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A_3-PROOF OF STRUCTURE THEOREMS FOR CHEVALLEY GROUPS OF TYPES E_6 AND E_7 II. MAIN LEMMA

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Abstract. The author and Mikhail Gavrilovich proposed a geometric proof of the structure theorems for Chevalley groups of types $\Phi = \mathrm{E}_6, \mathrm{E}_7$, based on the following fact. There are nontrivial root unipotents of type A_2 stabilizing columns of a root element. In the present paper it is shown that *two* adjacent columns of a root element can be stabilized simultaneously by a nontrivial root unipotent of type A_3. This makes it possible to prove structure theorems for Chevalley groups of types E_6 and E_7 and their forms, by using only the presence of split classical subgroups of very small ranks.

In the present paper, which is a sequel of [43, 6] and [45], we prove the main lemma of the A_3-proof for Chevalley groups $G = \mathrm{G}(\Phi, R)$ of types $\Phi = \mathrm{E}_6, \mathrm{E}_7$ over a commutative ring R. Thus, here we complete the proof of Theorem 2 stated in [45]. That theorem asserts that by a root type element sitting in a fundamental subgroup of type A_3, one can *simultaneously* stabilize in a nontrivial way two *adjacent* columns of a *root* element $g \in G$, in the minimal representation $V = V(\omega)$.

To put this result in context, let us recall some preceding statements in the same spirit.

• In the paper [17] by the author, Eugene Plotkin, and Alexei Stepanov, it was observed that, by an element of root type sitting in a subgroup of type A_5 for $\Phi = \mathrm{E}_6$ and of type A_7 for $\Phi = \mathrm{E}_7$, one can stabilize a column of an *arbitrary* element $g \in G$ in a minimal representation. A sketch of the proof, with a reference to computer calculations, was reproduced in [42]. In [43] we produced a complete *human* proof. In those papers, and in [7, 41, 46, 47], one can find a much more detailed discussion of the general context and the role of these results in the proof of the main structure theorems.

• In the paper [6] by the author and Mikhail Gavrilovich, we proposed an A_2-proof of the main structure theorems for Chevalley groups of types E_6 and E_7, based on the following fact. Commuting a noncentral root unipotent $g \in G$ with an appropriate element of root type sitting in a fundamental subgroup of type A_2 — in the sequel such elements are simply called elements of root type A_2 — one can get a *nontrivial* element in the *maximal* parabolic subgroup of type P_1 or P_7, respectively. This makes it possible

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to give a very easy proof of structure theorems at the level of K_1, in particular, of the standard description of normal subgroups.

• Later, in the paper [16] by the author and Sergei Nikolenko, this proof was extended also to Chevalley groups of type F_4, where one can only use elements of *long* root type A_2, to get into a parabolic subgroup of type P_4. On the other hand, in the paper [14] by the author and Alexander Luzgarev, a similar proof was carried through for the last remaining case of Chevalley groups of type E_8. In this last case, in general, one needs *two* commutations with elements of root type A_2, rather than one such commutation, to get into a proper parabolic subgroup of type P_8.

Throughout the rest of this paper $\Phi = \mathrm{E}_6, \mathrm{E}_7$. To indicate that the resulting answer depends on whether $\Phi = \mathrm{E}_6$ or E_7, we write "A resp B". Such a record should be interpreted as follows: the outcome A holds for E_6, whereas the outcome B holds for E_7. Recall that the group G acts on the microweight module $V = V(\varpi_1)$ resp $V = V(\varpi_7)$ of dimension 27 resp 56. Denote by Λ the set of weights of V. Two weights $\lambda, \mu \in \Lambda$ are said to be adjacent if their difference is a root, in this case we write $d(\lambda, \mu) = 1$. In general, the distance $d(\lambda, \mu)$ between two weights λ and μ is the length of a shortest chain of weights joining λ with μ in which every two consecutive terms are adjacent. Thus, if two distinct weights $\lambda, \mu \in \Lambda$ are not adjacent, then $d(\lambda, \mu) \geq 2$. All other notation pertaining to Chevalley groups and their representations will be recalled in §1.

It is well known that, in the minimal representations, Chevalley groups of types E_6 and E_7 are defined by equations of degrees 3 resp 4, with respect to the natural coordinates; see references in [11, 12, 42, 43, 47]. The first of the above-mentioned series of papers allows us, when working with one column, to take into account only *quadratic* equations on its components. The second goes much further, since it allows us, modulo some easy group theoretic arguments, to use only *linear* equations defining the corresponding Lie algebra. In the present paper we persist in the same line of research, and demonstrate that with the same tools one can simultaneously calculate with two columns of a root element.

Theorem. *Let $\Phi = \mathrm{E}_6, \mathrm{E}_7$. Fix two adjacent weights $\lambda, \mu \in \Lambda$, $d(\lambda, \mu) = 1$, and assume that for some $z \in G(\Phi, R)$ the commutator $g = [z, x_\rho(1)]$ is noncentral. Then there exists an element*

$$x = x_{\beta_1}(\xi) x_{\beta_2}(\zeta) x_{\beta_3}(\eta)$$

of root type A_3 that stabilizes the columns of g with indices λ and μ,

$$(xg)_{*\lambda} = g_{*\lambda}, \qquad (xg)_{*\mu} = g_{*\mu}$$

such that $[x, g] \neq e$.

Originally, this theorem, with a *sketch* of the proof for the case of E_6, was stated in my paper [45]. There, in terms of second order minors, I constructed some unipotents x that *obviously* stabilize two columns of a matrix g subject to some linear equations. However, due to length limitations, the proof of what was called the *main lemma* in [6, 16], was omitted. In other words, in [45] we have wittingly left out the analysis of the situation where all such obvious unipotents x that stabilize columns of g commute with g. Incidentally, this is the most interesting and by far the hardest part of the proof, which depends on important new technical artifices. Next, in [45] we skipped many further important details, not as deep, but still rather tricky. All these omissions will find complete explanations in the present paper.

We recall that each column of a root element $g = z x_\rho(1) z^{-1}$, in its action on a minimal module, has 10 resp 28 zero components. Namely, $g_{\lambda\mu} = 0$ for all pairs of weights $\lambda, \mu \in \Lambda$

such that $d(\lambda,\mu)\geq 2$. Since right multiplication by an elementary root unipotent changes at most 6 resp 12 columns, the remaining 21 resp 44 columns of the commutator

$$[z,x_\rho(1)]=\big(zx_\rho(1)z^{-1}\big)x_\rho(-1)$$

are not affected by this action, and enjoy the same property. Therefore, in the present paper we shall actually prove the following fairly technical but more precise statement.

The main lemma. *Let $\Phi=\mathrm{E}_6,\mathrm{E}_7$, and let two weights $\lambda,\mu\in\Lambda$, $d(\lambda,\mu)=1$, be fixed. Suppose that the following relations are valid for a noncentral element $g\in G(\Phi,R)$:*

- $g_{\sigma\lambda}=0$ *for all* $\sigma\in\Lambda$, $d(\lambda,\sigma)\geq 2$,
- $g_{\tau\mu}=0$ *for all* $\tau\in\Lambda$, $d(\mu,\tau)\geq 2$.

Then either g itself is contained in a proper parabolic subgroup of type P_2 resp P_1, or there exists an element x of root type A_3 such that $[g^{-1},x]\neq e$ is contained in a proper parabolic subgroup of type P_3 resp P_6.

The proof is based on two main ideas. First, it is the procedure of simultaneous stabilization of several columns by root elements whose parameters are specified in terms of minors. For the first time, this procedure was used in 1987 in the Theses of Alexei Stepanov and the present author; see [41]. Second, it is an observation from the paper [6] by the author and Mikhail Gavrilovich. Namely, we noticed that to hit a proper parabolic subgroup upon commutation, one does not have to stabilize a column of a matrix *verbatim*. This column may change slightly, provided that the extra summands belong to the annihilators of the appropriate entries of the inverse matrix. This easy consideration allows us to consecutively lower the degrees of the occurring relations.

The present paper addresses exclusively the practical and *technical* aspects of calculations in minimal modules. Our goal here is to provide evidence that such calculations afford a much wider technical freedom than it is customary to believe. The philosophy, history, and further applications of related techniques were discussed in much more detail in [7, 42, 43, 46].

Initially, the trick of simultaneous stabilization of two columns was developed with the hope to prove structure results at the level of K_2, in particular, to prove the centrality of the extension $\mathrm{St}(\Phi,R)\longrightarrow E(\Phi,R)$. This project was described in [45], but over the last three years I failed to obtain any tangible further progress in this direction.

However, the *facility* itself, with which we can now perform *by hand* calculations with *several* columns of matrices from exceptional groups of types E_6 and E_7 in their minimal representations, is very suggestive and tempting as such. An additional important aspect is that we invoke no equations whatsoever, apart from the obvious linear equations defining the corresponding Lie algebra. In the long run, this seems to be much more interesting and important than the majority of specific structure results.

The paper is organized as follows. In §1 we recall some requisite notation, and in §2 state the necessary technical lemmas from [6] and some further facts in the same spirit. The rest of the paper is devoted to a detailed proof of the above main lemma. In §3 we construct unipotents of root type A_2 that simultaneously stabilize two adjacent columns of a root element g of the group of type E_6 in the 27-dimensional representation. In §4 we examine what happens when all such unipotents commute with g. In §5 and §6 we do the same for two adjacent columns of a root element g of the Chevalley group of type E_7 in the 56-dimensional representation. Finally, in §7 we discuss further prospects and state several unsolved problems.

§1. Principal notation

The present paper is embedded into a certain general context and it is not feasible to try to recall all necessary background here. In fact, it is an *immediate* sequel of [42, 43, 6, 7, 16, 45] and we assume that the reader has seen at least *some* of the above papers. Thus, we limit ourselves to a few generic references and summon the basic notation used in the sequel.

• The works [2, 25, 26, 33] may serve as background references on Chevalley groups and their representations.

• General facts pertaining to Chevalley groups over rings were discussed in [1, 28, 29, 30, 31, 35, 39, 40, 42, 47]. There one can also find an extensive bibliography.

• All necessary technical information regarding weight diagrams and signs of structure constants can be found in [4, 5, 12, 38, 43, 44, 47].

• The Chevalley groups of types E_6 and E_7 were studied in related aspects in [6, 7, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 37, 42, 43, 45].

1°. Chevalley groups. Let Φ be a reduced irreducible root system of rank l; in the sequel we usually assume that $\Phi = \mathrm{E}_6$ or E_7. Further, let P be a lattice lying between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$. We fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+, and Φ^- the corresponding sets of fundamental, positive, and negative roots, respectively. Our numbering of fundamental roots follows [3]. By ρ we denote the maximal root of Φ with respect to this order. Thus, for $\Phi = \mathrm{E}_6$, E_7, and E_8 the root ρ equals

$$\begin{matrix}12321\\2\end{matrix} \qquad \begin{matrix}234321\\2\end{matrix} \qquad \begin{matrix}2465432\\3\end{matrix}$$

respectively. Denoting by $P(\Phi)_{++}$ the set of dominant weights for this order, we recall that it consists of all nonnegative integer linear combinations of the fundamental weights $\varpi_1, \dots, \varpi_l$. By $W = W(\Phi)$ we denote the Weyl group of Φ.

Next, let R be a commutative ring with 1. It is classically known that from this data one can construct the Chevalley group $G = G_P(\Phi, R)$. This group is the group of R-points of a certain affine group scheme $G = G_P(\Phi, -)$, known as the Chevalley–Demazure scheme. Usually we consider simply connected groups, for which $P = P(\Phi)$. Mostly, for simply connected groups we drop any mention of P and write simply $G(\Phi, R)$. Sometimes, when it is important to emphasize that the group is simply connected, we write $G_{\mathrm{sc}}(\Phi, R)$. The adjoint group for which $P = Q(\Phi)$ is denoted by $G_{\mathrm{ad}}(\Phi, R)$.

We fix a split maximal torus $T(\Phi, R)$ in the Chevalley group $G(\Phi, R)$ and a parametrization of the unipotent root subgroups X_α, $\alpha \in \Phi$, elementary with respect to this torus. Let $x_\alpha(\xi)$ be the elementary root unipotent corresponding to $\alpha \in \Phi$, $\xi \in R$. Recall that $X_\alpha = \{x_\alpha(\xi), \xi \in R\}$, and that the absolute elementary group $E(\Phi, R) = \langle X_\alpha, \alpha \in \Phi\rangle$ is spanned by all elementary root subgroups.

A paramount role in the sequel is played by parabolic subgroups of the Chevalley group G. We are mostly interested in maximal parabolic subgroups. Fix an r, $1 \le r \le l$; the corresponding maximal parabolic subgroup, its Levi subgroup, and its unipotent radical will be denoted by P_r, L_r, and U_r, respectively. Levi decomposition asserts that P_r decomposes into the semidirect product $P_r = L_r \leftthreetimes U_r$. Any subgroup conjugate to P_r is called a parabolic subgroup of type P_r.

2°. Weyl modules. We always fancy a Chevalley group together with its action on the Weyl module $V = V(\omega)$ for a dominant weight ω. Commonly, we assume that the highest weight ω of the module V is fundamental, $\omega = \varpi_r$. Below, when we talk of Chevalley groups of types E_6 or E_7, these groups are considered together with their action on one of the two contragredient 27-dimensional modules $V(\varpi_1)$ or $V(\varpi_6)$ resp the 56-dimensional module $V(\varpi_7)$. Observe that these modules are microweight, see [35, 38, 40, 42] and the references there. We denote by $\Lambda = \Lambda(\omega)$ the set of weights of the module $V = V(\omega)$ *with multiplicities*. For a microweight representation, all weights are extremal and thus have multiplicity 1, so that in this case Λ coincides with the Weyl orbit of the highest weight, $\Lambda = W\omega$.

In the sequel we fix an admissible base v^λ, $\lambda \in \Lambda$, of the module V. Recall that a base is said to be admissible if it consists of weight vectors and $x_\alpha(\xi)v^\lambda$ can be expressed as an integral linear combination of the base vectors v^μ, $\mu \in \lambda$. For a microweight representation, an admissible base can be normed in such a way that $x_\alpha(\xi)v^\lambda = v^\lambda + c_{\lambda\alpha}\xi v^{\lambda+\alpha}$, where all action structure constants $c_{\lambda\alpha}$ are equal to ± 1. As a matter of fact, we always choose a crystal base, where all structure constants $c_{\lambda\alpha}$ are equal to $+1$ for fundamental and negative fundamental roots, $c_{\lambda\alpha} = +1$ if $\alpha \in \pm\Pi$, see the references in [43, 4].

We conceive a vector $a \in V$, $a = \sum a_\lambda v^\lambda$, as a coordinate *column* $a = (a_\lambda)$, $\lambda \in \Lambda$. In this setting, it is natural to think of an element b of the contragredient module V^* as a coordinate *row* $b = (b_\lambda)$, $\lambda \in \Lambda$. Obviously, with respect to the weights Λ^* of the contragredient module V^*, the picture is reversed: the elements of V^* should be represented by *columns* $b = (b_\lambda)$, $\lambda \in \Lambda^*$, whereas the elements of V are conceived as *rows* $a = (a_\lambda)$, $\lambda \in \Lambda^*$.

It is essential to emphasize that in the present paper we index *both* columns and rows of all occurring matrices by the weights of V itself. In particular, the indices λ, μ, ν always belong to Λ. In other words, it is convenient for us to index the coordinates of a vector from the dual module V^* by the weights of the module V and visualize them as rows. Ordinarily, in other sources they are indexed by the weights of the module V^* itself and visualized as columns. This is precisely why in our paper the formulas describing the action of elements of G look differently for rows and for columns.

An important technical aspect is that the components of these rows and columns are not linearly ordered but rather partially ordered, in accordance with the order on Λ determined by the choice of the fundamental system Π. Namely, we set $\lambda \geq \mu$ if $\lambda - \mu = \sum m_i\alpha_i$, where $m_i \geq 0$. Under the above interpretation of the elements of V, it is natural to conceive elements of the Chevalley group as matrices $g = (g_{\lambda\mu})$, $\lambda, \mu \in \Lambda$, with respect to the base v^λ. As usual, the columns of this matrix are the coordinate columns of the vectors gv^μ, $\mu \in \Lambda$, with respect to the base v^λ, $\lambda \in \Lambda$. We constantly use the following piece of notation: the μth column of the matrix g is denoted by $g_{*\mu}$, whereas the λth row of this matrix is denoted by $g_{\lambda *}$.

There are several ways to introduce a linear order on the weights of V that is compatible with the above partial order. Quite often in the literature, one encounters the *natural* ordering, also known as the height-lexicographic order, where weights are ordered by height, and the weights of the same height are ordered lexicographically. However, for our purposes it is usually more convenient to order weights in accordance with branching relative to certain subsystems $\Delta \subseteq \Phi$, specifically, subsystems of types D_5, or A_5 resp A_6, or E_6. In other words, first we subdivide the weights of V into the irreducible components of the restriction of V to $G(\Delta, R)$, and then number them naturally inside the components. The tables of weights with respect to these orderings were reproduced in [15, 13].

Throughout the present paper we strictly observe the following alphabetic coding: $\alpha, \beta, \gamma, \delta \in \Phi$ denote roots; $\lambda, \mu, \nu, \rho, \sigma, \tau \in \Lambda$ denote weights of the module V; $\xi, \zeta, \eta \in R$ denote elements of the ground ring; x, y, z, g, h denote elements of the group $G(\Phi, R)$; and finally, u, v denote vectors in V.

§2. Some ancillary facts

As in [6], all proofs in the present paper are based on the following observation; see [6, Proposition 3].

Lemma 1. *If $g \in G(\Phi, R)$ is a root element of a Chevalley group in a microweight representation V, then $g_{\lambda\mu} = 0$ for all weights $\lambda, \mu \in \Lambda$ such that $d(\lambda, \mu) \geq 2$.*

The following statement is Lemma 8 of [6].

Lemma 2. *If an element $g \in \mathrm{GL}(n, R)$ commutes with a root element $x_\alpha(\xi)$ for some root $\alpha \in \Phi$, then*

- $\xi g_{\lambda\mu} = 0$ *if* $\lambda + \alpha \in \Lambda$ *but* $\mu + \alpha \notin \Lambda$;
- $\xi g_{\lambda\mu} = 0$ *if* $\mu - \alpha \in \Lambda$ *but* $\lambda - \alpha \notin \Lambda$;
- $\xi(g_{\lambda\mu} - g_{\lambda+\alpha,\mu+\alpha}) = 0$ *if* $\lambda + \alpha, \mu + \alpha \in \Lambda$.

In particular, this immediately implies the following corollary, which is Proposition 4 of [6].

Lemma 3. *If $[g, x_\alpha(1)] = e$, then g lies in a proper parabolic subgroup of type P_2 resp P_1.*

We shall also need the following Proposition 5 of [6].

Lemma 4. *Let $g \in G(\Phi, R)$, where $\Phi = \mathrm{E}_6, \mathrm{E}_7$. If for some roots $\alpha, \beta \in \Phi$ and some ring elements $\xi, \zeta \in R$ the commutator $[g, z]$, where $z = x_\alpha(\xi)x_\beta(\eta)$, is central, then it equals e.*

The proof is based on the fact that multiplication by two root elements changes at most $12 < 27/2$ resp $24 < 56/2$ rows or columns of the matrix g. At the same time, multiplication by three root elements may change $18 > 27/2$ resp $36 > 56/2$ rows or columns of g. Thus, there is no *obvious* reason why an analog of Lemma 4 should be true for an *arbitrary* product of three root elements.

However, we are mostly interested in the products of root elements forming mutual angles of $\pi/3$. With this additional assumption, we can use essentially the same argument as in the proof of Proposition 5 in [6], with some minor complications.

Proposition 1. *Let $g \in G(\Phi, R)$, where $\Phi = \mathrm{E}_6, \mathrm{E}_7$. If for some roots $\alpha, \beta, \gamma \in \Phi$ forming mutual angles of $\pi/3$, and some ring elements $\xi, \zeta, \theta \in R$, the commutator $[g, z]$, where $z = x_\alpha(\xi)x_\beta(\eta)x_\gamma(\theta)$, is central, then it equals e.*

Proof. As in [6], we can in fact prove this for an arbitrary element of $g \in \mathrm{GL}(27, R)$ resp $g \in \mathrm{GL}(56, R)$. We claim that if the commutator $[g, z]$ is central, then it equals e. Let $zg = \varepsilon gz$ for some $\varepsilon \in R^*$. Lemma 7 of [6] implies that

$$(x_\alpha(\xi)x_\beta(\eta)x_\gamma(\theta)g)_{\lambda\mu} = g_{\lambda\mu} = (gx_\alpha(\xi)x_\beta(\eta)x_\gamma(\theta))_{\lambda\mu}$$

for all $\lambda, \mu \in \Lambda$ such that

$$\lambda - \alpha, \lambda - \beta, \lambda - \gamma \notin \Lambda, \qquad \mu + \alpha, \mu + \beta, \mu + \gamma \notin \Lambda.$$

Thus, $g_{\lambda\mu} = \varepsilon g_{\lambda\mu}$ for all such pairs of weights.

Now we estimate how many weights do not satisfy this condition. Recall that

- all subsystems of type A_3 in E_6 and E_7 are conjugate;
- the Weyl group $W(\mathrm{E}_6)$ — and, *a fortiori*, $W(\mathrm{E}_7)$ — induces a contragredient module on A_3; see, for instance, [19] and [32]. Thus, all triples of weights with mutual angles of $\pi/3$ form a single orbit under the action of W. To calculate the number of such weights λ and μ, it suffices to take any such triple for the role of α, β, γ, say $\alpha_1, \alpha_1+\alpha_3, \alpha_1+\alpha_2+\alpha_4$. A straightforward calculation (or contemplation of weight diagrams, for that matter)

shows that such a triple affects 18 resp 36 additions of 11 resp 22 weights to 11 resp 22 weights. This is explained by the fact that there are 2 resp 4 weights to which the roots α, β, γ affect 3 additions. In other words, for these weights all three differences $\lambda - \alpha, \lambda - \beta, \lambda - \gamma$ are weights. Moreover, there are 3 resp 6 weights to which the roots α, β, γ affect 2 additions. In other words, for these weights two of the three differences $\lambda - \alpha, \lambda - \beta, \lambda - \gamma$ are weights (distinct pairs for distinct weights, in the case of E_6).

This means that the number of weights λ and μ not affected by additions with the roots α, β, γ, equals 16 resp 34. Thus, the elements $g_{\lambda\mu}$ for which $g_{\lambda\mu} = \varepsilon g_{\lambda\mu}$, generate the unit ideal of R. Indeed, should they belong to a maximal ideal M of R, the projection $\bar{g} \in \mathrm{GL}(n, R/M)$ would have a zero block of size 16×16 resp 34×34 and thus would be degenerate (because $16 > 27/2$ resp $34 > 56/2$). This immediately implies that $\varepsilon = 1$. □

§3. Simultaneous stabilization of two columns in E_6

In the present section $\Phi = \mathrm{E}_6$. We reproduce some parts of the general context of the A_5-proof in [43]. In somewhat less detail, this general context was also described in [7].

That proof runs inside the 27-dimensional module $V(\varpi_1)$ with the highest weight ϖ_1 of the simply connected Chevalley group $G = G(\mathrm{E}_6, R)$. This module itself is conceived as the unipotent radical $V = U_7$ of the standard parabolic subgroup P_7 of the simply connected Chevalley group $G(\mathrm{E}_7, R)$. The representation of G is then described as the conjugation action of the (commutator subgroup of the) Levi subgroup L_7 on the unipotent radical U_7.

Thus, the roots of E_6 are represented by their Dynkin form in E_6, whereas the weights of the 27-dimensional module V are represented by their Dynkin form in E_7. In this realization, the weights of V are precisely the roots of E_7 whose linear expansion with respect to the fundamental roots contains α_7 with the coefficient 1. As usual, we denote the set of all such weights by Λ. As a base of V, now one can take the vectors $v^\alpha = x_\alpha(1)$, $\alpha \in \Lambda$.

Recall that the maximal number of roots in E_6 that form mutual angles of $\pi/3$ equals five. Pick up the senior among all such root sets:

$$\beta_1 = \begin{smallmatrix}12321\\2\end{smallmatrix}, \quad \beta_2 = \begin{smallmatrix}12321\\1\end{smallmatrix}, \quad \beta_3 = \begin{smallmatrix}12221\\1\end{smallmatrix}, \quad \beta_4 = \begin{smallmatrix}12211\\1\end{smallmatrix}, \quad \beta_5 = \begin{smallmatrix}12210\\1\end{smallmatrix}.$$

Recall that we have fixed an element $g \in G(\mathrm{E}_6, R)$ that has 10 zeros in each of the two adjacent columns $g_{*\lambda}$ and $g_{*\mu}$, where $\lambda, \mu \in \Lambda$, $d(\lambda, \mu) = 1$. Since the Weyl group $W = W(\mathrm{E}_6)$ acts transitively on the set of all such pairs, in the sequel as (λ, μ) we can take the senior among all such pairs, namely, $\lambda = \omega$, $\mu = \omega - \alpha_1$.

Then by the condition of the main lemma, the first column of the matrix g — in other words, the column $g_{*\omega}$ — has zeros at the positions corresponding to the following two chains of weights:

$$\gamma_1 = \begin{smallmatrix}001111\\1\end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix}001111\\0\end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix}000111\\0\end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix}000011\\0\end{smallmatrix}, \quad \gamma_5 = \begin{smallmatrix}000001\\0\end{smallmatrix},$$
$$\delta_1 = \begin{smallmatrix}011111\\0\end{smallmatrix}, \quad \delta_2 = \begin{smallmatrix}011111\\1\end{smallmatrix}, \quad \delta_3 = \begin{smallmatrix}012111\\1\end{smallmatrix}, \quad \delta_4 = \begin{smallmatrix}012211\\1\end{smallmatrix}, \quad \delta_5 = \begin{smallmatrix}012221\\1\end{smallmatrix}.$$

In Figure 1, the corresponding vertices of the weight diagram are depicted by solid black dots. Next, $\gamma_5 = -\varpi_6$ is the minimal weight of the representation $V(\varpi_1)$, whereas δ_5 is the uppermost weight in the middle of the diagram:

We consider another chain of weights:

$$\varepsilon_1 = \begin{smallmatrix}111111\\0\end{smallmatrix}, \quad \varepsilon_2 = \begin{smallmatrix}111111\\1\end{smallmatrix}, \quad \varepsilon_3 = \begin{smallmatrix}112111\\1\end{smallmatrix}, \quad \varepsilon_4 = \begin{smallmatrix}112211\\1\end{smallmatrix}, \quad \varepsilon_5 = \begin{smallmatrix}112221\\1\end{smallmatrix}.$$

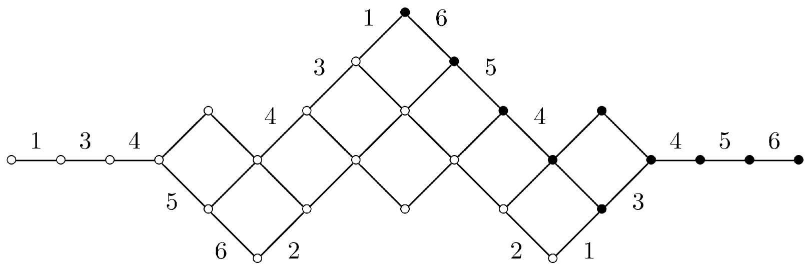

FIGURE 1. The first column of the matrix g in (E_6, ϖ_1).

Together with the weights $\gamma_1, \dots, \gamma_5$, these weights are depicted by solid black dots in Figure 2. By the condition of the main lemma, the second column of the matrix g — in other words, the column $g_{*,\omega-\alpha_1}$ — has zeros at the positions corresponding to the weights $\gamma_1, \dots, \gamma_5, \varepsilon_1, \dots, \varepsilon_5$.

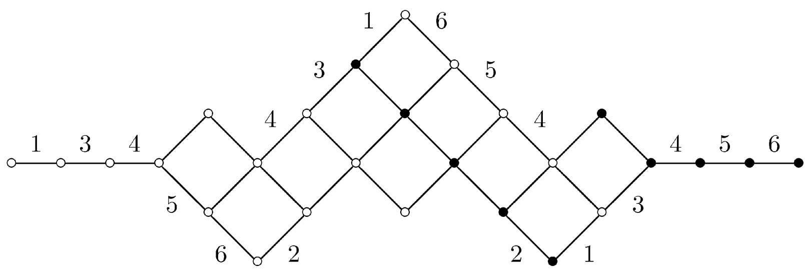

FIGURE 2. The second column of the matrix g in (E_6, ϖ_1).

We shall need the following description of the action of root unipotents $x_{\beta_i}(\xi)$ on the module V, stated in [43, §5–2].

Lemma 5. *The sum of a root β_i and a weight $\gamma_j, \delta_j, \varepsilon_j$ is a weight in the following cases*:

- $\beta_i + \gamma_j \in \Lambda$ *if and only if* $i \neq j$;
- $\gamma_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$, $i \neq j$;
- $\beta_i + \delta_j, \beta_i + \varepsilon_j \in \Lambda$ *if and only if* $i = j$;
- $\beta_i + \varepsilon_i = \beta_j + \varepsilon_j = \omega$;
- $\beta_i + \delta_i = \beta_j + \delta_j = \omega - \alpha_1$;
- $\beta_i + \lambda \notin \Lambda$ *if* $\lambda \neq \gamma_j, \delta_i, \varepsilon_i$, $j \neq i$.

Thus,

$$\Lambda = \big\{\omega,\ \omega - \alpha_1,\ \gamma_{ij} = \beta_i + \gamma_j,\ i < j,\ \varepsilon_i,\ \delta_i,\ \gamma_i\big\}.$$

Henceforth, the weights are always considered with respect to this order. For instance, speaking of the first 12 rows of a matrix $g \in G$, we mean the rows with the indices $\omega, \omega - \alpha_1, \gamma_{ij}$. These are precisely the 12 vertices to the left of the solid black vertices in Figure 2.

The above six series of weights correspond to the branching of the 27-dimensional module V upon restriction from E_6 to A_4. They can be described as follows:

- ◦ γ_i are precisely the roots $\gamma \in \Lambda$ such that $m_1(\gamma) = m_3(\gamma) = 0$;

∘ δ_i are precisely the roots such that $m_1(\gamma)=0$, $m_3(\gamma)=1$;
∘ ε_i are precisely the roots such that $m_1(\gamma)=m_3(\gamma)=1$;
∘ γ_{ij} are precisely the roots such that $m_1(\gamma)=1$, $m_3(\gamma)=2$;
∘ $\omega-\alpha_1$ is precisely the (unique) root such that $m_1(\gamma)=1$, $m_3(\gamma)=3$;
∘ ω is precisely the (unique) root such that $m_1(\gamma)=2$, $m_3(\gamma)=3$.

Now we construct unipotent elements that stabilize the first two columns of g. Namely, for $1\le j,h\le 5$ we consider the minor

$$\Delta^{jh}(g)=g_{\varepsilon_j,\omega}g_{\delta_h,\omega-\alpha_1}-g_{\varepsilon_h,\omega}g_{\delta_j,\omega-\alpha_1}$$

of the (5×2)-matrix consisting of the piece of the first column of g at the positions 13 through 17, and the piece of its second columns at the positions 18 through 22.

Next, we choose *three* distinct indices $1\le i,j,h\le 5$ and set

$$x=x(i,j,h)=x_{\beta_i}(\Delta^{jh}(g))\,x_{\beta_j}(\pm\Delta^{hi}(g))\,x_{\beta_h}(\pm\Delta^{ij}(g)).$$

In the course of our calculations, a specific choice of signs does not really matter. The only important point is that it can be performed consistently, as will be explained below. In other words[1],

$$\begin{aligned}x(i,j,h)&=x_{\beta_i}(g_{\varepsilon_j,\omega}g_{\delta_h,\omega-\alpha_1}-g_{\varepsilon_h,\omega}g_{\delta_j,\omega-\alpha_1})\\&\quad\times x_{\beta_j}(\pm g_{\varepsilon_i,\omega}g_{\delta_h,\omega-\alpha_1}\mp g_{\varepsilon_h,\omega}g_{\delta_i,\omega-\alpha_1})\\&\quad\times x_{\beta_h}(\pm g_{\varepsilon_i,\omega}g_{\delta_j,\omega-\alpha_1}\mp g_{\varepsilon_j,\omega}g_{\delta_i,\omega-\alpha_1}).\end{aligned}$$

Since α_1 occurs in the expansion of the roots $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5$ with coefficient 1, the action of the corresponding root unipotents passes through exactly one edge with mark 1.

• Contemplating Figure 1, we see that — with the only exception of the leftmost edge with mark 1 — the entries $g_{*\omega}$ to the right of any edge with mark 1 are all equal to zero! This means that $x(i,j,h)$ affects exactly *three* additions in the first column of g, all of them to the diagonal entry $g_{\omega\omega}$.

• Contemplating Figure 2 in the same spirit, we see that $x(i,j,h)$ affects three matching additions also in the second column of g, all of them to the diagonal entry $g_{\omega-\alpha_1,\omega-\alpha_1}$. Much as above, all other entries in the second column affected by $x(i,j,h)$ are equal to 0.

Appealing to Lemma 5, we see that the element added to $g_{\omega\omega}$ in the first column is

$$\Delta^{jh}(g)g_{\varepsilon_i,\omega}\pm\Delta^{hi}(g)g_{\varepsilon_j,\omega}\pm\Delta^{ij}(g)g_{\varepsilon_h,\omega},$$

whereas the element added to $g_{\omega-\alpha_1,\omega-\alpha_1}$ in the second column is

$$\Delta^{jh}(g)g_{\delta_i,\omega-\alpha_1}\pm\Delta^{hi}(g)g_{\delta_j,\omega-\alpha_1}\pm\Delta^{ij}(g)g_{\delta_h,\omega-\alpha_1}.$$

Clearly, the parameters of the elementary root unipotents constituting $x(i,j,h)$ were expressly chosen in such a way that these modifying additions at *both* positions cancel, *provided that* their signs match.

That the signs match indeed can be derived by the algorithm described in [43, 4], as we do at the end of §5. Otherwise, one can verify that simply by looking at Table 15 of [15]. The last five nonzero entries in the second column of that table, at the positions 18 through 22, are equal to $--+-+$. In the first column of that table one should look at the entries at the positions 11, 13, 15, 16, 17, which are also equal to $--+-+$. Thus, the signs of additions affected by $x(i,j,h)$ in the first and the second column coincide, as claimed.

[1]Unfortunately, in [45] a systematic mistake cropped up at this key spot. Namely, $g_{\delta_i,\omega-\alpha_1}$ was everywhere replaced by $g_{\delta_i,\omega}$. Clearly, this does not make any sense whatsoever, because $g_{\delta_i,\omega}=0$.

§4. The proof of the main lemma for E_6

Now we approach the most intricate part of the proof. What happens when g commutes with all $x(i,j,h)$? First, we show that this condition imposes many algebraic equations on the entries of g. Using these equations, we can construct new unipotent elements stabilizing the first two columns of g. If these new unipotents also commute with g, then we obtain further algebraic equations on the entries of g. Proceeding in the same vein, we can finally ensure that g itself falls into a proper parabolic subgroup. In this last case we can instantly invoke the calculations from [6].

• First, recall that if the element $[g, x(i,j,h)]$ is central, then it is equal to e by Proposition 1. In other words, g commutes with $x(i,j,h)$. Comparing the corresponding entries of the matrices $gx(i,j,h)$ and $x(i,j,h)g$, we obtain algebraic equations on the entries of g.

For this we observe that from the proof of Proposition 1 we know that left multiplication by

$$x = x_{\beta_i}(\xi)x_{\beta_j}(\zeta)x_{\beta_h}(\theta)$$

modifies 11 rows of g, out of the *first* 12 rows, while right multiplication by x modifies 11 columns of g, out of the *last* 12 columns. These additions are grouped as follows:

∘ there are three additions to each of two rows/columns,

∘ there are two additions to each of three rows/columns,

∘ finally, there is exactly one addition to each of the 6 remaining rows/columns affected by the action of x.

These 6 additions are precisely the additions of the first and the second columns multiplied by ξ, ζ, θ, respectively. Since the last 15 rows of the matrix g are not affected by the left multiplication by x, the relation $xg = gx$ implies that

$$g_{\varepsilon_k,\omega}\xi = g_{\varepsilon_k,\omega}\zeta = g_{\varepsilon_k,\omega}\theta = g_{\delta_k,\omega-\alpha_1}\xi = g_{\delta_k,\omega-\alpha_1}\zeta = g_{\delta_k,\omega-\alpha_1}\theta = 0.$$

Returning to the case of $x = x(i,j,h)$, we see that if x commutes with g, then

$$g_{\varepsilon_i,\omega}\Delta^{jh}(g) = g_{\delta_i,\omega-\alpha_1}\Delta^{jh}(g) = 0, \qquad 1 \le i,j,h \le 5,$$

and we even have no need to stipulate that $i \ne j,h$.

• The equations obtained in the previous item allow us to vary x. Namely, now we set

$$x = x_{\beta_i}(\Delta^{jh}(g)), \qquad 1 \le i,j,h \le 5.$$

By the above, left multiplication by x does not change the first two columns of the matrix g,

$$(xg)_{*\omega} = g_{*\omega}, \qquad (xg)_{*,\omega-\alpha_1} = g_{*,\omega-\alpha_1}.$$

Indeed, $g_{\gamma_i,\omega} = g_{\delta_i,\omega} = g_{\gamma_i,\omega-\alpha_1} = g_{\varepsilon_i,\omega-\alpha_1} = 0$ by the very choice of g and, as we have shown, $\Delta^{jh}(g)g_{\varepsilon_i,\omega} = \Delta^{jh}(g)g_{\delta_i,\omega-\alpha_1} = 0$.

Now, if g does not commute with some of the above root elements x, we are done again. Thus, we can assume that $gx = xg$ for all such x. Invoking Lemma 2, we see that

$$g_{\lambda,\omega}\Delta^{jh}(g) = g_{\lambda,\omega-\alpha_1}\Delta^{jh}(g) = 0, \qquad 1 \le j,h \le 5,$$

for all $\lambda \ne \omega, \omega-\alpha_1$.

• Now, we are all set to apply the second main stratagem from [6], degree reduction for the occurring algebraic equations. To this end, we observe that, since the ring R is commutative, the ideal generated by the entries

$$g'_{\lambda,\omega},\ g'_{\lambda,\omega-\alpha_1}, \qquad \lambda \ne \omega, \omega-\alpha_1,$$

coincides with the ideal generated by the entries

$$g_{\lambda,\omega},\ g_{\lambda,\omega-\alpha_1}, \qquad \lambda \neq \omega, \omega-\alpha_1.$$

In particular,

$$g'_{\lambda,\omega}\Delta^{jh}(g) = g'_{\lambda,\omega-\alpha_1}\Delta^{jh}(g) = 0, \qquad 1 \leq j, h \leq 5,$$

for all $\lambda \neq \omega, \omega-\alpha_1$.

• The additional algebraic equations we obtained in the previous item allow us to reduce the degree of the parameters of x, with respect to the entries of g. Namely, now we construct new elements of root type x that *essentially* stabilize the first two columns of g. The advantage of these new elements as compared to those we succeeded to construct earlier, is that the parameters of all occurring elementary root unipotents are linear rather than quadratic, with respect to the entries of g.

To be exact, we set

$$x_\varepsilon(i,j) = x_{\beta_i}(g_{\varepsilon_j,\omega})x_{\beta_j}(\pm g_{\varepsilon_i,\omega}), \qquad x_\delta(i,j) = x_{\beta_i}(g_{\delta_j,\omega-\alpha_1})x_{\beta_j}(\pm g_{\delta_i,\omega-\alpha_1}),$$

where $1 \leq i, j \leq 5$, and the signs are chosen in such a way that left multiplication of g by $x_\varepsilon(i,j)$ stabilizes the first column of g, whereas left multiplication of g by $x_\delta(i,j)$ stabilizes its second column. Observe that — up to replacing g^{-1} by g — these are precisely the matrices $x(\lambda,\mu)$ that appeared in the proof of the main lemma of [6].

Incidentally, left multiplication of g by $x_\varepsilon(i,j)$ *does* change the second column of g, in general. Namely, $\pm\Delta^{ij}(g)$ is added to the diagonal entry $g_{\omega-\alpha_1,\omega-\alpha_1}$. Similarly, left multiplication of g by $x_\delta(i,j)$ *does* change the first column of g, in general. Namely, $\pm\Delta^{ij}(g)$ is added to the diagonal entry $g_{\omega,\omega}$.

Nevertheless, the equations we established in the previous item ensure that all elements $g^{-1}x_\varepsilon(i,j)g$ and $g^{-1}x_\delta(i,j)g$ fall into proper parabolic subgroups of type P_3. If some of these elements are noncentral, we are done again.

• On the other hand, if g commutes with all $x_\varepsilon(i,j)$ and $x_\delta(i,j)$, the following situation occurs. Left multiplication by $x = x_{\beta_i}(\xi)x_{\beta_j}(\zeta)$ modifies 9 rows of the matrix g, out of the *first* 12 rows, while right multiplication by x modifies 9 columns of g, out of the *last* 12 columns. These additions are grouped as follows:

∘ there are two additions to each of three rows/columns,

∘ there is exactly one addition to each of the 6 remaining rows/columns affected by the action of x.

Among these 6 additions, there are additions of the first and the second columns, multiplied by ξ, ζ, respectively. Since the last 15 rows of the matrix g are not affected by left multiplication by x, as before, the relation $xg = gx$ implies that

$$g_{\varepsilon_h,\omega}\xi = g_{\varepsilon_h,\omega}\zeta = g_{\delta_h,\omega-\alpha_1}\xi = g_{\delta_h,\omega-\alpha_1}\zeta = 0.$$

Applying these equations to the matrices $x_\varepsilon(i,j)$ and $x_\delta(i,j)$, we see that

$$g_{\varepsilon_i,\omega}g_{\varepsilon_j,\omega} = g_{\varepsilon_i,\omega}g_{\delta_j,\omega-\alpha_1} = g_{\delta_i,\omega-\alpha_1}g_{\delta_j,\omega-\alpha_1} = 0.$$

• We persist in our attempts to reduce the degree of equations. The equations obtained in the previous item allow us to return afresh to the second item of the present section, replacing there quadratic parameters by linear ones. To be exact, we set

$$x = x_{\beta_j}(\xi), \qquad \xi = g_{\varepsilon_i,\omega}, g_{\delta_i,\omega-\alpha_1}, \quad 1 \leq i, j \leq 5.$$

The equations established in the previous section ensure that left multiplication by x does not modify the first two columns of g.

If g does not commute with some of these root elements x, we are done again. Thus, we can assume that $gx = xg$ for all such x. Invoking Lemma 2, we see that

$$g_{\lambda,\omega}g_{\varepsilon_i,\omega} = g_{\lambda,\omega-\alpha_1}g_{\varepsilon_i,\omega} = g_{\lambda,\omega}g_{\delta_i,\omega-\alpha_1} = g_{\lambda,\omega-\alpha_1}g_{\delta_i,\omega-\alpha_1} = 0,$$

for all $1 \leq i \leq 5$ and all $\lambda \neq \omega, \omega - \alpha_1$. As above, using the commutativity of the ring R, we see that then also

$$g'_{\lambda,\omega}g_{\varepsilon_i,\omega} = g'_{\lambda,\omega-\alpha_1}g_{\varepsilon_i,\omega} = g'_{\lambda,\omega}g_{\delta_i,\omega-\alpha_1} = g'_{\lambda,\omega-\alpha_1}g_{\delta_i,\omega-\alpha_1} = 0,$$

for all such i and λ.

• Now we are all set for another round of degree reduction — as it happens, the last one! Namely, set $x = x_{\beta_i}(1)$. Left multiplication of g by x modifies the diagonal entries in the first two columns of the matrix g as follows:

$$(x_{\beta_i}(1)g)_{\omega,\omega} = g_{\omega,\omega} + g_{\varepsilon_i,\omega}, \qquad (x_{\beta_i}(1)g)_{\omega-\alpha_1,\omega-\alpha_1} = g_{\omega-\alpha_1,\omega-\alpha_1} + g_{\delta_i,\omega-\alpha_1},$$

and all other entries of these columns remain unchanged.

The equations of the previous item ensure that all elements $g^{-1}x_{\beta_i}(1)g$ fall into proper parabolic subgroups P_3. If some of these elements are noncentral, we are done again.

On the other hand, if at least one of these elements is central, then by Lemma 3 the initial element g lies in a proper parabolic subgroup of type P_2. This completes the proof of the main lemma for the case where $\Phi = \mathrm{E}_6$.

§5. Simultaneous stabilization of two columns in E_7

The proof for the group of type E_7 is totally similar. Due to the higher dimension, it may be slightly harder to devise. Nevertheless, *from a technical viewpoint* it is actually even slightly easier, due to the fact that the module $V(\varpi_7)$ is symplectic, which leads to a greater symmetry in the description of the action of root unipotents.

In this proof, the 56-dimensional module $V = V(\varpi_7)$ of the simply connected Chevalley group $G(\mathrm{E}_7, R)$ is realized inside the standard parabolic subgroup P_8 of the Chevalley group $G(\mathrm{E}_8, R)$. More precisely, we conceive this representation as the conjugation action of the (commutator of the) Levi subgroup L_8 on the abelianization (= central quotient, in this case) $V = U_8/[U_8, U_8]$ of the unipotent radical U_8 of P_8.

Thus, the roots of E_7 are represented by their Dynkin form in E_7, whereas the weights of V are represented by their Dynkin form in E_8. In this realization, the weights of the 56-dimensional module are precisely the roots of E_8 whose linear expansion with respect to the fundamental roots involves α_8 with the coefficient 1. As in the case of E_6, we denote the set of all such roots by Λ. Since $[U_8, U_8] = X_\rho$, where, as above,

$$\rho = \begin{smallmatrix}2465432\\3\end{smallmatrix}$$

is the maximal root of E_8, we can take the vectors $v^\alpha = x_\alpha(1)X_\rho$, $\alpha \in \Lambda$, as a base of V.

We describe the weights of $V = V(\varpi_7)$. Consider the following chain of weights:

$$\gamma_1 = \begin{smallmatrix}1111111\\0\end{smallmatrix}, \quad \gamma_2 = \begin{smallmatrix}0111111\\0\end{smallmatrix}, \quad \gamma_3 = \begin{smallmatrix}0011111\\0\end{smallmatrix}, \quad \gamma_4 = \begin{smallmatrix}0001111\\0\end{smallmatrix},$$
$$\gamma_5 = \begin{smallmatrix}0000111\\0\end{smallmatrix}, \quad \gamma_6 = \begin{smallmatrix}0000011\\0\end{smallmatrix}, \quad \gamma_7 = \begin{smallmatrix}0000001\\0\end{smallmatrix}.$$

Next, consider the senior among all chains of seven roots of E_7 that form mutual angles of $\pi/3$,

$$\beta_1 = \begin{smallmatrix}234321\\2\end{smallmatrix}, \quad \beta_2 = \begin{smallmatrix}134321\\2\end{smallmatrix}, \quad \beta_3 = \begin{smallmatrix}124321\\2\end{smallmatrix}, \quad \beta_4 = \begin{smallmatrix}123321\\2\end{smallmatrix},$$
$$\beta_5 = \begin{smallmatrix}123221\\2\end{smallmatrix}, \quad \beta_6 = \begin{smallmatrix}123211\\2\end{smallmatrix}, \quad \beta_7 = \begin{smallmatrix}123210\\2\end{smallmatrix}.$$

Now we reproduce the description of the action of the root unipotents $x_{\beta_i}(\varepsilon)$ on the module V from [43, §5–3].

Lemma 6. *The sum of a root β_i and a weight γ_j is a weight in the following cases*:

- $\beta_i + \gamma_j \in \Sigma$ *if and only if* $i \neq j$;
- $\gamma_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$;
- *set* $\gamma^*_{ij} = \rho - \gamma_{ij}$; *then* $\beta_i + \gamma^*_{jh} \in \Lambda$ *if and only if* $i = j, h$;
- *set* $\gamma^*_j = \rho - \gamma_j$; *then* $\beta_i + \gamma^*_{ij} = \gamma^*_j$;
- $\beta_i + \lambda \notin \Lambda$ *if* $\lambda \neq \gamma_j, \gamma^*_{ij}$.

All 56 weights constructed in this lemma are pairwise distinct, and thus

$$\Lambda = \{\gamma_i,\ \gamma^*_{ij}, i<j,\ \gamma_{ij}, i<j,\ \gamma^*_i\}.$$

The above four series of weights correspond to the branching of the 56-dimensional module V upon restriction from E_7 to A_6. They can be described as follows:

- ∘ γ_i are precisely the roots $\gamma \in \Sigma$ such that $m_2(\gamma) = 0$;
- ∘ γ^*_{ij} are precisely the roots such that $m_2(\gamma) = 1$;
- ∘ γ_{ij} are precisely the roots such that $m_2(\gamma) = 2$;
- ∘ γ^*_i are precisely the roots $\gamma \in \Sigma$ such that $m_2(\gamma) = 3$.

This completely describes the action of $x_{\beta_i}(\xi)$ on an arbitrary column $u = (u_\lambda) \in V$ up to signs. Indeed, $x_{\beta_i}(\xi)$ multiplies u_{γ_j} by ξ and adds it to or subtracts it from $u_{\gamma_{ij}}$. Moreover, $x_{\beta_i}(\xi)$ multiplies $u_{\gamma^*_{ij}}$ by ξ and adds it to or subtracts it from $u_{\gamma^*_j}$.

To facilitate the tracing of the action of root unipotents in Figures 3 and 4, as also for reference purposes, below we reproduce a table that establishes the correspondence of our new notation for the weights of $V(\varpi_7)$ with their notation as roots of E_8. In this table the weights are ordered by height from 1 through 28, and lexicographically within a given height.

Now, we can return to the construction of a root type element that simultaneously stabilizes two columns of an element g. Recall that in the main lemma we fixed an element $g \in G(\mathrm{E}_7, R)$ that has 28 zeros in each of the two adjacent columns $g_{*\lambda}$ and $g_{*\mu}$, where $\lambda, \mu \in \Lambda$, $d(\lambda, \mu) = 1$. Since the Weyl group $W = W(\mathrm{E}_7)$ acts transitively on the set of all such pairs, as (λ, μ) we take the senior among all such pairs, namely $\lambda = \omega$, $\mu = \omega - \alpha_7$.

Then by the condition of the main lemma, the first column of g — in other words, the column $g_{*\omega}$ — has zeros at the positions corresponding to the vertices depicted by solid black dots in Figure 3. Not to overload the pictures we left only a fraction of marks on their edges. These marks correspond to the fundamental roots of the system E_7. The remaining marks can be recovered uniquely with the help of the following three facts: the marks on two opposite sides of a parallelogram coincide; upon removing the rightmost node, the lower component of the diagram coincides with the weight diagram (E_6, ϖ_1), as depicted in Figures 1 and 2; upon removing the leftmost node, the upper component of the diagram coincides with the weight diagram (E_6, ϖ_6), which is symmetric to (E_6, ϖ_1).

Moreover, by the conditions of the Main Lemma, the second column of g, i.e., the column indexed by $\omega - \alpha_7$, has zeros at the positions corresponding to weights that are at least 2-distant from $\omega - \alpha_7$. These positions are shown by black bold dots in Figure 4.

Now we pass to the construction of root type unipotents that stabilize the first two columns of g. The highest weight ω of the module V and the next one by height are of the form

$$\omega = \gamma^*_7 = \begin{matrix}2465431\\3\end{matrix}, \qquad \omega - \alpha_7 = \gamma^*_6 = \begin{matrix}2465421\\3\end{matrix}.$$

TABLE 1. Weights of $V(\varpi_7)$ as roots of E_8.

1	$\gamma_7 = \substack{0000001\\0}$	**2**	$\gamma_6 = \substack{0000011\\0}$	**3**	$\gamma_5 = \substack{0000111\\0}$
		4	$\gamma_4 = \substack{0001111\\0}$	**5**	$\gamma_3 = \substack{0011111\\0}$
6	$\gamma^*_{12} = \substack{0011111\\1}$		$\gamma_2 = \substack{0111111\\0}$		
7	$\gamma_1 = \substack{1111111\\0}$		$\gamma^*_{13} = \substack{0111111\\1}$		
8	$\gamma^*_{23} = \substack{1111111\\1}$		$\gamma^*_{14} = \substack{0121111\\1}$		
9	$\gamma^*_{24} = \substack{1121111\\1}$		$\gamma^*_{15} = \substack{0122111\\1}$		
10	$\gamma^*_{34} = \substack{1221111\\1}$		$\gamma^*_{25} = \substack{1122111\\1}$		$\gamma^*_{16} = \substack{0122211\\1}$
11	$\gamma^*_{35} = \substack{1222111\\1}$		$\gamma^*_{26} = \substack{1122211\\1}$		$\gamma^*_{17} = \substack{0122221\\1}$
12	$\gamma^*_{45} = \substack{1232111\\1}$		$\gamma^*_{36} = \substack{1222211\\1}$		$\gamma^*_{27} = \substack{1122221\\1}$
13	$\gamma_{67} = \substack{1232111\\2}$		$\gamma^*_{46} = \substack{1232211\\1}$		$\gamma^*_{37} = \substack{1222221\\1}$
14	$\gamma_{57} = \substack{1232211\\2}$		$\gamma^*_{56} = \substack{1233211\\1}$		$\gamma^*_{47} = \substack{1232221\\1}$
15	$\gamma_{47} = \substack{1233211\\2}$		$\gamma_{56} = \substack{1232221\\2}$		$\gamma^*_{57} = \substack{1233221\\1}$
16	$\gamma_{37} = \substack{1243211\\2}$		$\gamma_{46} = \substack{1233221\\2}$		$\gamma^*_{67} = \substack{1233321\\1}$
17	$\gamma_{27} = \substack{1343211\\2}$		$\gamma_{36} = \substack{1243221\\2}$		$\gamma_{45} = \substack{1233321\\2}$
18	$\gamma_{17} = \substack{2343211\\2}$		$\gamma_{26} = \substack{1343221\\2}$		$\gamma_{35} = \substack{1243321\\2}$
19	$\gamma_{16} = \substack{2343221\\2}$		$\gamma_{25} = \substack{1343321\\2}$		$\gamma_{34} = \substack{1244321\\2}$
20	$\gamma_{15} = \substack{2343321\\2}$		$\gamma_{24} = \substack{1344321\\2}$		
21	$\gamma_{14} = \substack{2344321\\2}$		$\gamma_{23} = \substack{1354321\\2}$		
22	$\gamma_{13} = \substack{2354321\\2}$		$\gamma^*_{1} = \substack{1354321\\3}$		
23	$\gamma^*_{2} = \substack{2354321\\3}$		$\gamma_{12} = \substack{2454321\\2}$		
		24	$\gamma^*_{3} = \substack{2454321\\3}$	**25**	$\gamma^*_{4} = \substack{2464321\\3}$
26	$\gamma^*_{5} = \substack{2465321\\3}$	**27**	$\gamma^*_{6} = \substack{2465421\\3}$	**28**	$\gamma^*_{7} = \substack{2465431\\3}$

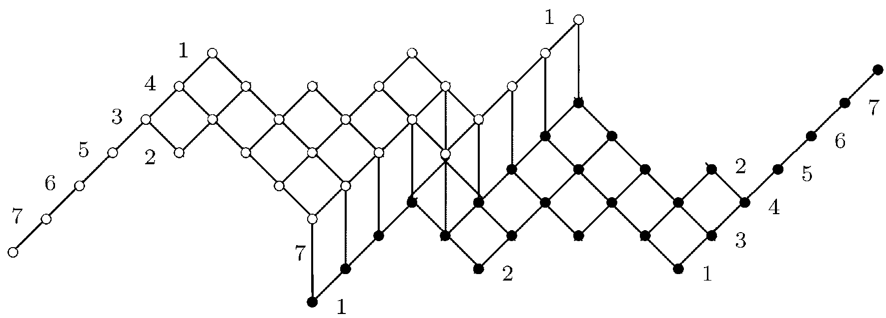

FIGURE 3. The first column of the matrix g in (E_7, ϖ_7).

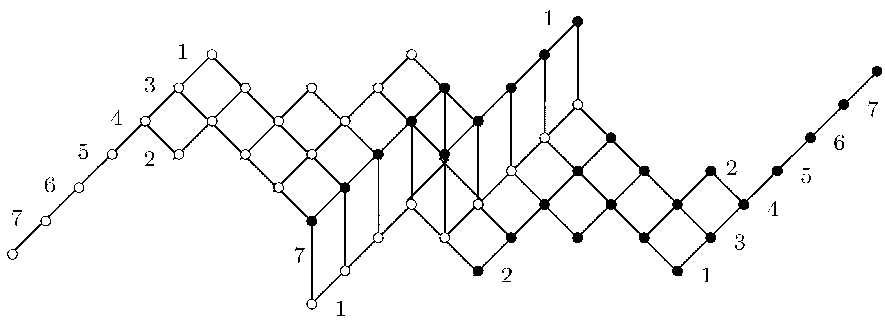

FIGURE 4. The second column of the matrix g in (E_7, ϖ_7).

For $1 \leq j, h \leq 5$, we consider the minor

$$\Delta^{jh}(g) = g_{\gamma^*_{j7},\omega} g_{\gamma^*_{h6},\omega-\alpha_7} - g_{\gamma^*_{h7},\omega} g_{\gamma^*_{j6},\omega-\alpha_7}$$

of the (5×2)-matrix consisting of a piece of the first column of g and a piece of its second column. In the A$_6$-numbering, this is the piece of the first column at the positions 29 through 33 and the piece of the second column at the positions 34 through 38. On the other hand, in the E$_6$-numbering this is the piece of the first column at the positions 24 through 28 and the piece of the second column at the positions 29 through 33.

Next, we choose *three* distinct indices $1 \leq i, j, h \leq 5$ and set

$$x = x(i,j,h) = x_{\beta_i}(\Delta^{jh}(g))\, x_{\beta_j}(\pm\Delta^{hi}(g))\, x_{\beta_h}(\pm\Delta^{ij}(g)).$$

As in the case of E$_6$, the only important point is that a specific choice of signs can be performed consistently.

Since α_7 occurs in the expansion of the roots $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ with coefficient 1, the action of the corresponding root unipotents passes through exactly one edge with mark 7.

• Contemplating Figure 3, we see that — with the only exception of the leftmost occurrence of an edge with mark 7 — the entries of $g_{*\omega}$ underneath any edge with mark 7 are all equal to zero! This means that $x(i,j,h)$ affects exactly *three* additions in the first columns, all of them to the diagonal entry $g_{\omega\omega}$.

• Contemplating Figure 4 in the same mood, we see that multiplication by $x(i,j,h)$ affects three matching additions in the second column, all of them to the diagonal entry $g_{\omega-\alpha_7,\omega-\alpha_7}$. As above, all other entries in the second column affected by $x(i,j,h)$ are equal to 0.

Resorting to Lemma 6, we see that the element added to $g_{\omega\omega}$ in the first column is

$$\Delta^{jh}(g) g_{\gamma^*_{i7},\omega} \pm \Delta^{hi}(g) g_{\gamma^*_{j7},\omega} \pm \Delta^{ij}(g) g_{\gamma^*_{h7},\omega},$$

whereas the element added to $g_{\omega-\alpha_7,\omega-\alpha_7}$ in the second column is

$$\Delta^{jh}(g)g_{\gamma^*_{i6},\omega-\alpha_7} \pm \Delta^{hi}(g)g_{\gamma^*_{j6},\omega-\alpha_7} \pm \Delta^{ij}(g)g_{\gamma^*_{h6},\omega-\alpha_7}.$$

As in the case of E_6, the parameters of the elementary root unipotents constituting $x(i,j,h)$ have been deliberately chosen in such a way that these modifying additions in *both* columns cancel, *provided* their signs match.

Thus, it only remains to verify that these signs indeed match. To be sure, in this case too we would much prefer to directly refer to the corresponding tables. Unfortunately, the respective tables for the case of E_7 are not yet available[2]. This is why we have to calculate signs with the help of the algorithm described in [43, 4]. For this, we recall that the canonical strings of the roots $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are equal to

$$\begin{array}{lll} 13425431654234567, & 3425431654234567, & 425431654234567, \\ & 25431654234567, & 2431654234567, \end{array}$$

respectively. For the weight ω, all marks in these sequences, except for the last mark 7, are nasty, so that altogether there are $16, 15, 14, 13, 12$ nasty marks, respectively. In other words, the signs with which the root unipotents $x_{\beta_i}(\xi)$ affect additions to the ωth coordinate are equal to $+-+-+$. On the other hand, for the weight $\omega-\alpha_7$ all initial marks up to the first occurrence of the mark 6 are nasty, whereas all subsequent marks, starting with 6, are not. Thus, altogether there are $8, 7, 6, 5, 4$ nasty marks, respectively, which again gives the same sequence of signs $+-+-+$. We can conclude that the signs of the additions affected by $x(i,j,h)$ in the first and the second column coincide, as claimed.

§6. The proof of the main lemma for E_7

The proof of the main lemma for the case of E_7 is completely similar to that for the case of E_6. Actually, it may be even somewhat easier technically, due to a more symmetric description of the action of root unipotents. The general line of arguments is exactly the same, so we concentrate on the instances where noticeable combinatorial differences emerge.

• If an element $[g, x(i,j,h)]$ is central, then it equals e by Proposition 1. Comparing the respective entries of the matrices $gx(i,j,h)$ and $x(i,j,h)g$, we get algebraic equations on the entries of g. As we know from the proof of Proposition 1, left multiplication by

$$x = x_{\beta_i}(\xi)x_{\beta_j}(\zeta)x_{\beta_h}(\theta)$$

modifies 22 rows of the matrix g, out of the *first* 28 rows, with respect to the A_6-ordering, whereas right multiplication by x modifies 22 columns of the matrix g, out of the *last* 28 columns. These additions are grouped as follows:

∘ there are three additions to each of four rows/columns,

∘ there are two additions to each of six rows/columns,

∘ finally, there is exactly one addition to each of the remaining 12 rows/columns, affected by the action of x.

Among these 12 simple additions there are, in particular, additions of the first and the second columns, multiplied by ξ, ζ, θ, respectively. Since the last 28 rows of the matrix g are not affected by left multiplication by x, the relation $xg = gx$ implies that

$$g_{\gamma^*_{i7},\omega}\xi = g_{\gamma^*_{i7},\omega}\zeta = g_{\gamma^*_{i7},\omega}\theta = g_{\gamma^*_{i6},\omega-\alpha_7}\xi = g_{\gamma^*_{i6},\omega-\alpha_7}\zeta = g_{\gamma^*_{i6},\omega-\alpha_7}\theta = 0.$$

[2] *Note added in proof.* As of the submission date of the Russian original. Presently these tables are published in [13].

Returning to the case of $x = x(i,j,h)$, we see that if x commutes with g, then

$$g_{\gamma^*_{i7},\omega}\Delta^{jh}(g) = g_{\gamma^*_{i6},\omega-\alpha_7}\Delta^{jh}(g) = 0, \qquad 1 \le i,j,h \le 5;$$

we even have no need to stipulate that $i \ne j,h$.

• Now we can set

$$x = x_{\beta_i}(\Delta^{jh}(g)), \qquad 1 \le i,j,h \le 5.$$

By the previous item, left multiplication by x does not modify the first two columns of the matrix g,

$$(xg)_{*\omega} = g_{*\omega}, \qquad (xg)_{*,\omega-\alpha_7} = g_{*,\omega-\alpha_7}.$$

As always, we can assume that $gx = xg$ for all such x, for otherwise we are done. For each row, with the exception of the first and the second row, left multiplication of g by some of the elements $x_{\beta_i}(\xi)$, $1 \le i \le 5$, does not modify that row. Indeed, the row $g_{\gamma^*_i,*}$, $1 \le i \le 5$, is not modified by the left multiplication by $x_{\beta_i}(\xi)$. On the other hand, the row $g_{\gamma_{ij},*}$, $1 \le i,j \le 7$, is not modified by the left multiplication by $x_{\beta_h}(\xi)$, $h \ne i,j$. Thus, invoking Lemma 2, we see that

$$g_{\lambda,\omega}\Delta^{jh}(g) = g_{\lambda,\omega-\alpha_7}\Delta^{jh}(g) = 0, \qquad 1 \le j,h \le 5,$$

for all $\lambda \ne \omega, \omega-\alpha_7$. As usual, a reference to the commutativity of the ground ring ensures that then also

$$g'_{\lambda,\omega}\Delta^{jh}(g) = g'_{\lambda,\omega-\alpha_7}\Delta^{jh}(g) = 0, \qquad 1 \le j,h \le 5,$$

for all $\lambda \ne \omega, \omega-\alpha_7$.

• As in the case of E_6, now we can start reducing the degrees of the parameters of x, with respect to the entries of g. Namely, set

$$x_7(i,j) = x_{\beta_i}(g_{\gamma^*_{j7},\omega})x_{\beta_j}(\pm g_{\gamma^*_{i7},\omega}), \qquad x_6(i,j) = x_{\beta_i}(g_{\gamma^*_{j6},\omega-\alpha_7})x_{\beta_j}(\pm g_{\gamma^*_{i6},\omega-\alpha_7}),$$

where $1 \le i,j \le 5$, and the signs are chosen in such a way that left multiplication of g by $x_7(i,j)$ stabilizes the first column of the matrix g, whereas left multiplication of g by $x_6(i,j)$ stabilizes its second column. Observe that again — up to replacing g^{-1} by g — these are precisely the matrices $x(\lambda,\mu)$ that appeared in the proof of the main lemma of [6].

Again, left multiplication of g by $x_7(i,j)$, does, in general, change its second column. But this modification is not very essential, namely, $\pm\Delta^{ij}(g)$ is added to the diagonal entry $g_{\omega-\alpha_7,\omega-\alpha_7}$. Similarly, left multiplication of g by $x_6(i,j)$, does, in general, change the first column of g, namely, $\pm\Delta^{ij}(g)$ is added to the diagonal entry $g_{\omega,\omega}$.

Thus, the equations from the previous item ensure that all elements $g^{-1}x_7(i,j)g$ and $g^{-1}x_6(i,j)g$ fall into a proper parabolic subgroups P_6. If some of these elements are noncentral, we have achieved our goal.

• On the other hand, if g commutes with all $x_7(i,j)$ and $x_6(i,j)$, the following situation occurs. A usual calculation convinces us that if the matrix g commutes with $x = x_{\beta_i}(\xi)x_{\beta_j}(\zeta)$, then

$$g_{\gamma^*_{i7},\omega}\xi = g_{\gamma^*_{i7},\omega}\zeta = g_{\gamma^*_{i6},\omega-\alpha_7}\xi = g_{\gamma^*_{i6},\omega-\alpha_7}\zeta = 0.$$

Substituting here $x = x_7(i,j), x_6(i,j)$, we get the equations

$$g_{\gamma^*_{i7},\omega}g_{\gamma^*_{j7},\omega} = g_{\gamma^*_{i7},\omega}g_{\gamma^*_{j6},\omega-\alpha_7} = g_{\gamma^*_{i6},\omega-\alpha_7}g_{\gamma^*_{j6},\omega-\alpha_7} = 0.$$

• Now, we can again return to additions by one root β_i, with a linear parameter instead of a quadratic one,

$$x = x_{\beta_j}(\xi), \qquad \xi = g_{\gamma^*_{i7},\omega}, g_{\gamma^*_{i6},\omega-\alpha_7}, \quad 1 \le i,j \le 5.$$

The equations established in the previous item ensure that left multiplication by x does not modify the first two columns of the matrix g.

If g commutes with all resulting elements, then the usual argument shows that

$$g_{\lambda,\omega}g_{\gamma^*_{i7},\omega} = g_{\lambda,\omega-\alpha_7}g_{\gamma^*_{i7},\omega} = g_{\lambda,\omega}g_{\gamma^*_{i6},\omega-\alpha_7} = g_{\lambda,\omega-\alpha_7}g_{\gamma^*_{i6},\omega-\alpha_7} = 0$$

for all $1 \leq i \leq 5$ and all $\lambda \neq \omega, \omega - \alpha_7$, and thus

$$g'_{\lambda,\omega}g_{\gamma^*_{i7},\omega} = g'_{\lambda,\omega-\alpha_7}g_{\gamma^*_{i7},\omega} = g'_{\lambda,\omega}g_{\gamma^*_{i6},\omega-\alpha_7} = g'_{\lambda,\omega-\alpha_7}g_{\gamma^*_{i6},\omega-\alpha_7} = 0$$

for all such i and λ.

• Now we are all set for the last round of degree reduction. Namely, we can set $x = x_{\beta_i}(1)$. Left multiplication of g by x modifies the diagonal entries in the first two columns of the matrix g as follows:

$$(x_{\beta_i}(1)g)_{\omega,\omega} = g_{\omega,\omega} + g_{\gamma^*_{i7},\omega}, \qquad (x_{\beta_i}(1)g)_{\omega-\alpha_7,\omega-\alpha_7} = g_{\omega-\alpha_7,\omega-\alpha_7} + g_{\gamma^*_{i6},\omega-\alpha_7},$$

and all other entries in these columns remain unchanged.

The equations from the previous item ensure that the elements $g^{-1}x_{\beta_i}(1)g$ fall into a proper parabolic subgroup P_6. If some of these elements are noncentral, we are done again.

On the other hand, if at least one of these elements is central, Lemma 3 implies that the initial element g lies in a proper parabolic subgroup of type P_1. This completes the proof of the main lemma for the case where $\Phi = \mathrm{E}_7$.

§7. Concluding remarks

In conclusion, let us mention immediate further prospects of calculations in minimal modules, and some unsolved problems. First of all, recall that the idea of simultaneous stabilization of two columns was proposed to assail the proof of structure results at the level of K_2, where it is not enough to be able to hit *some* proper parabolic subgroup, see [45].

One of the outstanding unsolved problems[3] at the level of K_2 is the proof of the centrality of the extension $\mathrm{St}(\Phi, R) \longrightarrow E(\Phi, R)$. Ability to stabilize one column makes it possible to *construct* the modified Steinberg group $\mathrm{St}^*(\Phi, R)$ and to *haphazardly* model its generators in the usual Steinberg group $\mathrm{St}(\Phi, R)$. This is precisely the starting point of van der Kallen's *another presentation* [34, 27], for groups of type A_l. However, to verify that the generators of $\mathrm{St}^*(\Phi, R)$ are *correctly* modeled in $\mathrm{St}(\Phi, R)$, one needs larger freedom. Namely, one must be able to hit *submaximal* parabolic subgroups.

Initially, the author cherished the hope that the computational trick proposed in [45] would allow to imitate van der Kallen's proof for Chevalley groups of types E_6 and E_7. However, later it turned out that the resulting model of the modified Steinberg group is far too complicated to successfully complete this program. Presently, I would put more trust in the two simpler ideas described below.

• First, as we understand it now, the method of simultaneous stabilization of two columns, as presented in this paper, is an (A_3, P_1)-proof. In the recent papers [8, 9, 10] by the author and Victoria Kazakevich, we proposed another, much simpler trick to simultaneously stabilize *a column and a row* of an *arbitrary* matrix in the group $\mathrm{GL}(n, R)$, namely, an (A_3, P_2)-proof.

I am convinced that in Chevalley groups of types E_6 and E_7 one can carry through a similar calculation for *root* elements. Furthermore, I expect it to be technically *easier*

[3] *Note added in proof.* It is impossible not to mention the following recent amazing breakthrough in this direction. Namely, Matthias Wendt [51] constructed examples of rings for which $\mathrm{K}_2(3, R)$ is not central. This sheds an entirely new light on the whole area.

than the calculations carried out in the present paper. In fact, (A_3, P_2)-proofs work in terms of tensor products, whereas (A_3, P_1)-proofs work in terms of exterior squares, as exemplified by the calculation in the present paper. Everyone knows that tensor products are tangibly easier to handle, classically it is called separation of variables. To give an idea, this new proof would use one third fewer summands: 4 quadratic parameters of the form ab, instead of 3 quadratic parameters of the form $ab - cd$.

Problem 1. *Carry through an* (A_3, P_2)*-proof for the simultaneous stabilization of a column and a row of a root element in minimal representations of Chevalley groups of types* E_6 *and* E_7*.*

• Second, the proofs in [6, 7, 16, 45], as also the proofs in the present paper use a *small fraction of* linear equations satisfied by a root element $g \in G(\Phi, R)$ in minimal representations of Chevalley groups of types $\Phi = \mathrm{E}_6, \mathrm{E}_7$. Meanwhile, in Lemma 6 of [6] one can find further linear equations of this sort on matrix entries of g. In particular, the pairs of last *nonzero* entries in the first three columns of g coincide.

In the present paper we used an element of root type A_3 to stabilize *two* columns of (a matrix subject to linear equations on columns of) a root element. Below, we propose to do something *truly* amazing: to use an element of root type A_2 to simultaneously stabilize *three* columns of a root element. At first glance, the following problem looks like a clearance sale: `paghi uno`, `prendi tre`. Nevertheless, presently I see no obstacles which would block the implementation of this catchword.

Problem 2. *Carry through simultaneous stabilization of* three *columns of a root element in minimal representations of Chevalley groups of types* E_6 *and* E_7*, by an element of root type* A_2*.*

I am convinced that these new gadgets would lead to *drastic* simplifications of the existing proofs of structure results.[4] This would then open *immense* prospects of further applications. Let us state some specific problems which we are about to assail by means of these new devices.

First of all, this would allow us to construct simpler models of the Steinberg group, which would then *reanimate* hopes of a solution of the following problem.

Problem 3. *Prove the centrality of the extension* $\mathrm{St}(\Phi, R) \longrightarrow E(\Phi, R)$ *for types* $\Phi = \mathrm{E}_6, \mathrm{E}_7$*.*

Furthermore, the possibility to simultaneously stabilize several columns — or else, a column and a row — of a root element would *dramatically* simplify the proofs in the recent papers by the author and Sergei Nikolenko [16] and by Alexander Luzgarev [21], concerning the structure of the Chevalley group $G(\mathrm{F}_4, R)$ and description of its overgroups. After that, it would be much easier to obtain similar results for *twisted* Chevalley groups ${}^2G(\mathrm{E}_6, R)$, also in the case where the involution on R is nontrivial.

Problem 4. *Devise an* (A_3, P_2)*-proof of the main structure theorems for the twisted Chevalley group* ${}^2G(\mathrm{E}_6, R)$ *over an arbitrary commutative ring* R*.*

Problem 5. *Devise an* (A_3, P_2)*-proof of the description of overgroups of the twisted Chevalley group* ${}^2G(\mathrm{E}_6, R)$ *in the Chevalley group* $G(\mathrm{E}_6, R)$ *over an arbitrary commutative ring* R*.*

We take the liberty of reiterating some problems, which were already stated as such in [6]. However, presently their solution seems to be much more imminent.

[4] *Note added in proof.* In January 2012, the author solved the above Problem 1 and Problem 2; see [48]. The possible applications mentioned in the text are still pending.

Problem 6. *Establish the standard description of normal subgroups in twisted forms of Chevalley groups of types* E_6 *and* E_7 *over an arbitrary commutative ring* R*, provided that they contain a split subgroup of type* A_2*.*

We list the forms ${}_{\varphi}G(\Phi, R)$ in question; see [36].

• For E_6, this is the quasisplit group of type ${}^2\mathrm{E}_6$ already mentioned in Problem 3 plus two inner forms of relative rank 2, with Tits indices $\mathrm{E}^{28}_{6,2}$ and $\mathrm{E}^{16}_{6,2}$, plus two outer forms of relative rank 2, with Tits indices ${}^2\mathrm{E}^{16'}_{6,2}$ and ${}^2\mathrm{E}^{16''}_{6,2}$.

• For E_7, these are the forms of relative ranks 2, 3, and 4, with Tits indices $\mathrm{E}^{31}_{7,2}$, $\mathrm{E}^{28}_{7,3}$ and $\mathrm{E}^{9}_{7,4}$, respectively.

It is well known that the solution of the following problem would be a decisive step towards a complete description of subnormal subgroups of $G = G(\Phi, R)$.

Problem 7. *Describe the subgroups of* $G(\Phi, R)$*,* $\Phi = \mathrm{E}_6, \mathrm{E}_7$*, normalized by the relative elementary subgroup* $E(\Phi, R, A)$ *for some ideal* $A \trianglelefteq R$*.*

Recall that the standard answer to this problem can be stated as follows. There exists a natural m that depends on Φ alone and has the property that for each subgroup H normalized by $E(\Phi, R, A)$ there exists an ideal $I \trianglelefteq R$ with

$$\mathrm{E}(\Phi, R, A^m I) \leq H \leq \mathrm{C}(\Phi, R, I),$$

where $C(\Phi, R, I)$ is the full congruence subgroup of level I. Such an ideal I is unique up to a certain natural equivalence relation $\Diamond_A$. From a technical viewpoint, the most important aspect of this problem is to find the *minimal* value of m for which these inclusions occur. Recently Zuhong Zhang and the present author have established some preliminary results in this direction. We expect that the solution of the above Problem 2 would allow to give realistic bounds for m, close to the known bounds for classical groups.[5]

Moreover, in view of this new gear it would make sense to reconsider the description of overgroups of subsystem subgroups in exceptional Chevalley groups[6], at least for some examples of subsystems $\Delta \subseteq \mathrm{E}_6, \mathrm{E}_7$.

Problem 8. *Describe the subgroups of* $G(\Phi, R)$ *containing an elementary subsystem subgroup* $E(\Delta, R)$*, provided that* $\Delta^{\perp} = \varnothing$ *and all irreducible components of* Δ*, except maybe one of them, have ranks at least* 2*.*

Apart from the problems listed above, the solution of Problems 1 and 2 would offer a new approach to many further important questions of structure theory, both to those already settled and to some open ones.

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[5] *Note added in proof.* After publication of the Russian original, Zuhong Zhang and the author [50] completely solved Problem 7, with the bound $m = 7$.

[6] *Note added in proof.* The first important step in this direction was taken by Alexander Shchegolev and the author in [49]. In particular, we completely determine the levels of all overgroups of subsystem subgroups, under the assumption $\Delta^{\perp} = \varnothing$. For exceptional groups, the second condition on Δ in Problem 8 seems to be immaterial.

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