Алгебра и анализ
Том 24 (2012), № 1

St. Petersburg Math. J.
Vol. 24 (2013), No. 1, Pages 39–69
S 1061-0022(2012)01231-8
Article electronically published on November 15, 2012

ON FULLY NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS WITH VMO COEFFICIENTS IN DOMAINS

HONGJIE DONG, N. V. KRYLOV, AND XU LI

Abstract. The solvability in the Sobolev spaces $W_p^{1,2}$, $p > d+1$, of the terminal-boundary value problem is proved for a class of fully nonlinear parabolic equations, including parabolic Bellman's equations, in bounded cylindrical domains, in the case of VMO "coefficients". The solvability in W_p^2, $p > d$, of the corresponding elliptic boundary-value problem is also obtained.

§1. Introduction and main results

In this paper, we consider parabolic equations

$$\partial_t u(t,x) + F(D^2u(t,x),t,x) + G(D^2u(t,x),Du(t,x),u(t,x),t,x) = 0 \tag{1.1}$$

in subdomains of $\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$, where

$$\mathbb{R}^d = \{x = (x^1,\dots,x^d) : x^1,\dots,x^d \in \mathbb{R} = (-\infty,\infty)\}.$$

Here

$$D^2u = (D_{ij}u), \quad Du = (D_iu), \quad D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_iD_j, \quad \partial_t = \frac{\partial}{\partial t}.$$

We introduce $\mathcal{S}$ as the set of symmetric $(d\times d)$-matrices, fix some constants $\delta \in (0,1)$ and $K \in \mathbb{R}_+ := (0,\infty)$, and assume throughout that the following three conditions (H_1)–(H_3) are satisfied.

(H_1) $F(u'',t,x)$ is convex and positive homogeneous of degree one with respect to $u'' \in \mathcal{S}$, and

$$\delta|\xi|^2 \le F(u''+\xi\xi^*,t,x) - F(u'',t,x) \le \delta^{-1}|\xi|^2$$

for all values of the arguments and $\xi \in \mathbb{R}^d$.

Remark 1.1. Observe that condition (H_1) implies that $F(u'',t,x)$ is Lipschitz continuous with respect to u'' with constant independent of t and x. As such, for each (t,x), it is differentiable a.e., and, since it is positive homogeneous of degree one in u'', at any point u'' where it is differentiable we have

$$F(u'',t,x) = u''_{ij}F_{u''_{ij}}(u'',t,x).$$

(H_2) $G(u'',u',u,t,x)$, $u'' \in \mathcal{S}, u' \in \mathbb{R}^d, u \in \mathbb{R}$, is monotone nonincreasing in u, and for all values of the arguments (notice u'' and not v'') we have

$$|G(u'',u',u,t,x) - G(u'',v',v,t,x)| \le K\big(|u'-v'| + |u-v|\big),$$
$$|G(u'',u',u,t,x)| \le \chi(|u''|)|u''| + K(|u'|+|u|) + \bar{G}(t,x),$$

2010 *Mathematics Subject Classification.* Primary 35K61, 35B65, 35R05.

Key words and phrases. Vanishing mean oscillation, fully nonlinear elliptic and parabolic equations, Bellman's equations.

The first author was partially supported by NSF grant DMS-0800129. The second author was partially supported by NSF grant DMS-0653121.

where $\bar{G}$ and χ are given functions such that $\chi : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ is bounded and monotone decreasing, and $\chi(r) \to 0$ as $r \to \infty$.

Remark 1.2. As one can see from our proofs, we do not need $\chi(\infty)$ be 0 but rather $\chi(\infty)$ be sufficiently small.

(H_3) (i) For all values of the arguments and $\xi \in \mathbb{R}^d$, we have

$$\delta|\xi|^2 \le F(u''+\xi\xi^*,t,x)+G(u''+\xi\xi^*,u',u,t,x)-F(u'',t,x)-G(u'',u',u,t,x) \le \delta^{-1}|\xi|^2;$$

(ii) $F(u'',t,x)+G(u'',u',u,t,x)$ is convex with respect to $u'' \in \mathcal{S}$ and, for any u',u,t,x, at each point u'' where $G(u'',u',u,t,x)$ is differentiable with respect to u'' we have

$$|G(u'',u',u,t,x) - u''_{ij}G_{u''_{ij}}(u'',u',u,t,x)| \le \widehat{G}(t,x)M(|u|)(1+|u'|), \tag{1.2}$$

where $M(s)$ is a continuous function on $\mathbb{R}$ and $\widehat{G}$ is a locally integrable function on $\mathbb{R}^{d+1}$.

Remark 1.3. As in Remark 1.1, the function $F+G$ is Lipschitz with respect to u'' and, since F is Lipschitz, G is also Lipschitz continuous with Lipschitz constant independent of other arguments, so that (1.2) makes sense. Sometimes it is helpful to have in mind that

$$G(u'',u',u,t,x) - u''_{ij}G_{u_{ij}}(u'',u',u,t,x)$$

is the value at $\mu = 1$ of the first derivative of $\mu G(\mu^{-1}u'',u',u,t,x)$ with respect to μ.

We shall derive *a priori* estimates only using conditions (H_1) and (H_2). It is in the proofs of solvability where condition (H_3) plays its role. However, we have the following

Conjecture. Assumption (H_3) (ii) can be dropped.

To state our main results, we introduce some notation. For $r > 0$, $x \in \mathbb{R}^d$, and $t \in \mathbb{R}$, we denote

$$B_r(x) = \{y \in \mathbb{R}^d : |x-y| < r\}, \quad Q_r(t,x) = (t, t+r^2) \times B_r(x).$$

If $\mathcal{D}$ is a domain in $\mathbb{R}^d$ and $-\infty \le S < T < \infty$, we denote the parabolic boundary of the cylinder $(S,T) \times \mathcal{D}$ by

$$\partial'((S,T)\times\mathcal{D}) = (\{T\}\times\mathcal{D}) \cup ((S,T]\times\partial\mathcal{D}).$$

Finally, for any $T > 0$, we define $\mathcal{D}_T = (0,T)\times\mathcal{D}$.

The following VMO (vanishing mean oscillation) assumption is imposed on the leading term in (1.1) with a constant $\theta \in (0,1]$ to be specified later.

Assumption 1.1. There exists $R_0 \in (0,1]$ such that for any $r \in (0,R_0]$, $\tau \in \mathbb{R}$, and $z \in \mathcal{D}$ there exists a function $\bar{F}(u'')$ (independent of (t,x)) satisfying condition (H_1) and such that for any $u'' \in \mathcal{S}$ with $|u''| = 1$ we have

$$\int_{Q_r(\tau,z)} |F(u'',t,x) - \bar{F}(u'')|\,dx\,dt \le \theta r^{d+2}. \tag{1.3}$$

The first main result of the paper is about the terminal-boundary value problem for fully nonlinear parabolic equations with "VMO coefficients" in bounded cylinders.

Theorem 1.1. *Let $p > d+1$ be a constant, let $T \in \mathbb{R}_+$, and let $\mathcal{D}$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Assume that $\bar{G} \in L_p(\mathcal{D}_T)$. Then there exists a constant $\theta \in (0,1]$ depending only on d, p, δ, and the $C^{1,1}$ norm of $\partial\mathcal{D}$ such that if Assumption 1.1 is satisfied with this θ, then the following assertions hold true. For any $g \in W^{1,2}_p(\mathcal{D}_T)$, there is a unique solution $u \in W^{1,2}_p(\mathcal{D}_T)$ of (1.1) such that $u - g \in \mathring{W}^{1,2}_p(\mathcal{D}_T)$. Moreover, we have*

$$\|u\|_{W^{1,2}_p(\mathcal{D}_T)} \le N\|\bar{G}\|_{L_p(\mathcal{D}_T)} + N\|g\|_{W^{1,2}_p(\mathcal{D}_T)} + N_0, \tag{1.4}$$

where N depends only on d, p, δ, K, R_0, the $C^{1,1}$ norm of $\partial\mathcal{D}$, and $\operatorname{diam}(\mathcal{D})$*, and N_0 depends only on the same objects, T, and χ. In particular, $N_0 = 0$ if $\chi \equiv 0$.*

Here $W^{1,2}_p(\mathcal{D}_T)$ denotes the set of functions v defined on $\mathcal{D}_T$ and such that v, Dv, D^2v, and $\partial_t v$ are in $L_p(\mathcal{D}_T)$, and $\mathring{W}^{1,2}_p(\mathcal{D}_T)$ is the set of all functions $v \in W^{1,2}_p(\mathcal{D}_T)$ such that v vanishes on $\partial'\mathcal{D}_T$.

If F and G are independent of t, we also consider elliptic equations

$$F(D^2u(x), x) + G(D^2u(x), Du(x), u(x), x) = 0 \tag{1.5}$$

with Dirichlet boundary condition in subdomains of $\mathbb{R}^d$. In that case, Assumption 1.1 becomes the following.

Assumption 1.2. There exists $R_0 \in (0, 1]$ such that, for any $r \in (0, R_0]$ and $z \in \mathcal{D}$, there exists a function $\bar{F}(u'')$ (independent of x) satisfying condition (H_1) and such that for any $u'' \in \mathcal{S}$ with $|u''| = 1$ we have

$$\int_{B_r(z)} |F(u'', x) - \bar{F}(u'')|\, dx \le \theta r^d. \tag{1.6}$$

Our next theorem is about the boundary-value problem for elliptic equations with VMO coefficients in bounded domains.

Theorem 1.2. *Let $p > d$ be a constant and $\mathcal{D}$ a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Assume that $\bar{G}$ is independent of t and $\bar{G} \in L_p(\mathcal{D})$. Then there exists a constant $\theta \in (0, 1]$, depending only on d, p, δ, and the $C^{1,1}$ norm of $\partial\mathcal{D}$, such that if Assumption 1.2 is satisfied with this θ, then the following assertions hold true. For any $g \in W^2_p(\mathcal{D})$, there is a unique solution $u \in W^2_p(\mathcal{D})$ of (1.5) such that $u - g \in \mathring{W}^2_p(\mathcal{D})$. Moreover, we have*

$$\|u\|_{W^2_p(\mathcal{D})} \le N\|\bar{G}\|_{L_p(\mathcal{D})} + N\|g\|_{W^2_p(\mathcal{D})} + N_0, \tag{1.7}$$

where N depends only on d, p, δ, R_0, K, the $C^{1,1}$ norm of $\partial\mathcal{D}$, and $\operatorname{diam}(\mathcal{D})$*, and N_0 depends only on the same objects and χ. In particular, $N_0 = 0$ if $\chi \equiv 0$.*

Here $W^2_p(\mathcal{D})$ denotes the set of all functions v defined in $\mathcal{D}$ and such that v, Dv, and D^2v are in $L_p(\mathcal{D})$, and $\mathring{W}^2_p(\mathcal{D})$ is the set of all functions $v \in W^2_p(\mathcal{D})$ such that v vanishes on $\partial\mathcal{D}$.

In the literature, the interior $W^2_p, p > d$, estimates for a class of fully nonlinear uniformly elliptic equations of the form

$$F(D^2u, x) = f(x)$$

were obtained for the first time by Caffarelli in [2] (see also [3]). His proof is geometric and is based on the Aleksandrov–Bakel′man–Pucci *a priori* estimate, the Krylov–Safonov Harnack inequality and a covering argument that can be found in [16] and [23]. Adapting this technique, similar interior estimates were proved by Wang [28] for parabolic equations. In the same paper, a boundary estimate is stated but without a proof; see Theorem 5.8 there. By exploiting a weak reverse Hölder's inequality, the result of [2] was sharpened by Escauriaza in [8], who obtained the interior W^2_p-estimate for the same equations allowing $p > d - \varepsilon$, with a small constant ε depending only on the ellipticity constant and d. Very recently, Winter [29] further extended this technique to establish the corresponding boundary estimate as well as the W^2_p*-solvability* of the associated boundary-value problem. It is also worth noting that a solvability theorem in the space $W^{1,2}_{p,\mathrm{loc}}(Q) \cap C(\bar{Q})$ can be found in [6] for the boundary-value problem for fully nonlinear parabolic equations. In these papers, a small oscillation assumption in the integral sense is imposed on the operators; see, for instance, [2, Theorem 1]. However, as was pointed out in [29, Remark 2.3] and in [15] (see also [6, Example 8.3] for a relevant discussion),

this assumption turns out to be equivalent to a small oscillation condition in the L_∞ sense, which, particularly in the *linear* case, is the same as what is required in the classical L_p theory based on the Calderón–Zygmund estimates. Thus, it seems to us that the results in [2, 28, 8, 6, 29] mentioned above are in general not formally applicable to the operators under Assumption 1.1 or 1.2, in which local oscillations are measured in the average sense, so that huge jumps in the L_∞ norm are allowed. It is still possible that the *methods* developed in the above cited articles can be used to obtain our results. In our opinion, our method is somewhat simpler and leads to the results faster.

The results obtained in this paper contain and generalize the Sobolev space theory of linear equations with VMO coefficients, which was developed about twenty years ago by Chiarenza, Frasca, and Longo in [4, 5] for non-divergence form elliptic equations, and later in [1] by Bramanti and Cerutti for parabolic equations. The proofs in these references are based on the Calderón–Zygmund theorem and the Coifman–Rochberg–Weiss commutator theorem. For further related results, we refer the reader to the book [22] and the references therein.

However, remarkably, not all known results related to VMO coefficients and second-order elliptic and parabolic *linear* equations can be obtained from the results of the present paper.

In [12, 13] the reader can find a unified approach to investigating the L_p (and $L_q - L_p$) solvability of both divergence and non-divergence form parabolic and elliptic equations with leading coefficients that are in VMO in the spatial variables and only *measurable* in the time variable in the parabolic case. In the nonlinear setting, it is an extremely challenging problem whether or not the F's that are only measurable in t can be treated. The proofs in [12, 13] rely mainly on pointwise estimates of sharp functions of spatial derivatives of solutions, so that the VMO coefficients are treated in a rather straightforward manner. This approach is rather flexible: it has been applied to both divergence and non-divergence form linear equations/systems whose coefficients are very irregular in some of the independent variables. For example, in [9, 10] Kim and Krylov established the solvability in Sobolev spaces of non-divergence elliptic and parabolic equations with leading coefficients measurable in a space variable and VMO in the other variables; in [7] Dong and Kim considered both divergence and non-divergence form higher-order elliptic and parabolic systems in the whole space, the half-space, and bounded domains, with coefficients in the same class as in [12, 13]; see also the references in [7] for other results in this line of research.

Here we follow the general scheme outlined in [12, 13] to study fully nonlinear elliptic and parabolic equations with VMO coefficients in bounded domains or cylinders. This paper is a continuation of [15], where interior estimates for elliptic Bellman's equations were obtained. The key ingredients in our proofs are the Evans–Krylov theorem applied to homogeneous equations with constant coefficients and a W^2_ε-estimate for elliptic equations with measurable coefficients, which was originally obtained by Lin in [21] and extended to the parabolic case in [15]. We also remark that, as in [8, 6, 29], by making use of a refined Aleksandrov–Bakel′man–Pucci estimate instead of the classical estimate, the range of p in our results can be extended to $p > d - \varepsilon$ in the elliptic case and to $p > d + 1 - \varepsilon$ in the parabolic case, where ε is a small constant depending only on d and δ. These ranges are sharp, as is seen from the examples in §I.2 of [17].

Remark 1.4. A few comments on the structures of (1.1) and (1.5) are in order. Usually, the last two terms on the left-hand side of (1.1) are combined into one $H = F + G$. However, if we are given a function $H(u'', u', u, t, x)$, we can always represent it as $F + G$ with $F = H(u'', 0, 0, t, x) - H(0, 0, 0, t, x)$ and $G = H - F$. Then usual ellipticity, convexity in u'', Lipschitz continuity, and the growth conditions with respect to (u'', u', u)

from the theory of fully nonlinear equations (see, for instance, [11]) will transform into our conditions even with $\chi \equiv 0$. Our form may look more attractive in the sense that no convexity condition with respect to u'' is imposed on G. The above decomposition of H lacks however the requirement that F be positive homogeneous of degree one. Then we can define

$$\widehat{F}(u'',t,x) = \lim_{\lambda\to\infty} \frac{1}{\lambda} F(\lambda u'',t,x), \quad \widehat{G} = F - \widehat{F},$$

and combine $\widehat{G}$ with G. The fact that $\widehat{F}$ is well defined follows from the Lipschitz continuity and convexity of F in u''. Obviously, $\widehat{F}$ is positive homogeneous of degree one. Furthermore, for each (t,x), the functions $\frac{1}{\lambda}F(\lambda u'',t,x)$ are equicontinuous in u'', and hence converge uniformly on compact sets, which means exactly that

$$\chi(u'',t,x) := \frac{1}{|u''|}|\widehat{F}(u'',t,x) - F(u'',t,x)| \to 0$$

as $|u''| \to \infty$.

Remark 1.5. There are natural and essentially unique candidates for the functions $\bar{F}$ in Assumptions 1.1 and 1.2. To show them for a function f defined on a Borel set $\mathcal{U} \subset \mathbb{R}^{d+1}$, we set

$$(f)_{\mathcal{U}} = \frac{1}{|\mathcal{U}|}\int_{\mathcal{U}} f(t,x)\,dx\,dt = \fint_{\mathcal{U}} f(t,x)\,dx\,dt,$$

where $|\mathcal{U}|$ is the $(d+1)$-dimensional Lebesgue measure of $\mathcal{U}$. In case $\mathcal{U}$ is a Borel subset of $\mathbb{R}^d$, we define $|\mathcal{U}|$ and $(f)_{\mathcal{U}}$ in a similar way. The reader understands that if f also depends on $u'' : f(u'',t,x)$, then after averaging with respect to (t,x) we get the result depending on u'' as well, which we denote $(f)_{\mathcal{U}}(u'')$. Now it is easy to show that if (1.3) is fulfilled with an $\bar{F}$, then it is also fulfilled with

$$\bar{F}(u'') = (F)_{Q_r(\tau,z)}(u''),$$

provided that we multiply the right-hand side of (1.3) by 2. Then $\bar{F}(u'')$ satisfies (H_1) as long as F does.

Remark 1.6. A typical example when it is relatively easy to verify our hypotheses is given by the following Bellman's equation:

$$\begin{aligned}\partial_t u(t,x) + \sup_{\omega\in\Omega}[a^{ij}(\omega,t,x)D_{ij}u(t,x) + b^i(\omega,t,x)D_i u(t,x)\\ -c(\omega,t,x)u(x) + f(\omega,t,x)] = 0,\end{aligned} \tag{1.8}$$

where the set Ω is a separable metric space, $a = (a^{ij})$, $b = (b^i)$, $c \geq 0$, and f are given functions measurable in (t,x) for each $\omega \in \Omega$ and continuous in ω for each (t,x).

As usual, the summation convention is enforced throughout the paper, and summation in (1.8) and in similar situations is performed before the supremum is taken. Equations of that type appear in many applications; in particular, in the theory of optimal control of diffusion type processes they are called Bellman's equations.

We introduce

$$F(u'',t,x) = \sup_{\omega\in\Omega} a^{ij}(\omega,t,x)u''_{ij},$$

$$G(u'',u',u,t,x) = \sup_{\omega\in\Omega}\left[a^{ij}(\omega,t,x)u''_{ij} + b^i(\omega,t,x)u'_i - c(\omega,t,x)u + f(\omega,t,x)\right] - F(u'',t,x)$$

and assume that for any ω the function $a^{ij}(\omega,t,x)u''_{ij}$ satisfies (H_1) and the function $b^i(\omega,t,x)u'_i - c(\omega,t,x)u + f(\omega,t,x)$ satisfies (H_2). Then F and G satisfy (H_1)–(H_3) with $\chi \equiv 0$.

Several conditions in terms of a^{ij} can be given that are sufficient for (1.3) to hold. For instance, (1.3) is satisfied if for any $r \in (0, R_0]$, $t \in \mathbb{R}$, and $z \in \mathcal{D}$ one can find functions $\bar{a}^{ij}(\omega)$ such that the functions $\bar{a}^{ij}(\omega)u''_{ij}$ satisfy (H_1) and for any $u'' \in \mathcal{S}$ with $|u''| = 1$ we have

$$\int_{Q_r(\tau,z)} |\sup_\omega a^{ij}(\omega,t,x)u''_{ij} - \sup_\omega \bar{a}^{ij}(\omega)u''_{ij}|\,dx\,dt \le \theta r^{d+2},$$

or, since the difference of suprema is less than the supremum of the absolute values of the differences, if for all i, j we have

$$\int_{Q_r(\tau,z)} \sup_\omega |a^{ij}(\omega,t,x) - \bar{a}^{ij}(\omega)|\,dx\,dt \le \theta r^{d+2}. \tag{1.9}$$

In addition, if Ω is a finite set, then we can drop the last supremum and require the condition to hold for each ω. As in Remark 1.5, the latter condition is satisfied with some $\bar{a}$ if and only if so it does (with slightly modified right-hand side) with $\bar{a} = a_{Q_r(\tau,z)}$.

The remainder of the paper is organized as follows. We consider elliptic equations in the half-space with constant coefficients in §2 and with VMO coefficients in §3. With these preparations, the proof of Theorem 1.2 is given in §4. Then we turn to parabolic equations in the entire space with constant coefficients in §5 and with VMO coefficients in §6, as well as parabolic equations in the half-space in §7 and 8. Finally, the proof of Theorem 1.1 is presented at the end of §8. The reader may notice that we could have somewhat shortened the paper by deriving some results for elliptic equations from their parabolic counterparts. We do not do that because it is much easier and shorter to explain the main ideas in the elliptic case.

A few times, in the paper we shall be using known results from the $C^{2+\alpha}$-theory of elliptic and parabolic fully nonlinear equations. Part of these results is proved for H concave in u'' and part for convex H. The reader understands that results for concave H also apply to equations with convex H, because the transformation $H(u'') \to -H(-u'')$ changes the direction of convexity and does not affect the ellipticity condition.

The authors are sincerely grateful to the referee for several comments concerning some lacking explanations, which led to introducing a missing assumption and generally helped improve the presentation.

§2. Elliptic equations with constant coefficients in $\mathbb{R}^d_+$

First, we introduce more notation. Set $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$. For $r > 0$ and $x = (x^1, x') \in \mathbb{R}^d_+$, denote

$$B_r = B_r(0), \quad B_r(x^1) = B_r(x^1, 0),$$
$$B^+_r(x) = B_r(x) \cap \mathbb{R}^d_+, \quad B^+_r = B^+_r(0), \quad B^+_r(x^1) = B^+_r(x^1, 0).$$

Recall that by Du and D^2u we denote the gradient and the Hessian of u, respectively.

In this section, we are interested in the equation

$$F(D^2u) = f(x), \tag{2.1}$$

in the half-space $\mathbb{R}^d_+$, with $F = F(u'', x)$ independent of x. Since F is convex and positive homogeneous of degree one, it can be represented as in Remark 1.6, so that we are dealing with Bellman's equations.

Lemma 2.1. *For any $r \in (0, \infty)$ and $u \in W^2_d(B^+_r)$ vanishing on $x^1 = 0$, we have*

$$\sup_{B^+_r} |u(x) - x^1(D_1u)_{B^+_r}|^d \le N r^{2d} \fint_{B^+_r} |D^2u|^d\,dx,$$

where N depends only on d.

Proof. Let $\widetilde{u}$ be the odd extension of u with respect to x^1, i.e.,

$$\widetilde{u}(x^1, x') := u(|x^1|, x')\,\mathrm{sgn}(x^1).$$

By Lemma 8.2.1 in [14], $\widetilde{u} \in W^2_d$. Note that

$$(\widetilde{u})_{B_r} = 0, \quad (D_1\widetilde{u})_{B_r} = (D_1 u)_{B_r^+}, \quad (D_i\widetilde{u})_{B_r} = 0 \quad \text{for} \quad i \geq 2.$$

Now the lemma follows from Lemma 2.1 of [15]. □

Lemma 2.2. *Let $r \in (0,\infty)$, let $\kappa \geq 2$, and let $v \in C(\bar{B}^+_{\kappa r}) \cap C^2_b(B^+_{\kappa\rho})$ for any $\rho \in (0,r)$. Assume that v is a solution of (2.1) in $B^+_{\kappa r}$ with $f \equiv 0$ and $v = 0$ on $x^1 = 0$. Then there are constants $\alpha \in (0,1)$ and N, depending only on d and δ, such that*

$$[D^2 v]_{C^\alpha(B^+_r)} \leq N(\kappa r)^{-2-\alpha} \sup_{\partial B^+_{\kappa r}} |v|.$$

Proof. Dilations show that it suffices to prove the inequality for $\kappa r = 1$. In this case, the result follows from Theorem 7.1 in [24] (or in [25]), which states that

$$[D^2 v]_{C^\alpha(B^+_{1/2})} \leq N \sup_{B^+_1} |v|.$$

Due to the maximum principle, the lemma is proved. □

Denote by $\mathcal{S}_\delta$ the set of symmetric $(d \times d)$-matrices $\alpha = (\alpha^{ij})$ satisfying

$$\delta|\xi|^2 \leq \alpha^{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2, \quad \xi \in \mathbb{R}^d.$$

We introduce $\mathbb{L}_\delta$ as the collection of operators $Lu = a^{ij}D_{ij}u$ with $a(x) = (a^{ij}(x)) \in \mathcal{S}_\delta$ for all $x \in \mathbb{R}^d$.

We need a slight generalization of the main result of [21] (stated as Lemma 2.3 in [15]) which can be proved in the same way as in [21] by using dilations and the standard approximation argument.

Lemma 2.3. *Let $r \in (0,\infty)$, and let $u \in C(\bar{B}_r) \cap W^2_d(B_\rho)$ for any $\rho \in (0,r)$. Then there are constants $\gamma \in (0,1]$ and N, depending only on δ and d, such that for any $L \in \mathbb{L}_\delta$ we have*

$$\fint_{B_r} |D^2 u|^\gamma \, dx \leq N \left(\fint_{B_r} |Lu|^d \, dx \right)^{\gamma/d} + N r^{-2\gamma} \sup_{\partial B_r} |u|^\gamma.$$

Lemma 2.4. *Let $r \in (0,\infty)$, and let $w \in W^2_d(B^+_\rho) \cap C(\bar{B}^+_r)$ for any $\rho \in (0,r)$. Assume that $w = 0$ on ∂B^+_r. Then there are constants $\gamma \in (0,1]$ and N, depending only on δ and d, such that for any $L \in \mathbb{L}_\delta$ we have*

$$\fint_{B^+_r} |D^2 w|^\gamma \, dx \leq N \left(\fint_{B^+_r} |Lw|^d \, dx \right)^{\gamma/d}.$$

Proof. Denote $f = Lw$. Let $\widetilde{w}$ and $\widetilde{f}$ (respectively) be the odd extensions of w and f with respect to x^1. We denote by $\widetilde{L} \in \mathbb{L}_\delta$ the operator with the coefficients

$$\begin{aligned} &\widetilde{a}^{ij}(x) = \mathrm{sgn}(x^1) a^{ij}(|x^1|, x') && \text{for } \ i = 1, j \geq 2 \ \text{ or } \ j = 1, i \geq 2, \\ &\widetilde{a}^{ij}(x) = a^{ij}(|x^1|, x') && \text{otherwise.} \end{aligned}$$

Clearly, $\widetilde{w} \in C(\bar{B}_r) \cap W^2_d(B_\rho)$ for any $\rho < r$, $\widetilde{w} = 0$ on ∂B_r, and $\widetilde{L}\widetilde{w} = \widetilde{f}$ in B_r. Now Lemma 2.3 yields

$$\fint_{B_r} |D^2 \widetilde{w}|^\gamma \, dx \leq N \left(\fint_{B_r} |\widetilde{f}|^d \, dx \right)^{\gamma/d}.$$

To finish the proof of the lemma, it suffices to recall the definitions of $\widetilde{w}$ and $\widetilde{f}$. □

Everywhere below in this section α and γ are the constants which appear in Lemmas 2.2 and 2.4, respectively.

Lemma 2.5. *Take $r \in (0,\infty)$, $\kappa \geq 16$, and $x^1_0 \geq 0$. Let $u \in W^2_d(B^+_{\kappa r}(x^1_0))$ be a solution of (2.1) in $B^+_{\kappa r}(x^1_0)$ vanishing on $B_{\kappa r}(x^1_0) \cap \partial\mathbb{R}^d_+$. Then*

$$\begin{aligned}&\fint_{B^+_r(x^1_0)}\fint_{B^+_r(x^1_0)} |D^2u(x)-D^2u(y)|^\gamma\,dx\,dy\\ &\qquad\leq N\kappa^d\left(\fint_{B^+_{\kappa r}(x^1_0)}|f|^d\,dx\right)^{\gamma/d}+N\kappa^{-\gamma\alpha}\left(\fint_{B^+_{\kappa r}(x^1_0)}|D^2u|^d\,dx\right)^{\gamma/d},\end{aligned}\tag{2.2}$$

where the constant N depends only on d and δ.

Proof. Dilations show that it suffices to prove the lemma for $\kappa r = 8$. We consider two cases.

Case 1: $x^1_0 > 1$**.** Then $B^+_{\kappa r/8}(x^1_0) = B_{r\kappa/8}(x^1_0) \subset \mathbb{R}^d_+$. Therefore, inequality (2.2) is an immediate consequence of Lemma 2.4 in [15] because $\kappa/8 \geq 2$ (cf. the comment at the beginning of the section).

Case 2: $x^1_0 \in [0,1]$**.** Since $r = 8/\kappa \leq 1/2$, we have

$$B^+_r(x^1_0) \subset B^+_2 \subset B^+_4 \subset B^+_{\kappa r}(x^1_0).$$

By a standard density argument, we may assume $u \in C^\infty_b(\bar{B}^+_{\kappa r}(x^1_0))$. Define $\widehat{u}(x) := u(x) - x^1(D_1u)_{B^+_4}$. Let v be a classical solution of (2.1) in B^+_4 with $f \equiv 0$ and the boundary condition $v = \widehat{u}$ on ∂B^+_4. Such a solution exists by Theorems 7.1 in [24] (or in [25]). Then by Lemmas 2.2 and 2.1,

$$\begin{aligned}&\fint_{B^+_r(x^1_0)}\fint_{B^+_r(x^1_0)} |D^2v(x)-D^2v(y)|\,dx\,dy \leq Nr^\alpha[D^2v]_{C^\alpha(B^+_2)}\\ &\qquad\leq Nr^\alpha\sup_{\partial B^+_4}|v| = Nr^\alpha\sup_{\partial B^+_4}|\widehat{u}| \leq N\kappa^{-\alpha}\left(\fint_{B^+_4}|D^2u|^d\,dx\right)^{1/d}.\end{aligned}$$

Recall that $\gamma \in (0,1]$. By Hölder's inequality, we get

$$\fint_{B^+_r(x^1_0)}\fint_{B^+_r(x^1_0)} |D^2v(x)-D^2v(y)|^\gamma\,dx\,dy \leq N\kappa^{-\gamma\alpha}\left(\fint_{B^+_{\kappa r}(x^1_0)}|D^2u|^d\,dx\right)^{\gamma/d}.\tag{2.3}$$

Next we recall a simple and well-known fact that condition (H_1) implies that for any $\mathcal{S}$-valued functions $u''(x)$ and $v''(x)$ there is an operator $L = a^{ij}D_{ij} \in \mathbb{L}_\delta$ such that $F(u''(x)) - F(v''(x)) = a^{ij}[u''_{ij} - v''_{ij}](x)$. Then define $w := \widehat{u} - v$ in B^+_4 and observe that $w \in W^2_d(B^+_\rho) \cap C(\bar{B}^+_4)$ for any $\rho < 4$, $w = 0$ on ∂B^+_4, and $F(D^2\widehat{u}) = f$.

The said above shows that there exists an operator $L \in \mathbb{L}_\delta$ such that $Lw = f$ in B^+_4. By Lemma 2.4 and the fact that $\kappa r = 8$, we get

$$\fint_{B^+_r(x^1_0)}|D^2w|^\gamma\,dx \leq N\kappa^d\fint_{B^+_4}|D^2w|^\gamma\,dx \leq N\kappa^d\left(\fint_{B^+_{\kappa r}(x^1_0)}|f|^d\,dx\right)^{\gamma/d}$$

and

$$\fint_{B^+_r(x^1_0)}\fint_{B^+_r(x^1_0)} |D^2w(x)-D^2w(y)|^\gamma\,dx\,dy \leq N\kappa^d\left(\fint_{B^+_{\kappa r}(x^1_0)}|f|^d\,dx\right)^{\gamma/d}.$$

Combining this with (2.3) and observing that $D^2u = D^2v + D^2w$, we arrive at (2.2). The lemma is proved. □

If g is a measurable function in $\mathbb{R}^d$, we define its maximal function by

$$\mathbb{M}(g)(x) = \sup_{B_r(y)\ni x} \fint_{B_r(y)} |g(z)|\,dz.$$

It is easily seen that, for any $r>0$ and $x\in\mathbb{R}^d_+$,

$$\fint_{B_r^+(x)} |g(z)|\,dz \le 2\fint_{B_r(x)} |g(z)I_{\mathbb{R}^d_+}(z)|\,dz \le 2\mathbb{M}(gI_{\mathbb{R}^d_+})(x). \tag{2.4}$$

Next, in the measure space $\mathbb{R}^d_+$ endowed with the Borel σ-field and Lebesgue measure, consider the filtration of dyadic cubes $\mathfrak{C}=\{\mathbb{C}_n, n\in\mathbb{Z}\}$, where $\mathbb{Z}=\{0,\pm1,\pm2,\dots\}$ and $\mathbb{C}_n$ is the collection of cubes

$$(i_1 2^{-n},(i_1+1)2^{-n}]\times\cdots\times(i_d 2^{-n},(i_d+1)2^{-n}], \quad i_1,\dots,i_d\in\mathbb{Z},\ \ i_1\ge 0.$$

For $x\in\mathbb{R}^d_+$, introduce

$$g^{\#}_{\gamma}(x) = \sup_{C\in\mathfrak{C}\,:\,x\in C}\left(\fint_C\fint_C |g(y)-g(z)|^{\gamma}\,dydz\right)^{1/\gamma}.$$

Note that if $x\in C\in\mathfrak{C}$, then for the smallest $r>0$ such that $C\subset B_r(x)$ we have

$$\fint_C\fint_C |g(y)-g(z)|^{\gamma}\,dy\,dz \le N(d)\fint_{B_r^+(x)}\fint_{B_r^+(x)} |g(y)-g(z)|^{\gamma}\,dy\,dz.$$

Combining this with (2.4) and Lemma 2.5 leads to the following.

Corollary 2.6. *Let $u\in\mathring{W}^2_d(\mathbb{R}^d_+)$ be a solution of (2.1) in $\mathbb{R}^d_+$. Then, for any $x\in\mathbb{R}^d_+$ and $\kappa\ge 16$, we have*

$$(D^2u)^{\#}_{\gamma}(x) \le N\kappa^{d/\gamma}\mathbb{M}^{1/d}(|f|^dI_{\mathbb{R}^d_+})(x) + N\kappa^{-\alpha}\mathbb{M}^{1/d}(|D^2u|^dI_{\mathbb{R}^d_+})(x),$$

where the constant N depends only on d and δ.

Now we recall Theorem 5.3 of [15], which is a version of the Fefferman–Stein theorem: let $p\in(1,\infty)$ and $\gamma\in(0,1]$; then for any $g\in L_p(\mathbb{R}^d_+)$, we have

$$\|g\|_{L_p(\mathbb{R}^d_+)} \le N\|g^{\#}_{\gamma}\|_{L_p(\mathbb{R}^d_+)}, \tag{2.5}$$

where N depends on p, γ, and d only.

Theorem 2.7. *Let $p>d$.*

(i) *Suppose $u\in\mathring{W}^2_p(\mathbb{R}^d_+)$ satisfies (2.1). Then there exists $N=N(d,\delta,p)$ such that*

$$\|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N\|f\|_{L_p(\mathbb{R}^d_+)}.$$

(ii) *For any $\lambda>0$ and $u\in\mathring{W}^2_p(\mathbb{R}^d_+)$, we have*

$$\lambda\|u\|_{L_p(\mathbb{R}^d_+)} + \|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N\|F(D^2u)-\lambda u\|_{L_p(\mathbb{R}^d_+)}, \tag{2.6}$$

where N depends only on d,p, and δ.

(iii) *For any $f\in L_p(\mathbb{R}^d_+)$ and $\lambda>0$, there is a unique solution $u\in\mathring{W}^2_p(\mathbb{R}^d_+)$ of the equation*

$$F(D^2u)-\lambda u = f. \tag{2.7}$$

Proof. (i) First, we fix $\kappa\ge 16$. Using Corollary 2.6, inequality (2.5), and the Hardy–Littlewood theorem on maximal functions, we see that

$$\|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N\kappa^{d/\gamma}\|f\|_{L_p(\mathbb{R}^d_+)} + N\kappa^{-\alpha}\|D^2u\|_{L_p(\mathbb{R}^d_+)}, \tag{2.8}$$

where $N=N(d,\delta,p)$. Assertion (i) is proved at once by observing that inequality (2.8) is true for arbitrary $\kappa\ge 16$.

(ii) Assertion (i) implies that, to prove (2.6), it suffices to prove the estimate

$$\lambda\|u\|_{L_p(\mathbb{R}^d_+)} \leq N\|f\|_{L_p(\mathbb{R}^d_+)}, \tag{2.9}$$

where $f = F(D^2u) - \lambda u$.

We may assume that u is smooth in $\bar{\mathbb{R}}^d_+$ and vanishes for x large and for $x^1 = 0$. Take an operator $L \in \mathbb{L}_\delta$ such that and $Lu - \lambda u = f$. Then we obtain (2.9) by Theorem 3.5.15 and the proof of Lemma 3.5.5 in [11] with N depending only on d, p, and δ.

(iii) The proof of the solvability of equation (2.7) relies on its solvability in $C^{2+\alpha}(\bar{\mathbb{R}}^d_+)$ with zero boundary condition ($\alpha \in (0,1)$ is perhaps different from the one above). First we assume that $f \in C_0^\infty(\mathbb{R}^d_+)$ and use the classical results (see, e.g., [11] or [26]) to find a function $u \in C^{2+\alpha}(\bar{\mathbb{R}}^d_+)$ with $u(0,\cdot) = 0$ satisfying (2.7). Simple barriers show that $u(x) \to 0$ exponentially fast as $|x| \to \infty, x^1 \geq 0$.

Furthermore, there is a well-known and standard procedure (see, for instance, the proof of Lemma 2.4.4 in [14]) to derive the following inequality from (2.6):

$$\|u\|_{W^2_p(B^+_1(x))} \leq N\|u\|_{L_p(B^+_2(x))} + N\|f\|_{L_p(B^+_2(x))}, \quad x \in \mathbb{R}^d_+, \tag{2.10}$$

where N is independent of x. To start this procedure it suffices to notice that for any nonnegative $\zeta \in C_0^\infty(\mathbb{R}^d)$ we have

$$F(D^2(\zeta u)) - \lambda\zeta u = \zeta f + g,$$

where

$$g = F(D^2(\zeta u)) - \zeta F(D^2u),$$

and, by the homogeneity and Lipschitz continuity of F,

$$|g| \leq N|D^2(\zeta u) - \zeta D^2 u| \leq N(|Du| + |u|).$$

Upon combining (2.10) and the fact that $u, f \in L_p(\mathbb{R}^d_+)$, we conclude that $u \in \mathring{W}^2_p(\mathbb{R}^d_+)$, so that estimate (2.6) is applicable.

Having done this step, we approximate the given $f \in L_p(\mathbb{R}^d_+)$ in the $L_p(\mathbb{R}^d_+)$ norm by functions $f_n \in C_0^\infty(\mathbb{R}^d_+)$, which would give us a sequence of $u_n \in \mathring{W}^2_p(\mathbb{R}^d_+)$ with the $\mathring{W}^2_p(\mathbb{R}^d_+)$ norms bounded and such that $F(D^2u_n) - \lambda u_n = f_n$. Then a subsequence $u_{n'}$ converges uniformly on compact subsets of $\bar{\mathbb{R}}^d_+$ to a function $u \in \mathring{W}^2_p(\mathbb{R}^d_+)$. The fact that u satisfies (2.7) is now a consequence of Theorems 3.5.15 and 3.5.6 (a) of [11]. This proves the existence of solutions.

As usual, uniqueness follows from the fact that $F(D^2u) - F(D^2v) = L(u - v)$, where $L \in \mathbb{L}_\delta$. The theorem is proved. □

§3. Elliptic equations with VMO coefficients in $\mathbb{R}^d_+$

We are about to deal with the equation

$$F(D^2u, x) - \lambda u = f(x) \tag{3.1}$$

in the half-space $\mathbb{R}^d_+$. Of course, we suppose that Assumption 1.2 is satisfied with $\mathcal{D} = \mathbb{R}^d_+$.

Remark 3.1. We are going to use the following fact. For any $\mu > 0$, there exists $\theta = \theta(\mu, d, \delta) > 0$ such that if (1.6) is true with this θ for any $u'' \in \mathcal{S}$ with $|u''| = 1$, then

$$\int_{B_r(z)} \sup_{u'' \in \mathcal{S}\,:\,|u''|=1} |F(u'', x) - \bar{F}(u'')|\,dx \leq \mu r^d. \tag{3.2}$$

To prove this, observe that one can find $n = n(d, \delta, \mu)$ points $u''_1, \ldots, u''_n$ such that, for any x and any u'' with $|u''| = 1$, at least one of the quantities $|F(u'', x) - F(u''_i, x)| + |\bar{F}(u'') - \bar{F}(u''_i)|$ is less than $\mu/(4|B_1|)$. This follows from the Lipschitz continuity of

F and $\bar{F}$ in u'' (uniform with respect to x). After that, it obviously suffices to choose $\theta = \mu/(4n)$.

We are also going to use the fact that the supremum in (3.2) is bounded by a constant depending only on δ and d.

Everywhere below in this section, α and γ are the constants appearing in Lemmas 2.2 and 2.4, respectively.

Lemma 3.1. *Take $\beta \in (1,\infty)$, $\lambda = 0$, $\kappa \geq 16$, $\mu, r > 0$, $x_0^1 \geq 0$, and $z \in \mathbb{R}^d_+$. Also, suppose that $\theta = \theta(\mu, d, \delta)$ is as in Remark 3.1. Let $u \in \mathring{W}^2_d(\mathbb{R}^d_+)$ be a solution of (3.1) vanishing outside $B^+_{R_0}(z)$. Then*

$$\begin{aligned}
&\fint_{B^+_r(x_0^1)} \fint_{B^+_r(x_0^1)} |D^2u(x) - D^2u(y)|^\gamma \, dx\, dy \\
&\quad \leq N\kappa^d \left(\fint_{B^+_{\kappa r}(x_0^1)} |f|^d \, dx \right)^{\gamma/d} + N\kappa^d \left(\fint_{B^+_{\kappa r}(x_0^1)} |D^2u|^{\beta d} \, dx \right)^{\gamma/(\beta d)} \mu^{\gamma/(\beta' d)} \\
&\qquad + N\kappa^{-\gamma\alpha} \left(\fint_{B^+_{\kappa r}(x_0^1)} |D^2u|^d \, dx \right)^{\gamma/d},
\end{aligned}$$

where $N = N(d, \delta, \beta)$ and $\beta' = \beta/(\beta - 1)$.

Proof. We introduce

$$\bar{F}(u'') = \begin{cases} (F)_{B^+_{R_0}(z)}(u'') & \text{if } \kappa r \geq R_0, \\ (F)_{B^+_{\kappa r}(x_0^1)}(u'') & \text{otherwise.} \end{cases}$$

Observe that

$$\bar{F}(D^2u) = \widehat{f}(x),$$

where

$$\widehat{f}(x) = \bar{F}(D^2u) - F(D^2u, x) + f(x).$$

By Lemma 2.5,

$$\begin{aligned}
&\fint_{B^+_r(x_0^1)} \fint_{B^+_r(x_0^1)} |D^2u(x) - D^2u(y)|^\gamma \, dx\, dy \\
&\quad \leq N\kappa^d \left(\fint_{B^+_{\kappa r}(x_0^1)} |\widehat{f}|^d \, dx \right)^{\gamma/d} + N\kappa^{-\gamma\alpha} \left(\fint_{B^+_{\kappa r}(x_0^1)} |D^2u|^d \, dx \right)^{\gamma/d}.
\end{aligned}$$

The triangle inequality yields

$$\fint_{B^+_{\kappa r}(x_0^1)} |\widehat{f}|^d \, dx \leq N \fint_{B^+_{\kappa r}(x_0^1)} |f|^d \, dx + N \fint_{B^+_{\kappa r}(x_0^1)} |\bar{F}(D^2u) - F(D^2u, x)|^d \, dx.$$

For any $x \in \mathbb{R}^d_+$, denote

$$h(x) = \sup_{u'' \in \mathcal{S} \,:\, |u''| = 1} |F(u'', x) - \bar{F}(u'')|.$$

By Hölder's inequality,

$$\fint_{B^+_{\kappa r}(x_0^1)} |\bar{F}(D^2u) - F(D^2u, x)|^d \, dx \leq \fint_{B^+_{\kappa r}(x_0^1)} h^d(x) |D^2u|^d \, dx \leq J_1^{1/\beta} J_2^{1/\beta'},$$

where

$$J_1 = \fint_{B^+_{\kappa r}(x^1_0)} |D^2u|^{\beta d}\,dx,$$
$$J_2 = \fint_{B^+_{\kappa r}(x^1_0)} h^{\beta' d}(x) I_{B^+_{R_0}(z)}\,dx \le N \fint_{B^+_{\kappa r}(x^1_0)} h(x) I_{B^+_{R_0}(z)}\,dx.$$

If $\kappa r \ge R_0$, then, by Remark 3.1,

$$J_2 \le N(\kappa r)^{-d} \int_{B^+_{R_0}(z)} h(x)\,dx \le N(\kappa r)^{-d} R_0^d \fint_{B^+_{R_0}(z)} h(x)\,dx \le N\mu.$$

If $\kappa r < R_0$, then

$$J_2 \le N \fint_{B^+_{\kappa r}(x^1_0)} h(x)\,dx \le N\mu.$$

Therefore,

$$\fint_{B^+_{\kappa r}(x^1_0)} |\bar F(D^2u) - F(D^2u, x)|^d\,dx \le N \left(\fint_{B^+_{\kappa r}(x^1_0)} |D^2u(x)|^{\beta d}\,dx \right)^{1/\beta} \mu^{1/\beta'},$$

and this leads to the desired result. The lemma is proved. □

Corollary 3.2. *Under the assumptions of Lemma* 3.1*, let $p > \beta d$. Then there is a constant N_0 depending only on δ, β, d, p and such that*

$$\|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N_0 \kappa^{d/\gamma} \|f\|_{L_p(\mathbb{R}^d_+)} + N_0(\kappa^{d/\gamma}\mu^{1/(\beta' d)} + \kappa^{-\alpha}) \|D^2u\|_{L_p(\mathbb{R}^d_+)}.$$

Indeed it suffices to proceed as in the derivation of (2.8).

By taking $2\beta = 1 + p/d$ and choosing κ and θ in such a way that

$$N_0(\kappa^{d/\gamma}\mu^{1/(\beta' d)} + \kappa^{-\alpha}) \le \frac{1}{2},$$

we arrive at the following corollary.

Corollary 3.3. *Let $p > d$, and let $u \in \mathring{W}^2_d(\mathbb{R}^d_+)$ be a solution of* (3.1) *with $\lambda = 0$ vanishing outside $B^+_{R_0}(z)$, where $z \in \mathbb{R}^d_+$. Then there exists $\theta = \theta(d, p, \delta) \in (0, 1]$ and $N = N(d, p, \delta)$ such that if Assumption* 1.2 *is satisfied with this θ, then $\|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N\|f\|_{L_p(\mathbb{R}^d_+)}$.*

The next theorem is the main result of this section.

Theorem 3.4. *Let $p > d$. Then there exists a constant $\theta \in (0, 1]$ depending only on d, p, δ and a constant λ_0 depending only on d, p, δ, and R_0 such that if Assumption* 1.2 *is satisfied with this θ, then the following is true.*

(i) *For any $\lambda \ge \lambda_0$ and any $u \in \mathring{W}^2_p(\mathbb{R}^d_+)$ satisfying* (3.1)*, we have*

$$\lambda\|u\|_{L_p(\mathbb{R}^d_+)} + \|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N\|f\|_{L_p(\mathbb{R}^d_+)}, \tag{3.3}$$

where $N = N(d, p, \delta)$.

(ii) *For any $\lambda > 0$, there exists a constant $N = N(d, p, \delta, R_0, \lambda)$ such that if $u, v \in W^2_p(\mathbb{R}^d_+)$ and $u - v \in \mathring{W}^2_p(\mathbb{R}^d_+)$, then*

$$\|u\|_{W^2_p(\mathbb{R}^d_+)} \le N\|F(D^2u, \cdot) - \lambda u\|_{L_p(\mathbb{R}^d_+)} + N\|v\|_{W^2_p(\mathbb{R}^d_+)}. \tag{3.4}$$

Proof. We suppose that Assumption 1.2 is satisfied with θ as in Corollary 3.3.

(i) Take a nonnegative $\zeta \in C_0^\infty$ that has support in $B_{R_0}^+$ and is such that ζ^p integrates to one. For the parameter $z \in \mathbb{R}^d_+$, we define

$$u_z(x) = u(x)\zeta(z-x)$$

and observe that

$$\int_{\mathbb{R}^d_+} \zeta^p(z-x)\,dz = 1 \tag{3.5}$$

for any $x \in \mathbb{R}^d_+$. By the homogeneity of F, for any $z \in \mathbb{R}^d_+$ we have

$$F(D^2u_z(x), x) = f_z(x) + \lambda u_z(x),$$

where

$$f_z(x) = f(x)\zeta(z-x) + F(D^2u_z(x), x) - F(\zeta(z-x)D^2u, x).$$

By Corollary 3.3 and the Lipschitz continuity of F in u'',

$$\|\zeta(z-\cdot)|D^2u|\|^p_{L_p(\mathbb{R}^d_+)} \le N\|\zeta(z-\cdot)f\|^p_{L_p(\mathbb{R}^d_+)} + N\|\,|D\zeta(z-\cdot)|\,|Du|\,\|^p_{L_p(\mathbb{R}^d_+)} + N\|(|D^2\zeta(z-\cdot)| + \lambda\zeta(z-\cdot))u\|^p_{L_p(\mathbb{R}^d_+)}.$$

Integrating with respect to $z \in \mathbb{R}^d_+$ and using (3.5), we get

$$\|D^2u\|^p_{L_p(\mathbb{R}^d_+)} \le N_1(\|f\|^p_{L_p(\mathbb{R}^d_+)} + \lambda^p\|u\|^p_{L_p(\mathbb{R}^d_+)}) + N_2(\|Du\|^p_{L_p(\mathbb{R}^d_+)} + \|u\|^p_{L_p(\mathbb{R}^d_+)}),$$

where $N_1 = N_1(d, \delta, p)$ and $N_2 = N_2(d, \delta, p, R_0)$. Furthermore, as in the proof of Theorem 2.7, by analyzing the proof of Lemma 3.5.5 of [11], for any $\lambda > 0$ we have

$$\lambda\|u\|_{L_p(\mathbb{R}^d_+)} \le N\|f\|_{L_p(\mathbb{R}^d_+)},$$

where N depends only on d, p, and δ. Hence,

$$\lambda^p\|u\|^p_{L_p(\mathbb{R}^d_+)} + \|D^2u\|^p_{L_p(\mathbb{R}^d_+)} \le N_1\|f\|^p_{L_p(\mathbb{R}^d_+)} + N_2(\|Du\|^p_{L_p(\mathbb{R}^d_+)} + \|u\|^p_{L_p(\mathbb{R}^d_+)}),$$

and to finish the proof of (3.3) with $N = 2N_1$ it only remains to use the multiplicative inequalities and choose $\lambda_0(d, \delta, p, R_0)$ sufficiently large.

(ii) Set $w = u - v$ and $f = F(D^2u, x) - \lambda u$. Observe that

$$F(D^2w, x) - \lambda w = f + [F(D^2w, x) - F(D^2w + D^2v, x)] + \lambda v$$

and $|F(D^2w, x) - F(D^2w + D^2v, x)| \le N|D^2v|$. Then we see that (3.4) follows from the proof of assertion (i). The theorem is proved. □

The following solvability theorem is a standard result, which however will not be used later in the paper. The main emphasis here is on the method of proof.

Theorem 3.5. *Let $p > d$. Then there exists a constant $\theta \in (0, 1]$ depending only on d, p, δ such that if Assumption 1.2 is satisfied with this θ, then for any $v \in W^2_p(\mathbb{R}^d_+)$, $\lambda > 0$, and $f \in L_p(\mathbb{R}^d_+)$, there exists a unique $u \in W^2_p(\mathbb{R}^d_+)$ satisfying (3.1) and such that $u - v \in \mathring{W}^2_p(\mathbb{R}^d_+)$.*

Proof. We take θ as in Theorem 3.4 and, first, assume that $F(u'', x)$ is infinitely differentiable with respect to $x \in \mathbb{R}^d$ and each of its derivatives is continuous in (u'', x) and Lipschitz continuous in u'' uniformly with respect to x (in particular, the F_{x^k} are differentiable with respect to u'' (a.e) on $\mathcal{S}$ and the $F_{x^k u''_{ij}}$ are bounded). Then we take a function $\eta \in C_0^\infty(\mathcal{S})$ with unit integral and for $n > 0$ set

$$F^n(u'', x) = \int_{\mathcal{S}} F(u'' + v''/n, x)\eta(v'')\,dv''.$$

In this way we obtain a sequence $F^n(u'',x)$ of functions infinitely differentiable in (u'',x), converging to F as $n\to\infty$, and convex in u''. Furthermore, by (H_1), the function $F(u'',x)$ is Lipschitz continuous in u'' with a constant independent of x and, therefore, has bounded generalized derivatives with respect to u'', and for any $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^d$ we have

$$\delta|\xi|^2\leq F_{u''_{ij}}(u'',x)\xi^i\xi^j\leq\delta^{-1}|\xi|^2\quad\text{a.e. on }\mathcal{S}.$$

It follows that

$$F^n_{u''_{ij}}(u'',x)=\int_{\mathcal{S}}F_{u''_{ij}}(u''+v''/n,x)\eta(v'')\,dv''$$

and, for all n and all values of the arguments and $\xi\in\mathbb{R}^d$ and $v''\in\mathcal{S}$, we have

$$\delta|\xi|^2\leq F^n_{u''_{ij}}\xi^i\xi^j\leq\delta^{-1}|\xi|^2,\quad -F^n_{u''_{ij}x^k}v''_{ij}\xi^k-F^n_{x^kx^r}\xi^k\xi^r\leq N(|v''|+|\xi|)|\xi|,$$

where N is independent of n. Also, since $F(u'',x)$ is positive homogeneous of degree one, we have $F(u'',x)=F_{u''_{ij}}(u'',x)u''_{ij}$ at all points where F is differentiable in u'', that is, almost everywhere on $\mathcal{S}$. Hence,

$$F^n_{u''_{ij}}(u'',x)u''_{ij}=F^n(u'',x)-(1/n)\int_S F_{u''_{ij}}(u''+v''/n,x)v''_{ij}\eta(v'')\,dv''$$

implying that,

$$|F^n-F^n_{u''_{ij}}u''_{ij}|\leq 1,$$

for sufficiently large n and all values of the arguments.

It follows that the functions $-F^n(-u'',x)-\lambda u$ belong to the class $\mathcal{F}$ introduced in [11, §5.5]. We also take $f\in C_0^\infty(\mathbb{R}^d_+)$ and $v\in C_0^\infty(\mathbb{R}^d)$. Then we claim that, for each n, equation (3.1) with F^n in place of F and with the boundary condition $u=v$ on $\partial\mathbb{R}^d_+$ has a unique solution u^n such that it is twice continuously differentiable in $\bar{\mathbb{R}}^d_+$ and the $C^{2+\varepsilon}(\bar{\mathbb{R}}^d_+)$-norm of u^n is bounded by a constant independent of n, where $\varepsilon>0$ is independent of n.

We explain how this claim can be proved on the basis of the classical results found in [11] or [26], where the results are stated for smooth bounded domains only. We approximate $\mathbb{R}^d_+$ with a sequence of expanding smooth domains $\mathcal{D}_m$ such that $\partial\mathbb{R}^d_+\cap\partial\mathcal{D}_m$ expands to $\partial\mathbb{R}^d_+$. By Theorem 6.2.7 and Remark 6.2.8 in [11], equation (3.1) with F^n in place of F in $\mathcal{D}_m$ and with the boundary condition $u=v$ on $\partial\mathcal{D}_m$ has a unique solution $u^{nm}\in C^{2+\varepsilon}(\bar{\mathcal{D}}_m)$ and Du^{nm},D^2u^{nm} are in $C^{2+\varepsilon}_{loc}(\mathcal{D}_m)$. By Theorem 5.5.8 of [11], for any $x\in\mathbb{R}^d_+$, any $m\geq 1$, and all $k>m$ such that $\mathcal{D}_k$ engulfs the 1-neighborhood of $\mathcal{D}_m$ intersected with $\bar{\mathbb{R}}^d_+$, the $C^{2+\varepsilon}(\bar{\mathcal{D}}_m)$-norms of u^{nk} are bounded by a constant independent of k and m. Then a subsequence of $\{u^{nm},m=1,2,\dots\}$ converges uniformly on compact subsets of $\bar{\mathbb{R}}^d_+$ along with its first and second order derivatives to a function u^n we need and to its corresponding derivatives.

This proves our claim, which implies that, along a subsequence, u^n converges uniformly on compact subsets of $\bar{\mathbb{R}}^d_+$ along with its first and second order derivatives to a function u and its corresponding derivatives. As in the proof of Theorem 2.7 we have $u-v\in\mathring{W}^2_p(\mathbb{R}^d_+)$ and estimate (3.4) holds (with $F(D^2u,x)-\lambda u=f$). This proves the theorem in our particular case.

Next, we consider the general situation. Take the function ζ as in the proof of assertion (i) of Theorem 3.4 but such that (3.5) is fulfilled with $p=1$. For $n=1,2,\dots$, define

$$F_n(u'',x)=\int_{\mathbb{R}^d_+}F(u'',x+y/n)\zeta(y)\,dy.$$

Obviously, these infinitely differentiable functions of x are positive homogeneous of degree one and satisfy (H_1) and Assumption 1.2 with the same parameters as F does.

Then we approximate f and v in appropriate norms with functions f_n and v_n possessing the properties described above. This yields a sequence of $u_n \in W^2_p(\mathbb{R}^d_+)$ with uniformly bounded $W^2_p(\mathbb{R}^d_+)$-norms and such that $u_n - v_n \in \mathring{W}^2_p(\mathbb{R}^d_+)$ and

$$F_n(D^2u_n, x) - \lambda u_n(x) = f_n(x).$$

By embedding theorems, there exists $u \in W^2_p(\mathbb{R}^d_+)$ and a subsequence, still denoted by $\{u_n\}$, such that $u_n \to u$ uniformly on compact subsets of $\bar{\mathbb{R}}^d_+$. In particular, $u = v$ on $\partial\mathbb{R}^d_+$, so that $u - v \in \mathring{W}^2_p(\mathbb{R}^d_+)$.

Since $f_n \to f$ in $L_p(\mathbb{R}^d_+)$, we may assume that $f_n \to f$ a.e.. Therefore, if we define

$$\bar{F}_{n_0}(u'', x) = \sup_{n \geq n_0} F_n(u'', x), \quad \underline{F}_{n_0}(u'', x) = \inf_{n \geq n_0} F_n(u'', x),$$

then, for any n_0, almost everywhere we have

$$\liminf_{n\to\infty} \bar{F}_{n_0}(D^2u_n, x) \geq \liminf_{n\to\infty} F_n(D^2u_n, x) = f(x) + \lambda u(x),$$
$$\limsup_{n\to\infty} \underline{F}_{n_0}(D^2u_n, x) \leq \limsup_{n\to\infty} F_n(D^2u_n, x) = f(x) + \lambda u(x).$$

It follows from the above by Theorems 3.5.15 and 3.5.6 of [11] that for any n_0 (a.e.)

$$\bar{F}_{n_0}(D^2u, x) \geq f(x) + \lambda u(x) \geq \underline{F}_{n_0}(D^2u, x). \tag{3.6}$$

Now observe that, for each $u'' \in \mathcal{S}$, $F_n(u'', x) \to F(u'', x)$ a.e. Since both parts are positive homogeneous and Lipschitz continuous in u'' with constant independent of n, we also have

$$\bar{F}_{n_0}(u'', x) - \underline{F}_{n_0}(u'', x) \leq \varepsilon_{n_0}(x)|u''|,$$

where $\varepsilon_{n_0}(x) \to 0$ a.e. as $n_0 \to \infty$. After that, to finish the proof of existence it only remains to pass to the limit in (3.6).

Uniqueness is proved in the same way as in Theorem 2.7. This completes the proof of Theorem 3.5. □

§4. Proof of Theorem 1.2

First, we state a generalization of a result of [15]. The point is that in that paper the counterpart of our Assumption 1.2 is formulated as (1.9) (in its elliptic version).

Theorem 4.1. *Let $p > d$. Then there exists a constant $\theta \in (0, 1]$ depending only on d, p, δ and a constant λ_0 depending only on d, p, δ, and R_0 such that if Assumption 1.2 is satisfied with this θ and $\mathcal{D} = \mathbb{R}^d$, then the following is true.*

(i) *For any $u \in W^2_p(\mathbb{R}^d)$ satisfying (3.1), we have*

$$\lambda\|u\|_{L_p(\mathbb{R}^d)} + \|D^2u\|_{L_p(\mathbb{R}^d)} \leq N\|f\|_{L_p(\mathbb{R}^d)} \tag{4.1}$$

if $\lambda \geq \lambda_0$, where $N = N(d, p, \delta)$, and

$$\|u\|_{W^2_p(\mathbb{R}^d)} \leq N\|f\|_{L_p(\mathbb{R}^d)} \tag{4.2}$$

if $\lambda > 0$, where $N = N(d, p, \delta, R_0, \lambda)$.

(iii) *For any $\lambda > 0$ and $f \in L_p(\mathbb{R}^d)$, there exists a unique $u \in W^2_p(\mathbb{R}^d)$ satisfying (3.1).*

We only give a few comments on the proof of this theorem. In case F is independent of x, the *a priori* estimate (4.1) was obtained in [15] for all $\lambda > 0$. If F depends on x, estimates (4.1) for $\lambda \geq \lambda_0$ and (4.2) for $\lambda > 0$ can be obtained as in the proof of Theorem 3.4. After the necessary *a priori* estimates are obtained, in [15] it is stated that the solvability theorems are derived in a standard way without giving any specific details. This standard way is presented in the proof of Theorem 3.5.

Proof of Theorem 1.2. As in the proof of Theorem 3.4, first we establish (1.7) as an *a priori* estimate and the case of general g is reduced to the case where $g \equiv 0$ by replacing the unknown function u with $u - g$. We shall see that to obtain the *a priori* estimate we do not need condition (H_3).

Observe that Theorems 3.4 and 4.1 with $\lambda = \lambda_0$ imply that

$$\begin{aligned} &\|D^2u\|_{L_p(\mathbb{R}^d_+)} \le N(\|F(D^2u)\|_{L_p(\mathbb{R}^d_+)} + \|u\|_{L_p(\mathbb{R}^d_+)}), \quad u \in \mathring{W}^2_p(\mathbb{R}^d_+), \\ &\|D^2v\|_{L_p(\mathbb{R}^d)} \le N(\|F(D^2v)\|_{L_p(\mathbb{R}^d)} + \|v\|_{L_p(\mathbb{R}^d)}), \quad v \in W^2_p(\mathbb{R}^d), \end{aligned} \tag{4.3}$$

where $N = N(d, p, \delta, R_0)$ (provided that $\theta = \theta(d, p, \delta)$ is chosen appropriately).

Now assume that $u \in \mathring{W}^2_p(\mathcal{D})$ satisfies

$$F(D^2u(x), x) + G(D^2u(x), Du(x), u(x), x) = 0. \tag{4.4}$$

We move the term G to the right-hand side and define

$$f(x) := -G(D^2u(x), Du(x), u(x), x).$$

After that, by using the technique based on flattening the boundary, partitions of unity, and interpolation inequalities allowing one to estimate Du through D^2u and u, and also using (4.3), we obtain

$$\|D^2u\|_{L_p(\mathcal{D})} \le N_1(\|f\|_{L_p(\mathcal{D})} + \|u\|_{L_p(\mathcal{D})}), \tag{4.5}$$

provided that θ is sufficiently small, depending only on d, p, δ, and the $C^{1,1}$ norm of ∂D. Here N_1 depends only on d, p, δ, R_0, and the $C^{1,1}$-norm of $\partial\mathcal{D}$. Below by N_i we denote the same type of constants as N_1. The definition of f and condition (H_2) imply that, for any $s > 0$,

$$\begin{aligned} \|f\|_{L_p(\mathcal{D})} &\le \|\chi(|D^2u|)D^2u\|_{L_p(\mathcal{D})} + K(\|Du\|_{L_p(\mathcal{D})} + \|u\|_{L_p(\mathcal{D})}) + \|\bar{G}\|_{L_p(\mathcal{D})} \\ &\le \chi(s)\|D^2u\|_{L_p(\mathcal{D})} + \|\chi\|_{L_\infty} s|\mathcal{D}|^{1/p} \\ &\qquad + K(\|Du\|_{L_p(\mathcal{D})} + \|u\|_{L_p(\mathcal{D})}) + \|\bar{G}\|_{L_p(\mathcal{D})}. \end{aligned} \tag{4.6}$$

Upon taking s large so that $N_1\chi(s) \le 1/2$, we deduce from (4.5), (4.6), and the interpolation inequality that

$$\|u\|_{W^2_p(\mathcal{D})} \le N_2(\|u\|_{L_p(\mathcal{D})} + \|\bar{G}\|_{L_p(\mathcal{D})} + \|\chi\|_{L_\infty} s|\mathcal{D}|^{1/p}). \tag{4.7}$$

To estimate the term $\|u\|_{L_p(\mathcal{D})}$ on the right-hand side of (4.7), we rewrite (4.4) as

$$\begin{aligned} F(D^2u(x), x) + G(D^2u(x), Du(x), u(x), x) - G(D^2u(x), 0, 0, x) \\ = - G(D^2u(x), 0, 0, x). \end{aligned} \tag{4.8}$$

Note that, by conditions (H_1) and (H_2), there exists $L \in \mathbb{L}_\delta$ and bounded measurable functions $b = (b^1, \dots, b^d)$ and c such that the left-hand side of (4.8) can be represented as $Lu + b^iD_iu - cu$. Since G is monotone nonincreasing in u, we have $c \ge 0$. Therefore, by Alexandrov's estimate,

$$\sup_{\mathcal{D}} |u|, \quad \|u\|_{L_p(\mathcal{D})} \le N\|G(D^2u(\cdot), 0, 0, \cdot)\|_{L_p(\mathcal{D})},$$

where $N = N(d, p, \delta, \mathrm{diam}(\mathcal{D}))$. Again by condition ($\mathrm{H}_2$), for any $s_1 > 0$, we have

$$\begin{aligned} \|u\|_{L_p(\mathcal{D})} &\le N\|\chi(|D^2u|)D^2u\|_{L_p(\mathcal{D})} + N\|\bar{G}\|_{L_p(\mathcal{D})} \\ &\le N_3(\chi(s_1)\|D^2u\|_{L_p(\mathcal{D})} + \|\chi\|_{L_\infty} s_1|\mathcal{D}|^{1/p} + \|\bar{G}\|_{L_p(\mathcal{D})}). \end{aligned} \tag{4.9}$$

Combining (4.7) with (4.9) and taking s_1 so large that $N_2N_3\chi(s_1) \le 1/2$, we arrive at

$$\|u\|_{W^2_p(\mathcal{D})} \le N_4\|\bar{G}\|_{L_p(\mathcal{D})} + N_4(s + s_1)\|\chi\|_{L_\infty}|\mathcal{D}|^{1/p},$$

which is (1.7) in the case where $g = 0$.

To prove the existence and uniqueness of solutions, first we assume that the functions $H := F + G$ and g are smooth in x, the function $\widehat{G}$ in (1.2) is bounded in $\mathcal{D}$, and the domain is of class $C^{2+\alpha}$. Then we mollify them with respect to (u'', u', u) like this was done in the proof of Theorem 3.5, obtaining an equation determined by a function of class $\mathcal{F}$. Here condition (1.2) becomes necessary. Then, by referring to the same results of [11] as in the proof of Theorem 3.5, we obtain solutions of approximating problems with uniformly bounded $C^{2+\varepsilon}(\bar{D})$ norms. Passing to the limit, we get the existence of a $C^{2+\varepsilon}(\bar{D})$-solution of the original problem. Its uniqueness follows from the classical maximum principle.

This solution is certainly in $W^2_p(\mathcal{D})$, and we have an estimate of its $W^2_p(\mathcal{D})$ norm. After that, we mollify the original F and G in x, mollify g, and approximate $\mathcal{D}$ by a sequence of increasing smooth domains $\mathcal{D}_n$ with the $C^{1,1}$-norm under control. Moreover, after mollifications, the corresponding $\widehat{G}$ is bounded in $\mathcal{D}_n$. We take these domains because otherwise after mollifications F may fail to satisfy (1.6) for all $x \in \mathcal{D}$. To construct the domains $\mathcal{D}_n$ we can use the so-called regularized distance function $\rho(x)$ such that it is equivalent to $\operatorname{dist}(x, \partial\mathcal{D})$ for $x \in \mathcal{D}$ close to $\partial\mathcal{D}$, is of class C^∞_{loc} in $\mathcal{D}$, its gradient is strictly larger than zero near $\partial\mathcal{D}$, and it has its $C^{1,1}(\bar{\mathcal{D}})$-norm under control (the existence and the listed properties of ρ can be found, for instance, in [19], specifically, see Theorem 2.1 therein). One then can set $\mathcal{D}_n = \{x \in \mathcal{D} : \rho(x) > 1/n\}$. After that, it suffices to repeat the last part of the proof of Theorem 3.5. The limiting function u satisfies $u - g \in \mathring{W}^2_p(\mathcal{D})$ because $u_n, g_n \in W^2_p(\mathcal{D}_n)$ with uniformly bounded norms and $(u_n - g_n)I_{\mathcal{D}_n} \in \mathring{W}^1_p(\mathcal{D})$ with uniformly bounded norms. Of course, while passing to the limits and proving uniqueness we use the fact that for any $u, v \in W^2_p(\mathcal{D})$ there is an operator $L \in \mathbb{L}_\delta$ and bounded measurable functions $b = (b^1, \dots, b^d)$ and c satisfying $|b| \le K, \quad 0 \le c \le K$, such that

$$\begin{aligned} &H(D^2u, Du, u, x) - H(D^2v, Dv, v, x) \\ &\quad = H(D^2u, Du, u, x) - H(D^2v, Du, u, x) + H(D^2v, Du, u, x) - H(D^2v, Dv, v, x) \\ &\quad = L(u - v) + b^i D_i(u - v) - c(u - v). \end{aligned}$$

The theorem is proved. □

§5. Parabolic Bellman's equations in $\mathbb{R}^{d+1}$ with constant coefficients

In this section we consider the equation

$$\partial_t u(t, x) + F(D^2u) = f(t, x), \tag{5.1}$$

in the entire space. For the same reasons as in §2, equation (5.1) can be written as a parabolic Bellman's equation

$$\partial_t u(t, x) + \sup_{\omega \in \Omega}[a^{ij}(\omega)D_{ij}u(t, x)] = f(t, x).$$

For $r > 0$, introduce $Q_r := Q_r(0, 0)$. The following is Lemma 4.2.2 in [14].

Lemma 5.1. *Let $p \in [1, \infty)$. Then there is a constant $N = N(d, p)$ such that for any $r \in (0, \infty)$ and $u \in C^\infty_{\text{loc}}(\mathbb{R}^{d+1})$ we have*

$$\fint_{Q_r} |Du - (Du)_{Q_r}|^p \, dx\, dt \le N r^p \fint_{Q_r} (|D^2u| + |\partial_t u|)^p \, dx\, dt,$$

$$\fint_{Q_r} |u(t, x) - (u)_{Q_r} - x^i (D_i u)_{Q_r}|^p \, dx\, dt \le N r^{2p} \fint_{Q_r} (|D^2u| + |\partial_t u|)^p \, dx\, dt.$$

The second lemma is a parabolic embedding theorem proved as Lemma II.3.3 in [18].

Lemma 5.2. *Suppose $p > (d+2)/2$ and $u \in W^{1,2}_p(Q_1)$. Then for any $(t,x) \in Q_1$, we have*

$$|u(t,x)| \le N\|u\|_{W^{1,2}_p(Q_1)},$$

where $N = N(d,p)$.

Let $v = u(t,x) - (u)_{Q_r} - x^i(D_iu)_{Q_r}$, then v belongs to $W^{1,2}_p(Q_r)$ whenever u does. Noting that

$$D_{ij}v = D_{ij}u, \quad \partial_t u = \partial_t v, \quad D_i v = D_i u - (D_i u)_{Q_r},$$

we get the following corollary by dilations, combining Lemmas 5.1 and 5.2.

Corollary 5.3. *Take $p > (d+2)/2$ and $r \in (0,\infty)$. Then for any $u \in W^{1,2}_p(Q_r)$, we have*

$$\sup_{(t,x)\in Q_r} |u(t,x) - (u)_{Q_r} - x^i(D_iu)_{Q_r}|^p \le Nr^{2p} \fint_{Q_r} (|D^2u| + |\partial_t u|)^p \, dx\, dt,$$

where $N = N(d,p)$.

Lemma 5.4. *Take $r \in (0,\infty)$ and $\kappa \ge 2$. Let $v \in C^{1,2}_b(Q_{\kappa r})$ be a solution of (5.1) with $f \equiv 0$. Then there are constants $\alpha \in (0,1)$ and N depending only on d and δ such that*

$$\fint_{Q_r}\fint_{Q_r} |D^2v(t,x) - D^2v(s,y)| \, dx\, dt\, dy\, ds \le N\kappa^{-2-\alpha} r^{-2} \sup_{\partial' Q_{\kappa r}} |v|.$$

Proof. Dilations show that we may concentrate on the case where $r = 1/\kappa$. In this case, from Theorem 5.5.2 in [11] one can derive routinely that there exist constants α and N depending only on δ and d such that for any $(t,x),(s,y) \in Q_{1/2}$, we have

$$|D^2v(t,x) - D^2v(s,y)| \le N(|x-y|^\alpha + |t-s|^{\alpha/2}) \sup_{Q_1} |v|. \tag{5.2}$$

Applying the maximum principle, we complete the proof of the lemma. □

Remark 5.1. By "derive routinely" we mean the following. First, observe that we may assume that

$$\sup_{Q_1} |v| = 1.$$

Indeed, if this supremum is zero, we have nothing to prove. However if it is different from zero, we can replace v with the ratio of v and the supremum.

Then we approximate $F(u'')$ by smooth convex functions $F^n(u'')$ so that $F^n \to F$ as $n \to \infty$ uniformly on compact sets and, for all values of variables,

$$\delta|\xi|^2 \le F^n_{u''_{ij}} \xi^i\xi^j \le \delta^{-1}|\xi|^2, \quad |F^n - F^n_{u''_{ij}} u''_{ij}| \le 1$$

(see the proof of Theorem 3.5). Then we approximate v on $\partial' Q_1$ uniformly by infinitely differentiable functions ϕ^n such that $|\phi^n| \le 1$. Next, we apply Theorem 6.2.5 of [11] to find a unique $u^n \in C^{1,2}(Q_1) \cap C(\bar{Q}_1)$ such that

$$\partial_t u^n + F^n(D^2u^n) - \tfrac{1}{n}u^n = 0 \quad \text{in} \quad Q_1$$

and $u^n = \phi^n$ on $\partial' Q_1$.

This theorem also guarantees that

$$u^n, Du^n, D^2u^n, \partial_t u^n \in C^{1,2}([\varepsilon, 1-\varepsilon] \times \bar{B}_\varepsilon)$$

for any $\varepsilon \in (0,1/2)$. By the maximum principle, the u^n are uniformly bounded in Q_1. Since

$$\partial_t v + F^n(D^2v) - \tfrac{1}{n}v = F^n(D^2v) - F(D^2v) - \tfrac{1}{n}v$$

and the latter expression tends to zero uniformly in Q_1, by the maximum principle we have $u^n \to v$ uniformly in Q_1.

Now, formally applying Theorem 5.5.2 of [11], we see that the norms of u^n in the space $C^{1+\alpha/2,2+\alpha}(Q_{1/2})$ are uniformly bounded; in particular, for any $(t,x),(s,y) \in Q_{1/2}$, we have

$$|D^2u^n(t,x) - D^2u^n(s,y)| \le N(|x-y|^\alpha + |t-s|^{\alpha/2}),$$

where N depends only on d and δ. Since $u^n \to v$ uniformly and the D^2u^n are uniformly equicontinuous in $Q_{1/2}$, we have $D^2u^n \to D^2v$ in $Q_{1/2}$, which yields

$$|D^2v(t,x) - D^2v(s,y)| \le N(|x-y|^\alpha + |t-s|^{\alpha/2}),$$

and this coincides with (5.2).

We introduce $\mathbb{L}_\delta$ as before Lemma 2.4, but allow the dependence of the coefficients on (t,x) rather than on x only.

Lemma 5.5. *Let $r \in (0,\infty)$, and let $u \in C(\bar{Q}_r) \cap W^{1,2}_{d+1}(Q_\rho)$ for any $\rho \in (0,r)$. Then there are constants $\gamma \in (0,1]$ and N, depending only on δ, d, such that for any $L \in \mathbb{L}_\delta$ we have*

$$\fint_{Q_r} |D^2u|^\gamma \,dx\,dt \le Nr^{-2\gamma} \sup_{\partial' Q_r} |u|^\gamma + N\left(\fint_{Q_r} |\partial_t u + Lu|^{d+1}\,dx\,dt\right)^{\gamma/(d+1)}. \tag{5.3}$$

Proof. If we prove (5.3) with ρ in place of r for any $\rho \in (0,r)$, then by passing to the limit we shall obtain (5.3) as is. Hence, we may assume that $u \in W^{1,2}_{d+1}(Q_r)$. As usual, we may also assume that $r=1$. Then we may also assume that the coefficients $a^{ij}(t,x)$ of L are infinitely differentiable in $\mathbb{R}^{d+1}$. Now we set $f = \partial_t u + Lu$ in Q_1 and extend $f(t,x)$ for $t \le 0$ as zero. Also, set $u(t,x) = u(-t,x)$ for $t \le 0$. Observe that the new u belongs to $W^{1,2}_{d+1}((-1,1)\times B_1)$. After that, we define $v(t,x)$ as a unique $W^{1,2}_{d+1}((-1,1)\times B_1) \cap C([-1,1]\times \bar{B}_1)$-solution of $\partial_t v + Lv = f$ with terminal and lateral conditions being u. The existence and uniqueness of such a solution is a classical result (see, for instance, Theorem IV.9.1 in [18] or Theorem 7.17 in [20]). By uniqueness, $v = u$ in Q_1, so that by Corollary 4.2 in [15], we have

$$\begin{aligned}
\int_{Q_1} |D^2u|^\gamma\,dx\,dt &= \int_{Q_1} |D^2v|^\gamma\,dx\,dt \\
&\le N\left(\int_{(-1,1)\times B_1} |f|^{d+1}\,dx\,dt\right)^{\gamma/(d+1)} + N \sup_{\partial'(-1,1)\times B_1} |v|^\gamma \\
&= N\left(\int_{Q_1} |f|^{d+1}\,dx\,dt\right)^{\gamma/(d+1)} + N \sup_{\partial' Q_1} |u|^\gamma.
\end{aligned}$$

The lemma is proved. □

We note that a slightly weaker version of Lemma 5.5 can be found in [27], where for the proof the reader is referred to [28].

Everywhere below in this section, α and γ are the constants appearing in Lemmas 5.4 and 5.5, respectively.

Lemma 5.6. *Take $r \in (0,\infty)$ and $\kappa \ge 2$. Let $u \in W^{1,2}_{d+1}(Q_{\kappa r})$ be a solution of (5.1). Then*

$$\begin{aligned}
&\fint_{Q_r}\fint_{Q_r} |D^2u(t,x) - D^2u(s,y)|^\gamma\,dx\,dt\,dy\,ds \\
&\qquad \le N\kappa^{d+2}(|f|^{d+1})^{\gamma/(d+1)}_{Q_{\kappa r}} + N\kappa^{-\alpha\gamma}(|D^2u|^{d+1})^{\gamma/(d+1)}_{Q_{\kappa r}},
\end{aligned}$$

where N depends only on d and δ.

Proof. As usual, it suffices to prove the lemma for $r = 1$. We follow the proof of Lemma 2.4 in [15] and, as there, without trouble reduce the general case to the one where $u \in C_b^\infty(\bar{Q}_\kappa)$. Define $\widehat{u} := u - (u)_{Q_\kappa} - x^i(D_i u)_{Q_\kappa}$, and let $v \in C_b^{1,2}(Q_\kappa) \cap C(\bar{Q}_\kappa)$ be a solution of (5.1) in Q_κ with $f \equiv 0$ and $v = \widehat{u}$ on $\partial' Q_\kappa$. Such a solution v exists by Theorem 6.4.1 of [11]. By Lemma 5.4, Hölder's inequality, and Corollary 5.3, we have

$$\begin{aligned} &\fint_{Q_1}\fint_{Q_1} |D^2 v(t,x) - D^2 v(s,y)|^\gamma \, dx\, dt\, dy\, ds \\ &\qquad \le N\kappa^{-\gamma(2+\alpha)} \sup_{\partial' Q_\kappa} |v|^\gamma \le N\kappa^{-\alpha\gamma}(|D^2 u|^{d+1} + |\partial_t u|^{d+1})^{\gamma/(d+1)}_{Q_\kappa}. \end{aligned} \tag{5.4}$$

Let $w := \widehat{u} - v$ in $\bar{Q}_\kappa$. Then, by the same argument as in the proof of Lemma 2.4 in [15] or our Lemma 2.5, we deduce that there exists an operator $L \in \mathbb{L}_\delta$ such that $\partial_t w + Lw = f$. Then by Lemma 5.5,

$$\fint_{Q_1} |D^2 w|^\gamma \, dx\, dt \le N\kappa^{d+2} \fint_{Q_\kappa} |D^2 w|^\gamma \, dx\, dt \le N\kappa^{d+2} \left(\fint_{Q_\kappa} |f|^{d+1} \, dx\, dt \right)^{\gamma/(d+1)}$$

and

$$\fint_{Q_1}\fint_{Q_1} |D^2 w(t,x) - D^2 w(s,y)|^\gamma \le N\kappa^{d+2} \left(\fint_{Q_\kappa} |f|^{d+1} \, dx\, dt \right)^{\gamma/(d+1)}.$$

Combining this inequality and (5.4) and observing that $D^2 u = D^2 v + D^2 w$ and

$$|\partial_t u| = |f - F(D^2 u)| \le |f| + N|D^2 u|, \tag{5.5}$$

we get the desired result. The lemma is proved. □

The next theorem is the main result of this section. For simplicity of notation, we set

$$L_p = L_p(\mathbb{R}^{d+1}), \quad W_p^{1,2} = W_p^{1,2}(\mathbb{R}^{d+1}). \tag{5.6}$$

Theorem 5.7. *Let $p > d + 1$.*

(i) *Let $u \in W_p^{1,2}$ be a solution of* (5.1). *Then*

$$\|D^2 u\|_{L_p} + \|\partial_t u\|_{L_p} \le N\|f\|_{L_p}, \tag{5.7}$$

where N depends only on p, d, and δ.

(ii) *For any $\lambda > 0$ and $f \in L_p$, there exists a unique solution $u \in W_p^{1,2}$ of the equation*

$$\partial_t u + F(D^2 u) - \lambda u = f. \tag{5.8}$$

Furthermore,

$$\lambda\|u\|_{L_p} + \|D^2 u\|_{L_p} + \|\partial_t u\|_{L_p} \le N\|f\|_{L_p}, \tag{5.9}$$

where N depends only on p, d, and δ.

Proof. (i) The estimate of the $D^2 u$ term on the left-hand side of (5.7) is derived from Theorem 5.3 of [15] and Lemma 5.6 in the same way as Theorem 2.5(i) of [15] or Theorem 2.7(i). Of course, this time we use the filtration of parabolic dyadic cubes. The estimate of $\partial_t u$ follows from that of $D^2 u$ and (5.5).

(ii) To prove the *a priori* estimate (5.9), we replace f with $\lambda u + f$ in the above estimates, obtaining

$$\|\partial_t u\|_{L_p} + \|D^2 u\|_{L_p} \le \lambda\|u\|_{L_p} + \|f\|_{L_p}.$$

Hence, it suffices to prove that

$$\lambda\|u\|_{L_p} \le N\|f\|_{L_p},$$

which is done in the same way as in the elliptic case. After that, the solvability of (5.8) is proved in the same way as in Theorem 2.7. □

§6. Parabolic equations in $\mathbb{R}^{d+1}$ with VMO coefficients

In this section, we consider the parabolic equation

$$\partial_t u(t,x) + F(D^2u(t,x),t,x) - \lambda u(t,x) = f(t,x). \tag{6.1}$$

Everywhere below in this section, we suppose that Assumption 1.1 is satisfied with $\mathcal{D} = \mathbb{R}^{d+1}$, and the constants α and γ are as in Lemmas 5.4 and 5.5, respectively. We use the notation (5.6) and recall that $\theta(\mu,d,\delta)$ is introduced in Remark 3.1.

Lemma 6.1. *Take $\beta \in (1,\infty)$, $\lambda = 0$, $\mu, r \in (0,\infty)$, $\kappa \geq 2$, and $(t_0,x_0) \in \mathbb{R}^{d+1}$. Also, suppose that $\theta = \theta(\mu,d,\delta)$. Let $u \in W^{1,2}_{d+1}$ be a solution of (6.1) vanishing outside $Q_{R_0}(t_0,x_0)$. Then*

$$\begin{aligned} &\fint_{Q_r}\fint_{Q_r} |D^2u(t,x) - D^2u(s,y)|^\gamma \,dx\,dt\,dy\,ds \\ &\quad\leq N\kappa^{d+2}\big(|f|^d\big)^{\gamma/(d+1)}_{Q_{\kappa r}} + N\kappa^{d+2}\big(|D^2u|^{\beta(d+1)}\big)^{\gamma/(\beta d+\beta)}_{Q_{\kappa r}}\mu^{\gamma/(\beta' d+\beta')} \\ &\qquad\qquad + N\kappa^{-\alpha\gamma}\big(|D^2u|^{d+1}\big)^{\gamma/(d+1)}_{Q_{\kappa r}}, \end{aligned} \tag{6.2}$$

where $N = N(d,\delta,\beta)$ and $\beta' = \beta/(\beta-1)$.

Proof. We will basically repeat the proof of Lemma 3.1, adapting it to the parabolic case and the whole space. Introduce

$$\bar{F}(u'') = \begin{cases} (F)_{Q_{R_0}(t_0,x_0)}(u'') & \text{if } \kappa r \geq R_0, \\ (F)_{Q_{\kappa r}}(u'') & \text{otherwise,} \end{cases}$$

and

$$h(t,x) = \sup_{u''\in\mathcal{S}\,:\,|u''|=1} |F(u'',t,x) - \bar{F}(u'')|.$$

Note that

$$\partial_t u + \bar{F}(D^2u) = \widetilde{f},$$

where

$$\widetilde{f}(t,x) = f(t,x) + \bar{F}(D^2u) - F(D^2u,t,x).$$

By Lemma 5.6 and the triangle inequality,

$$\begin{aligned} &\fint_{Q_r}\fint_{Q_r} |D^2u(t,x) - D^2u(s,y)|^\gamma \,dx\,dt\,dy\,ds \\ &\quad\leq N\kappa^{d+2}\left(\big(|f|^{d+1}\big)^{\gamma/(d+1)}_{Q_{\kappa r}} + J^{\gamma/(d+1)}\right) + N\kappa^{-\alpha\gamma}\left(|D^2u|^{d+1}\right)^{\gamma/(d+1)}_{Q_{\kappa r}}, \end{aligned} \tag{6.3}$$

where $N = N(d,\delta)$, and

$$J = \fint_{Q_{\kappa r}} |\bar{F}(D^2u) - F(D^2u,t,x)|^{d+1} I_{Q_{R_0}}(t_0,x_0)\,dx\,dt \leq J_1^{1/\beta} J_2^{1/\beta'},$$

with

$$\begin{aligned} J_1 &= \fint_{Q_{\kappa r}} |D^2u|^{\beta(d+1)}\,dx\,dt, \\ J_2 &= \fint_{Q_{\kappa r}} h^{\beta'(d+1)} I_{Q_{R_0}(t_0,x_0)}\,dx\,dt \leq N\fint_{Q_{\kappa r}} h I_{Q_{R_0}(t_0,x_0)}\,dx\,dt. \end{aligned}$$

If $\kappa r < R_0$, we have

$$J_2 \le N \fint_{Q_{\kappa r}} h\,dx\,dt \le N\mu.$$

If $\kappa r \ge R_0$, we have

$$J_2 \le N(\kappa r)^{-d-2} \int_{Q_{R_0}(t_0,x_0)} h\,dx\,dt \le N(\kappa r)^{-d-2} R_0^{d+2} \fint_{Q_{R_0}(t_0,x_0)} h\,dx\,dt \le N\mu.$$

Therefore, in any case,

$$J \le N \left(\fint_{Q_{\kappa r}} |D^2 u(x)|^{\beta(d+1)}\,dx\,dt \right)^{1/\beta} \mu^{1/\beta'}.$$

Substituting the above inequality back in (6.3), we get (6.2). The lemma is proved. □

From Lemma 6.1, by a standard argument using Theorem 5.3 of [15] and the Hardy–Littlewood theorem, we get the following corollary.

Corollary 6.2. *Let $p > d+1$, and let $u \in W^{1,2}_{d+1}$ be a solution of* (6.1) *with $\lambda = 0$ vanishing outside Q_{R_0}. Then there exist constants N and θ depending only on p, d, and δ, such that if Assumption* 1.1 *is satisfied with this θ, then*

$$\|D^2 u\|_{L_p} + \|\partial_t u\|_{L_p} \le N\|f\|_{L_p}.$$

For any $T \in [-\infty, \infty)$, we denote

$$\mathbb{R}^{d+1}_T = \{(t,x) \in \mathbb{R}^{d+1} \,:\, t > T\}.$$

The main result of this section is the following theorem.

Theorem 6.3. *Take $p > d+1$ and $T \in [-\infty, \infty)$. Then there exists $\theta \in (0,1]$, depending only on d, δ, p, and a constant λ_0 depending only on d, δ, p, and R_0, such that if Assumption* 1.1 *is satisfied with this θ, then the following is true.*

(i) *For any $\lambda \ge \lambda_0$ and any $u \in W^{1,2}_{d+1}(\mathbb{R}^{d+1}_T)$ satisfying* (6.1)*, we have*

$$\lambda\|u\|_{L_p(\mathbb{R}^{d+1}_T)} + \|\partial_t u\|_{L_p(\mathbb{R}^{d+1}_T)} + \|D^2 u\|_{L_p(\mathbb{R}^{d+1}_T)} \le N\|f\|_{L_p(\mathbb{R}^{d+1}_T)}, \tag{6.4}$$

where $N = N(d,\delta,p)$.

(ii) *For any $\lambda > 0$, there exists a constant $N = N(d,p,\delta,R_0,\lambda)$ such that for any $u \in W^{1,2}_p(\mathbb{R}^{d+1}_T)$ satisfying* (6.1) *we have*

$$\|u\|_{W^{1,2}_p(\mathbb{R}^{d+1}_T)} \le N\|f\|_{L_p(\mathbb{R}^{d+1}_T)}. \tag{6.5}$$

(iii) *For any $\lambda > 0$ and $f \in L_p(\mathbb{R}^{d+1}_T)$, there exists a unique solution of* (6.1) *in $W^{1,2}_p(\mathbb{R}^{d+1}_T)$.*

Proof. First, we assume $T = -\infty$. The proof of Theorem 3.5 shows that assertion (iii) follows from (ii). We suppose that Assumption 1.1 is satisfied with θ as in Corollary 6.2.

Take a nonnegative function $\zeta \in C^\infty$ with support in $-Q_{R_0}$ and such that ζ^p integrates to one. Fixing $(s,y) \in \mathbb{R}^{d+1}$, we define

$$u_{(s,y)}(t,x) = u(t,x)\zeta(s-t, y-x).$$

Then $u_{(s,y)}(t,x)$ is supported in $Q_{R_0}(s,y)$, and

$$\partial_t u_{(s,y)} + F(u_{(s,y)}, t, x) = f_{(s,y)},$$

where

$$f_{(s,y)}(t,x) = f(t,x)\zeta(s-t,y-x) + F(u_{(s,y)},t,x) - F(\zeta(s-t,y-x)D^2u,t,x) \\ - (\partial_t\zeta)(s-t,y-x)u + \lambda u_{(s,y)}.$$

By Corollary 6.2 and condition (H_1),

$$\begin{aligned}\|\zeta(s-\cdot,y-\cdot)\partial_t u\|^p_{L_p}+\|\zeta(s-\cdot,y-\cdot)D^2u\|^p_{L_p}\\ \le N\|\zeta(s-\cdot,y-\cdot)f\|^p_{L_p}+N\big\||D\zeta(s-\cdot,y-\cdot)|Du\big\|^p_{L_p}\\ +\big\|(|\partial_t\zeta|+|D^2\zeta|+\lambda|\zeta|)(s-\cdot,y-\cdot)u\big\|^p_{L_p}.\end{aligned}$$

Integrating the above inequality over $(s,y)\in\mathbb{R}^{d+1}$, we get

$$\|\partial_t u\|^p_{L_p}+\|D^2u\|^p_{L_p}\le N_1(\|f\|^p_{L_p}+\lambda^p\|u\|^p_{L_p})+N_2(\|Du\|^p_{L_p}+\|u\|^p_{L_p}),$$

where $N_1=N_1(d,\delta,p)$ and $N_2=N_2(d,\delta,p,R_0)$. Now to obtain (6.4) and (6.5), it suffices to use the proof of Lemma 3.5.5 of [11] once again, as in Theorem 3.4. This completes the proof of the theorem in the special case where $T=-\infty$.

For $T>-\infty$, we extend f to be zero for $t\le T$, and then find a unique solution $\widetilde{u}\in W^{1,2}_p(\mathbb{R}^{d+1})$ of (6.1) in $\mathbb{R}^{d+1}$, the existence of which is guaranteed by the argument above. In its turn, this also yields the existence of a solution of (6.1) in $\mathbb{R}^{d+1}_T$ satisfying (6.4) or (6.5) as appropriate. Its uniqueness in $W^{1,2}_p(\mathbb{R}^{d+1}_T)$ follows as usual from uniqueness in the case of linear equations (with measurable coefficients) and parabolic Alexandrov's estimates. The theorem is proved. □

We finish the section by proving the following result about the Cauchy problem. Denote by $\mathring{W}^{1,2}_p((0,T)\times\mathbb{R}^d)$ the set of functions of class $W^{1,2}_p((0,T)\times\mathbb{R}^d)$ having zero trace on the plane $\{(T,x):x\in\mathbb{R}^d\}$.

Theorem 6.4. *Let $p>d+1$ and $T>0$. Then there exists $\theta\in(0,1]$ depending only on d, δ, p and such that if Assumption 1.1 is satisfied with this θ, then the following assertions hold true.*

(i) *For any $v\in W^{1,2}_p((0,T)\times\mathbb{R}^d)$ and $f\in L_p((0,T)\times\mathbb{R}^d)$, there exists a unique solution $u\in W^{1,2}_p((0,T)\times\mathbb{R}^d)$ of (6.1) in $(0,T)\times\mathbb{R}^d$ with $\lambda=0$ satisfying $u-v\in\mathring{W}^{1,2}_p((0,T)\times\mathbb{R}^d)$.*

(ii) *Moreover,*

$$\|u\|_{W^{1,2}_p((0,T)\times\mathbb{R}^d)}\le N\|v\|_{W^{1,2}_p((0,T)\times\mathbb{R}^d)}+N\|f\|_{L_p((0,T)\times\mathbb{R}^d)},\tag{6.6}$$

where $N=N(d,\delta,p,T,R_0)$.

Proof. As in the proof of Theorem 3.5, it suffices to prove (6.6) as an *a priori* estimate. By considering $u-v$ instead of the unknown function u, without loss of generality we may assume that $v\equiv 0$. Furthermore, having in mind the possibility of substitution $\widehat{u}=e^tu$, we see that it suffices to consider equation (1.1) with $\lambda=1$. We extend u to be zero for $t>T$. It is easily seen that the extended $u\in W^{1,2}_p(\mathbb{R}^{d+1}_0)$ satisfies (6.1) in $\mathbb{R}^{d+1}_0$ with $f(t,x)=0$ for $t\ge T$. Then, estimate (6.6) follows from Theorem 6.3 (ii). □

§7. Parabolic Bellman's equations in $\mathbb{R}^{d+1}_+$ with constant coefficients

In this section, we consider equation (5.1) in the half-space

$$\mathbb{R}^{d+1}_+:=\mathbb{R}\times\mathbb{R}^d_+.$$

For $r>0$, $t\in\mathbb{R}$, and $x=(x^1,x')\in\mathbb{R}^d_+$, denote

$$Q^+_r(t,x)=Q_r(t,x)\cap\mathbb{R}^{d+1}_+,\quad Q^+_r=Q^+_r(0,0),\quad Q^+_r(x^1)=Q^+_r(0,x^1,0).$$

The following lemma can be deduced from Corollary 5.3 in the same way as in the proof of Lemma 2.1.

Lemma 7.1. *Take $p > (d+2)/2$ and $r \in (0,\infty)$. Then for any $u \in W^{1,2}_p(Q^+_r)$ vanishing on $x^1 = 0$, we have*

$$\sup_{(t,x)\in Q^+_r} |u - x^1(D_1u)_{Q^+_r}|^p \le Nr^{2p} \fint_{Q^+_r} (|D^2u| + |\partial_t u|)^p \, dx\, dt,$$

where N depends only on d and p.

Lemma 7.2. *Suppose that $r \in (0,\infty)$, $\kappa \ge 2$, and $v \in C(\bar{Q}^+_{\kappa r}) \cap C^{1,2}_b(Q^+_{\kappa\rho})$ for any $\rho \in (0,r)$. Assume that v is a solution of (5.1) with $f \equiv 0$ and $v = 0$ on $x^1 = 0$. Then there are constants $\alpha \in (0,1)$ and N depending only on d and δ and such that*

$$[D^2v]_{C^\alpha(Q^+_r)} \le N(\kappa r)^{-2-\alpha} \sup_{\partial' Q^+_{\kappa r}} |v|.$$

Proof. Dilations show that it suffices to prove this inequality for $\kappa r = 1$. We take a smooth domain $\mathcal{D}_1$ such that $B^+_{3/4} \subset \mathcal{D}_1 \subset B^+_1$. As in Lemma 5.4, from Theorem 5.5.2 in [11] it follows that

$$[D^2v]_{C^\alpha(Q^+_{1/2})} \le N \sup_{(0,3/4)\times\mathcal{D}_1} |v|.$$

Applying the maximum principle, we complete the proof of the lemma. □

The next lemma is a consequence of Lemma 5.5 and can be proved in the same way as Lemma 2.4.

Lemma 7.3. *Let $r \in (0,\infty)$, and let u be a function satisfying $u \in C(\bar{Q}^+_r) \cap W^{1,2}_{d+1}(Q^+_\rho)$ for any $\rho \in (0,r)$ and $u = 0$ on $\partial' Q^+_r$. Then there are constants $\gamma \in (0,1]$ and N, depending only on δ and d, such that for any $L \in \mathbb{L}_\delta$ we have*

$$\fint_{Q^+_r} |D^2u|^\gamma \, dx\, dt \le N \left(\fint_{Q^+_r} |\partial_t u + Lu|^{d+1} \, dx\, dt \right)^{\gamma/(d+1)}.$$

Everywhere below in this section α is the smallest of the constants called α in Lemmas 5.4 and 7.2, and γ is the smallest of the corresponding constants in Lemmas 5.5 and 7.3.

Lemma 7.4. *Take $r \in (0,\infty)$, $\kappa \ge 16$, and $x^1_0 \ge 0$. Let $u \in W^{1,2}_{d+1}(Q^+_{\kappa r}(x^1_0))$ be a solution of (5.1) in $Q^+_{\kappa r}(x^1_0)$ vanishing on $Q^+_{\kappa r}(x^1_0) \cap \partial\mathbb{R}^{d+1}_+$. Then*

$$\begin{aligned} &\fint_{Q^+_r(x^1_0)} \fint_{Q^+_r(x^1_0)} |D^2u(t,x) - D^2u(s,y)|^\gamma \, dx\, dt\, dy\, ds \\ &\quad \le N\kappa^{d+2} \left(\fint_{Q^+_{\kappa r}(x^1_0)} |f|^{d+1} \, dx\, dt \right)^{\gamma/(d+1)} + N\kappa^{-\alpha\gamma} \left(\fint_{Q^+_{\kappa r}(x^1_0)} |D^2u|^{d+1} \, dx\, dt \right)^{\gamma/(d+1)}, \end{aligned} \tag{7.1}$$

where the constant N depends only on d and δ.

Proof. As in the proof of Lemma 2.5, dilations show that we only need to consider the case where $\kappa r = 8$. Again, we consider the following two cases.

Case 1: $x^1_0 > 1$. In this case, we have $Q_{r\kappa/8}(x^1_0) \subset \mathbb{R}^{d+1}_+$ and inequality (7.1) is an immediate consequence of Lemma 5.6, because $\kappa/8 \ge 2$.

Case 2: $x^1_0 \in [0,1]$. Since $r = 8/\kappa \le 1/2$, we have

$$Q^+_r(x^1_0) \subset Q^+_{3/2} \subset Q^+_4 \subset Q^+_{\kappa r}(x^1_0).$$

By using a standard approximation argument, we may assume that $u \in C^\infty_b(\bar{Q}^+_{\kappa r}(x^1_0))$. Define $\widehat{u} := u - x^1(D_1u)_{Q^+_4}$. We claim that there exists a function v such that

(i) $v \in C(\bar{Q}_4)$, $v = \widehat{u}$ on $\partial' Q_4^+$;
(ii) $v \in C_b^{1,2}(\bar{Q}_\rho^+)$ for any $\rho < 4$;
(iii) v satisfies (5.1) in Q_4^+ with $f \equiv 0$.

The proof of this claim is obtained as follows. First, we take smooth domains $\mathcal{D}_n$ such that $B_{4-1/n}^+ \subset \mathcal{D}_n \subset B_4^+$, set $\mathcal{Q}_n = (0,16) \times \mathcal{D}_n$, and, by applying Theorem 6.4.1 of [11], find unique $v_n \in C_b^{1,2}(\mathcal{Q}_n) \cap C(\bar{\mathcal{Q}}_n)$ satisfying (5.1) with $f \equiv 0$, and boundary condition $v_n = \widehat{u}$ on $\partial' \mathcal{Q}_n$. Then from Theorem 5.5.2 in [11] we can derive routinely (cf. Remark 5.1) that there exists $\beta \in (0,1)$ such that for any $\rho < 4$ the $C^{1+\beta/2,2+\beta}(Q_\rho^+)$-norms of v_n are bounded for all large n. After that, taking a subsequence of v_n if necessary, we find a function v possessing the above properties (ii) and (iii). The fact that v also satisfies (i) is proved in the same way as a similar statement was proved in Theorem 6.3.1 of [11].

Now Lemmas 7.2 and 7.1 and the maximum principle easily imply

$$\fint_{Q_r^+(x_0^1)} \fint_{Q_r^+(x_0^1)} |D^2 v(t,x) - D^2 v(s,y)| \, dx \, dt \, dy \, ds \le N r^\alpha [D^2 v]_{C^\alpha(Q_{3/2}^+)}$$
$$\le N r^\alpha \sup_{\partial' Q_4^+} |v| \le N \kappa^{-\alpha} \left(\fint_{Q_4^+} (|D^2 u| + |\partial_t u|)^{d+1} \, dx \, dt \right)^{1/(d+1)}.$$

Recall that $\gamma \in (0,1]$. By Hölder's inequality,

$$\fint_{Q_r^+(x_0^1)} \fint_{Q_r^+(x_0^1)} |D^2 v(t,x) - D^2 v(s,y)|^\gamma \, dx \, dt \, dy \, ds$$
$$\le N \kappa^{-\alpha\gamma} \left(\fint_{Q_{\kappa r}^+(x_0^1)} (|D^2 u| + |\partial_t u|)^{d+1} \, dx \, dt \right)^{\gamma/(d+1)}. \tag{7.2}$$

Next for $w := \widehat{u} - v$ in Q_4^+, we have $w \in W_{d+1}^{1,2}(Q_\rho^+)$ for any $\rho < 4$ and $w = 0$ on $\partial' Q_4^+$. By the same argument as in the proof of Lemma 2.5, we know that there exists an operator $L \in \mathbb{L}_\delta$ such that $\partial_t w + L w = f$ in Q_4^+. By Lemma 7.3 and the fact that $\kappa r = 8$, we get

$$\fint_{Q_r^+(x_0^1)} |D^2 w|^\gamma \, dx \, dt \le N \kappa^{d+2} \fint_{Q_4^+} |D^2 w|^\gamma \, dx \, dt \le N \kappa^{d+2} \left(\fint_{Q_4^+} |f|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)}$$
$$\le N \kappa^{d+2} \left(\fint_{Q_{\kappa r}^+(x_0^1)} |f|^{d+1} \, dx \, dt \right)^{\gamma/(d+1)}$$

and

$$\fint_{Q_r^+(x_0^1)} \fint_{Q_r^+(x_0^1)} |D^2 w(t,x) - D^2 w(s,y)|^\gamma \, dx \, dt \, dy \, ds$$
$$\le N \kappa^{d+2} \left(\fint_{Q_{\kappa r}^+(x_0^1)} |f|^{d+1} \, dx \, dt \right)^{\gamma/d+1}.$$

Combining this inequality with (7.2), observing that $D^2 u = D^2 v + D^2 w$, and using (5.5), we get (7.1). The lemma is proved. □

As in the proof of Theorem 2.7, the following theorem can be deduced from Lemma 7.4, the Hardy–Littlewood theorem, and Theorem 5.3 of [15], which we apply to the filtration of dyadic parabolic cubes belonging to $\mathbb{R}_+^{d+1}$. We denote by $\mathring{W}_p^{1,2}(\mathbb{R}_+^{d+1})$ the set of functions in $W_p^{1,2}(\mathbb{R}_+^{d+1})$ with zero trace at $x^1 = 0$.

Theorem 7.5. *Let $p > d+1$.*

(i) *If $u \in \mathring{W}^{1,2}_p(\mathbb{R}^{d+1}_+)$ satisfies (5.1) in $\mathbb{R}^{d+1}_+$, then*

$$\|D^2u\|_{L_p(\mathbb{R}^{d+1}_+)} + \|\partial_t u\|_{L_p(\mathbb{R}^{d+1}_+)} \le N\|f\|_{L_p(\mathbb{R}^{d+1}_+)},$$

where N depends only on d, δ, and p.

(ii) *For any $f \in L_p(\mathbb{R}^{d+1}_+)$ and $\lambda > 0$, there exists a unique solution $u \in \mathring{W}^{1,2}_p(\mathbb{R}^{d+1}_+)$ of the equation*

$$\partial_t u(t,x) + F(D^2u(t,x)) - \lambda u(t,x) = f(t,x).$$

Furthermore,

$$\lambda\|u\|_{L_p(\mathbb{R}^{d+1}_+)} + \|D^2u\|_{L_p(\mathbb{R}^{d+1}_+)} + \|\partial_t u\|_{L_p(\mathbb{R}^{d+1}_+)} \le N\|f\|_{L_p(\mathbb{R}^{d+1}_+)},$$

where N depends only on d, δ, and p.

§8. Parabolic equations in $\mathbb{R}^{d+1}_+$ with VMO coefficients

In this section, we consider parabolic equations in $\mathbb{R}^{d+1}_+$ with variable coefficients

$$\partial_t u(t,x) + F(D^2u(t,x),t,x) - \lambda u(t,x) = f(t,x). \tag{8.1}$$

In the sequel, we suppose that Assumption 1.1 is satisfied with $\mathcal{D} = \mathbb{R}^{d+1}_+$, and the constants α and γ in Lemma 8.1 are as in §7. Recall that $\theta(\mu,d,\delta)$ was introduced in Remark 3.1.

Lemma 8.1. *Take $\beta \in (1,\infty)$, $\lambda = 0$, $\mu, r > 0$, $\kappa \ge 16$, $x^1_0 \ge 0$, and $(\tau,z) \in \mathbb{R}^{d+1}_+$. Also, suppose that $\theta = \theta(\mu,d,\delta)$. Let $u \in \mathring{W}^{1,2}_{d+1}(\mathbb{R}^{d+1}_+)$ be a solution of (8.1) vanishing outside $Q^+_{R_0}(\tau,z)$. Then*

$$\begin{aligned}
&\fint_{Q^+_r(x^1_0)}\fint_{Q^+_r(x^1_0)} |D^2u(t,x) - D^2u(s,y)|^\gamma \,dx\,dt\,dy\,ds \\
&\quad\le N\kappa^{d+2}\left(\fint_{Q^+_{\kappa r}(x^1_0)} |f|^d\,dx\,dt\right)^{\gamma/(d+1)} \\
&\qquad + N\kappa^{d+2}\left(\fint_{Q^+_{\kappa r}(x^1_0)} |D^2u|^{\beta(d+1)}\,dx\,dt\right)^{\gamma/(\beta d+\beta)} \mu^{\gamma/(\beta' d+\beta')} \\
&\qquad + N\kappa^{-\alpha\gamma}\left(\fint_{Q^+_{\kappa r}(x^1_0)} |D^2u|^{d+1}\,dx\,dt\right)^{\gamma/(d+1)},
\end{aligned} \tag{8.2}$$

where $N = N(\delta,d,\beta)$ and $\beta' = \beta/(\beta-1)$.

Proof. We introduce

$$\bar{F}(u'') = \begin{cases} (F)_{Q^+_{R_0}(\tau,z)}(u'') & \text{if } \kappa r \ge R_0, \\ (F)_{Q^+_{\kappa r}(x^1_0)}(u'') & \text{otherwise} \end{cases}$$

and

$$h(t,x) = \sup_{u''\in\mathcal{S}\,:\,|u''|=1} |F(u'',t,x) - \bar{F}(u'')|.$$

Note that

$$\partial_t u(t,x) + \bar{F}(D^2u) = \widetilde{f},$$

where

$$\widetilde{f}(t,x) = f(t,x) + \bar{F}(D^2u) - F(D^2u,t,x).$$

By Lemma 7.4 and the triangle inequality,

$$\begin{aligned}
&\fint_{Q_r^+(x_0^1)}\fint_{Q_r^+(x_0^1)} |D^2u(t,x)-D^2u(s,y)|^\gamma\,dx\,dt\,dy\,ds \\
&\quad\le N\kappa^{d+2}\left(\fint_{Q_{\kappa r}^+(x_0^1)}|f|^{d+1}\,dx\,dt\right)^{\gamma/(d+1)} + N\kappa^{d+2}J^{\gamma/(d+1)} \\
&\qquad + N\kappa^{-\alpha\gamma}\left(\fint_{Q_{\kappa r}^+(x_0^1)}|D^2u|^{d+1}\,dx\,dt\right)^{\gamma/(d+1)},
\end{aligned} \tag{8.3}$$

where $N=N(d,\delta)$,

$$J=\fint_{Q_{\kappa r}^+(x_0^1)}|\bar F(D^2u)-F(D^2u,t,x)|^{d+1}I_{Q_{R_0}^+(\tau,z)}\,dx\,dt \le J_1^{1/\beta}J_2^{1/\beta'}.$$

Here

$$\begin{aligned}
J_1&=\fint_{Q_{\kappa r}^+(x_0^1)}|D^2u|^{\beta(d+1)}\,dx\,dt,\\
J_2&=\fint_{Q_{\kappa r}^+(x_0^1)}h^{\beta'(d+1)}I_{Q_{R_0}^+(\tau,z)}\,dx\,dt \le N\fint_{Q_{\kappa r}^+(x_0^1)}hI_{Q_{R_0}^+(\tau,z)}\,dx\,dt.
\end{aligned}$$

If $\kappa r<R_0$, we have

$$J_2\le N\fint_{Q_{\kappa r}^+(x_0^1)}h(t,x)\,dx\,dt\le N\mu.$$

If $\kappa r\ge R_0$, we have

$$\begin{aligned}
J_2&\le N(\kappa r)^{-d-2}\int_{Q_{R_0}^+(\tau,z)}h(t,x)\,dx\,dt\\
&\le N(\kappa r)^{-d-2}R_0^{d+2}\fint_{Q_{R_0}^+(\tau,z)}h(t,x)\,dx\,dt\le N\mu.
\end{aligned}$$

Therefore, in any case,

$$J\le N\left(\fint_{Q_{\kappa r}^+(x_0^1)}|D^2u(x)|^{\beta(d+1)}\,dx\,dt\right)^{1/\beta}\mu^{1/\beta'}.$$

Substituting the above inequality back in (8.3), we get (8.2). The lemma is proved. □

The proof of Lemma 8.1 is just a rather dull repetition of already given proofs of similar facts. The following corollary is obtained in the same way as similar assertions were obtained before.

Corollary 8.2. *Let $p>d+1$, and let $u\in \mathring{W}^{1,2}_{d+1}(\mathbb{R}^{d+1}_+)$ be a solution of* (6.1) *with $\lambda=0$ vanishing outside $Q_{R_0}^+(\tau,z)$, where $(\tau,z)\in\mathbb{R}^{d+1}_+$. Then there exist constants $\theta\in(0,1]$ and N depending only on p, d, and δ, such that if Assumption* 1.1 *is satisfied with this θ, then*

$$\|D^2u\|_{L_p}+\|\partial_t u\|_{L_p}\le N\|f\|_{L_p}.$$

Next we state the main result of this section, which is deduced from Corollary 8.2 by modifying the proof of Theorem 3.5. By $\mathring{W}^{1,2}_p((T,\infty)\times\mathbb{R}^d_+)$ we denote the set of functions of class $W^{1,2}_p((T,\infty)\times\mathbb{R}^d_+)$ with zero trace on $x^1=0$.

Theorem 8.3. *Let $p > d+1$ and $T \in [-\infty, \infty)$. There exist constants $\theta = \theta(d, \delta, p) \in (0, 1]$ and λ_0 depending only on d, p, δ and R_0, such that if Assumption 1.1 is satisfied with this θ, then the following is true.*

(i) *For any $\lambda \geq \lambda_0$ and $u \in \mathring{W}^{1,2}_p((T,\infty) \times \mathbb{R}^d_+)$ satisfying (8.1), we have*

$$\lambda \|u\|_{L_p((T,\infty)\times\mathbb{R}^d_+)} + \|\partial_t u\|_{L_p((T,\infty)\times\mathbb{R}^d_+)} + \|D^2 u\|_{L_p((T,\infty)\times\mathbb{R}^d_+)} \leq N \|f\|_{L_p((T,\infty)\times\mathbb{R}^d_+)},$$

where $N = N(d, \delta, p)$.

(ii) *For any $\lambda > 0$, there exists a constant $N = N(d, p, \delta, R_0, \lambda)$ such that for any $u \in \mathring{W}^{1,2}_p((T,\infty) \times \mathbb{R}^d_+)$ satisfying (8.1), we have*

$$\|u\|_{W^{1,2}_p((T,\infty)\times\mathbb{R}^d_+)} \leq N \|f\|_{L_p((T,\infty)\times\mathbb{R}^d_+)}.$$

(iii) *For any $\lambda > 0$ and $f \in L_p((T,\infty) \times \mathbb{R}^d_+)$, there exists a unique solution $u \in \mathring{W}^{1,2}_p((T,\infty) \times \mathbb{R}^d_+)$ of (8.1).*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is similar to that of Theorem 1.2 in §4 with some minor modifications. As before, first we establish (1.4) as an *a priori* estimate, and we may assume that $g \equiv 0$.

Again, we shall see that to obtain the *a priori* estimate we do not need condition (H_3).

Observe that Theorems 6.3 and 8.3 with $\lambda = \lambda_0$ imply that

$$\begin{aligned}
&\|\partial_t u\|_{L_p(\mathbb{R}^{d+1}_0)} + \|D^2 u\|_{L_p(\mathbb{R}^{d+1}_0)} \\
&\qquad \leq N\big(\|\partial_t u + F(D^2 u)\|_{L_p(\mathbb{R}^{d+1}_0)} + \|u\|_{L_p(\mathbb{R}^{d+1}_0)}\big), \quad u \in W^{1,2}_p(\mathbb{R}^{d+1}_0), \\
&\|\partial_t v\|_{L_p(\mathbb{R}_+\times\mathbb{R}^d_+)} + \|D^2 v\|_{L_p(\mathbb{R}_+\times\mathbb{R}^d_+)} \\
&\qquad \leq N\big(\|\partial_t v + F(D^2 v)\|_{L_p(\mathbb{R}_+\times\mathbb{R}^d_+)} + \|v\|_{L_p(\mathbb{R}_+\times\mathbb{R}^d_+)}\big), \quad v \in \mathring{W}^{1,2}_p(\mathbb{R}_+ \times \mathbb{R}^d_+),
\end{aligned} \tag{8.4}$$

where $N = N(d, p, \delta, R_0)$ (provided that $\theta = \theta(d, p, \delta)$ is chosen appropriately).

Now suppose that $u \in \mathring{W}^{1,2}_p(\mathcal{D}_T)$ satisfies

$$\partial_t u + F(D^2 u, t, x) + G(D^2 u, Du, u(t,x), t, x) = 0 \tag{8.5}$$

in $\mathcal{D}_T$. We extend u and G to be zero for $t > T$. It is easily seen that the extended $u \in \mathring{W}^{1,2}_p(\mathcal{D}_\infty)$ satisfies (8.5) in $\mathcal{D}_\infty$. Define

$$f(t,x) = -G(D^2 u(t,x), Du(t,x), u(t,x), t, x).$$

After that, by using the technique based on flattening the boundary, partitions of unity, and interpolation inequalities allowing one to estimate Du through $D^2 u$ and u, and also using (8.4), we obtain

$$\|\partial_t u\|_{L_p(\mathcal{D}_\infty)} + \|D^2 u\|_{L_p(\mathcal{D}_\infty)} \leq N_1\big(\|f\|_{L_p(\mathcal{D}_\infty)} + \|u\|_{L_p(\mathcal{D}_\infty)}\big),$$

which is the same as

$$\|\partial_t u\|_{L_p(\mathcal{D}_T)} + \|D^2 u\|_{L_p(\mathcal{D}_T)} \leq N_1\big(\|f\|_{L_p(\mathcal{D}_T)} + \|u\|_{L_p(\mathcal{D}_T)}\big), \tag{8.6}$$

provided that θ is sufficiently small, depending only on d, p, δ, and the $C^{1,1}$ norm of ∂D. Here N_1 depends only on d, p, δ, R_0, and the $C^{1,1}$-norm of $\partial\mathcal{D}$.

The definition of f and condition (H_2) show that, for any $s > 0$,

$$\begin{aligned}
\|f\|_{L_p(\mathcal{D}_T)} \leq \chi(s)\|D^2 u\|_{L_p(\mathcal{D}_T)} &+ \|\chi\|_{L_\infty} s T^{1/p} |\mathcal{D}|^{1/p} \\
&+ K\big(\|Du\|_{L_p(\mathcal{D}_T)} + \|u\|_{L_p(\mathcal{D}_T)}\big) + \|\bar{G}\|_{L_p(\mathcal{D}_T)}.
\end{aligned} \tag{8.7}$$

Taking s so large that $N_1\chi(s) \le 1/2$, we use (8.6), (8.7), and the interpolation inequality to obtain the estimate

$$\|u\|_{W^{1,2}_p(\mathcal{D}_T)} \le N_2\big(\|u\|_{L_p(\mathcal{D}_T)} + \|\bar{G}\|_{L_p(\mathcal{D}_T)} + \|\chi\|_{L_\infty} sT^{1/p}|\mathcal{D}|^{1/p}\big), \tag{8.8}$$

where N_2 is a constant of the same type as N_1.

Next, the $L_p(\mathcal{D}_T)$-norm of u can be estimated by rewriting (8.5), like (4.8), as

$$\partial_t u + Lu + b^i D_i u - cu = -G(D^2u, 0, 0, t, x)$$

and using the parabolic Alexandrov estimates. This will lead to an *a priori* estimate (1.4) as in the proof of Theorem 1.2, with N depending also on T. To see that N can be chosen independent of T, we suppose without loss of generality that $\mathcal{D} \subset B_{R/2}$, where $R = 4\,\mathrm{diam}(\mathcal{D})$, and take the barrier function v_0 defined on $\mathbb{R}^d$ in Lemma 11.1.2 of [14]; in B_R this v_0 satisfies

$$v_0 > 0, \quad Lv_0 + b^i D_i v_0 - cv_0 \le -1.$$

Denote $v = u/v_0$. Then $v \in \mathring{W}^{1,2}_p(\mathcal{D}_T)$ and

$$\partial_t v + Lv + \widetilde{b}^i D_i v - \widetilde{c}v = -v_0^{-1}G(D^2(v_0 v), 0, 0, t, x)$$

in $\mathcal{D}_T$, where

$$\widetilde{b}^i = b^i + 2a^{ij}v_0^{-1}D_j v_0, \quad \widetilde{c} = -v_0^{-1}(Lv_0 + b^i D_i v_0 - cv_0).$$

It is easily seen that we can find constants $\widetilde{K} > 0$ and $\nu > 0$ depending only on d, δ, K, and R and such that

$$|\widetilde{b}| \le \widetilde{K}, \quad \nu \le \widetilde{c} \le \widetilde{K}.$$

Then we write $\widetilde{c} = \widehat{c} + \nu$, so that $\widehat{c} \ge 0$. As in the proof of Theorem 2.7 (ii), we have

$$\nu\|v\|_{L_p(\mathcal{D}_T)} \le N(d, \delta, p)\|v_0^{-1}G(D^2(v_0 v), 0, 0, t, x)\|_{L_p(\mathcal{D}_T)},$$

which gives

$$\|u\|_{L_p(\mathcal{D}_T)} \le N(d, \delta, p, K, R)\|G(D^2u, 0, 0, t, x)\|_{L_p(\mathcal{D}_T)}, \tag{8.9}$$

owing to properties of v_0. Combining (8.9) and (8.8), we finish the proof of the *a priori* estimate as in the proof of Theorem 1.2.

With the *a priori* estimate (1.4) in hand, the existence and uniqueness claims are obtained by the same argument as at the end of §4, relying on condition (H_3). The theorem is proved. □

References

[1] M. Bramanti and M. Cerutti, *$W^{1,2}_p$ solvability for the Cauchy–Dirichlet problem for parabolic equations with VMO coefficients*, Comm. Partial Differential Equations **18** (1993), no. 9-10, 1735–1763. MR1239929 (94j:35180)

[2] L. A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2) **130** (1989), 189–213. MR1005611 (90i:35046)

[3] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, Amer. Math. Soc. Colloq. Publ., vol. 43, Amer. Math. Soc., Providence, RI, 1995. MR1351007 (96h:35046)

[4] F. Chiarenza, M. Frasca, and P. Longo, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), no. 1, 149–168. MR1191890 (93k:35051)

[5] ———, *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), no. 2, 841–853. MR1088476 (93f:35232)

[6] M. G. Crandall, M. Kocan, and A. Święch, *L^p-theory for fully nonlinear uniformly parabolic equations*, Comm. Partial Differential Equations **25** (2000), no. 11–12, 1997–2053. MR1789919 (2003b:35093)

[7] Hongjie Dong and Doyoon Kim, *On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients*, Arch. Rational Mech. Anal. **199** (2011), no. 3, 889–941. MR2771670 (2012h:35152)

[8] L. Escauriaza, *$W^{2,n}$ a priori estimates for solutions to fully non-linear equations*, Indiana Univ. Math. J. **42** (1993), no. 2, 413–423. MR1237053 (94h:35022)

[9] Doyoon Kim and N. V. Krylov, *Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others*, SIAM J. Math. Anal. **39** (2007), no. 2, 489–506. MR2338417 (2008j:35031)

[10] ———, *Parabolic equations with measurable coefficients*, Potential Anal. **26** (2007), no. 4, 345–361. MR2300337 (2008f:35161)

[11] N. V. Krylov, *Nonlinear elliptic and parabolic equations second order*, Nauka, Moscow, 1985; English transl., Math. Appl. (Soviet Ser.), vol. 7, D. Reidel Publ. Co., Dordrecht, 1987. MR0815513 (87h:35002); MR0901759 (88d:35005)

[12] ———, *Parabolic and elliptic equations with VMO coefficients*, Comm. Partial Differential Equations **32** (2007), no. 1–3, 453–475. MR2304157 (2008a:35125)

[13] ———, *Parabolic equations with VMO coefficients in Sobolev spaces with mixed norms*, J. Funct. Anal. **250** (2007), no. 2, 521–558. MR2352490 (2008f:35164)

[14] ———, *Lectures on elliptic and parabolic equations in Sobolev spaces*, Grad. Stud. in Math., vol. 96, Amer. Math. Soc., Providence, RI, 2008. MR2435520 (2009k:35001)

[15] ———, *On Bellman's equations with VMO coefficients*, Methods Appl. Anal. **17** (2010), no. 1, 105–121. MR2735102 (2011j:35062)

[16] N. V. Krylov and M. V. Safonov, *A property of solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 1, 161–175; English transl., Math. USSR-Izv. **16** (1981), no. 1, 151–164. MR0563790 (83c:35059)

[17] O. A. Ladyzhenskaya and N. N. Ural′tseva, *Linear and quasilinear elliptic equations*, Nauka, Moscow, 1964; English transl., Acad. Press, New York–London, 1968. MR0211073 (35:1955); MR0244627 (39:5941)

[18] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural′tseva, *Linear and quasilinear parabolic equations*, Nauka, Moscow, 1967; English transl., Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968. MR0241821 (39:3159a); MR0241822 (39:3159b)

[19] G. M. Lieberman, *Regularized distance and its applications*, Pacific J. Math. **117** (1985), no. 2, 329–352. MR0779924 (87j:35101)

[20] ———, *Second order parabolic differential equations*, World Sci. Publ. Co., Inc., River Edge, NJ, 1996. MR1465184 (98k:35003)

[21] Fang-Hua Lin, *Second derivative Lp-estimates for elliptic equations of nondivergent type*, Proc. Amer. Math. Soc. **96** (1986), no. 3, 447–451. MR0822437 (88b:35058)

[22] A. Maugeri, D. Palagachev, and L. Softova, *Elliptic and parabolic equations with discontinuous coefficients*, Math. Res., vol. 109, Wiley–VCH, Berlin, 2000. MR2260015 (2007f:35001)

[23] M. V. Safonov, *Harnack's inequality for elliptic equations and Hölder property of their solutions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **96** (1980), 272–287; English transl., J. Soviet Math. **21** (1983), no. 5, 851–863. MR0579490 (82b:35045)

[24] ———, *Nonlinear elliptic equations of second order*, Lecture Notes Dip. Mat. Appl. "G. Sansone", Univ. Degli Studi Firenze, 1991.

[25] ———, *On the boundary value problems for fully nonlinear elliptic equations of second order*, Math. Res. Report No. MRR 049-94, Austral. Nat. Univ., Canberra, 1994. `http://www.math.umn.edu/~safonov/NOTES/PDE_94/PDE.pdf`

[26] N. S. Trudinger, *Fully nonlinear, uniformly elliptic equations under natural structure conditions*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 751–769. MR0701522 (85b:35016)

[27] L. Wang, *On the regularity theory of fully nonlinear parabolic equations*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), no. 1, 107–114. MR0999619 (90g:35028)

[28] ———, *On the regularity of fully nonlinear parabolic equations.* I, Comm. Pure Appl. Math. **45** (1992), 27–76. MR1135923 (92m:35126)

[29] N. Winter, *$W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations*, Z. Anal. Anwend. **28** (2009), no. 2, 129–164. MR2486925 (2010i:35102)

Division of Applied Mathematics, Brown University, 182 George Street, Providence, Rhode Island 02912
E-mail address: hongjie_dong@brown.edu

University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455
E-mail address: krylov@math.umn.edu

University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455
E-mail address: lixxx489@umn.edu

Received 12/DEC/2010
Originally published in English