REDUCIBILITY OF FUNCTION PAIRS IN $H^\infty_R$

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Abstract. A short proof of a result by Brett Wick on the reducibility of function pairs in $H^\infty_R$ is presented, and some unusual properties of the solutions to the associated Bézout equations are unveiled.

§1. The problem

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and let $H^\infty$ be the (complex) algebra of all bounded holomorphic functions on $\mathbb{D}$. Moreover, let $H^\infty_R$ be the (real) algebra of all bounded holomorphic functions in $\mathbb{D}$ that satisfy the symmetry condition $f(z) = \overline{f(\bar{z})}$ for $z \in \mathbb{D}$. It is well known that $H^\infty_R$ is the set of functions in $H^\infty$ with real Fourier coefficients, and that $H^\infty_R$ coincides with the set of all functions in $H^\infty$ that are real-valued on $]-1,1[$. This algebra gained a lot of interest the last years in view of its appearance in control theory (see [13, 14, 9]). It is a direct consequence of the corona theorem for $H^\infty$ that if $(f, g)$ is a function pair in $H^\infty_R$ with $\delta := \inf_{z \in \mathbb{D}} |f(z)| + |g(z)| > 0$, then there exist $p, q \in H^\infty_R$ such that $pf + qg = 1$ (see [14]). On the other hand, in contrast to the $H^\infty$-setting (see [12]), it is not always possible to choose the first factor $p$ to be invertible in $H^\infty_R$. In fact, as was already mentioned in [13], if $f(z) = z$ and $g(z) = 1 - z^2$, then each function of the form $F := z + h(1 - z^2)$ with $h \in H^\infty_R$ must have a zero within $]-1,1[$, because $F$ is real-valued on $]-1,1[$, $F(-1) < 0$ and $F(1) > 0$.

It is therefore a natural problem to characterize the pairs $(f, g)$ in $H^\infty_R$ that are reducible. Recall that a pair $(f, g)$ in a commutative, unital Banach algebra $A$ is said to be invertible if $af + bg = 1$ for some $a, b \in A$ and that $(f, g)$ is reducible if there exists $h \in A$ such that $f + hg$ is invertible in $A$. In other words, $(f, g)$ is reducible if there exists a solution $(u, h)$ of the Bézout equation $uf + hg = 1$ such that the first factor $u$ is invertible.

Motivated by his earlier result [13] for the algebra $A_R(\mathbb{D})$ of functions in $H^\infty_R$ that admit continuous extensions to the Euclidean closure of $\mathbb{D}$, in a series of papers [14, 15] Brett Wick developed the following theorem.

Theorem W (see [15]). Let $(f, g) \in (H^\infty_R)^2$ be a function pair satisfying $\|f\|_\infty \leq 1$, $\|g\|_\infty \leq 1$ and $|f| + |g| \geq \delta > 0$. Suppose that there exists $\varepsilon > 0$ such that $f$ has constant sign on the set $\{x \in ]-1,1[, |g(x)| < \varepsilon\}$. Then there exists a constant $C = C(\delta, \varepsilon)$ depending only on $\delta$ and $\varepsilon$ such that the Bézout equation

$$uf + hg = 1$$

admits a solution $(u, h) \in (H^\infty_R)^2$ with $u$ invertible and

$$\Delta(u, u^{-1}, h) := \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta, \varepsilon).$$

Note that the hypothesis on the sign is satisfied automatically whenever $|g|$ is bounded away from zero on $]-1,1[$ (in that case for $\varepsilon$ we can take any number strictly smaller than

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if \(x \in \mathbb{R} \setminus \{-1,1\}\). Also, if \((f,g)\) is reducible in \(H^\infty_\mathbb{R}\), then the sign condition is fulfilled necessarily (see below).

The intention of our paper is two-fold. First, in §2 we give a new short proof of the reducibility part of this theorem. Our proof will be based on the maximal ideal space theory for \(H^\infty\) and on Treil’s theorem [12] on the reducibility of pairs in \(H^\infty\). The original proof by Wick involved the \(\delta\)-calculus and depended on the methods developed by Treil to solve Bézout equations.

Then, in §3 we present a new proof of Theorem W with norm estimates that depends only on the corresponding result by Treil [12], but is otherwise quite elementary.

In §4 we provide some counterexamples to the original statement of Wick’s result [14]. First, we give examples of invertible pairs \((f,g)\) that show that, in general, the weaker hypothesis of \(f\) having constant sign on the set of real zeros of \(g\) is not sufficient in order to get a solution of the Bézout equation

\[
uf + hg = 1, \quad u \text{ invertible in } H^\infty_\mathbb{R}.
\]

We mention that in [9, Proposition 2.8] it was shown that if \(g\) is a Blaschke product having only real and simple zeros, then the hypothesis of \(f\) being positive on the set of real zeros of \(g\) is indeed sufficient to guarantee the reducibility of the pair (whenever \((f,g)\) is invertible).

Then we show that there exist reducible pairs \((f_n,g_n)\) of functions in the unit ball of \(H^\infty_\mathbb{R}\) with \(|f_n| + |g_n| \geq \delta > 0\) such that, quite surprisingly,

\[
\sup_n \Delta(u_n,u_n^{-1},h_n) = \infty
\]

for any solutions \((u_n,h_n)\) of the equation \(uf_n + hg_n = 1\) with \(u\) invertible in \(H^\infty_\mathbb{R}\).

This shows that in Wick’s Theorem W the parameter \(\varepsilon\) plays an essential role. To the best of our knowledge, this is the first occurrence of a corona-type theorem for a uniform algebra where the uniform boundedness of solutions fails.

§2. A maximal ideal space view of Wick’s result

Let \(M(H^\infty)\) be the spectrum (or maximal ideal) space of \(H^\infty\); that is, \(M(H^\infty)\) is the space of all (nonzero) multiplicative linear functionals on \(H^\infty\) endowed with the trace of the weak*-topology \(\sigma((H^\infty)^*,H^\infty)\), also called the Gelfand topology. Identifying each point \(z_0\) in \(\mathbb{D}\) with the evaluation functional \(\Phi_{z_0} : f \mapsto f(z_0)\), we can identify \(\mathbb{D}\) with a subset of \(M(H^\infty)\). By Carleson’s famous corona theorem, \(\mathbb{D}\) is dense in \(M(H^\infty)\). The Gelfand transform \(\hat{f} : M(H^\infty) \to \mathbb{C}\) of an element \(f \in H^\infty\) is given by \(\hat{f}(m) = m(f)\). It can be viewed as a unique continuous extension of \(f\) from \(\mathbb{D}\) to \(M(H^\infty)\). As usual, we shall identify \(\hat{f}\) with \(f\).

A corona pair in \(A = H^\infty\) (or \(A = H^\infty_\mathbb{R}\)) is a pair \((f,g)\) of functions in \(A\) such that \(\inf_{z \in \mathbb{D}} |f(z)| + |g(z)| \geq \delta > 0\). The zero set of \(f \in H^\infty\) is given by \(Z(f) = \{m \in M(H^\infty) : f(m) = 0\}\). Moreover, let \(Z_\mathbb{D}(f) = \{z \in \mathbb{D} : f(z) = 0\}\). It is well known that if \(b\) is an interpolating Blaschke product, then \(Z(b)\) is the closure of \(Z_\mathbb{D}(b)\) (see [3, p. 379]); in general, for arbitrary Blaschke products \(B\), the set \(Z_\mathbb{D}(B)\) is much smaller than \(Z(B)\) (see [4] for a study of the inner functions \(\Theta\) for which \(Z(\Theta) = Z_\mathbb{D}(\Theta)\)).

In the sequel, we may identify \(M(H^\infty)\) with \(M(H^\infty_\mathbb{R})\) (because \(H^\infty\) is the complexification of \(H^\infty_\mathbb{R}\); see [6, pp. 37–38]), and we look upon the functions in \(H^\infty_\mathbb{R}\) as being defined on \(M(H^\infty)\).

A prominent role in our approach to Wick’s result will be played by the closure \(E\) of the interval \([-1,1]\) in \(M(H^\infty)\). If \(g \in H^\infty_\mathbb{R}\), then the set of extended real zeros of \(g\) is the set \(Z(g) \cap E\).
The following topological observation will give a relationship between the hypotheses in Theorem W and our approach (see [9, p. 178]).

**Observation 2.1.** Let \((f, g) \in H_\mathbb{R}^\infty\) be a corona pair. The following assertions are equivalent:

1. There exists \(\varepsilon > 0\) such that \(f\) has constant sign on the set \(\{x \in ]-1, 1[: |g(x)| < \varepsilon\}\);
2. \(f\) has constant sign on the set of extended real zeros of \(g\).

**Proof.** Suppose that \(|f| + |g| \geq \delta > 0\).

1. \(\Rightarrow\) (2). We may assume that \(\varepsilon < \delta/2\). Then \(|f(z)| > \delta/2\) whenever \(|g(z)| < \varepsilon\). Say that \(f\) is positive at each \(x \in ]-1, 1[\) with \(|g(x)| < \varepsilon\). Let \(m_1, m_2 \in Z(g) \cap E\). Choose a net \(z_\alpha\) in \(]-1, 1[\) that converges to \(m_1\). Hence, \(g(z_\alpha) \to 0\). In particular, there is \(\alpha_0\) such that \(|g(z_\alpha)| < \varepsilon\) for all \(\alpha > \alpha_0\), whence \(f(z_\alpha) > \delta/2\). Therefore, \(f(m_1) \geq \delta/2 > 0\).

By the same argument, \(f(m_2) \geq \delta/2 > 0\). Thus, \(f\) has constant sign on the set of extended real zeros of \(g\).

2. \(\Rightarrow\) (1). Since \(|f| + |g| \geq \delta > 0\), by (2) we may assume that \(f > \delta/2\) on \(Z(g) \cap E\). Now,

\[Z(g) \cap E = \bigcap_{n=1}^{\infty} \{m \in E : |g(m)| \leq 1/n\}.\]

The finite intersection property for compact spaces (here \(E\)) implies that there is \(n_0\) such that

\[\{m \in E : |g(m)| < 1/n_0\} \subseteq \{m \in E : f(m) > \delta/2\}.\]

Thus, \(f(x) > 0\) at each point \(x \in ]-1, 1[\) with \(|g(x)| \leq 1/n_0\). \(\square\)

The following lemma is due to Corach and Suárez [1, 2].

**Lemma 2.2** (see [1, p. 636] and [2, p. 608]). Let \(A\) be a real or complex commutative unital Banach algebra with unit element denoted by 1.

a) Suppose that \((f, g) \in A^2\) is an invertible pair. If \((f_n)\) converges to \(f\) in the norm and if the pairs \((f_n, g)\) are reducible, then \((f, g)\) is also reducible.

b) For \(g \in A\), the set

\[R(g) = \{f \in A : (f, g) \text{ is reducible}\}\]

is open-closed inside

\[I(g) = \{f \in A : (f, g) \text{ is invertible}\}\].

c) If \(\phi : [0, 1] \to I(g)\) is a continuous curve and \((\phi(0), g)\) is reducible, then \((\phi(1), g)\) is reducible.

Note that part a) follows from the identity

\[f_n + u_n g = f(x f_n + y g) + (y f_n - f) + u_n g,\]

where \(f_n + u_n g \in A^{-1}\) and \(x f + y g = 1\) (see [10] or [8]).

One of our tools will be the following result from [7]. Recall that a compact set \(C \subseteq M(A)\) is said to be \(A\)-convex if \(C\) coincides with its \(A\)-convex hull

\[\tilde{C} = \{m \in M(A) : |\hat{f}(m)| \leq \max_{\tilde{C}} |\hat{f}| \text{ for all } f \in A\}.\]

Here we represent the short proof for the reader’s convenience. In the sequel, let \(f^*\) be the function defined by \(f^*(z) = \hat{f}(z)\).

**Theorem 2.3** (see [7]). The set \(E\) is \(H^\infty\) and \(H_\mathbb{R}^\infty\)-convex.
Then there exist \( h \in H_\infty \) such that \( f(m) = 0 \) and \( f \neq 0 \) on \( E \). We may assume that \( \|f\|_\infty \leq 1 \). Let \( g = ff^* \). Then \( g \in H_\infty^\infty \) and \( g \) is real-valued on \( E \). By the definition of \( g \), actually we have \( 1 \geq g \geq 0 \) on \( E \). Since \( f \neq 0 \) on \( E \) and \( f^* = \bar{f} \) on \( E \), we see that \( \sigma := \min_{E} g > 0 \). Now, let \( h = 1 - g \). Then \( h \in H_\infty^\infty \subseteq H_\infty \) and \( h(m) = 1 \), whence

\[
\max_{E} |h| = \max_{E} h \leq 1 - \sigma < |h(m)|.
\]

Thus, \( m \) does not belong to the \( H_\infty^\infty \)-convex closure of \( E \). Therefore, \( E \) is \( H_\infty^\infty \)-convex as well as \( H_\infty^\infty \)-convex.

Now we are ready to give the announced short proof of the reducibility part of Wick’s result. It is essentially based on Treil’s result that in \( H_\infty \) every corona pair is reducible \(^{[12]}\).

**Theorem 2.4** (Treil’s theorem). Let \((f, g) \in (H_\infty)^2 \) be a corona pair satisfying

\[
\|f\|_\infty \leq 1, \quad \|g\|_\infty \leq 1, \quad \text{and} \quad |f| + |g| \geq \delta > 0.
\]

Then there exist \( h \in H_\infty \) and a constant \( \kappa = \kappa(\delta) \) depending only on \( \delta \) such that \( v := f + hg \) is invertible in \( H_\infty \) and

\[
\|h\|_\infty + \|v^{-1}\|_\infty \leq \kappa(\delta).
\]

**Theorem 2.5** (see \(^{[14] [15]}\)). Let \((f, g) \) be a corona pair in \( H_\infty^\infty \). Then \((f, g)\) is reducible if and only if \( f \) has constant sign on the set of extended real zeros of \( g \).

**Proof.** First, we prove the “only if” part. Let \( u = f + hg \) be invertible in \( H_\infty^\infty \). Here \( h \in H_\infty^\infty \). Suppose that there exist \( m_1, m_2 \in Z(g) \cap E \) such that \( f(m_1) < 0 \) and \( f(m_2) > 0 \). In particular \( u(m_1) < 0 \) and \( u(m_2) > 0 \). Since \( E \) is a connected set, the continuity of the (real-valued) function \( u \) on \( E \) implies that \( u \) has a zero on \( E \). Hence, \( u \) is no longer invertible. (A finer analysis actually would yield a zero of \( u \) in \([-1, 1] \).)

To prove the sufficiency of the sign condition, consider the (real) trace-algebra \( A = H_\infty^\infty |_E \), where the closure is taken with respect to the supremum norm in the space \( C(E, \mathbb{R}) \) of all real-valued continuous functions on \( E \). Since \( E \) is \( H_\infty^\infty \)-convex by Theorem 2.3, the spectrum \( M(A) \) of \( A \) can be identified with \( E \) (see \(^{[6]} \) p. 125)). By our hypothesis, we may assume that \( f > 0 \) on \( Z(g) \cap E \). This implies that for every \( t \geq 0 \) the pairs \((f|E + t, g|E)\) are invertible in \( A \). Since \( f|E + N \) is invertible in \( A \) for large \( N \in \mathbb{N} \), the pair \((f|E + N, g|E)\) is reducible in \( A \). By Lemma 2.2, this implies that \((f|E, g|E)\) is also reducible in \( A \). This means that there exists \( q \in A \) such that \( f + qg \neq 0 \) on \( E \). Approximating \( q \) uniformly on \( E \) by a function \( h \in H_\infty^\infty \), we see that \( v := f + hg \in H_\infty^\infty \) and that \( v \) does not vanish on \( E \). In particular, \( v \) has no real zeros. Now, \((v, g)\) is also a corona pair. Therefore, by \(^{[9]}\), Proposition 2.6, \((v, g)\) is reducible in \( H_\infty^\infty \). Thus, there exists \( H \in H_\infty^\infty \) so that

\[
(f + hg) + Hg = f + (h + H)g
\]

is invertible in \( H_\infty^\infty \). In other words, \((f, g)\) is reducible in \( H_\infty^\infty \). \( \square \)

§3. Wick’s Theorem with norm estimates

In this section we present a new proof of Theorem W with norm estimate. Our proof depends only on Treil’s result, i.e., Theorem 2.4, but does not need the quite delicate construction from \(^{[12]}\).

For this, we shall need the following extensions of \(^{[9]}\) Proposition 2.2 and \(^{[9]}\) Proposition 2.6. The proofs work exactly in the same way as those in \(^{[9]}\): at each occurrence

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\(^{1}\) Here Treil’s result is used.
of Treil’s theorem, we additionally must take into account his norm estimates. Upon the referee’s request, we present here the details.

**Proposition 3.1.** Let \((f, g)\) be a corona pair in \(H^\infty_R\) with \(\|f\|_\infty \leq 1, \|g\|_\infty \leq 1,\) and \(|f| + |g| \geq \delta > 0\). Then the following assertions hold.

1. There exist a constant \(C(\delta)\) (depending only on \(\delta\)) and functions \(u, h \in H^\infty_R\) with \(u\) invertible such that \(1 = uf^2 + hg\) and
   \[
   \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta).
   \]

2. Suppose that \(f\) has no real zeros. Then, again, there exist a constant \(C(\delta)\)
   (depending only on \(\delta\)) and functions \(u, h \in H^\infty_R\) with \(u\) invertible such that
   \(1 = uf^2 + hg\) and
   \[
   \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta).
   \]

**Proof.** (1) By Theorem 2.4 there exist a constant \(\kappa\) depending only on \(\delta\), and \(k \in H^\infty\) such that \(v := f + kg\) is invertible in \(H^\infty\) and \(\|k\|_\infty + \|v^{-1}\|_\infty \leq \kappa\). In particular, \(v^* = f + k^*g\) is invertible in \(H^\infty\). Multiplying the two equations, we see that
   \[
   f^2 + g(f + k^* + kk^*g) = vv^*
   \]
   is invertible in \(H^\infty\). But the factors \(k + k^*\) and \(kk^*\) belong to \(H^\infty_R\). Thus, with \(u = (vv^*)^{-1}\) and \(h = (f(k + k^*) + kk^*g)(vv^*)^{-1}\) we get \(1 = uf^2 + hg\) and
   \[
   \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \leq C(\delta).
   \]

(2) Write \(f = BF\) where \(F\) is zero-free and \(B\) is the Blaschke product associated with the zeros of \(f\). Since the zeros of \(B\) are symmetric with respect to the real axis, \(B\) and \(F\) are in \(H^\infty_R\). Without loss of generality, we may assume that \(F > 0\) on \([-1, 1]\). Moreover, \(F = G^2\) for some \(G \in H^\infty_R\) (see [9, p. 180]). Thus, \((B, g)\) and \((G, g)\) are invertible pairs in \((H^\infty_R)^2\) satisfying \(|B| + |g| \geq \delta\) and \(|G| + |g| \geq \delta\). By (1), there exists a constant \(\kappa_1 > 0\) independent of \(\delta\) such that
   \[
   w := G^2 + k_1g = F + k_1g
   \]
is invertible in \(H^\infty_R\) for some \(k_1 \in H^\infty_R\) with \(\|k_1\|_\infty + \|w^{-1}\|_\infty \leq \kappa_1\).

It remains to prove the same property for the pair \((B, g)\). Let \(C\) be the Blaschke product formed with the zeros of \(B\) on the upper-half plane, and let \(C^*\) be defined by \(C^*(z) := C(\overline{z})\). Since \(f\) (and with it, \(B\)) has no real zeros, we see that \(B = CC^*\). Since \((C, g)\) is an invertible pair in \((H^\infty_R)^2\) with \(|C| + |g| \geq \delta\), Treil’s Theorem 2.4 yields the existence of \(k_2 \in H^\infty\) and \(\kappa_2 > 0\) depending only on \(\delta\) such that \(v := C + k_2g\) is invertible in \(H^\infty\) and \(\|k_2\|_\infty + \|v^{-1}\| \leq \kappa_2\). But \(v^* = C^* + k_2^*g\). Multiplying the two equations, we get
   \[
   CC^* + ((Ck_2^* + C^*k_2) + k_2k_2^*g)g = vv^*.
   \]
Thus, with
   \[
   K = (Ck_2^* + C^*k_2) + k_2k_2^*g \in H^\infty_R,
   \]
the function \(\tilde{w} := vv^* = B + Kg\) is invertible in \(H^\infty_R\) and
   \[
   \|K\|_\infty + \|\tilde{w}^{-1}\|_\infty \leq \kappa(\delta)
   \]
for some constant \(\kappa(\delta)\).

To summarize, we have shown that
   \[
   R := w\tilde{w} = (F + k_1g)(B + Kg) = BF + g(KF + k_1B + k_1Kg)
   \]
is invertible in \(H^\infty_R\) and that \(R = f + hg\), where \(h := KF + k_1B + k_1Kg\) and \(\|h\|_\infty + \|R^{-1}\|_\infty \leq c(\delta)\).

It is now easy to deduce the claim. \(\square\)
Theorem 3.2. Let \((f, g) \in (H^\infty_{\mathbb{R}})^2\) be a corona pair satisfying \(\|f\|_{\infty} \leq 1\), \(\|g\|_{\infty} \leq 1\), and \(|f| + |g| \geq \delta > 0\). Suppose that there exists \(\varepsilon > 0\) such that \(f\) has constant sign on the set \(\{x \in ]-1, 1[\colon |g(x)| < \varepsilon\}\). Then there exists a constant \(C = C(\delta, \varepsilon)\) depending only on \(\delta\) and \(\varepsilon\) such that the Bézout equation

\[ uf + hg = 1 \]

admits a solution \((u, h) \in (H^\infty_{\mathbb{R}})^2\) with \(u\) invertible and

\[ \Delta(u, u^{-1}, h) := \|u\|_{\infty} + \|u^{-1}\|_{\infty} + \|h\|_{\infty} \leq C(\delta, \varepsilon). \]

Proof. The idea is to replace the function \(g\) by \(g^2\), so that for large \(N\), the function \(f + Ng^2\) is positive on the set of all \(x \in ]-1, 1[\) where \(|g(x)|\) is small.

So, suppose that \(f(x) > 0\) for all \(x \in ]-1, 1[\) with \(|g(x)| < \varepsilon\). We may assume that \(\varepsilon < \delta/2\). Then \(|f| + |g| \geq \delta > 0\) implies that \(f(x) > \delta/2\) for those \(x\). Let \(N = \max\left\{\frac{4}{\delta}, \frac{1}{\varepsilon^2}\right\}\).

Then \(N = \frac{1}{\varepsilon^2}\). We claim that \(f + Ng^2\) has no zeros on \(]-1, 1[\). In fact, if \(|g(x)| \leq \varepsilon\), then

\[ |f(x) + Ng^2(x)| = f(x) + Ng^2(x) > \frac{\delta}{2} > 0. \]

If \(|g(x)| > \varepsilon\), then \(g^2(x) > \varepsilon^2\), whence

\[ |f(x) + Ng^2(x)| \geq Ng^2(x) - |f(x)| > N\varepsilon^2 - 1 \geq 0. \]

Let \(F := \frac{f + Ng^2}{1 + N}\). Then \((F, g)\) is a corona pair. Next we show that

\[ |F| + |g| \geq \frac{\delta}{4N}. \]

Indeed, if \(|g| \geq \sqrt{\frac{\delta}{4N}}\), then we are done. Now, let \(|g| \leq \sqrt{\frac{\delta}{4N}}\). By the choice of \(N\), we have

\[ \sqrt{\frac{\delta}{4N}} \leq \frac{\delta}{4}. \]

Hence, since \(|f| + |g| \geq \delta\), we see that \(|f| \geq \frac{3}{4}\delta\). Accordingly,

\[ |F| \geq \frac{|f| - |g|^2}{N + 1} \geq \frac{3}{4}\delta - N \frac{\delta}{4N} = \frac{\delta}{2} \frac{1}{1 + N} \geq \frac{\delta}{4N}. \]

This establishes (3.1).

Since \(F\) has no real zeros, we may apply Proposition 3.1 to obtain functions \(u\) and \(h\) in \(H^\infty_{\mathbb{R}}\) such that \(u\) is invertible,

\[ 1 = uF + hg, \]

and

\[ \|u\|_{\infty} + \|u^{-1}\|_{\infty} + \|h\|_{\infty} \leq C(\delta/4N) = C((\varepsilon^2\delta)/4) =: C. \]

Thus,

\[ 1 = \frac{u}{1 + N} f + \left(\frac{Nu g}{1 + N} + h\right)g \]

is the solution we were looking for; simply put

\[ C(\delta, \varepsilon) = 2C + \left(1 + \frac{1}{\varepsilon^2}\right)C. \]
A natural question is whether it is possible to get bounds on the solution triple
\((u, u^{-1}, h)\) of the Bézout equation \(uf + hg = 1\), \(u\) invertible in \(H^\infty\), that depend only on
\[\delta = \inf_z (|f(z)| + |g(z)|).\]

For example, this is the case under the hypotheses of Proposition 3.1. The answer is
provided in §3.

§4. The sign condition on \([-1, 1]\) is not sufficient

In [9] Proposition 2.8 it was shown that the pair \((f, B)\) is reducible in \(H^\infty\) whenever
\(B\) is a Blaschke product having only real zeros (of multiplicity one) and \(f \in H^\infty\) is such
that \(f\) is positive on the real zeros of \(B\) and \(|f| + |B| \geq \delta > 0\).

If \(B\) is replaced by an outer function \(g\) or a singular inner function, then \(g\) has no
zeros at all in \(\mathbb{D}\) and the hypothesis of \(f\) being positive at the real zeros of \(g\) is true
by convention. We show that in that case reducibility is not necessarily forced by this
vacuous condition.

Example 4.1. Let \(f(z) = z\) and \(g(z) = \exp(-\frac{1+z^2}{1-z^2}).\) Then \(g\) has no zeros in \([-1, 1]\),
but nevertheless \((f, g)\) is not reducible in \(H^\infty\).

Proof. Let \(h \in H^\infty\) be given, and set \(L(z) = z + h(z)g(z)\). Since \(\lim_{r \to 1} g(r) = 0\) and
\(\lim_{x \to -1} g(x) = 0\), we see that \(\lim_{r \to 1} L(r) = 1\) and \(\lim_{x \to -1} L(x) = -1\). Thus, being
real valued on \([-1, 1]\), \(L\) has a zero within \([-1, 1]\), so that \(L\) is not invertible.

We could have worked also with the pair \((z, 1 - z^2)\). But the example above has the
additional property that \(g\) has no zeros in \(\mathbb{C}\) at all (apart from the essential singularities
at \(-1\) and 1).

Here is yet another example; this time we give an example such that the second
member \(g\) does have real zeros (note that conditions involving empty sets are sometimes
arguable). It is based on the fact that \(f\) is positive on the real zeros of \(g\) if and only
if between two real zeros of \(g\) there is an even number of zeros of \(f\) (or none). In the
following, let \(\mathbb{N} = \{0, 1, 2, \ldots\}\).

Example 4.2. Let \(B\) be the interpolating Blaschke product with the zeros \(z_n = 1 - 2^{-n}\)
\((n = 1, 2, \ldots)\), let \(B_1\) be the subproduct of \(B\) with the zero set
\[\{z_n : n = 3j + 2 \text{ or } n = 3(j + 1), \ j \in \mathbb{N}\},\]
that is, \(Z\mathbb{D}(B_1) = \{z_2, z_3, z_5, z_6, \ldots\}\), and let \(B_2\) be the subproduct of \(B\) with the zero set
\[\{z_n : n = 3j + 1, \ j \in \mathbb{N}\}.

We introduce two functions \(f\) and \(g\) by
\[f(z) = zB_1(z), \quad g(z) = B_2(z) \exp \left(-\frac{1-z}{1+z}\right).\]

We are going to show that \((f, g)\) is a corona pair, \(f\) has constant sign on the real zeros
of \(g\), but \((f, g)\) is not reducible in \(H^\infty\).

Proof. Since \(B = B_1B_2\) is an interpolating Blaschke product, from [5, p. 208] we see that
the ideal generated by \(B_1\) and \(B_2\) in \(H^\infty\) equals \(H^\infty\). Hence \(B_1\) and \(B_2\) have no common
zeros on the spectrum of \(H^\infty\). Moreover, since the zeros of \(\exp(-\frac{1-z}{1+z})\) in \(M(H^\infty)\) are
in the fiber over the point \(z = -1\), we conclude that \(f\) and \(g\) have no zeros in common
on \(M(H^\infty)\). Thus \((f, g)\) is a corona pair in \(H^\infty\), and consequently also in \(H^\infty\).
By construction, between two distinct real zeros of $B_2$ there are exactly two zeros of $f$. Thus $f$ has constant sign on the real zeros of $g$. Now all the zeros of $B_1$ are larger than the first zero $z_1$ of $g$. Recall that $B_1$ has the form

\begin{equation}
B_1(x) = \prod_j \frac{a_j - x}{1 - a_j x}.
\end{equation}

Hence, $f(z_1) > 0$, and so this constant sign is the positive sign. Moreover, $B_1$ is bounded away from zero on the zeros of $B_2$ (because $(B_1, B_2)$ is a corona pair). So, we can conclude that for given $h \in H^\infty_{\mathbb{R}}$, the function $W(z) = zB_1(z) + h(z)g(z)$ satisfies

$$\liminf_n W(z_{3n+1}) = \liminf_n z_{3n+1}B_1(z_{3n+1}) > 0.$$ 

On the other hand, $B_1$ is unimodular and continuous at $-1$ and so, by (4.1), $B_1(-1) = 1 > 0$. Thus,

$$\lim_{x \to -1} W(x) = -1.$$ 

Again we see that $W$ has a zero in $] -1, 1[$. \hfill $\Box$

§5. NON-UNIFORM BOUNDED SOLUTIONS

In contrast to the situation in $H^\infty$ (see [12]), we will show here that the reducibility of sequences of corona pairs $(f_n, g_n)$ in the unit ball of $H^\infty_{\mathbb{R}}$ with $|f_n| + |g_n| \geq \delta > 0$ does not imply the existence of uniformly bounded solutions of the Bézout equation $u_nf_n + h_ng_n = 1$ with $u_n$ invertible, where the bounds of the norms of $u_n, u_n^{-1}$ and $h_n$ depend only on $\delta$. Recall that

$$\Delta(u, u^{-1}, h) = \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty.$$ 

**Example 5.1.** Put $f(z) = z$ and $g(z) = \exp(-\frac{1+z}{1-z})$. We choose a sequence $(r_n)$ of positive numbers converging to 1 ($0 < r_n < 1$); let $f_n(z) = f(r_nz)$ and $g_n(z) = g(r_nz)$. Then the pairs $(f_n, g_n)$ are reducible in $H^\infty_{\mathbb{R}}$ and satisfy $|f_n| + |g_n| \geq \delta > 0$, but

$$\sup_n \Delta(u_n, u_n^{-1}, h_n) = \infty$$

for any solutions $(u_n, h_n) \in (H^\infty_{\mathbb{R}})^2$ of the equation $u_nf_n + h_ng_n = 1$ with $u_n$ invertible in $H^\infty_{\mathbb{R}}$.

**Proof.** Suppose to the contrary that there exist solutions $(u_n, h_n)$ satisfying

$$\sup_n \Delta(u_n, u_n^{-1}, h_n) < \infty.$$ 

Using a normal families argument, we can find subsequences of $u_n$ and $h_n$ converging locally uniformly to some functions $u$ and $h$ in $H^\infty_{\mathbb{R}}$. Since $|u_n|$ is bounded away from zero by a constant independent of $n$, the limit function $u$ is also invertible. Thus,

$$uf + hg = 1,$$

but this contradicts Example 4.1. \hfill $\Box$

We can also create counterexamples to this norm-estimate problem by working with finite Blaschke products. Note that this is the class of functions Brett Wick worked with in [14].

**Example 5.2.** There exists a sequence $(B_n)$ of finite Blaschke products in $H^\infty_{\mathbb{R}}$ such that the pairs $(z, B_n)$ are reducible in $H^\infty_{\mathbb{R}}$ and satisfy $|z| + |B_n| \geq \eta > 0$, but

$$\sup_n \Delta(u_n, u_n^{-1}, h_n) = \infty$$

for any solutions $(u_n, h_n) \in (H^\infty_{\mathbb{R}})^2$ of $u_nz + h_n B_n = 1$ with $u_n$ invertible in $H^\infty_{\mathbb{R}}$. 

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Reducibility of function pairs in $H^\infty_R$

Proof. Let $g^*(z) = \overline{g(z)}$. Note that a function $g \in H^\infty$ belongs to $H^\infty_R$ if and only if $g = g^*$. Let $f(z) = z, g(z) = (1 - z^2)/2$. By the Carathéodory theorem (see [3, p. 6]), we can choose a sequence of finite Blaschke products $b_n$ converging locally uniformly to $g$ in $D$. If $b_n$ has a real zero, we move it vertically a little to the upper half disk. Since $g = g^*$, $B_n := b_n b_n^*$ tends locally uniformly to $g^2$. Moreover, $B_n \in A(D)_R$ and has no real zeros. Obviously, $(z, g^2)$ is a corona pair, say $|z| + |g_2| \geq \delta > 0$, where we may choose $\delta < 1$. Since $B_n$ converges to $g^2$, uniformly on $|z| \leq 1/2$, we see that $|z| + |B_n| \geq \delta/2$ on $|z| \leq 1/2$ whenever $n$ is large. For $1/2 \leq |z| \leq 1$ we obtain $|z| + |B_n| \geq 1/2 \geq \delta/2$.

Thus, the $(z, B_n)$ are corona pairs with $|z| + |B_n| \geq \delta/2$. Using Lemma 2.2, we show that these pairs are reducible in $A(D)_R$. Indeed, since $B_n > 0$ on $[-1, 1]$, the pairs $(z + t, B_n)$ are invertible for any $t \geq 0$. For $t = 2$, e.g., $z + 2$ is an invertible function in $A(D)_R$; hence, $(z + 2, B_n)$ is reducible. Thus, by Lemma 2.2, $(z, B_n)$ is also reducible.

So, let

$$1 = u_n z + h_n B_n,$$

where $u_n$ is invertible in $A_R(D)$.

As above, no bound on the norms of $u_n, u_n^{-1}$ and $h_n$ is possible, because otherwise $1 = uz + hg^2$ for some $u \in H^\infty_R$, with $u$ invertible and $h \in H^\infty_R$. But the pair $(z, g^2)$ is not reducible in $H^\infty_R$, because for every $k \in H^\infty_R$ the function $z + k(1 - z^2)^2$ takes the value $-1$ at $-1$ and $1$ at $1$, so that it admits a zero on $]-1, 1[^\infty$. ☐

Examples 5.1 and 5.2 show that for fixed $\delta$, the constants $C(\delta, \varepsilon)$ in Wick’s theorem may tend to infinity as $\varepsilon \to 0$. This is the case, e.g., if $\varepsilon := \varepsilon_n = \frac{1}{2} \min_{[-1, 1]} B_n$, where $B_n$ is the finite Blaschke product in Example 5.2. (Indeed, fix $\varepsilon' > 0$ and choose $r$ so close to 1 that $g^2(r) < \varepsilon'$. Since $B_n(r)$ converges to $g^2(r)$, we see that $\varepsilon_n \leq B_n(r) < \varepsilon'$ for $n$ sufficiently large. Hence, $\varepsilon_n \to 0$ as $n \to \infty$.)

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