field, i.e., every polynomial of degree at most 6 has a root in K. Then every element in $G_{\mathrm{ad}}(\mathrm{E}_6, K)$ can be expressed as a product of at most eight root elements.*

Proof. 1. As was noted in §1, $G_{\mathrm{ad}}(\mathrm{E}_6, L) = G_{\mathrm{sc}}(\mathrm{E}_6, L)/\mu_3$ for every 3-closed field L. For each element of $G_{\mathrm{ad}}(\mathrm{E}_6, K)$ there are three elements of $G_{\mathrm{sc}}(\mathrm{E}_6, K)$; they differ from each other by a factor of $\sqrt[3]{1}$. Let $g \in G_{\mathrm{sc}}(\mathrm{E}_6, K)$ be any element corresponding to our element in $G_{\mathrm{ad}}(\mathrm{E}_6, K)$. By Theorem 6 we may assume that $\det \overline{g} \neq 0$. Furthermore, by the corollary to Theorem 5, there exists a product h of five root elements such that $\overline{h} = \overline{g}$. By Theorem 3, $g = v_1 A' w_1$ and $h = v_2 B' w_2$, where $v_1, w_1, v_2, w_2 \in U_2^-$ and $A', B' \in L'$. Moreover, it is clear that $\overline{A'} = \overline{g} = \overline{h} = \overline{B'}$.

2. Suppose $A = A' J^{-1}$ and $B = B' J^{-1}$, where $J = x_\delta(1) x_{-\delta}(-1) x_\delta(1)$. Then, by definition, both A and B lie in L. By the definition of L, A (respectively, B) is a block diagonal matrix with three blocks $A_1 = A|_{V_1}$ (respectively, $B_1 = B|_{V_1}$), $A_2 = A|_{V_2}$ (respectively, $B_2 = B|_{V_2}$), and $A_3 = A|_{V_3}$ (respectively, $B_3 = B|_{V_3}$). Therefore, A' and B' consist of three blocks on the antidiagonal, and $A_1 = \overline{A'} = \overline{B'} = B_1$ (as before, here we use the standard isomorphism $V_3 \to V_1$, see §3). By Theorem 1, either $A = B$ (and, therefore, $A' = B'$), or there exists $\xi = \sqrt[3]{1}$ such that $B_2 = \xi A_2$ and $B_3 = \xi^2 A_3$. Assume that $A' \neq B'$. Conjugate h by a diagonal matrix f such that $f_{\rho\rho} = \xi^2$ for $\rho \in I_1$, $f_{\rho\rho} = 1$ for $\rho \in I_2$, and $f_{\rho\rho} = \xi$ for $\rho \in I_3$. Notice that f belongs to $G_{\mathrm{sc}}(\mathrm{E}_6, K)$ by Lemma 2.1 and [22, Prop. 5]. It is easily seen that $v_3 = f v_2 f^{-1}$ and $w_3 = f w_2 f^{-1}$ still belong to U_2^-, whereas $C' = f B' f^{-1}$ belongs to L'. Put $C = C' J^{-1}$. Clearly, $C_1 = C|_{V_1} = \xi B_1 = \xi A_1$, $C_2 = C|_{V_2} = B_2 = \xi A_2$, and $C_3 = C|_{V_3} = \xi^2 B_3 = \xi A_3$. Thus, $C = \xi A$ and $C' = \xi A'$.

Furthermore, since we are interested in elements of $G_{\mathrm{ad}}(\mathrm{E}_6, K)$, we can replace g with $\xi g = v_1(\xi A')w_1 = v_1 C' w_1$. Therefore, we may assume that $A' = B'$.

3. The first two parts of the proof imply that $g = v_1 A' w_1$ and $h = v_2 B' w_2 = v_2 A' w_2$, whereas h is a product of five root elements. Conjugating g with the unipotent w_1, and h with w_2, we may assume that $g = v_1 A'$ and $h = v_2 A'$. Thus, $gh^{-1} = v_1 v_2^{-1} \in U_2^-$ is a product of three root elements by Theorem 2. Therefore, g is a product of eight root elements. $\square$

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