ON SPATIAL MAPPINGS WITH INTEGRAL RESTRICTIONS
ON THE CHARACTERISTIC

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Abstract. For a given domain \( D \subset \mathbb{R}^n \), some families \( \mathcal{F} \) of mappings \( f : D \to \mathbb{R}^n \), \( n \geq 2 \) are studied; such families are more general than the mappings with bounded distortion. It is proved that a family is equicontinuous if \( \int_{\delta_0}^{\infty} \frac{dx}{\tau(\Phi^{-1}(\tau))^{\frac{2}{n+1}}} = \infty \), where the integral depends on each mapping \( f \in \mathcal{F} \), \( \Phi \) is a special function, and \( \delta_0 > 0 \) is fixed. Under similar restrictions, removability results are obtained for isolated singularities of \( f \). Also, analogs of the well-known Sokhotsky–Weierstrass and Liouville theorems are proved.

§1. Introduction

In this paper, we use the following notation: \( B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \} \), \( \mathbb{B}^n := B(0, 1) \), \( S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \} \), \( S^{n-1} := S(0, 1) \); \( \Omega_n \) denotes the volume of the unit ball \( \mathbb{B}^n \) of \( \mathbb{R}^n \), \( \omega_{n-1} \) is the area of the sphere \( S^{n-1} \) in \( \mathbb{R}^n \), \( D \) is a domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( m \) is Lebesgue measure on \( \mathbb{R}^n \), \( \text{dist}(A, B) \) is the Euclidean distance between sets \( A, B \subset \mathbb{R}^n \), \( |A| \) denotes the linear measure of a set \( A \subset \mathbb{R} \) whenever there is no confusion. The notation \( f : D \to \mathbb{R}^n \) assumes that the mapping \( f \) is continuous on the domain \( D \). As usual, we write \( f \in W_1^1(D) \) if the coordinate functions \( f = (f_1, \ldots, f_n) \) have the first order generalized partial derivatives the \( n \)th power of which are locally integrable on \( D \). A mapping \( f : D \to \mathbb{R}^n \) is said to be discrete if, for every point \( y \in \mathbb{R}^n \), the preimage \( f^{-1}(y) \) consists of isolated points; the mapping is open if the image of any open set \( U \subset D \) is an open set in \( \mathbb{R}^n \).

Conditions of the following form are important in the theory of spacial mappings:

\[
\int_D \Phi(Q(x)) \, dm(x) < \infty,
\]

where \( \Phi : [0, \infty] \to [0, \infty] \) and \( Q : D \to [0, \infty] \) are fixed measurable functions. Note that properties of the form \( (1) \) arise in various problems; see, e.g., \([12][6][3][13][14][18][20]\) and \([24]\). In the present paper, we investigate how the restriction \( (1) \) can affect the properties of spacial mappings.

First, we give several definitions. Here and in what follows, a curve \( \gamma \) is a continuous mapping of a segment \( [a, b] \) (or an open interval \( (a, b) \)) into \( \mathbb{R}^n \), that is, \( \gamma : [a, b] \to \mathbb{R}^n \). We use the symbol \( \Gamma \) to denote a fixed family of curves \( \gamma \); by definition,

\[
f(\Gamma) = \{ f \circ \gamma \mid \gamma \in \Gamma \}.
\]

The following definitions can be found, for example, in \([22]\) Chapter 1, \( \S\S\S 1–6 \). A Borel function \( \rho : \mathbb{R}^n \to [0, \infty] \) is said to be admissible for a family \( \Gamma \) of curves \( \gamma \) in \( \mathbb{R}^n \) if the inequality \( \int_{\gamma} \rho(x) \, |dx| \geq 1 \) is true for the curvilinear integral of the first kind and for all

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curves $\gamma \in \Gamma$. If this is the case, then we write $\rho \in \text{adm} \Gamma$. The modulus of a family $\Gamma$ of curves is defined by

$$M(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_{D} \rho^{n}(x) \, dm(x).$$

In a sense, the properties of the modulus are similar to those of Lebesgue measure $m$ on $\mathbb{R}^{n}$. Namely, the modulus of the empty family is equal to zero: $M(\emptyset) = 0$; the modulus is monotone with respect to families of curves: $\Gamma_{1} \subset \Gamma_{2} \Rightarrow M(\Gamma_{1}) \leq M(\Gamma_{2})$. Also, the modulus is semiadditive: $M(\bigcup_{i=1}^{\infty} \Gamma_{i}) \leq \sum_{i=1}^{\infty} M(\Gamma_{i})$; see [22] Theorem 6.2. Recall that $f : D \rightarrow \mathbb{R}^{n}$ is called a mapping with bounded distortion if the following conditions are satisfied: 1) $f \in W^{1,1}_{\text{loc}}$; 2) the Jacobian $J(x, f)$ of the mapping $f$ at $x \in D$ has one and the same sign almost everywhere in $D$; 3) $\|f'(x)\|^{n} \leq K \cdot |J(x, f)|$ for almost all $x \in D$ and for some constant $K < \infty$, where, as usual, $\|f'(x)\| := \sup_{h \in \mathbb{R}^{n} : |h| = 1} |f'(x)h|$ (see, e.g., [16] Chapter I, §3 or [17] Chapter I, §2, Definition 2.1). For us, it is important that each mapping with bounded distortion satisfies the inequality of E. A. Poletsky (see [15] §4, Theorem 1)). Namely, if $f : D \rightarrow \mathbb{R}^{n}$ is a mapping with bounded distortion, then

$$M(f(\Gamma)) \leq K' \cdot M(\Gamma)$$

for an arbitrary family $\Gamma$ of curves $\gamma$ in the domain $D$, where $M$ is the conformal modulus introduced above for the family of curves (i.e., an outer measure defined on the family of curves in $\mathbb{R}^{n}$) and $K' < \infty$ is a constant (see also [17] Chapter II, §8, Theorem 8.1). A homeomorphism $f : D \rightarrow \mathbb{R}^{n}$ or, depending on the context, a homeomorphism $f : D \rightarrow \mathbb{R}^{n}$, $\mathbb{R}^{n} = \mathbb{R}^{n} \cup \{\infty\}$, is called a $K'$-quasiconformal mapping on $D$ if $f$ satisfies (3) for an arbitrary family $\Gamma$ of curves $\gamma$ (see, e.g., [22] Chapter 2, §13; Chapter IV, Theorem 34.3).

In the present paper, we investigate mappings $f$ that are more general than those with bounded distortion. More precisely, given a measurable function $Q : D \rightarrow [0, \infty]$ that maps a domain $D$ in $\mathbb{R}^{n}$ into the real numbers, instead of (3) we assume that

$$M(f(\Gamma(S_{1}, S_{2}, A))) \leq \int_{A} Q(x) \cdot \eta^{n}(|x - x_{0}|) \, dm(x),$$

where

$$A = A(r_{1}, r_{2}, x_{0}) = \{x \in \mathbb{R}^{n} : r_{1} < |x - x_{0}| < r_{2}\}$$

denotes the spherical ring with center $x_{0}$ and with radii $r_{1}$, $r_{2}$; $S_{i} = S(x_{0}, r_{i})$ denotes the sphere with center $x_{0}$ and with radius $r_{i}$, $i = 1, 2$; $\Gamma(S_{1}, S_{2}, A)$ denotes the family of all curves joining $S_{1}$ and $S_{2}$ inside the domain $A$. The real-valued function $\eta : (r_{1}, r_{2}) \rightarrow [0, \infty]$ in inequality (4) is assumed to be measurable and to satisfy the condition

$$\int_{r_{1}}^{r_{2}} \eta(r) \, dr \geq 1.$$

See, e.g., [3] [10] [11] [12] and [20] for further details about inequalities of type (4). Note that, in the case where $Q(x) \leq K$ a.e., an expression like $K \cdot M(\Gamma(S_{1}, S_{2}, A))$ arises on the right-hand side of inequality (4), see inequalities (4) and (3). It follows that an arbitrary mapping with bounded distortion satisfies (4) for a constant $Q$. In general, if the function $Q(x)$ is merely measurable, then the right-hand side of inequality (4) is, in a sense, the modulus $M$ of the family $\Gamma(S_{1}, S_{2}, A)$ of curves with weight $Q(x)$. We are primarily interested in the case where the function $Q$ in (4) is unbounded. Suppose that an open discrete mapping $f$ satisfies inequality (4) and condition (1) is fulfilled for the function $Q(x)$, where $\Phi$ is a function such that

$$\int_{\delta_{0}}^{\infty} \frac{d\tau}{\tau^{\Phi^{-1}(\tau)^{-1}}} = \infty$$
for some \( \delta_0 > 0 \). In the present paper, we show that the mapping \( f \) extends by continuity to an isolated point of the boundary of \( D \). Also we prove the equicontinuity of the families under consideration, assuming that \( f \) satisfies (1), where the function \( Q(x) \) satisfies (1), and the function \( \Phi \) occurring in (1) satisfies condition (1). Also, we shall show that the restriction on the function \( \Phi \) mentioned above, that is, property (7), is not only sufficient, but, in an appropriate sense, also necessary. We assume that \( f \) is an open and discrete mapping, \( Q \) is merely a measurable function, and \( \Phi \) is a monotone nondecreasing and convex function.

We say that \( f : D \to \mathbb{R}^n \) is a ring \( Q \)-mapping at a point \( x_0 \in D \) if \( f \) satisfies (4) for any annulus \( A = A(r_1, r_2, x_0) \), \( 0 < r_1 < r_2 < r_0 = \text{dist}(x_0, \partial D) \), and for every measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) with property (8). If \( f \) is a homeomorphism, then \( f \) is called a ring \( Q \)-homeomorphism at the point \( x_0 \). Finally, \( f \) is called a ring \( Q \)-mapping (a ring \( Q \)-homeomorphism, respectively) if (4) is true at each point \( x_0 \in D \). Similarly, in the studies of questions related to the boundary behavior of mappings, one may introduce the notion of a ring \( Q \)-mapping \( f : D \setminus \{ x_0 \} \to \mathbb{R}^n \) that is defined \( a \) priori in a neighborhood of an isolated point \( x_0 \) of the boundary \( D \setminus \{ x_0 \} \).

Let \( E \) be a compact set in \( \mathbb{R}^n \) with a positive conformal capacity: \( \text{cap} \, E > 0 \). Assume that we are given a function \( \Phi : [0, \infty] \to [0, \infty] \), a Lebesgue measurable function \( Q : D \to [0, \infty] \), and a number \( M > 0 \). Denote by \( \mathcal{R}^{\Phi, Q}_{D, E} \) the family of all open discrete ring \( Q \)-mappings \( f : D \to \mathbb{R}^n \setminus E \) such that

\[
\int_D \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n} \leq M.
\]

To mention explicitly the domain \( D \), we sometimes use the notation \( \mathcal{R}^{\Phi, Q}_{D, E}(D) \). The following assertions are among the main results of the present paper.

**Theorem 1.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a monotone nondecreasing convex function. If \( (\text{1}) \) is true for some \( \delta_0 > \tau_0 := \Phi(0) \), then the class \( \mathcal{R}^{\Phi, Q}_{D, E} \) is equicontinuous; hence, it forms a normal family of mappings for all \( M \in (0, \infty) \).

For a Lebesgue measurable function \( Q : D \to [0, \infty] \) and a compact set \( E \subset \mathbb{R}^n \) with \( \text{cap} \, E > 0 \), denote by \( K^{\Phi, Q}_{D, E}(D \setminus \{ x_0 \}) \) the family of all open and discrete \( Q \)-mappings \( f : D \setminus \{ x_0 \} \to \mathbb{R}^n \setminus E \) that are ring at the point \( x_0 \) and satisfy (5).

**Theorem 2.** Assume that \( Q(x) \geq 1 \) almost everywhere, \( \Phi : [0, \infty] \to [0, \infty] \) is a monotone nondecreasing convex function, and \( x_0 \in D \). If (7) is true for some \( \delta_0 > \tau_0 := \Phi(0) \), then an arbitrary mapping \( f : D \setminus \{ x_0 \} \to \mathbb{R}^n \), \( f \in K^{\Phi, Q}_{D, E}(D \setminus \{ x_0 \}) \), extends by continuity to the point \( x_0 \in D \) up to an open discrete mapping \( f : D \to \mathbb{R}^n \).

**Theorem 3.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a monotone nondecreasing convex function. Assume that, for all sets \( E \) of positive capacity, all numbers \( M > 0 \), and all measurable functions \( Q : D \to [0, \infty] \), the class \( \mathcal{R}^{\Phi, Q}_{D, E} \) is equicontinuous (normal). Then, for all \( \delta_0 \in (\tau_0, \infty) \), \( \tau_0 := \Phi(0) \), we have

\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau^{\frac{1}{\Phi^{-1}(\tau)}}} = \infty.
\]

**Theorem 4.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a monotone nondecreasing convex function. Assume that, for all sets \( E \) of positive capacity, all numbers \( M > 0 \), and all measurable functions \( Q : D \to [0, \infty] \), an arbitrary mapping \( f : D \setminus \{ x_0 \} \to \mathbb{R}^n \), \( x_0 \in D \), \( f \in K^{\Phi, Q}_{D, E}(D \setminus \{ x_0 \}) \), extends by continuity to the point \( x_0 \). Then condition (9) is fulfilled for all \( \delta_0 \in (\tau_0, \infty) \), \( \tau_0 := \Phi(0) \).
Remark 1. Observe that the condition \( \int_D \Phi(Q(x)) \, dm(x) \leq M \) implies property (\( \mathbb{S} \)). Hence, property (\( \mathbb{S} \)) is more general, and the corresponding class of ring \( Q \)-mappings is a subclass of the family \( \mathcal{R}^{I,P,E} \). On the other hand, if the domain \( D \) is bounded, then (\( \mathbb{S} \)) implies the condition \( \int_D \Phi(Q(x)) \, dm(x) \leq M_*, \) where \( M_* = M \cdot (1 + \delta^2)^n \), \( \delta_* = \sup_{x \in D} |x| \).

In the next section, we accurately introduce the main definitions and notation used above.

\[ \text{§2. Preliminaries} \]

By definition, a condenser in \( \mathbb{R}^n \), \( n \geq 2 \), is a pair \( E = (A,C) \), where \( A \) is an open set in \( \mathbb{R}^n \) and \( C \) is a compact subset of \( A \). A mapping \( f : D \rightarrow \mathbb{R}^n \) is said to be absolutely continuous on lines, \( f \in \text{ACL} \), if in any \( n \)-dimensional parallelepiped \( P \) with edges parallel to the coordinate axes, \( \bar{P} \subset D \), all coordinate functions \( f = (f_1, \ldots, f_n) \) are absolutely continuous on almost all lines that are parallel to the coordinate axes. By the conformal capacity (or merely capacity) of a condenser \( E \) we mean the following quantity:

\[
\cap E = \cap(A,C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^n \, dm(x),
\]

where \( W_0(E) = W_0(A,C) \) is the family of all nonnegative continuous functions \( u : A \rightarrow \mathbb{R} \) compactly supported in \( A \) and such that \( u(x) \geq 1 \) for \( x \in C \) and \( u \in \text{ACL} \). We say that a compact set \( C \) in \( \mathbb{R}^n \), \( n \geq 2 \), is of zero capacity and we write \( \cap C = 0 \) if \( \cap(A,C) = 0 \) for at least one bounded open set \( A \) that contains \( C \). Otherwise, \( \cap C > 0 \).

It is known that the sets of capacity zero are discontinuous; see, e.g., Corollary 2.5 in Chapter 3 of [17]. In particular, the condition \( \cap C = 0 \) implies that \( \text{Int} C = \emptyset \). We say that a set \( A \) in \( \mathbb{R}^n \) is of capacity zero if an arbitrary compact subset of \( A \) is of capacity zero. The notions of a condenser and of a set of capacity zero in \( \mathbb{R}^n \) are defined similarly (see, e.g., [17]).

Let \((X,d)\) and \((X',d')\) be metric spaces with distances \( d, d' \), respectively. We say that a family \( \mathcal{F} \) of continuous mappings \( f : X \rightarrow X' \) is normal if the following is true: for any sequence of mappings \( f_n \in \mathcal{F} \), there is a subsequence \( f_{n_k} \) that converges locally uniformly on \( X \) to a continuous function \( f : X \rightarrow X' \). The above notion is closely related with the following one: a family \( \mathcal{F} \) of mappings \( f : X \rightarrow X' \) is equicontinuous at a point \( x_0 \in X \) if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d'(f(x), f(x_0)) < \varepsilon \) for all \( x \) such that \( d(x, x_0) < \delta \). We say that \( \mathcal{F} \) is equicontinuous if \( \mathcal{F} \) is equicontinuous at each point of \( X \). If \((X,d)\) is a separable metric space and \((X',d')\) is a compact metric space, then a family \( \mathcal{F} \) of mappings \( f : X \rightarrow X' \) is normal if and only if \( \mathcal{F} \) is equicontinuous. This is a version of the well-known Arzelà–Ascoli theorem (see, e.g., [22 §20.4]).

Recall that an isolated point \( x_0 \) of the boundary \( \partial D \) of a domain \( D \) is said to be removable for a mapping \( f \) if there exists a finite limit \( \lim_{x \to x_0} f(x) \). An isolated point \( x_0 \) of the boundary \( \partial D \) is called an essential singular point of a mapping \( f : D \rightarrow \mathbb{R}^n \) if the mapping has neither finite, nor infinite limit as \( x \to x_0 \).

Suppose that a mapping \( f : D \rightarrow \mathbb{R}^n \), \( n \geq 2 \), has the first partial derivatives for almost all \( x \in D \). Then the inner dilatation of \( f \) at a point \( x \in D \) is the quantity

\[
K_I(x,f) = \left| \frac{J(x,f)}{l(f'(x))} \right|^n
\]

if \( J(x,f) \neq 0 \), \( K_I(x,f) = 1 \) if \( f'(x) = 0 \), and \( K_I(x,f) = \infty \) for other points. Here \( l(f'(x)) = \inf_{h \in \mathbb{R}^n, : |h| = 1} |f'(x)h| \). We shall need the following proposition.
Proposition 1. Let \( f : D \to \mathbb{R}^n \), \( f \in W_{1}^{1,n} \), be a homeomorphism. If \( K_I(x, f) \in L_{1}^{1,\text{loc}} \), then \( f \) is a ring \( Q \)-homeomorphism at each point \( x_0 \in D \) with \( Q = K_I(x, f) \); see, e.g., [11] Theorems 8.1 and 8.6]. Moreover, \( f \) is a ring \( Q \)-mapping at every isolated point of the boundary of \( D \); see op. cit.

To compute inner dilatations, sometimes it is useful to apply the quantities introduced below. Assume that a mapping \( f : D \to \mathbb{R}^n \) is differentiable at a point \( x_0 \in D \) and the Jacobi matrix \( f'(x_0) \) is nonsingular: \( J(x_0, f) = \det f'(x_0) \neq 0 \). Then there exist vector systems \( e_1, \ldots, e_n \) and \( \tilde{e}_1, \ldots, \tilde{e}_n \) as well as positive numbers \( \lambda_1(x_0), \ldots, \lambda_n(x_0) \) with \( \lambda_1(x_0) \leq \cdots \leq \lambda_n(x_0) \) such that \( f'(x_0)e_i = \lambda_i(x_0)\tilde{e}_i \); see Theorem 2.1 in Chapter 1 of [16]. Moreover, \( \lambda^2_1(x_0), \ldots, \lambda^2_n(x_0) \) are the eigenvalues of the symmetric mapping \( f'(x_0)^T f'(x_0) \), see Theorem 2.2 in Chapter 1 of [16], and

\[
|J(x_0, f)| = \lambda_1(x_0) \cdots \lambda_n(x_0), \quad \|f'(x_0)\| = \lambda_n(x_0), \quad l(f'(x_0)) = \lambda_1(x_0),
\]

(10)

\[
K_I(x_0, f) = \frac{\lambda_1(x_0) \cdots \lambda_n(x_0)}{\lambda^2_1(x_0)};
\]

see property (2.5) and additional comments in §2.1 in Chapter 1 of [16]. The numbers \( \lambda_1(x_0), \ldots, \lambda_n(x_0) \) mentioned above are called the principal values, and the vectors \( e_1, \ldots, e_n \) and \( \tilde{e}_1, \ldots, \tilde{e}_n \) are called the principal vectors of the mapping \( f'(x_0) \). More details on this issue can be found in the corresponding comment after the proof of Theorem 2.2 in Chapter 1 of [16]. Clearly, the principal vectors and principal values depend on the point \( x_0 \) and on the mapping \( f \). However, to simplify the notation here and in what follows, we omit \( (x_0) \) if there is no confusion.

Recall that the spherical (chordal) metric \( h(x, y) \) is equal to \( |\pi(x) - \pi(y)| \), where \( \pi \) is the stereographic projection of \( \mathbb{R}^n \) onto the sphere \( S^n(\frac{1}{2}e_{n+1}, \frac{1}{2}) \) in \( \mathbb{R}^{n+1} \), that is,

\[
h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}, \quad h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y.
\]

Let \( Q : D \to [0, \infty) \) be a Lebesgue measurable function. Then \( q_{x_0}(r) \) denotes the integral mean of \( Q(x) \) over the sphere \( |x - x_0| = r \):

\[
q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q(x) dS,
\]

(11)

where \( dS \) is the area element of the surface \( S \). Below we agree that \( a/\infty = 0 \) for \( a \neq \infty \), \( a/0 = \infty \) for \( a > 0 \), and \( 0 \cdot \infty = 0 \).

The following results were formulated and proved in [25].

Proposition 2. Let \( Q : \mathbb{B}^n \to [1, \infty] \) be a Lebesgue measurable function, and let \( f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n \), \( n \geq 2 \), be an open, discrete, ring \( Q \)-mapping at the point \( x_0 = 0 \) such that \( \text{cap}(\mathbb{R}^n \setminus f(\mathbb{B}^n \setminus \{0\})) > 0 \). Assume that there exists \( \varepsilon_0 > 0 \), \( \varepsilon_0 < 1 \), such that

\[
\int_0^{\varepsilon_0} dt \frac{1}{t q_{x_0}(t)} = \infty.
\]

Then \( f \) has a continuous extension onto \( \mathbb{B}^n \), that is, \( f : \mathbb{B}^n \to \mathbb{R}^n \). Continuity is understood in the sense of the space \( \mathbb{R}^n \) with respect to the chordal metric \( h \). Moreover, the extended mapping \( f : \mathbb{B}^n \to \mathbb{R}^n \) is open and discrete; see Theorem 6 in [25].

In what follows, equicontinuity, normality, and other notions are considered with respect to the spherical (chordal) metric \( h \).

Proposition 3. Assume that \( x_0 \in D, E \subset \mathbb{R}^n \) is a compact set of positive capacity, and \( \mathcal{F}_Q \) is a family of open, discrete, ring \( Q \)-mappings \( f : D \to \mathbb{R}^n \setminus E \) at the point \( x_0 \).
Assume that there exists a number \( \varepsilon_0 < \text{dist}(x_0, \partial D) \) such that (12) is true. Then the family \( \mathfrak{F}_Q \) is equicontinuous and, hence, normal at the point \( x_0 \).

Recall that a function \( \Phi : [0, \infty] \to [0, \infty] \) is said to be convex if
\[
\Phi(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda \Phi(t_1) + (1 - \lambda)\Phi(t_2)
\]
for all \( t_1, t_2 \in [0, \infty] \) and \( \lambda \in [0, 1] \). The inverse function \( \Phi^{-1} \) is well defined for any monotone nondecreasing function \( \Phi : [0, \infty] \to [0, \infty] \) as follows:
\[
\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t.
\]
As usual, the infimum in (13) is equal to \( \infty \) if there is no \( t \in [0, \infty] \) for which \( \Phi(t) \geq \tau \).

Remark 2. Obviously, the above definition shows that
\[
\Phi^{-1}(\Phi(t)) \leq t, \quad t \in [0, \infty].
\]
Moreover, equality occurs in (14) outside of the intervals where \( \Phi(t) \) is constant.

For the proposition formulated below, see, e.g., [19, Theorem 2.1]. In (15) and (16), the integrals are assumed to be equal to infinity if \( \Phi_p(t) = \infty \) or if there exists \( T \in \mathbb{R} \) such that \( H_p(t) = \infty \) for all \( t \geq T \in [0, \infty) \). The integral in (16) is that of Lebesgue–Stieltjes, the integrals in (15) and in (17)–(20) are those of Lebesgue.

**Proposition 4.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a monotone nondecreasing function. Put \( H_p(t) = \log \Phi_p(t) \), \( \Phi_p(t) = \Phi(t^p) \), \( p \in (0, \infty) \). Then the relation
\[
\int_\delta^\infty \frac{H'_p(t)}{t} \, dt = \infty
\]
implies
\[
\int_\delta^\infty \frac{dH_p(t)}{t} = \infty.
\]
Identity (16) is equivalent to the following property:
\[
\int_\delta^\infty H_p(t) \frac{dt}{t^2} = \infty
\]
for some \( \delta > 0 \). Also, (17) is equivalent to each of the following relations:
\[
\int_0^\Delta H_p\left(\frac{1}{t}\right) \, dt = \infty
\]
for some \( \Delta > 0 \);
\[
\int_\delta^{\delta_*} \frac{d\eta}{H_p^{-1}(\eta)} = \infty
\]
for some \( \delta_* > H(+0) \);
\[
\int_\delta^{\delta_*} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} = \infty
\]
for some \( \delta_* > \Phi(+0) \).

Moreover, (15) is equivalent to (16); hence, properties (15)–(20) are equivalent to each other if \( \Phi \) is additionally assumed to be absolutely continuous. In particular, all conditions (15)–(20) are equivalent to each other whenever \( \Phi \) is convex and monotone nondecreasing.
It is easily seen that conditions (15)–(20) become weaker if \( p \) grows; see, e.g., (17). Yet one more clarification is necessary. The right-hand sides of conditions (15)–(20) should be understood as +\( \infty \). If \( \Phi_p(t) = 0 \) for \( t \in [0, t_0] \), then \( H_p(t) = -\infty \) for \( t \in [0, t_0] \); if this is the case, we put \( H_p'(t) := 0 \) for \( t \in [0, t_0] \). Note that conditions (16) and (17) exclude the case where \( t_0 \) belongs to the interval of integration in the above properties. Indeed, otherwise the left-hand sides of (16) and (17) are either simultaneously equal to \( -\infty \) or not defined. Hence, in (15)–(17) we may assume that \( \delta > t_0 \) and \( \Delta < 1/t_0 \) where \( t_0 := \sup_{\Phi_p(t) = 0} t \) and \( t_0 = 0 \) if \( \Phi_p(0) > 0 \).

\[\text{§3. Main lemma}\]

The following assertion generalizes and refines Lemma 3.1 in [19].

**Lemma 1.** Let \( Q : \mathbb{B}^n \to [0, \infty) \) be a measurable function, and let \( \Phi : [0, \infty) \to (0, \infty) \) be a monotone nondecreasing convex function. Then

\[
\int_\varepsilon^1 \frac{dr}{r q_0^\frac{1}{n}(r)} \geq \frac{1}{n} \int_{eM(\varepsilon)}^{M(\varepsilon)} \frac{d\tau}{\tau^{\Phi^{-1}(\tau)}} \tag{21}, \quad p \in (0, \infty), \quad \varepsilon \in (0, 1),
\]

where \( q_0(r) \) is defined by (11) for \( x_0 = 0 \), and

\[
M(\varepsilon) = \frac{1}{\Omega_n(1 - \varepsilon^n)} \int_{A(\varepsilon, 1, 0)} \Phi(Q(x)) \, dm(x)
\]

is the integral mean of the function \( \Phi \circ Q \) on the annulus \( A(\varepsilon, 1, 0) \) defined by (5) with \( x_0 = 0, r_1 = \varepsilon \) and \( r_2 = 1 \).

**Remark 3.** Observe that, for every \( p \in (0, \infty) \), property (21) is equivalent to the following inequality:

\[
\int_\varepsilon^1 \frac{dr}{r q_0^\frac{1}{n}(r)} \geq \frac{1}{n} \int_{eM(\varepsilon)}^{M(\varepsilon)} \frac{d\tau}{\tau^{\Phi^{-1}(\tau^p)}}, \quad \Phi_p(t) := \Phi(t^p).
\]

**Proof of Lemma 1.**

**Step 1.** Denote

\[
t_0 = \sup_{\Phi_p(t) = \tau_0} t, \quad \tau_0 = \Phi(0) > 0.
\]

Putting \( H_p(t) := \log \Phi_p(t) \), we see that

\[
H_p^{-1}(\eta) = \Phi_p^{-1}(e^\eta), \quad \Phi_p^{-1}(\tau) = H_p^{-1}(\log \tau).
\]

Hence, using Remark 2 and setting

\[
h(r) := r^n \Phi(q_0(r)) = r^n \Phi_pq_0^\frac{1}{n}(r), \quad R_* = \{ r \in (\varepsilon, 1) : q_0^\frac{1}{n}(r) > t_* \},
\]

we obtain

\[
q_0^\frac{1}{n}(r) = H_p^{-1}\left( \log \frac{h(r)}{r^n} \right) = H_p^{-1}\left( n \log \frac{1}{r} + \log h(r) \right), \quad r \in R_*.
\]

Then we also have

\[
q_0^\frac{1}{n}(e^{-s}) = H_p^{-1}(ns + \log h(e^{-s})), \quad s \in S_*,
\]

where

\[
S_* = \left\{ s \in \left(0, \log \frac{1}{\varepsilon}\right) : q_0^\frac{1}{n}(e^{-s}) > t_* \right\}.
\]
Step 2. Applying Jensen’s inequality and the convexity of the function $\Phi$, we have
\[
\int_0^{\log \frac{1}{\varepsilon}} h(e^{-s}) ds = \int_{\varepsilon}^{1} h(r) \frac{dr}{r} = \int_{\varepsilon}^{1} \Phi(q_0(r)) r^{n-1} dr
\]
where $M(\varepsilon)$ is defined by (22) and, as above, $ds$ is the area element of the surface of integration. Hence, (28) implies that the linear measure $|T|$ of the set $T = \{s \in (0, \log \frac{1}{\varepsilon}) : h(e^{-s}) > M(\varepsilon)\}$ satisfies
\[
|T| = \int_{T} ds \leq \frac{1}{n}.
\]
Step 3. We show that
\[
q_0(e^{-s}) \leq H_{p}^{-1}(ns + \log M(\varepsilon)), \quad s \in \left(0, \log \frac{1}{\varepsilon}\right) \setminus T_s,
\]
where $T_s := T \cap S_s$ and the set $S_s$ is defined by (27). Observe that
\[
\left(0, \log \frac{1}{\varepsilon}\right) \setminus T_s = \left(\left(0, \log \frac{1}{\varepsilon}\right) \setminus S_s\right) \cup \left(\left(0, \log \frac{1}{\varepsilon}\right) \setminus T\right) = \left(\left(0, \log \frac{1}{\varepsilon}\right) \setminus S_s\right) \cup \left[S_s \setminus T\right].
\]
Remark that inequality (30) is true for $s \in S_s \setminus T$, because (26) is true and the function $H_{p}^{-1}$ is monotone nondecreasing. Also, by (24),
\[
e^{ns} M(\varepsilon) > \Phi(0) + \varepsilon_1 = \tau_0 + \varepsilon_1, \quad s \in \left(0, \log \frac{1}{\varepsilon}\right),
\]
for some sufficiently small $\varepsilon_1 > 0$. Thus, since the function $H_{p}^{-1}(\eta)$ is strictly monotone increasing for $\eta > \tau_0$, (25) and (31) guarantee that
\[
t_s < \Phi_{p}^{-1}\left(e^{ns} M(\varepsilon)\right) = H_{p}^{-1}(ns + \log M(\varepsilon)), \quad s \in \left(0, \log \frac{1}{\varepsilon}\right).
\]
Relation (32) shows that (30) is also valid for $s \in \left(0, \log \frac{1}{\varepsilon}\right) \setminus S_s$, and hence, for all $s \in \left(0, \log \frac{1}{\varepsilon}\right) \setminus T_s$.

Step 4. Since the function $H_{p}^{-1}$ is monotone nondecreasing, (29) and (30) guarantee that
\[
\int_{\varepsilon}^{1} \frac{dr}{r q_0^\frac{1}{n}(r)} = \int_{0}^{\log \frac{1}{\varepsilon}} \frac{ds}{q_0^\frac{1}{n}(e^{-s})} \geq \int_{\left(0, \log \frac{1}{\varepsilon}\right) \setminus T_s} \frac{ds}{H_{p}^{-1}(ns + \log M(\varepsilon))}
\]
\[
\geq \frac{1}{n} \int_{|T_s|}^{\log \frac{1}{\varepsilon}} \frac{ds}{H_{p}^{-1}(ns + \log M(\varepsilon))} \geq \frac{1}{n} \int_{1 + \log M(\varepsilon)}^{\log \frac{1}{\varepsilon}} \frac{d\eta}{H_{p}^{-1}(\eta)}
\]
Note that $1 + \log M(\varepsilon) = \log e M(\varepsilon)$. Therefore, by (33),
\[
\int_{\varepsilon}^{1} \frac{dr}{r q_0^\frac{1}{n}(r)} \geq \frac{1}{n} \int_{\log \varepsilon M(\varepsilon)}^{\log \frac{M(\varepsilon)}{e n}} \frac{d\eta}{H_{p}^{-1}(\eta)}.
\]
Introducing the new variable $\eta = \log \tau$ and applying (25), we obtain inequality (23). Hence, (21) is true. \[\Box\]
Theorem 5. Let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that
\begin{equation}
\int_{\mathbb{R}^n} \Phi(Q(x)) \, dm(x) < \infty,
\end{equation}
where $\Phi : [0, \infty] \to [0, \infty]$ is a monotone nondecreasing convex function such that
\begin{equation}
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \frac{1}{p} = \infty, \quad p \in (0, \infty),
\end{equation}
for some $\delta_0 > \tau_0 := \Phi(0)$. Then
\begin{equation}
\int_{0}^{1} \frac{dr}{r q^\frac{1}{p}(r)} = \infty.
\end{equation}

Remark 4. Observe that, in Theorem 5, the assumptions about $\Phi$ are more general, namely, $\Phi$ is allowed to take the value 0. Indeed, the above case reduces to the function $\Phi : [0, \infty] \to (0, \infty]$ by consideration of the following auxiliary function:
\begin{equation}
\Phi_*(t) = \begin{cases} 
\Phi(t) & \text{if } \Phi(t) > \delta, \\
\delta & \text{if } \Phi(t) \leq \delta,
\end{cases}
\end{equation}
where $\delta$ is an arbitrary number such that $\delta \in (0, \delta_0)$. Observe that properties (34) and (35) remain valid for $\Phi_*(t)$.

Remark 5. We have $[\Phi^{-1}(\tau)]^\frac{1}{p} = \Phi^{-1}_p(\tau)$, where $\Phi_p(t) = \Phi(t^p)$; hence, (35) implies that
\begin{equation}
\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_p(\tau)} = \infty, \quad \delta \in [0, \infty).
\end{equation}
On the other hand, property (37) with some $\delta \in [0, \infty)$ does not imply (35), in general.

Indeed, property (35) with some $\delta_0 > \tau_0$ clearly implies (37) for $\delta \in (0, \delta_0)$. Also, for $\delta \in (\delta_0, \infty)$ we have
\begin{equation}
0 \leq \int_{\delta_0}^{\delta} \frac{d\tau}{\tau \Phi^{-1}_p(\tau)} \leq \frac{1}{\Phi^{-1}_p(\delta_0)} \log \frac{\delta}{\delta_0} < \infty,
\end{equation}
because the function $\Phi^{-1}_p$ is monotone nondecreasing and $\Phi^{-1}_p(\delta_0) > 0$. Therefore, (37) is true for all $\delta \in [0, \infty)$.

On the other hand, by the definition of the inverse function, $\Phi^{-1}_p(\tau) \equiv 0$ for all $\tau \in [0, \tau_0]$ with $\tau_0 = \Phi_p(0)$. Thus, in general, condition (37) with $\delta \in [0, \tau_0)$ does not imply (35). If $\tau_0 > 0$, then
\begin{equation}
\int_{\delta}^{\tau_0} \frac{d\tau}{\tau \Phi^{-1}_p(\tau)} = \infty, \quad \delta \in [0, \tau_0).
\end{equation}
However, property (39) contains no proper information about the function $Q(x)$. Hence, condition (37) with $\delta < \Phi(0)$ definitely does not imply (35).

By property (37), the proof of Theorem 5 reduces to the claim of Lemma 1.

\section{Conditions sufficient for equicontinuity}

Proof of Theorem 1. Without loss of generality, we may assume that $\Phi(0) > 0$. By Proposition 3, it suffices to show that the assumptions of Theorem 1 imply property (12) for every $x_0 \in D$ and for some $\varepsilon_0 < \text{dist}(x_0, \partial D)$. Introducing the variable $t = r/\varepsilon_0$, we obtain
\begin{equation}
\int_{\varepsilon}^{\varepsilon_0} \frac{dr}{r q^{\frac{1}{p-1}}(r)} = \int_{\varepsilon/\varepsilon_0}^{1} \frac{dt}{t q^{\frac{1}{p-1}}(t \varepsilon_0)} = \int_{\varepsilon/\varepsilon_0}^{1} \frac{dt}{t q^{\frac{1}{p-1}}(t)}.
\end{equation}
where \( \tilde{q}_0(t) \) is the integral mean value of the function \( \tilde{Q}(x) := Q(\varepsilon_0 x + x_0) \) on the sphere \( |x| = t \), see (11) for the definition. Now, applying Lemma 1 with \( p = n - 1 \), we obtain

\[
\int_0^1 \frac{dt}{t \tilde{q}_0^{-1}(t)} \geq \frac{1}{n} \int_{M_+(\varepsilon/\varepsilon_0)^n} \frac{d\tau}{\tau^{\Phi^{-1}(\tau)^{n-1}}},
\]

\[
M_+(\varepsilon/\varepsilon_0) = \frac{1}{\Omega_n(1 - (\varepsilon/\varepsilon_0)^n)} \int_{A(\varepsilon/\varepsilon_0,1,0)} \Phi(Q(\varepsilon_0 x + x_0)) \, dm(x)
\]

\[
= \frac{1}{\Omega_n(\varepsilon_0^n - \varepsilon^n)} \int_{A(\varepsilon,\varepsilon_0,0)} \Phi(Q(x)) \, dm(x).
\]

Observe that if \( x \in A(\varepsilon, \varepsilon_0, x_0) \), then \( |x| \leq |x - x_0| + |x_0| \leq \varepsilon_0 + |x_0| \); thus, (42) implies that

\[
M_+(\varepsilon/\varepsilon_0) \leq \frac{\beta_n(x_0)}{\Omega_n} \int_{A(\varepsilon,\varepsilon_0,0)} \Phi(Q(x)) \, dm(x),
\]

where \( \beta_n(x_0) = (1 + (\varepsilon + |x_0|)^2)^n \). Hence, for \( \varepsilon \leq 1/\sqrt{\varepsilon_0^n - 1/2} \) we have

\[
M_+(\varepsilon/\varepsilon_0) \leq \frac{2\beta_n(x_0)}{\Omega_n} M,
\]

where \( M \) is the constant on the right-hand side of (8). Also, observe that

\[
M_+(\varepsilon/\varepsilon_0) > \Phi(0) > 0.
\]

Therefore, by (40) and (41),

\[
\int_0^\varepsilon \frac{dr}{r q_0^{-\frac{1}{n}}(r)} \geq \frac{1}{n} \int_0^{\Phi(0) M} \frac{d\tau}{\tau^{\Phi^{-1}(\tau)^{n-1}}}.
\]

However, the right-hand side of the above inequality tends to infinity by (7); hence, \( \int_0^\varepsilon \frac{dr}{r q_0^{-\frac{1}{n}}(r)} = \infty \). Now, the required conclusion follows from Proposition 3. The theorem is proved. \( \square \)

**Corollary 1.** Each of conditions (15) - (20) with \( p \in (0, n - 1] \) implies the equicontinuity and normality of the class \( \mathcal{H}_{M,E}^{q_0,\Phi} \) for all \( M \in (0, \infty) \).

### §5. Conditions Sufficient for the Removability of Singularities

The proof of Theorem 2 is quite similar to that of Theorem 1; namely, first we verify that the integral \( \int_0^\varepsilon \frac{dr}{r q_0^{-\frac{1}{n}}(r)} \) diverges. Second, we apply Proposition 2. \( \square \)

For a function \( f : D \to \mathbb{R}^n \), a set \( E \subset D \), and a point \( y \in \mathbb{R}^n \), we define the multiplicity function \( N(y, f, E) \) as the number of preimages of \( y \) in the set \( E \), that is,

\[
N(y, f, E) = \text{card} \{ x \in E : f(x) = y \}.
\]

**Corollary 2** (a Sokhotsky–Weierstrass type theorem). Let \( \Phi : [0, \infty) \to [0, \infty] \) be a monotone nondecreasing convex function. Assume that property (7) is fulfilled for some \( \delta_0 > \delta_0 := \Phi(0) \), and that an open discrete \( Q \)-mapping \( f : D \setminus \{ x_0 \} \to \mathbb{R}^n \) has an essential singular point \( x_0 \in D \). Suppose \( Q \) satisfies condition (8), in the domain \( D \) and \( Q(x) \geq 1 \) almost everywhere. Then there exists an \( F_\sigma \)-set \( C \subset \mathbb{R}^n \) of zero capacity and such that

\[
N(y, f, U \setminus \{ x_0 \}) = \infty
\]

for any neighborhood \( U \) of the point \( x_0 \) and for all \( y \in \mathbb{R}^n \setminus C \).
Proof. Let $U$ be an arbitrary neighborhood of the point $x_0$. Without loss of generality, we may assume that $x_0 = 0$ and $U = \mathbb{B}^n$. Consider the sets $V_k = B(0, 1/k) \setminus \{0\}$, $k = 1, 2, \ldots$. Put

$$C = \bigcup_{k=1}^{\infty} \mathbb{R}^n \setminus f(V_k).$$

By Theorem 2, the union of each set $B_k := \mathbb{R}^n \setminus f(V_k)$ and the set on the right-hand side of (43) has zero capacity. Hence, $C$ also has zero capacity; see, e.g., [7]. Fix $y \in \mathbb{R}^n \setminus C$. Then

$$y \in \bigcap_{k=1}^{\infty} f(V_k).$$

By (43), there exists a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_i \to 0$ as $i \to \infty$ and $f(x_i) = y$, $i = 1, 2, \ldots$. Corollary 2 is proved. \[ \square \]

**Corollary 3** (a Liouville type theorem). Let $\Phi : [0, \infty] \to [0, \infty]$ be a monotone nondecreasing convex function. Assume that condition (7) is fulfilled for some $\delta_0 > \tau_0 := \Phi(0)$ and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is an open discrete $Q$-mapping at the point $x_0 = \infty$, that is, the mapping $\tilde{f} := f \circ \varphi$ with $\varphi(x) = \frac{x}{|x|^2}$ is a ring $Q'(x)$-mapping at zero, where $Q'(x) = Q(\frac{x}{|x|^2})$. Suppose $Q$ satisfies condition (3) in the domain $D = \mathbb{R}^n$ and $Q(x) \geq 1$ almost everywhere. Then $\text{cap}(\mathbb{R}^n \setminus f(\mathbb{R}^n)) = 0$. In particular, $f$ cannot map the space $\mathbb{R}^n$ into a bounded domain.

**Proof.** Suppose the contrary, that is,

$$\text{cap}(\mathbb{R}^n \setminus f(\mathbb{R}^n)) > 0.$$  

Then, for the auxiliary mapping $\tilde{f} = f(\frac{x}{|x|^2})$, $\tilde{f} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$, we have $\tilde{f}(\mathbb{R}^n \setminus \{0\}) = f(\mathbb{R}^n \setminus \{0\})$, so that

$$\text{cap}(\mathbb{R}^n \setminus \tilde{f}(\mathbb{R}^n \setminus \{0\})) = \text{cap}(\mathbb{R}^n \setminus \tilde{f}(\mathbb{R}^n \setminus \{0\})) \geq \text{cap}(\mathbb{R}^n \setminus f(\mathbb{R}^n)) > 0.$$  

Moreover, note that $\varphi(x) = \frac{x}{|x|^2}$ is a similarity map from the sphere $S(0, r)$ onto the sphere $S(0, 1/r)$, whence $|J(x, \psi)| = (1/|x|)^{2n}$. Using the above observations and changing the variable in the integral in (3), we have

$$\int_{\mathbb{R}^n} Q'(x) \frac{dm(x)}{(1 + |x|^2)^n} = \int_{\mathbb{R}^n} Q\left(\frac{x}{|x|^2}\right) \frac{dm(x)}{(1 + |x|^2)^n} = \int_{\mathbb{R}^n} Q(y) \frac{1}{|y|^{2n}} \frac{dm(y)}{(1 + |y|^2)^n} \leq M.$$  

So, by Theorem 2 the mapping $\tilde{f}$ extends by continuity to an open discrete mapping $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$. Equivalently, $f$ also extends by continuity to an open discrete mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. In this case, the set $\tilde{f}(\mathbb{R}^n)$ is simultaneously open and closed in $\mathbb{R}^n$, whence $f(\mathbb{R}^n) = \mathbb{R}^n$. However, this contradicts the following assumption made above: $\text{cap}(\mathbb{R}^n \setminus f(\mathbb{R}^n)) > 0$. \[ \square \]

§6. ON THE BRANCHING POINTS OF MAPPINGS
WITH INTEGRAL RESTRICTIONS ON THE CHARACTERISTIC

Recall that $y_0 \in D$ is a branching point of a mapping $f : D \to \mathbb{R}^n$ if the restriction $f|_U$ is not a homeomorphism for any neighborhood $U$ of $y_0$. Given $f$, the set of all branching points of $f$ is denoted by $B_f$. We say that a point $x_0 \in \mathbb{R}^n$ is an asymptotic limit of a mapping $f : D \to \mathbb{R}^n$ at a point $b \in \partial D$ if there exists a curve $\alpha : [0, 1) \to D$, $\alpha(t) \to b$
as \( t \to 1 \), such that \( f(\alpha(t)) \to z_0 \) as \( t \to 1 \); see §2 in Chapter 7 of [17]. Informally, a mapping \( f \) defined on a domain \( D \) has \( z_0 \in \mathbb{R}^n \) as an asymptotic limit at a point \( b \) on the boundary of \( D \) if there exists a curve that approaches \( b \) in \( D \) and is such that, along this curve, the mapping \( f \) approaches \( z_0 \). The present section contains several consequences of the following assertion obtained by the author earlier; see Theorem 1 in [26].

**Proposition 5.** Let \( x_0 \in D \). Assume that an open discrete mapping \( f : D \setminus \{x_0\} \to \mathbb{R}^n \), \( n \geq 2 \), satisfies (1) at the point \( x_0 \) for any measurable function \( \eta \) with property (6) and for some measurable function \( Q(x) \), \( Q : D \to [1, \infty] \). Suppose that \( x_0 \) is an essential singular point of \( f \) and (12) is true for some \( \delta(x_0) > 0 \), \( \delta(x_0) < \text{dist}(x_0, \partial D) \).

I. If \( n \geq 3 \) and a point \( z_0 \in \mathbb{R}^n \) is an asymptotic limit of \( f \) at \( x_0 \), then \( z_0 \in \overline{f(B_f \cap U)} \) for any neighborhood \( U \subset D \) of \( x_0 \).

II. Each point of the set \( \mathbb{R}^n \setminus \overline{f(D \setminus \{x_0\})} \) is an asymptotic limit of \( f \) at \( x_0 \).

III. If \( n \geq 3 \), then \( \overline{\mathbb{R}^n \setminus \overline{f(D \setminus \{x_0\})}} \subset \overline{f(B_f)} \).

IV. If \( n \geq 3 \) and \( \infty \notin f(D \setminus \{x_0\}) \), then

a) the set \( f(B_f) \) is unbounded in \( \mathbb{R}^n \);

b) \( x_0 \in B_f \).

**Theorem 6.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a monotone nondecreasing convex function. Assume that (7) is true for some \( \delta_0 > \tau_0 := \Phi(0) \), and that an open discrete \( Q \)-mapping \( f : D \setminus \{x_0\} \to \mathbb{R}^n \) has an essential singular point \( x_0 \in D \). Suppose \( Q \) satisfies condition (5) in the domain \( D \) and \( Q(x) \geq 1 \) almost everywhere. Then the claims I–IV in Proposition 5 are true.

**Proof.** As in the proof of Theorem 11 first we prove that the integral \( \int_0^{\tau_0} \frac{dr}{q_{x_0}(r)} \) in (12) diverges. Second, we apply Proposition 5 to obtain the required conclusion. \( \square \)

**Corollary 4.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a monotone nondecreasing convex function. Suppose that (7) is true for some \( \delta_0 > \tau_0 := \Phi(0) \), and that a \( Q \)-mapping \( f : D \setminus \{x_0\} \to \mathbb{R}^n \) is a local homeomorphism in a domain \( D \setminus \{x_0\} \). Assume that the function \( Q \) satisfies condition (5) in \( D \) and \( Q(x) \geq 1 \) almost everywhere. Then \( f \) extends to a local homeomorphism \( \tilde{f} : D \to \mathbb{R}^n \).

**Proof of Corollary 4** easily follows from Theorem 6. Indeed, the condition \( B_f = \emptyset \) implies that \( f \) extends continuously to the point \( x_0 \). It remains to show that the extended mapping \( \tilde{f} : D \to \mathbb{R}^n \) is a homeomorphism on a certain ball \( B(x_0, \varepsilon_1) \). Assume that this is not the case. Then \( x_0 \in B_f \). On the other hand, for an arbitrary open discrete mapping \( g : G \to \mathbb{R}^n \) of a domain \( G \subset \mathbb{R}^n \), \( n \geq 2 \), the following inequality for the Hausdorff dimension is known: \( \dim_H g(B_y) \geq n - 2 \); see, e.g., Proposition 5.3 in Chapter 3 of [17]. In the case under consideration, this means that the domain \( D \) contains certain points of the set \( B_f \) besides the point \( x_0 \), a contradiction. Indeed, by assumption, \( f \) is a local homeomorphism in \( D \setminus \{x_0\} \). \( \square \)

§7. CONDITIONS NECESSARY FOR EQUICONTINUITY AND FOR THE REMOVABILITY OF SINGULARITIES

Before proving Theorem 5 we make the following remark.

**Remark 6.** Without loss of generality, we may assume that, in the definition of the class \( \mathfrak{R}^{\Phi, Q}_{M,E} \) used in Theorem 5 the measurable function \( Q(x) \) related to \( \Phi \) by (8) is greater than or equal to 1.

Also, observe that the function \( \Phi(t) \) used in Theorem 5 is not constant. Indeed, otherwise there are no restrictions on \( Q \) in the theorem under consideration. The only
exception is the condition $\Phi(t) \equiv \infty$, which implies that the class $\mathfrak{N}^{\Phi}_{M,E}$ is empty. Moreover, by a well-known convexity criterion (see, e.g., Proposition 5 in § I.4.3 in [4]) the slope $[\Phi(t) - \Phi(0)]/t$ is a monotone nondecreasing function. Thus, the proof of Theorem 3 reduces to the following statement.

**Lemma 2.** Assume that $\Phi : [0, \infty) \to [0, \infty]$ is a monotone nondecreasing function, and that

$$
\Phi(t) \geq C \cdot t^{\frac{1}{\alpha - 1}}, \quad t \in [T, \infty],
$$

for some $C > 0$ and $T \in (0, \infty)$. If the class $\mathfrak{N}^{\Phi,Q}_{M,E}$ is equicontinuous (normal) for all $M \in (0, \infty)$, all sets $E$ of positive capacity, and all Lebesgue measurable functions $Q$, then (5) is true for all $\delta \in (\tau_0, \infty)$, where $\tau_0 := \Phi(+0)$.

Proof. It suffices to consider the case where $D = \mathbb{B}^n$. Suppose the contrary: let property (4) fail, that is,

$$
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_{n-1}(\tau)} < \infty
$$

for some $\delta_0 \in (\tau_0, \infty)$, where $\Phi_{n-1}(t) := \Phi(t^{n-1})$. Then we also have

$$
\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}_{n-1}(\tau)} < \infty, \quad \delta \in (\tau_0, \infty),
$$

because $\Phi^{-1}(\tau) > 0$ for all $\tau > \tau_0$, and $\Phi^{-1}(\tau)$ is a monotone nondecreasing function. Note that, by (45), the inequality $\Phi_{n-1}(t) \geq C \cdot t, \quad t \geq T$, is also valid for some $C > 0$ and $T \in (1, \infty)$. Moreover, applying the linear transformation $\alpha \Phi + \beta$, where $\alpha = 1/C$ and $\beta = T$ (see, e.g., (17)), we may assume that

$$
\Phi_{n-1}(t) \geq t, \quad t \in [0, \infty).
$$

Clearly, we may also assume that $\Phi(t) = t$ for all $t \in [0, 1)$ because the values of the function $\Phi$ on the half-open interval $[0, 1)$ provide no information about the set $Q(x) \geq 1$ in [5]. Also, property (17) implies the condition $\Phi(t) < \infty$ for all $t < \infty$; see the criterion (17) and also (20). Now, observe that the function $\Psi(t) := t \Phi_{n-1}(t)$ is strictly monotone increasing, $\Psi(1) = \Phi(1)$, and $\Psi(t) \to \infty$ as $t \to \infty$. Thus, the functional equation

$$
\Psi(K(r)) = \left(\frac{\gamma}{r}\right)^2, \quad r \in (0, 1],
$$

with $\gamma = \Phi^{1/2}(1) \geq 1$, is solvable if $K(1) = 1$, and $K(r)$ is a strictly monotone decreasing continuous function such that $K(r) < \infty, \quad r \in (0, 1]$, and $K(r) \to \infty$ as $r \to 0$. Taking the logarithm in (49), we have $\log K(r) + \log \Phi_{n-1}(K(r)) = 2\log \frac{\gamma}{r}$. So, by (48), we obtain $\log K(r) \leq \log \frac{\gamma}{r}$, that is,

$$
K(r) \leq \frac{\gamma}{r}.
$$

Hence, by (49), $\Phi_{n-1}(K(r)) \geq \frac{\gamma}{r}$ and, by (14),

$$
K(r) \geq \Phi^{-1}_{n-1}\left(\frac{\gamma}{r}\right).
$$

We define the following mappings in the punctured unit ball $\mathbb{B}^n \setminus \{0\}$:

$$
f(x) = \frac{x}{|x|} \rho(|x|), \quad f_m(x) = \frac{x}{|x|} \rho_m(|x|), \quad m = 1, 2, \ldots,
$$

where $\rho(t) = \exp\{I(0) - I(t)\}$, $\rho_m(t) = \exp\{I(0) - I_m(t)\}$, $t \in [0, 1]$, and put

$$
I(t) = \int_{t}^{1} \frac{dr}{rK(r)}, \quad I_m(t) = \int_{t}^{1} \frac{dr}{rK_m(r)},
$$

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and

\[ K_m(r) = \begin{cases} K(r) & \text{for } r \geq 1/m, \\ K\left(\frac{1}{m}\right) & \text{for } r \in (0, 1/m). \end{cases} \]

Using (51), we obtain

\[ I(0) - I(t) = \int_0^t \frac{dr}{rK(r)} \leq \int_0^t \frac{dr}{r\Phi^{-1}_{n-1}(\frac{1}{r})} = \int_{\tau}^\infty \frac{d\tau}{\tau\Phi^{-1}_{n-1}(\tau)}, \quad t \in (0, 1], \]

where \( \gamma/t \geq \gamma \geq 1 > \Phi(0) = 0 \). Thus, by (47),

\[ I(0) - I(t) \leq I(0) = \int_0^1 \frac{dr}{rK(r)} < \infty, \quad t \in (0, 1]. \]

Also, we have \( f_m, f \in C^1(\mathbb{B}^n \setminus \{0\}) \) because \( K_m(r) \) and \( K(r) \) are continuous. We show that the \( f_m \) are \( K_m \)-quasiconformal in \( \mathbb{B}^n \), where \( K_m = K^{n-1}(1/m) \), \( f_m(0) = 0 \).

Fix \( \rho \in (0, 1) \) and a point \( x \in \mathbb{B}^n \setminus \{0\} \) such that \( |x| = \rho \). Observe that the principal distortions \( \lambda_i(x), \ldots, \lambda_{n-1}(x) \) of the mapping \( f \) at the point \( x \) in \( n-1 \) directions \( i_1, \ldots, i_{n-1} \) tangential to the sphere \( |x| = \rho \) are all equal and are given by the following formula:

\[ \lambda_i(x) := \delta_r(x) = \frac{|f(x)|}{|x|} = \frac{\exp\left\{\int_0^\rho \frac{dr}{rK(r)}\right\}}{\rho}, \]

\( k = 1, \ldots, n-1 \). The value \( \lambda_i(x) \) that corresponds to the perpendicular (radial) direction is computed as follows:

\[ \lambda_i(x) := \delta_r(x) = \frac{\partial |f(x)|}{\partial |x|} = \frac{\exp\left\{\int_0^\rho \frac{dr}{rK(r)}\right\}}{\rho K(\rho)}. \]

For more details about the definition and computation of distortions, see, e.g., §5 in Chapter 1 of [16], see also Proposition 6.3 in Chapter 4 of [11]. Properties (53) and (54) imply that \( \delta_r(x) \leq \delta_r(x) \), because \( K(r) \geq 1 \). Hence, applying (10), we see that

\[ K_I(x, f) = \frac{\delta_{r^{-1}}(x) \cdot \delta_r(x)}{\delta_r(x)} = K^{n-1}(|x|) \]

at all points \( x \in \mathbb{B}^n \setminus \{0\} \); see also [16, § 1.2.1]. Note that

\[ f_m(x) \equiv f(x) \quad \text{for all } x \text{ with } \frac{1}{m} < |x| < 1, \quad m = 1, 2, \ldots. \]

Thus, \( K_I(x, f_m) = K_I(x, f) = K^{n-1}(|x|) \) for \( \frac{1}{m} < |x| < 1 \), and \( K_I(x, f_m) = K^{n-1}(1/m) \) for \( 0 < |x| < \frac{1}{m} \) can be computed similarly. Therefore, by Proposition II each \( f_m \) is an \( \omega \)-mapping in \( \mathbb{B}^n \) with \( K(x) = K_I(x, f) \), because \( K(r) \) is monotone nonincreasing. By (49),

\[ \int_{\mathbb{B}^n} \Phi(K_I(x, f_m)) \, dm(x) \leq \int_{\mathbb{B}^n} \Phi_{n-1}(K(|x|)) \, dm(x) = \omega_{n-1} \int_0^1 \frac{\Psi(K(r))}{rK(r)} \cdot r^n \, dr \]

\[ \leq \gamma^2 \omega_{n-1} \int_0^1 \frac{dr}{rK(r)} \leq M := \gamma^2 \omega_{n-1} I(0) < \infty. \]

Note that \( f_m \) maps the unit ball \( \mathbb{B}^n \) onto the ball with center at the origin and of radius \( R = e^{I(0)} < \infty \). Therefore, \( f_m \in \mathfrak{A}^{\Phi, Q}_{M, E} \), where \( M \) is defined above; so, we may put \( E = \mathbb{B}^n \setminus B(0, e^{I(0)}) \). On the other hand, it is easily seen that

\[ \lim_{x \to 0} |f(x)| = \lim_{t \to 0} \rho(t) = e^0 = 1, \]
that is, \( f \) maps the punctured ball \( \mathbb{B}^n \setminus \{0\} \) onto the annulus \( 1 < |y| < R = e^{l(0)} \). Hence, by (53) and (57),
\[
|f_m(x)| = |f(x)| \geq 1 \quad \text{for all} \quad x \quad \text{with} \quad |x| \geq 1/m, \quad m = 1, 2, \ldots,
\]
that is, the family \( \{f_m\}_{m=1}^{\infty} \) is not equicontinuous at zero. The contradiction obtained shows that (46) cannot be true. \( \square \)

**Remark 7.** Theorem 8 shows that (7) is not merely sufficient, but also necessary for the continuity (normality) of the mapping classes satisfying the integral conditions (8) with a convex monotone nondecreasing function \( \Phi \). By Proposition 4, this remark extends to conditions (15)–(20) for \( p = n - 1 \).

**Proof of Theorem 4.** Applying the construction described in the proof of Theorem 3 and Lemma 2 and retaining the above notation, consider the mapping
\[
f(x) = \frac{x}{|x|} \rho(|x|)
\]
on the domain \( D = \mathbb{B}^n \setminus \{0\} \). As was mentioned above, \( f \) maps the punctured ball \( \mathbb{B}^n \setminus \{0\} \) onto the ring \( 1 < |y| < R = e^{l(0)} \). This means that \( f \) has an essential singularity at zero. Also, as was proved above, \( f \) is a ring \( Q \)-mapping at zero for \( Q(x) = K_1(x,f) = K^{n-1}(|x|) \); moreover, \( Q \) satisfies (56), that is, \( f \in K^\Phi_Q(\mathbb{B}^n \setminus \{0\}) \) for some \( M > 0 \) and a set \( E \) such that \( \text{cap } E > 0 \). \( \square \)

§8. Applications to the Sobolev classes

Assume that the first order partial derivatives are defined for a mapping \( f : D \to \mathbb{R}^n, n \geq 2 \), at almost all points \( x \in D \). Then the outer dilatation of \( f \) at a point \( x \) is defined as
\[
K_0(x,f) = \frac{\|f'(x)\|^n}{|J(x,f)|},
\]
provided that \( J(x,f) \neq 0 \). If \( J(x,f) = 0 \), then \( K_0(x,f) = 1 \) when \( f'(x) = 0 \), otherwise \( K_0(x,f) = \infty \). It is possible to verify that, for almost all \( x \in D \), we have \( K_1(x,f) \leq K_0^{n-1}(x,f), K_0(x,f) \leq K_1^{n-1}(x,f) \); see, e.g., §3 in Chapter 1 of 16. The following theorem was obtained by the author earlier (see 27 Theorem 1):

**Theorem 7.** Let \( x_0 \in D \), and let \( f : D \to \mathbb{R}^n \) be an open discrete mapping of class \( W^{1,n}_{\text{loc}}(D) \) such that \( Q = K_0^{n-1}(x,f) \in L^1_{\text{loc}}(D) \) and \( m(B_f) = 0 \). Then \( f \) satisfies condition (41) at the point \( x_0 \) for every nonnegative measurable function \( \eta \) with property (5) and for \( Q = K_0^{n-1}(x,f) \). The case where \( x_0 \) is an essential singular point in \( D \) is admissible.

Therefore, the results mentioned above apply to the Sobolev classes. Also, such results have direct reformulations in terms of the Sobolev classes.

**Postscriptum.** The present work continues the studies initiated by the prominent mathematician G. D. Suvorov who described “the ideal (and the goal!) of the function theory as achieving the situation when we have a large number of function classes and, for each class, we have an elaborate catalog of properties (metric and topological)”.

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