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REDUCIBILITY OF FUNCTION PAIRS IN $H^\infty_{\mathbb{R}}$

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Abstract. A short proof of a result by Brett Wick on the reducibility of function pairs in $H^\infty_{\mathbb{R}}$ is presented, and some unusual properties of the solutions to the associated Bézout equations are unveiled.

§1. The problem

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and let H^∞ be the (complex) algebra of all bounded holomorphic functions on $\mathbb{D}$. Moreover, let $H^\infty_{\mathbb{R}}$ be the (real) algebra of all bounded holomorphic functions in $\mathbb{D}$ that satisfy the symmetry condition $f(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{D}$. It is well known that $H^\infty_{\mathbb{R}}$ is the set of functions in H^∞ with real Fourier coefficients, and that $H^\infty_{\mathbb{R}}$ coincides with the set of all functions in H^∞ that are real-valued on $]-1,1[$. This algebra gained a lot of interest the last years in view of its appearance in control theory (see [13, 14, 9]). It is a direct consequence of the corona theorem for H^∞ that if (f,g) is a function pair in $H^\infty_{\mathbb{R}}$ with $\delta := \inf_{z\in\mathbb{D}} |f(z)|+|g(z)| > 0$, then there exist $p, q \in H^\infty_{\mathbb{R}}$ such that $pf+qg = 1$ (see [14]). On the other hand, in contrast to the H^∞-setting (see [12]), it is not always possible to choose the first factor p to be invertible in $H^\infty_{\mathbb{R}}$. In fact, as was already mentioned in [13], if $f(z) = z$ and $g(z) = 1-z^2$, then each function of the form $F := z + h(1-z^2)$ with $h \in H^\infty_{\mathbb{R}}$ must have a zero within $]-1,1[$, because F is real-valued on $]-1,1[$, $F(-1) < 0$ and $F(1) > 0$.

It is therefore a natural problem to characterize the pairs (f,g) in $H^\infty_{\mathbb{R}}$ that are reducible. Recall that a pair (f,g) in a commutative, unital Banach algebra A is said to be *invertible* if $af + bg = 1$ for some $a, b \in A$ and that (f,g) is *reducible* if there exists $h \in A$ such that $f + hg$ is invertible in A. In other words, (f,g) is reducible if there exists a solution (u,h) of the Bézout equation $uf + hg = 1$ such that the first factor u is invertible.

Motivated by his earlier result [13] for the algebra $A_{\mathbb{R}}(\mathbb{D})$ of functions in $H^\infty_{\mathbb{R}}$ that admit continuous extensions to the Euclidean closure of $\mathbb{D}$, in a series of papers [14, 15] Brett Wick developed the following theorem.

Theorem W (see [15]). *Let $(f,g) \in (H^\infty_{\mathbb{R}})^2$ be a function pair satisfying $\|f\|_\infty \le 1$, $\|g\|_\infty \le 1$ and $|f|+|g| \ge \delta > 0$. Suppose that there exists $\varepsilon > 0$ such that f has constant sign on the set $\{x \in\,]-1,1[: |g(x)| < \varepsilon\}$. Then there exists a constant $C = C(\delta,\varepsilon)$ depending only on δ and ε such that the Bézout equation*

$$uf + hg = 1$$

admits a solution $(u,h) \in (H^\infty_{\mathbb{R}})^2$ with u invertible and

$$\Delta(u, u^{-1}, h) := \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta,\varepsilon).$$

Note that the hypothesis on the sign is satisfied automatically whenever $|g|$ is bounded away from zero on $]-1,1[$ (in that case for ε we can take any number strictly smaller than

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$\inf_{x\in]-1,1[}|g(x)|)$. Also, if (f,g) is reducible in $H^\infty_{\mathbb{R}}$, then the sign condition is fulfilled necessarily (see below).

The intention of our paper is two-fold. First, in §2 we give a new short proof of the reducibility part of this theorem. Our proof will be based on the maximal ideal space theory for H^∞ and on Treil's theorem [12] on the reducibility of pairs in H^∞. The original proof by Wick involved the $\bar\partial$-calculus and depended on the methods developed by Treil to solve Bézout equations.

Then, in §3, we present a new proof of Theorem **W** with norm estimates that depends only on the corresponding result by Treil [12], but is otherwise quite elementary.

In §4, we provide some counterexamples to the original statement of Wick's result [14]. First, we give examples of invertible pairs (f,g) that show that, in general, the weaker hypothesis of f having constant sign on the set of real zeros of g is not sufficient in order to get a solution of the Bézout equation

$$uf+hg=1,\quad u \text{ invertible in } H^\infty_{\mathbb{R}}.$$

We mention that in [9, Proposition 2.8] it was shown that if g is a Blaschke product having only real and simple zeros, then the hypothesis of f being positive on the set of real zeros of g is indeed sufficient to guarantee the reducibility of the pair (whenever (f,g) is invertible).

Then we show that there exist reducible pairs (f_n,g_n) of functions in the unit ball of $H^\infty_{\mathbb{R}}$ with $|f_n|+|g_n|\geq\delta>0$ such that, quite surprisingly,

$$\sup_n \Delta(u_n,u_n^{-1},h_n)=\infty$$

for any solutions (u_n,h_n) of the equation $uf_n+hg_n=1$ with u invertible in $H^\infty_{\mathbb{R}}$.

This shows that in Wick's Theorem **W** the parameter ε plays an essential role. To the best of our knowledge, this is the first occurrence of a corona-type theorem for a uniform algebra where the uniform boundedness of solutions fails.

§2. A maximal ideal space view of Wick's result

Let $M(H^\infty)$ be the spectrum (or maximal ideal) space of H^∞; that is, $M(H^\infty)$ is the space of all (nonzero) multiplicative linear functionals on H^∞ endowed with the trace of the weak-$*$-topology $\sigma((H^\infty)^*,H^\infty)$, also called the Gelfand topology. Identifying each point z_0 in $\mathbb{D}$ with the evaluation functional $\Phi_{z_0}:f\mapsto f(z_0)$, we can identify $\mathbb{D}$ with a subset of $M(H^\infty)$. By Carleson's famous corona theorem, $\mathbb{D}$ is dense in $M(H^\infty)$. The Gelfand transform $\widehat{f}:M(H^\infty)\to\mathbb{C}$ of an element $f\in H^\infty$ is given by $\widehat{f}(m)=m(f)$. It can be viewed as a unique continuous extension of f from $\mathbb{D}$ to $M(H^\infty)$. As usual, we shall identify $\widehat{f}$ with f.

A corona pair in $A=H^\infty$ (or $A=H^\infty_{\mathbb{R}}$) is a pair (f,g) of functions in A such that $\inf_{z\in\mathbb{D}}|f(z)|+|g(z)|\geq\delta>0$. The zero set of $f\in H^\infty$ is given by $Z(f)=\{m\in M(H^\infty):f(m)=0\}$. Moreover, let $Z_{\mathbb{D}}(f)=\{z\in\mathbb{D}:f(z)=0\}$. It is well known that if b is an interpolating Blaschke product, then $Z(b)$ is the closure of $Z_{\mathbb{D}}(b)$ (see [3, p. 379]); in general, for arbitrary Blaschke products B, the set $\overline{Z_{\mathbb{D}}(B)}$ is much smaller than $Z(B)$ (see [4] for a study of the inner functions Θ for which $Z(\Theta)=\overline{Z_{\mathbb{D}}(\Theta)}$).

In the sequel, we may identify $M(H^\infty)$ with $M(H^\infty_{\mathbb{R}})$ (because H^∞ is the complexification of $H^\infty_{\mathbb{R}}$; see [6, pp. 37–38]), and we look upon the functions in $H^\infty_{\mathbb{R}}$ as being defined on $M(H^\infty)$.

A prominent role in our approach to Wick's result will be played by the closure E of the interval $]-1,1[$ in $M(H^\infty)$. If $g\in H^\infty_{\mathbb{R}}$, then the set of *extended real zeros of* g is the set $Z(g)\cap E$.

The following topological observation will give a relationship between the hypotheses in Theorem **W** and our approach (see [9, p. 178]).

Observation 2.1. Let $(f,g) \in H^\infty_{\mathbb{R}}$ be a corona pair. The following assertions are equivalent:

(1) there exists $\varepsilon > 0$ such that f has constant sign on the set

$$\{x \in]-1,1[: |g(x)| < \varepsilon\};$$

(2) f has constant sign on the set of extended real zeros of g.

Proof. Suppose that $|f| + |g| \geq \delta > 0$.

(1) $\Longrightarrow$ (2). We may assume that $\varepsilon < \delta/2$. Then $|f(z)| > \delta/2$ whenever $|g(z)| < \varepsilon$. Say that f is positive at each $x \in]-1,1[$ with $|g(x)| < \varepsilon$. Let $m_1, m_2 \in Z(g) \cap E$. Choose a net z_α in $]-1,1[$ that converges to m_1. Hence, $g(z_\alpha) \to 0$. In particular, there is α_0 such that $|g(z_\alpha)| < \varepsilon$ for all $\alpha > \alpha_0$, whence $f(z_\alpha) > \delta/2$. Therefore, $f(m_1) \geq \delta/2 > 0$. By the same argument, $f(m_2) \geq \delta/2 > 0$. Thus, f has constant sign on the set of extended real zeros of g.

(2) $\Longrightarrow$ (1). Since $|f| + |g| \geq \delta > 0$, by (2) we may assume that $f > \delta/2$ on $Z(g) \cap E$. Now,

$$Z(g) \cap E = \bigcap_{n=1}^{\infty} \{m \in E : |g(m)| \leq 1/n\}.$$

The finite intersection property for compact spaces (here E) implies that there is n_0 such that

$$\{m \in E : |g(m)| < 1/n_0\} \subseteq \{m \in E : f(m) > \delta/2\}.$$

Thus, $f(x) > 0$ at each point $x \in]-1,1[$ with $|g(x)| \leq 1/n_0$. □

The following lemma is due to Corach and Suárez [1, 2].

Lemma 2.2 (see [1], p. 636] and [2], p. 608])**.** *Let A be a real or complex commutative unital Banach algebra with unit element denoted by* 1.

a) *Suppose that $(f,g) \in A^2$ is an invertible pair. If (f_n) converges to f in the norm and if the pairs (f_n, g) are reducible, then (f,g) is also reducible.*

b) *For $g \in A$, the set*

$$R(g) = \{f \in A : (f,g) \quad \textit{is reducible}\}$$

is open-closed inside

$$I(g) = \{f \in A : (f,g) \quad \textit{is invertible}\}.$$

c) *If $\phi : [0,1] \to I(g)$ is a continuous curve and $(\phi(0), g)$ is reducible, then $(\phi(1), g)$ is reducible.*

Note that part a) follows from the identity

$$f_n + u_n g = f(xf_n + yg) + \big(y(f_n - f) + u_n\big)g,$$

where $f_n + u_n g \in A^{-1}$ and $xf + yg = 1$ (see [10] or [8]).

One of our tools will be the following result from [7]. Recall that a compact set $C \subseteq M(A)$ is said to be *A-convex* if C coincides with its A-convex hull

$$\check{C} = \{m \in M(A) : \big|\widehat{f}(m)\big| \leq \max_C \big|\widehat{f}\big| \quad \text{for all} \quad f \in A\}.$$

Here we represent the short proof for the reader's convenience. In the sequel, let f^* be the function defined by $f^*(z) = \overline{f(\bar{z})}$.

Theorem 2.3 (see [7])**.** *The set E is H^∞ and $H^\infty_{\mathbb{R}}$-convex.*

Proof. Let $m \in M(H^\infty) \setminus E$. Since H^∞ is separating (see [11, p. 242]), there exists $f \in H^\infty$ such that $f(m) = 0$ and $f \neq 0$ on E. We may assume that $\|f\|_\infty \leq 1$. Let $g = ff^*$. Then $g \in H^\infty_{\mathbb{R}}$ and g is real-valued on E. By the definition of g, actually we have $1 \geq g \geq 0$ on E. Since $f \neq 0$ on E and $f^* = \bar{f}$ on E, we see that $\sigma := \min_E g > 0$. Now, let $h = 1 - g$. Then $h \in H^\infty_{\mathbb{R}} \subseteq H^\infty$ and $h(m) = 1$, whence

$$\max_E |h| = \max_E h \leq 1 - \sigma < |h(m)|.$$

Thus, m does not belong to the $H^\infty_{\mathbb{R}}$-convex closure of E. Therefore, E is $H^\infty_{\mathbb{R}}$-convex as well as H^∞-convex. □

Now we are ready to give the announced short proof of the reducibility part of Wick's result. It is essentially based on Treil's result that in H^∞ every corona pair is reducible [12].

Theorem 2.4 (Treil's theorem)**.** *Let $(f, g) \in (H^\infty)^2$ be a corona pair satisfying*

$$\|f\|_\infty \leq 1, \quad \|g\|_\infty \leq 1, \quad \textit{and} \quad |f| + |g| \geq \delta > 0.$$

Then there exist $h \in H^\infty$ and a constant $\kappa = \kappa(\delta)$ depending only on δ such that $v := f + hg$ is invertible in H^∞ and

$$\|h\|_\infty + \|v^{-1}\|_\infty \leq \kappa(\delta).$$

Theorem 2.5 (see [14, 15])**.** *Let (f, g) be a corona pair in $H^\infty_{\mathbb{R}}$. Then (f, g) is reducible if and only if f has constant sign on the set of extended real zeros of g.*

Proof. First, we prove the "only if" part. Let $u = f + hg$ be invertible in $H^\infty_{\mathbb{R}}$. Here $h \in H^\infty_{\mathbb{R}}$. Suppose that there exist $m_1, m_2 \in Z(g) \cap E$ such that $f(m_1) < 0$ and $f(m_2) > 0$. In particular $u(m_1) < 0$ and $u(m_2) > 0$. Since E is a connected set, the continuity of the (real-valued) function u on E implies that u has a zero on E. Hence, u is no longer invertible. (A finer analysis actually would yield a zero of u in $]-1, 1[$.)

To prove the sufficiency of the sign condition, consider the (real) trace-algebra $A = \overline{H^\infty_{\mathbb{R}}|_E}$, where the closure is taken with respect to the supremum norm in the space $C(E, \mathbb{R})$ of all real-valued continuous functions on E. Since E is $H^\infty_{\mathbb{R}}$-convex by Theorem 2.3, the spectrum $M(A)$ of A can be identified with E (see [6, p. 125]). By our hypothesis, we may assume that $f > 0$ on $Z(g) \cap E$. This implies that for every $t \geq 0$ the pairs $(f|_E + t, g|_E)$ are invertible in A. Since $f|_E + N$ is invertible in A for large $N \in \mathbb{N}$, the pair $(f|_E + N, g|_E)$ is reducible in A. By Lemma 2.2, this implies that $(f|_E, g|_E)$ is also reducible in A. This means that there exists $q \in A$ such that $f + qg \neq 0$ on E. Approximating q uniformly on E by a function $h \in H^\infty_{\mathbb{R}}$, we see that $v := f + hg \in H^\infty_{\mathbb{R}}$ and that v does not vanish on E. In particular, v has no real zeros. Now, (v, g) is also a corona pair. Therefore, by [9, Proposition 2.6][1] (or Proposition 3.1 below), (v, g) is reducible in $H^\infty_{\mathbb{R}}$. Thus, there exists $H \in H^\infty_{\mathbb{R}}$ so that

$$(f + hg) + Hg = f + (h + H)g$$

is invertible in $H^\infty_{\mathbb{R}}$. In other words, (f, g) is reducible in $H^\infty_{\mathbb{R}}$. □

§3. Wick's Theorem with norm estimates

In this section we present a new proof of Theorem **W** with norm estimate. Our proof depends only on Treil's result, i.e., Theorem 2.4, but does not need the quite delicate construction from [12].

For this, we shall need the following extensions of [9, Proposition 2.2] and [9, Proposition 2.6]. The proofs work exactly in the same way as those in [9]: at each occurrence

[1] Here Treil's result is used.

of Treil's theorem, we additionally must take into account his norm estimates. Upon the referee's request, we present here the details.

Proposition 3.1. *Let (f,g) be a corona pair in $H^\infty_{\mathbb{R}}$ with $\|f\|_\infty \le 1$, $\|g\|_\infty \le 1$, and $|f|+|g| \ge \delta > 0$. Then the following assertions hold.*

(1) *There exist a constant $C(\delta)$ (depending only on δ) and functions $u,h \in H^\infty_{\mathbb{R}}$ with u invertible such that $1 = uf^2 + hg$ and*

$$\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta).$$

(2) *Suppose that f has no real zeros. Then, again, there exist a constant $C(\delta)$ (depending only on δ) and functions $u,h \in H^\infty_{\mathbb{R}}$ with u invertible such that $1 = uf + hg$ and*

$$\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta).$$

Proof. (1) By Theorem 2.4, there exist a constant κ depending only on δ, and $k \in H^\infty$ such that $v := f + kg$ is invertible in H^∞ and $\|k\|_\infty + \|v^{-1}\|_\infty \le \kappa$. In particular, $v^* = f + k^*g$ is invertible in H^∞. Multiplying the two equations, we see that

$$f^2 + g\big(f(k+k^*) + kk^*g\big) = vv^*$$

is invertible in H^∞. But the factors $k+k^*$ and kk^* belong to $H^\infty_{\mathbb{R}}$. Thus, with $u = (vv^*)^{-1}$ and $h = \big(f(k+k^*) + kk^*g\big)(vv^*)^{-1}$ we get $1 = uf^2 + hg$ and

$$\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta).$$

(2) Write $f = BF$ where F is zero-free and B is the Blaschke product associated with the zeros of f. Since the zeros of B are symmetric with respect to the real axis, B and F are in $H^\infty_{\mathbb{R}}$. Without loss of generality, we may assume that $F > 0$ on $]-1,1[$. Moreover, $F = G^2$ for some $G \in H^\infty_{\mathbb{R}}$ (see [9, p. 180]). Thus, (B,g) and (G,g) are invertible pairs in $(H^\infty_{\mathbb{R}})^2$ satisfying $|B|+|g| \ge \delta$ and $|G|+|g| \ge \delta$. By (1), there exists a constant $\kappa_1 > 0$ independent of δ such that

$$w := G^2 + k_1 g = F + k_1 g$$

is invertible in $H^\infty_{\mathbb{R}}$ for some $k_1 \in H^\infty_{\mathbb{R}}$ with $\|k_1\|_\infty + \|w^{-1}\|_\infty \le \kappa_1$.

It remains to prove the same property for the pair (B,g). Let C be the Blaschke product formed with the zeros of B on the upper-half plane, and let C^* be defined by $C^*(z) := \overline{C(\overline{z})}$. Since f (and with it, B) has no real zeros, we see that $B = CC^*$. Since (C,g) is an invertible pair in $(H^\infty)^2$ with $|C|+|g| \ge \delta$, Treil's Theorem 2.4 yields the existence of $k_2 \in H^\infty$ and $\kappa_2 > 0$ depending only on δ such that $v := C + k_2 g$ is invertible in H^∞ and $\|k_2\|_\infty + \|v^{-1}\| \le \kappa_2$. But $v^* = C^* + k_2^* g$. Multiplying the two equations, we get

$$CC^* + \big((Ck_2^* + C^*k_2) + k_2k_2^*g\big)g = vv^*.$$

Thus, with

$$K = (Ck_2^* + C^*k_2) + k_2k_2^*g \in H^\infty_{\mathbb{R}},$$

the function $\widetilde{w} := vv^* = B + Kg$ is invertible in $H^\infty_{\mathbb{R}}$ and

$$\|K\|_\infty + \|\widetilde{w}^{-1}\|_\infty \le \kappa(\delta)$$

for some constant $\kappa(\delta)$.

To summarize, we have shown that

$$R := w\widetilde{w} = (F + k_1 g)(B + Kg) = BF + g(KF + k_1B + k_1Kg)$$

is invertible in $H^\infty_{\mathbb{R}}$ and that $R = f + hg$, where $h := KF + k_1B + k_1Kg$ and $\|h\|_\infty + \|R^{-1}\|_\infty \le c(\delta)$.

It is now easy to deduce the claim. $\square$

Now we are ready to prove Theorem **W**, which we restate here, for the sake of a better exposition.

Theorem 3.2. *Let $(f,g) \in (H^\infty_{\mathbb{R}})^2$ be a corona pair satisfying $\|f\|_\infty \le 1$, $\|g\|_\infty \le 1$, and $|f|+|g| \ge \delta > 0$. Suppose that there exists $\varepsilon > 0$ such that f has constant sign on the set $\{x \in]-1,1[: |g(x)| < \varepsilon\}$. Then there exists a constant $C = C(\delta,\varepsilon)$ depending only on δ and ε such that the Bézout equation*

$$uf + hg = 1$$

admits a solution $(u,h) \in (H^\infty_{\mathbb{R}})^2$ with u invertible and

$$\Delta(u,u^{-1},h) := \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta,\varepsilon).$$

Proof. The idea is to replace the function g by g^2, so that for large N, the function $f + Ng^2$ is positive on the set of all $x \in]-1,1[$ where $|g(x)|$ is small.

So, suppose that $f(x) > 0$ for all $x \in]-1,1[$ with $|g(x)| < \varepsilon$. We may assume that $\varepsilon < \delta/2$. Then $|f|+|g| \ge \delta > 0$ implies that $f(x) > \delta/2$ for those x. Let

$$N = \max\Big\{\frac{4}{\delta}, \frac{1}{\varepsilon^2}\Big\}.$$

Then $N = \frac{1}{\varepsilon^2}$. We claim that $f + Ng^2$ has no zeros on $]-1,1[$. In fact, if $|g(x)| \le \varepsilon$, then

$$|f(x) + Ng^2(x)| = f(x) + Ng^2(x) > \delta/2 > 0.$$

If $|g(x)| > \varepsilon$, then $g^2(x) > \varepsilon^2$, whence

$$|f(x) + Ng^2(x)| \ge Ng^2(x) - |f(x)| > N\varepsilon^2 - 1 \ge 0.$$

Let $F := \frac{f+Ng^2}{1+N}$. Then (F,g) is a corona pair. Next we show that

$$|F| + |g| \ge \frac{\delta}{4N}. \tag{3.1}$$

Indeed, if $|g| \ge \sqrt{\frac{\delta}{4N}}$, then we are done. Now, let $|g| \le \sqrt{\frac{\delta}{4N}}$. By the choice of N, we have

$$\sqrt{\frac{\delta}{4N}} \le \frac{\delta}{4}.$$

Hence, since $|f|+|g| \ge \delta$, we see that $|f| \ge \frac{3}{4}\delta$. Accordingly,

$$|F| \ge \frac{|f| - N|g|^2}{N+1} \ge \frac{\frac{3}{4}\delta - N\frac{\delta}{4N}}{1+N} = \frac{\frac{\delta}{2}}{1+N} \ge \frac{\delta}{4N}.$$

This establishes (3.1).

Since F has no real zeros, we may apply Proposition 3.1 to obtain functions u and h in $H^\infty_{\mathbb{R}}$ such that u is invertible,

$$1 = uF + hg,$$

and

$$\|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty \le C(\delta/4N) = C((\varepsilon^2\delta)/4) =: C.$$

Thus,

$$1 = \frac{u}{1+N} f + \Big(\frac{Nug}{1+N} + h\Big) g$$

is the solution we were looking for; simply put

$$C(\delta,\varepsilon) = 2C + \Big(1 + \frac{1}{\varepsilon^2}\Big) C. \qquad \square$$

A natural question is whether it is possible to get bounds on the solution triple (u, u^{-1}, h) of the Bézout equation $uf + hg = 1$, u invertible in $H^\infty_{\mathbb{R}}$, that depend only on

$$\delta = \inf_z (|f(z)| + |g(z)|).$$

For example, this is the case under the hypotheses of Proposition 3.1. The answer is provided in §5.

§4. The sign condition on $]-1,1[$ is not sufficient

In [9, Proposition 2.8] it was shown that the pair (f, B) is reducible in $H^\infty_{\mathbb{R}}$ whenever B is a Blaschke product having only real zeros (of multiplicity one) and $f \in H^\infty_{\mathbb{R}}$ is such that f is positive on the real zeros of B and $|f| + |B| \geq \delta > 0$.

If B is replaced by an outer function g or a singular inner function, then g has no zeros at all in $\mathbb{D}$ and the hypothesis of f being positive at the real zeros of g is true by convention. We show that in that case reducibility is not necessarily forced by this vacuous condition.

Example 4.1. Let $f(z) = z$ and $g(z) = \exp(-\frac{1+z^2}{1-z^2})$. Then g has no zeros in $]-1,1[$, but nevertheless (f, g) is not reducible in $H^\infty_{\mathbb{R}}$.

Proof. Let $h \in H^\infty_{\mathbb{R}}$ be given, and set $L(z) = z + h(z)g(z)$. Since $\lim_{r\to 1} g(r) = 0$ and $\lim_{x\to -1} g(x) = 0$, we see that $\lim_{r\to 1} L(r) = 1$ and $\lim_{x\to -1} L(x) = -1$. Thus, being real valued on $]-1,1[$, L has a zero within $]-1,1[$, so that L is not invertible. □

We could have worked also with the pair $(z, 1 - z^2)$. But the example above has the additional property that g has no zeros in $\mathbb{C}$ at all (apart from the essential singularities at -1 and 1).

Here is yet another example; this time we give an example such that the second member g does have real zeros (note that conditions envolving empty sets are sometimes arguable). It is based on the fact that f is positive on the real zeros of g if and only if between two real zeros of g there is an even number of zeros of f (or none). In the following, let $\mathbb{N} = \{0, 1, 2, \dots\}$.

Example 4.2. Let B be the interpolating Blaschke product with the zeros $z_n = 1 - 2^{-n}$ $(n = 1, 2, \dots)$, let B_1 be the subproduct of B with the zero set

$$\{z_n \,:\, n = 3j + 2 \ \text{ or } \ n = 3(j + 1),\ j \in \mathbb{N}\},$$

that is, $Z_{\mathbb{D}}(B_1) = \{z_2, z_3, z_5, z_6, \dots\}$, and let B_2 be the subproduct of B with the zero set

$$\{z_n \,:\, n = 3j + 1,\ j \in \mathbb{N}\}.$$

We introduce two functions f and g by

$$f(z) = zB_1(z), \quad g(z) = B_2(z) \exp\Big(-\frac{1-z}{1+z}\Big).$$

We are going to show that (f, g) is a corona pair, f has constant sign on the real zeros of g, but (f, g) is not reducible in $H^\infty_{\mathbb{R}}$.

Proof. Since $B = B_1B_2$ is an interpolating Blaschke product, from [5, p. 208] we see that the ideal generated by B_1 and B_2 in H^∞ equals H^∞. Hence B_1 and B_2 have no common zeros on the spectrum of H^∞. Moreover, since the zeros of $\exp\big(-\frac{1-z}{1+z}\big)$ in $M(H^\infty)$ are in the fiber over the point $z = -1$, we conclude that f and g have no zeros in common on $M(H^\infty)$. Thus (f, g) is a corona pair in H^∞, and consequently also in $H^\infty_{\mathbb{R}}$.

By construction, between two distinct real zeros of B_2 there are exactly two zeros of f. Thus f has constant sign on the real zeros of g. Now all the zeros of B_1 are larger than the first zero z_1 of g. Recall that B_1 has the form

$$B_1(x) = \prod_j \frac{a_j - x}{1 - a_j x}. \tag{4.1}$$

Hence, $f(z_1) > 0$, and so this constant sign is the positive sign. Moreover, B_1 is bounded away from zero on the zeros of B_2 (because (B_1, B_2) is a corona pair). So, we can conclude that for given $h \in H^\infty_{\mathbb{R}}$, the function $W(z) = zB_1(z) + h(z)g(z)$ satisfies

$$\liminf_n W(z_{3n+1}) = \liminf_n z_{3n+1} B_1(z_{3n+1}) > 0.$$

On the other hand, B_1 is unimodular and continuous at -1 and so, by (4.1), $B_1(-1) = 1 > 0$. Thus,

$$\lim_{x \to -1} W(x) = -1.$$

Again we see that W has a zero in $]-1, 1[$. □

§5. Non-uniformly bounded solutions

In contrast to the situation in H^∞ (see [12]), we will show here that the reducibility of sequences of corona pairs (f_n, g_n) in the unit ball of $H^\infty_{\mathbb{R}}$ with $|f_n| + |g_n| \geq \delta > 0$ does not imply the existence of uniformly bounded solutions of the Bézout equation $u_n f_n + h_n g_n = 1$ with u_n invertible, where the bounds of the norms of u_n, u_n^{-1} and h_n depend only on δ. Recall that

$$\Delta(u, u^{-1}, h) = \|u\|_\infty + \|u^{-1}\|_\infty + \|h\|_\infty.$$

Example 5.1. Put $f(z) = z$ and $g(z) = \exp\left(-\frac{1+z^2}{1-z^2}\right)$. We choose a sequence (r_n) of positive numbers converging to 1 ($0 < r_n < 1$); let $f_n(z) = f(r_n z)$ and $g_n(z) = g(r_n z)$. Then the pairs (f_n, g_n) are reducible in $H^\infty_{\mathbb{R}}$ and satisfy $|f_n| + |g_n| \geq \delta > 0$, but

$$\sup_n \Delta(u_n, u_n^{-1}, h_n) = \infty$$

for any solutions $(u_n, h_n) \in (H^\infty_{\mathbb{R}})^2$ of the equation $u_n f_n + h_n g_n = 1$ with u_n invertible in $H^\infty_{\mathbb{R}}$.

Proof. Suppose to the contrary that there exist solutions (u_n, h_n) satisfying

$$\sup_n \Delta(u_n, u_n^{-1}, h_n) < \infty.$$

Using a normal families argument, we can find subsequences of u_n and h_n converging locally uniformly to some functions u and h in $H^\infty_{\mathbb{R}}$. Since $|u_n|$ is bounded away from zero by a constant independent of n, the limit function u is also invertible. Thus,

$$uf + hg = 1,$$

but this contradicts Example 4.1. □

We can also create counterexamples to this norm-estimate problem by working with finite Blaschke products. Note that this is the class of functions Brett Wick worked with in [14].

Example 5.2. There exists a sequence (B_n) of finite Blaschke products in $H^\infty_{\mathbb{R}}$ such that the pairs (z, B_n) are reducible in $H^\infty_{\mathbb{R}}$ and satisfy $|z| + |B_n| \geq \eta > 0$, but

$$\sup_n \Delta(u_n, u_n^{-1}, h_n) = \infty$$

for any solutions $(u_n, h_n) \in (H^\infty_{\mathbb{R}})^2$ of $u_n z + h_n B_n = 1$ with u_n invertible in $H^\infty_{\mathbb{R}}$.

Proof. Let $g^*(z) = \overline{g(\bar{z})}$. Note that a function $g \in H^\infty$ belongs to $H^\infty_{\mathbb{R}}$ if and only if $g = g^*$. Let $f(z) = z, g(z) = (1 - z^2)/2$. By the Carathéodory theorem (see [3, p. 6]), we can choose a sequence of finite Blaschke products b_n converging locally uniformly to g in $\mathbb{D}$. If b_n has a real zero, we move it vertically a little to the upper half disk. Since $g = g^*$, $B_n := b_n b_n^*$ tends locally uniformly to g^2. Moreover, $B_n \in A(\mathbb{D})_{\mathbb{R}}$ and has no real zeros. Obviously, (z, g^2) is a corona pair, say $|z| + |g^2| \geq \delta > 0$, where we may choose $\delta < 1$. Since B_n converges to g^2, uniformly on $|z| \leq 1/2$, we see that $|z| + |B_n| \geq \delta/2$ on $|z| \leq 1/2$ whenever n is large. For $1/2 \leq |z| \leq 1$ we obtain $|z| + |B_n| \geq 1/2 \geq \delta/2$.

Thus, the (z, B_n) are corona pairs with $|z| + |B_n| \geq \delta/2$. Using Lemma 2.2, we show that these pairs are reducible in $A(\mathbb{D})_{\mathbb{R}}$. Indeed, since $B_n > 0$ on $[-1, 1]$, the pairs $(z + t, B_n)$ are invertible for any $t \geq 0$. For $t = 2$, e.g., $z + 2$ is an invertible function in $A(\mathbb{D})_{\mathbb{R}}$; hence, $(z + 2, B_n)$ is reducible. Thus, by Lemma 2.2, (z, B_n) is also reducible.

So, let

$$1 = u_n z + h_n B_n,$$

where u_n is invertible in $A_{\mathbb{R}}(\mathbb{D})$.

As above, no bound on the norms of u_n, u_n^{-1} and h_n is possible, because otherwise $1 = uz + hg^2$ for some $u \in H^\infty_{\mathbb{R}}$, with u invertible and $h \in H^\infty_{\mathbb{R}}$. But the pair (z, g^2) is not reducible in $H^\infty_{\mathbb{R}}$, because for every $k \in H^\infty_{\mathbb{R}}$ the function $z + k(1 - z^2)^2$ takes the value -1 at -1 and 1 at 1, so that it admits a zero on $]-1, 1[$. □

Examples 5.1 and 5.2 show that for fixed δ, the constants $C(\delta, \varepsilon)$ in Wick's theorem **W** may tend to infinity as $\varepsilon \to 0$. This is the case, e.g., if $\varepsilon := \varepsilon_n = \frac{1}{2} \min_{[-1,1]} B_n$, where B_n is the finite Blaschke product in Example 5.2. (Indeed, fix $\varepsilon' > 0$ and choose r so close to 1 that $g^2(r) < \varepsilon'$. Since $B_n(r)$ converges to $g^2(r)$, we see that $\varepsilon_n \leq B_n(r) < \varepsilon'$ for n sufficiently large. Hence, $\varepsilon_n \to 0$ as $n \to \infty$.)

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