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CONTINUOUS SYMMETRIZATION VIA POLARIZATION

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Abstract. We discuss a one-parameter family of transformations that changes sets and functions continuously into their (k,n)-Steiner symmetrizations. Our construction consists of two stages. First, we employ a continuous symmetrization introduced by the author in 1990 to transform sets and functions into their one-dimensional Steiner symmetrization. Some of our proofs at this stage rely on a simple rearrangement called *polarization.* At the second stage, we use an approximation theorem due to Blaschke and Sarvas to give an inductive definition of the continuous (k,n)-Steiner symmetrization for any $2 \le k \le n$. This transformation provides us with the desired continuous path along which all basic characteristics of sets and functions vary monotonically. In its turn, this leads to continuous versions of several convolution type inequalities and Dirichlet's type inequalities as well as to continuous versions of comparison theorems for solutions of some elliptic and parabolic partial differential equations.

§1. Introduction

The first geometric transformation bearing the name *symmetrization* was introduced by Jacob Steiner in 1836 [30] in an attempt to find a rigorous proof for the classical isoperimetric problem. Let C be a closed contour on $\mathbb{R}^2$ enclosing a domain D, and let $m_D(x)$ denote the Lebesgue measure of the intersection of D with the vertical line $v_x = \{(x,y) \in \mathbb{R}^2 : -\infty < y < \infty\}$. Then Steiner's symmetrization of D with respect to the x-axis is defined by

$$D^* = \{(x,y) \in \mathbb{R}^2 : |y| < (1/2)m_D(x)\}. \tag{1.1}$$

This implies, in particular, that D^* is symmetric with respect to the x-axis and convex in the y-direction. Let $C^* = \partial D^*$ be the boundary of D^*. Steiner used his symmetrization to show that

$$\text{(a)} \quad \operatorname{area} D = \operatorname{area} D^*, \qquad\qquad \text{(b)} \quad \operatorname{length} C^* \le \operatorname{length} C,$$

which implies the classical isoperimetric inequality

$$\frac{\operatorname{area} D}{(\operatorname{length} C)^2} \le \frac{1}{4\pi},$$

assuming the existence of a minimizer.

This ingenious idea of Steiner has turned out to be extremely fruitful and was exploited over the years by many authors, who proved numerous, so-called, *isoperimetric inequalities* for several important geometrical and physical quantities characterizing the shape of planar and solid regions. We want to mention the following four inequalities of this sort, for the transfinite diameter $d(\bar{D})$ that is equal to the logarithmic capacity

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$\mathrm{cap}(\bar{D})$, for the inner radius $r(D,a)$ of D at its point $a \in D$, for the torsional rigidity $P(D)$, and for the principal frequency $\lambda(D)$:

(c) $d(\overline{D^*}) \le d(\bar{D})$, (d) $r(D^*, a^*) \ge r(D,a)$ for every $a \in D$,

(e) $P(D^*) \ge P(D)$, (f) $\lambda(D^*) \le \lambda(D)$.

The first period of history of symmetrization was summarized in the classical monograph of George Pólya and Gabor Szegő "Isoperimetric inequalities in mathematical physics" [24]. This fundamental study of isoperimetric inequalities was filled with new ideas, results, and problems; some of these problems still remain open. The story of continuous symmetrization also has its source in this book. The question below was raised in Note B of [24].

Is it possible to define a transformation T_λ on D depending in a continuous way on the parameter λ, $0 \le \lambda \le 1$, such that the following conditions are satisfied:

I. T_0 is the identity.

II. T_1 is the transformation replacing C by the symmetrized curve C^*, that is, T_1 is Steiner's symmetrization with respect to the line l.

III. For every λ, $0 \le \lambda \le 1$, T_λ has the same effect as described under (a)–(d).

Although Pólya and Szegő mentioned only relations (a)–(d), a similar question about inequalities (e) and (f) falls into the same context.

The authors of [24] did not explain explicitly what motivated them to study this problem. Among obvious reasons we want to mention the following three.

- First, from the point of view of classical mechanics it is interesting to embed D and D^* into a continuous path along which all basic geometrical and physical characteristics of shape vary continuously and monotonically.
- Often estimates better than those provided by inequalities (b)–(f) are needed. Indeed, even for simple shapes, for example, for the rhombus having angle $\alpha = \pi/100$ centered at $a = 0$, the gap in each of the inequalities (b)–(f) exceeds 40%.
- Steiner's symmetrization changes a given shape globally into a symmetric one. So, this transformation will not work in problems where the minimizer does not possesses a global symmetry and in problems concerned with local minimality.

Pólya and Szegő themselves studied this problem. In collaboration with Schiffman [24, Note B], they presented one continuous transformation of the above sort and proved relations (a)–(d) for the case of convex domains. Suppose that D is a convex domain on $\mathbb{R}^2$ bounded from below and above by the graphs of functions $y = y_1(x)$ and $y = y_2(x)$, respectively, such that $y_1(x) < y_2(x)$ for all $a < x < b$. For a given continuous function $\phi : [\alpha, \beta] \to [0,1]$, let

$$y_1^t = y_1 - \phi(t)\frac{y_1 + y_2}{2}, \quad y_2^t = y_2 - \phi(t)\frac{y_1 + y_2}{2} \tag{1.2}$$

and

$$D^{t,\phi} = \left\{(x,y) \in \mathbb{R}^2 \,:\, a < x < b,\ y_1^t < y < y_2^t\right\}. \tag{1.3}$$

If ϕ is a monotone increasing homeomorphism from $[\alpha, \beta]$ onto $[0,1]$, then it is clear that formulas (1.2), (1.3) define a continuous path from D to D^*. Choosing $\phi(t) = t$, Pólya and Szegő [24] proved that relations (a)–(d) hold true for all convex domains. S. Abramovich [1] used a variant of Pólya–Szegő's continuous symmetrization to prove the monotonicity of eigenvalues of certain second order differential equations in one variable.

Another continuous transformation, again for smooth convex domains, was introducedby McNabb in 1967 [22]. His transformation, called the *partial Steiner symmetriza-*

tion, can be defined as follows. We quote from [22]:

"A partial Steiner symmetrization of D may be performed in the following way. If the constant α lies between certain limits ($\alpha_L < \alpha < \alpha_R$), the line $x = \alpha$ will intersect the curve defined by midpoints of the line segments composing D. If just those line segments which have their midpoints to the left of $x = \alpha$ are translated parallel to themselves until these central points lie on $x = \alpha$, the ends of the segments now define a curve C_α bounding a partially symmetrized region D_α. It is as though the line $x = t$ swept across the x-y-plane from $t = -\infty$ to $t = \alpha$ and the midpoints of the line segments became attached to the line as it passed over them. As t increases from $\alpha = \alpha_L$ to $\alpha = \alpha_R$, D continuously evolves through a sequence D_t of partially symmetrized regions to its Steiner symmetrization D^*".

As McNabb noted in [22], his goal was to demonstrate on simple examples how his transformation works. He also mentioned that his treatment of the problem in [22] was heuristic and that technical "difficulties were glossed over" there.

We also want to mention two interesting continuous transformations discovered in [31] and [23], but those are not related, at least directly, to the problem raised in [24].

The first continuous transformation into Steiner symmetrization, which works for nonconvex domains and satisfies all the requirements of the Pólya–Szegő problem, was introduced by this author [27]. Our continuous symmetrization, which will be abbreviated to *SC symmetrization*, can be viewed as an extension of McNabb's partial Steiner symmetrization for the case of nonconvex domains.[1] We want to emphasize here that the approaches used in [22] and [27] are different.

It is interesting to mention that, eventually, the original idea of Pólya and Szegő was developed by Brock [12, 13], who defined a continuous symmetrization, called *BC symmetrization* in this paper, which works for nonconvex domains. This was achieved by choosing a parametrization $\phi(t) = 1 - e^{-t}$, $-\infty < t < \infty$, in (1.2), (1.3), combined with some other innovations. Instead of abbreviations SC and BC, we sometimes write "S-continuous symmetrization" and "B-continuous symmetrization", respectively. One particular difference between SC symmetrization and BC symmetrization is that under the first transformation the change of the shape is localized near some boundary arcs while the second transformation changes the boundary globally.

Although the exposition in [27] was given for planar domains, in the final Remark 5 in [27], the author emphasized that all definitions and proofs can be extended without substantial changes to n-dimensional spaces and that all major results of the paper have n-dimensional counterparts.

Our primary goal in the present paper is to give a full scale account of S-continuous symmetrization in the n-dimensional setting. Since the paper [27] is practically unknown to the experts, we want to mention here the major innovations introduced in [27]. First of all, *polarization* was used for the first time in [27] in the context of continuous symmetrizations. Then, an analog of the semigroup property was applied to prove some results about the continuous SC symmetrization. Later on, Brock [13] used a similar property as a part of the definition of his continuous symmetrization. Uniqueness results were treated in [27] in their full generality. Under certain conditions, this to the strict monotonicity of the domain characteristics under consideration as functions of the parameter of symmetrization. Finally, the SC symmetrization was applied in [27] to prove local symmetry in some problems on Green's functions and harmonic measures. A similar approach to local symmetry in a more general context was also used in the papers [12] and [13].

[1]The paper [27] does not refer to McNabb's work in [22] because at that time the author was not aware of McNabb's publication.

This paper is organized as follows. §2 contains our basic notation. In particular, we introduce there necessary spaces of functions and classes of domains. In §3, we remind the reader of basic properties of the Steiner (k,n)-symmetrization and polarization. The exposition in §2 and 3 follows the lines of our paper [14] joint with Brock. Sections 4–8 are devoted to geometric aspects of the SC 1-symmetrization.

In §9, we give an inductive definition of the continuous (k,n)-Steiner symmetrization for any $2 \le k \le n$.

Sections 10 and 11, where we again follow the lines of the paper [14], contain our main applications. In §10, we show that many integral inequalities known for the Steiner symmetrization have their continuous counterparts for the continuous (k,n)-symmetrization as well. In §11, we give a similar treatment of the comparison theorems for solutions of some elliptic and parabolic PDE's. Many proofs in §10 and 11 related to the L^p and Sobolev classes are based on ideas suggested by Brock when we worked on §9 and 10 of our joint paper [14], and which he developed further in [13].

In the present paper we combined and extended the ideas and methods developed in [27, 28], and [14]. Preparing this article for publication, the author used his notes written in the Fall semester, 1995, during his stay at the Mathematisches Forschungsinstitut Oberwolfach under the financial support of Volkswagen–Stiftung, RiP-program for Friedemann Brock and Alexander Solynin. Our intention at that time was to present in a joint paper our results for both the SC k-dimensional continuous symmetrization and the BC k-dimensional continuous symmetrization. Since 1995, the B-continuous symmetrization and its applications have already been discussed in several publications by Brock and his collaborators. Thus, in this paper, we are concentrating on the S-continuous symmetrization only. Although the original plan for this paper has been changed, this work remains closely related to the paper [14] joint with Brock, where such a possible continuation was referred to as "An approach to continuous symmetrization via polarization".

§2. Preliminaries

The following notation will be used throughout the paper. Let $\mathbb{R}^n$ be the Euclidean space, $\mathbb{R}^n_+ = \big\{(x_1,\dots,x_n) \in \mathbb{R}^n \,:\, x_i > 0,\ 1 \le i \le n\big\}$.

For $A \subset \mathbb{R}^n$, let $\bar{A}$ and ∂A denote the closure and the boundary of A, respectively. If $A, B \subset \mathbb{R}^n$, then $A + B := \big\{z \,:\, z = x + y,\ x \in A,\ y \in B\big\}$ denotes the Minkowski sum of A and B. For $x, y \in \mathbb{R}^n$, by $|x|$ and $\langle x, y\rangle$ we denote the norm of x and the scalar product of x and y, respectively. Then $H(a,n)$ and $\Sigma(a,n)$ will denote the half-space $\big\{x \in \mathbb{R}^n \,:\, \langle (x-a), n\rangle > 0\big\}$ and the hyperplane $\big\{x \in \mathbb{R}^n \,:\, \langle (x-a), n\rangle = 0\big\}$ determined by the point $a \in \mathbb{R}^n$ and the unit vector $n \in \mathbb{R}^n$. For $M \subset \mathbb{R}^n$, we denote by $\mathcal{L}^n(M)$ the n-dimensional Lebesgue measure of M. By $\mathcal{M}_n$, $\mathcal{F}_n$, and $\mathcal{G}_n$ we denote the sets of all measurable, compact, and open subsets of $\mathbb{R}^n$, respectively. Then $\mathcal{M}_{n,b}$ and $\mathcal{G}_{n,b}$ will denote the collections of all bounded subsets of $\mathcal{M}_n$ and $\mathcal{G}_n$.

Generally, we treat measurable sets only in an a.e. sense, i.e., we write

$$M = N \quad \text{if and only if} \quad \mathcal{L}^n\big(M \triangle N\big) = 0,$$
$$M \subset N \quad \text{if and only if} \quad \mathcal{L}^n\big(M \setminus N\big) = 0.$$

By $B_r^{(n)}(x_0)$ we denote the open ball in $\mathbb{R}^n$ with radius $r > 0$ centered at x_0, and we write $B_r^{(n)} = B_r^{(n)}(0)$, $B^{(n)} = B_r 1^{(n)}$. If $A \subset \mathbb{R}^n$ and $\varepsilon > 0$, then we denote by $A_\varepsilon := A + \varepsilon \overline{B^{(n)}}$ the exterior parallel set of A. The Hausdorff distance between compact sets A and B is defined by

$$d(A,B) := \inf\big\{\varepsilon > 0 \,:\, A \subset B_\varepsilon,\ B \subset A_\varepsilon\big\}.$$

It is well known that d is a metric on $\mathcal{F}_n$. By using the metric d, we define the convergence of a sequence of sets $F_i \in \mathcal{F}_n$, $i = 1, 2, \dots$, to a set $F \in \mathcal{F}_n$ by

$$\lim_{i\to\infty} F_i = F \quad \text{if and only if} \quad d(F_i, F) \to 0 \quad \text{as} \quad i \to \infty.$$

If Ω is an open set in $\mathbb{R}^n$ and $p \in [1, \infty]$, then $\|\cdot\|_p$ denotes the usual norm in the space $L^p(\Omega)$. For functions $u \in C(\mathbb{R}^n)$, we define the modulus of continuity by

$$\omega_u(\delta) := \sup\big\{|u(x) - u(y)| \,:\, |x - y| < \delta\big\}, \quad \delta > 0.$$

By $W^{1,p}(\Omega)$ we denote the Sobolev space of functions $u \in L^p(\Omega)$ having generalized partial derivatives $u_{x_i} \in L^p(\Omega)$, $i = 1, \dots, n$, and we write

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_p + \sum_{i=1}^n \|u_{x_i}\|_p \tag{2.1}$$

for the norm in this space. By $W_0^{1,p}(\Omega)$ we denote the completion of the set of infinitely differentiable functions with compact support in Ω, denoted by $C_0^\infty(\Omega)$, under the norm (2.1). Usually we extend measurable functions $u : \Omega \to \mathbb{R}_0^+$ by zero outside Ω, so that $W_0^{1,p}(\Omega) \subset W^{1,p}(\mathbb{R}^n)$ in that sense. By $C_0^{0,1}(\Omega)$ we denote the space of Lipschitz functions with compact support in Ω. For any function space, the lower subscript "+" denotes the corresponding subspace of nonnegative functions, e.g., $L_+^p(\Omega)$, $W_{0+}^{1,p}(\Omega)$, etc.

Let $\mathcal{S}_n$ denote the class of real measurable functions u satisfying

$$\mathcal{L}^n\big(\{u > c\}\big) < \infty \quad \text{for all } c > \inf u.$$

Here and in the sequel we use the following abbreviation: $\{u > c\} = \{x \in \mathbb{R}^n : u(x) > c\}$. Note that the spaces $L_+^p(\mathbb{R}^n)$, $C_{0+}^{0,1}(\mathbb{R}^n)$, and the space $W_+^{1,p}(\mathbb{R}^n)$ with $1 \le p < \infty$ are subspaces of $\mathcal{S}_{n+}$. The space of measurable functions with bounded variation is denoted by $BV(\mathbb{R}^n)$ and we write

$$\|Du\|_{BV} := \sup\left\{\int_{\mathbb{R}^n} u \sum_{i=1}^n \frac{\partial \psi_i}{\partial x_i}\, dx \,:\, \sum_{i=1}^n \psi_i^2 \le 1,\ \psi_i \in C_0^\infty(\mathbb{R}^n),\ i = 1, \dots, n\right\}.$$

Recall also that if $u \in W^{1,1}(\mathbb{R}^n)$, then $\|Du\|_{BV} = \|\nabla u\|_1$. Furthermore, if M is a Caccioppoli set in $\mathbb{R}^n$, then $\|D\chi(M)\|_{BV}$ is the perimeter of M in the sense of De Giorgi, see [32].

Finally, a function $j : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a Young function if j is continuous and convex with $j(0) = 0$.

§3. Steiner symmetrization and polarization

First we discuss some general properties of rearrangements. We remind the reader that a set transformation T (defined on $\mathcal{M}_n$) is called *a rearrangement* if it is monotone and measure preserving, i.e., if $T(A) \subset T(B)$ for all A and B such that $A \subset B$ and $\mathcal{L}^n(T(A)) = \mathcal{L}^n(A)$ for every measurable set A.

The class $\mathcal{S}_n$ introduced in the previous section is the natural class of functions for which a rearrangement can be defined. If T is a rearrangement and u is in $\mathcal{S}_n$, then the relations

$$Tu(x) := \operatorname{ess\,sup}\big\{c > \inf u \,:\, x \in T(\{u > c\})\big\}, \quad \inf Tu := \inf u, \tag{3.1}$$

define a function Tu on $\mathbb{R}^n$. If $u \in \mathcal{S}_n$ is continuous, then "ess sup" in (3.1) can be replaced by "sup". Clearly the function Tu is uniquely determined almost everywhere. Since T is measure preserving,

$$\mathcal{L}^n\big(T(\{u > c\})\big) = \mathcal{L}^n\big(\{Tu > c\}\big) \quad \text{for all } c > \inf u.$$

Thus, $Tu \in \mathcal{S}_n$ if $u \in \mathcal{S}_n$. The mapping $T : \mathcal{S}_n \to \mathcal{S}_n$ constructed in this way is again called *a rearrangement*. The following nonexpansivity lemma will be very useful in §10 and 11, see [14, Theorem 3.1].

Lemma 3.1. *Let T be a rearrangement. Then for every Young function j, we have*

$$\int_{\mathbb{R}^n} j\left(|Tu - Tv|\right)\, dx \le \int_{\mathbb{R}^n} j\left(|u - v|\right)\, dx \quad \text{for all} \;\; u, v \in \mathcal{S}_n, \tag{3.2}$$

whenever one of the integrals in (3.2) *converges.*

Sometimes we say that two functions $u, v \in \mathcal{S}_n$ are *rearrangements of each other* if $\inf u = \inf v$ and $\mathcal{L}^n(\{u > c\}) = \mathcal{L}^n(\{v > c\})$ for all $c > \inf u$.

We shall also use some additional properties of rearrangements. A set transformation T is said to be *open or compact* if $T(A)$ is open or compact whenever A is of the same kind. We say that T is *continuous from the inside* if $\bigcup_i T(G_i) = T\big(\bigcup_i G_i\big)$ for every increasing sequence $\{G_i\} \subset \mathcal{G}$. Similarly, we say that T is *continuous from the outside* if $\bigcap_i T(F_i) = T\big(\bigcap_i F_i\big)$ for every decreasing sequence $\{F_i\} \subset \mathcal{F}$.

Finally, a rearrangement T is *smoothing* if $T(F_r) \supset (T(F))_r$ for every $F \in \mathcal{F}$ and $r > 0$. Smoothing rearrangements were introduced by Sarvas [25].

Now we recall the definitions of the (k, n)-Steiner symmetrizations (for further information, see [30, 21], and [25]).

Definition 3.1. Every $(n-k)$-plane $\Sigma \subset \mathbb{R}^n$ with $1 \le k \le n$ defines a (k, n)-*Steiner symmetrization* S as follows.

For every $x \in \Sigma$, let $\Lambda(x)$ denote the k-dimensional plane through x and orthogonal to Σ.

1) Let $M \in (\mathcal{F}_n \cup \mathcal{G}_n) \cap \mathcal{M}_n$. If $\mathcal{L}^k\big(M \cap \Lambda(x)\big) = 0$, then $S(M) \cap \Lambda(x)$ is empty or the point $\{x\}$ according to whether $M \cap \Lambda(x)$ is empty or nonempty. If $\mathcal{L}^k\big(M \cap \Lambda(x)\big) > 0$, then

$$S(M) \cap \Lambda(x) = \begin{cases} B_r(x) \cap \Lambda(x) & \text{if } M \text{ is open,} \\ \overline{B_r(x)} \cap \Lambda(x) & \text{if } M \text{ is compact,} \end{cases} \tag{3.3}$$

where $r > 0$ is defined by the condition $\mathcal{L}^k\big(B_r(x) \cap \Lambda(x)\big) = \mathcal{L}^k\big(M \cap \Lambda(x)\big)$.

2) Let $M \in \mathcal{M}_n$ where M is *neither open nor compact.* Then the sets $S(M) \cap \Lambda(x)$ are defined in an a.e. sense by either one of the equations in (3.3).

From Definition 3.1 one deduces immediately that the (k, n)-Steiner symmetrization is a rearrangement which is continuous from the inside and from the outside. Note also that in case **2)** Fubini's theorem implies that the sets $M \cap \Lambda(x)$ are measurable with finite $\mathcal{L}^k$-measure for a.e. $x \in \Sigma$.

The (n, n)-Steiner symmetrization is often called the *Schwarz symmetrization* or the *symmetric decreasing rearrangement*, and we shall denote it by $S^\star$.

For our purposes it will often be helpful to use a special coordinate system in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$, where $1 \le k \le n$, $m = n - k$, and

$$x = (x_1, \dots, x_n) = (x', y), \quad x' = (x_1, \dots, x_{n-k}), \quad y = (x_{n-k+1}, \dots, x_n),$$

in which the plane Σ of symmetry becomes simply $\{y = 0\}$. If $M \in \mathcal{M}_n$, we introduce the "k-slices" of M at x' by

$$M(x') = \big\{y \in \mathbb{R}^k \,:\, (x', y) \in M\big\}, \quad x' \in \mathbb{R}^{n-k}.$$

For instance, if $x' \in \mathbb{R}^{n-k}$ with $1 \le k < n$, then $B_r^{(n)}(x')$ will denote the k-slice of the ball $B_r^{(n)}$ at x' and not the ball in $\mathbb{R}^n$ centered at x'. Let $S^\star(M(x'))$ denote the Schwarz

symmetrization of $M(x')$, *taken in* $\mathbb{R}^k$. Then (3.3) becomes

$$S(M) := \big\{x = (x', y) \,:\, y \in S^\star(M(x')),\ x' \in \mathbb{R}^{n-k}\big\}. \tag{3.4}$$

If $u \in \mathcal{S}_n$, then from (3.3) we see that the (k, n)-Steiner symmetrization $\mathcal{S}u$ of u is given by the relations

$$\mathcal{S}u(x', y) = \sup\big\{c > \inf u \,:\, x \in S\big(\{u(x', \cdot) > c\}\big)\big\}. \tag{3.5}$$

(Here and in the following, for simplicity $\{u(x', \cdot) > c\}$ denotes $\{y \in \mathbb{R}^k \,:\, u(x', y) > c\}$.)

It should be mentioned again that equations (3.4) and (3.5) are understood in the pointwise sense if and only if u is continuous. Note also that $\mathcal{S}u$ is "radially symmetric and monotone decreasing in $|y|$", i.e.,

$$\mathcal{S}u(x', y) = \mathcal{S}u(x', z_1) \geq \mathcal{S}u(x', z_2) \quad \text{if } |y| = |z_1| \leq |z_2|, \tag{3.6}$$

where $x' \in \mathbb{R}^{n-k}$ and $y, z_1, z_2 \in \mathbb{R}^k$.

Sometimes we write $S(M) = M^*$ and $\mathcal{S}u = u^*$ for the symmetrized objects.

There is an approach due to Schwarz and Blaschke (see, for instance, [11]) reducing a k-dimensional symmetrization to $(k-1)$-dimensional symmetrizations, see Theorem 4.32 in [25]. We shall use a slightly refined version of that theorem.

Let S be a (k, n)-Steiner symmetrization in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ with the symmetry plane $\Sigma = \{y = 0\}$. Let $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ be unit vectors orthogonal to Σ that form an angle of $\gamma\pi$, where $\gamma \in (0, 1)$ is irrational. Let S_i be the $(k-1, n)$-Steiner symmetrization with the symmetry plane Σ_i defined by the plane Σ and the unit vector $\overrightarrow{v}_i$, $i = 1, 2$.

For positive integer j and $\Omega \in \mathcal{F}_n \cup \mathcal{G}_{n,b}$, let

$$\Omega_j = (S_2 \circ S_1)^m(\Omega) \qquad \text{if } j = 2m \text{ is even,} \tag{3.7}$$

$$\Omega_j = S_1 \circ (S_2 \circ S_1)^m(\Omega) \qquad \text{if } j = 2m + 1 \text{ is odd.} \tag{3.8}$$

Here $(S_2 \circ S_1)^0$ is the identity transformation.

Theorem 3.1. *Let S, S_1, and S_2 be the symmetrizations defined above, and let $\Omega^* = S(\Omega)$. Then:*

$$\lim_{j\to\infty} d(\Omega_j, \Omega^*) = 0 \quad \textit{for every compact set } \Omega \in \mathcal{F}_n, \tag{3.9}$$

$$\lim_{j\to\infty} d(\partial\Omega_j, \partial\Omega^*) = 0 \quad \textit{for every bounded open set } \Omega \in \mathcal{G}_{n,b}, \tag{3.10}$$

and

$$\lim_{j\to\infty} \mathcal{L}^k\big(\Omega_j(x') \triangle \Omega^*(x')\big) = 0 \tag{3.11}$$

for every $\Omega \in \mathcal{F}_n \cup \mathcal{G}_{n,b}$ and every $x' \in \mathbb{R}^{n-k}$.

For compact sets this theorem is a part of Theorem 4.32 in [25]. For bounded open sets the proof will be given in the Appendix. In §9, we shall use the approximation pattern of Theorem 3.1 to give an inductive definition of our continuous (k, n)-Steiner symmetrization.

Through the last decade, an increase of activity has been observed in the theory of symmetrization, partly triggered by the paper [14], and related to polarization. This simplest rearrangement was introduced for sets by Wolontis [33] in 1952, who attributed some ideas of his paper to L. Ahlfors. Ahlfors himself used polarization in [2], where he introduced this transformation for functions. The term *polarization* was suggested by Dubinin in [15].

Let Σ be a hyperplane in $\mathbb{R}^n$, and let H be one of the open half-spaces into which $\mathbb{R}^n$ is split by Σ. Let σ_H denote the reflection in Σ. We write $\bar{x} = \sigma_H(x)$ for points $x \in \mathbb{R}^n$ and $\sigma_H(u) = u(\bar{x})$ for all $x \in \mathbb{R}^n$ for functions $u \in \mathcal{S}$.

Definition 3.2. If $u \in \mathcal{S}_n$, then its polarization Pu with the *polarizer* H is given by

$$Pu(x) := \begin{cases} \max\{u(x), u(\overline{x})\} & \text{if } \ x \in H, \\ \min\{u(x), u(\overline{x})\} & \text{if } \ x \in \mathbb{R}^n \setminus H. \end{cases} \tag{3.12}$$

If $M \in \mathcal{M}_n$, then the polarization $P(M)$ is given by its characteristic function via (3.12), i.e.,

$$\chi(P(M)) := P(\chi(M)). \tag{3.13}$$

In the case where u is continuous and M is open or closed, equations (3.12) and (3.13) must be understood in the pointwise sense.

Equations (3.12) and (3.13) can also be written in the following more precise form:

$$P(M) = \big((M \cup \sigma_H(M)) \cap H\big) \cup \big(M \cap \sigma_H(M)\big), \quad M \in \mathcal{M}_n, \tag{3.14}$$

and

$$Pu(x) = \operatorname{ess\,sup}\big\{c > \inf u \,:\, x \in P(\{u > c\})\big\}, \quad x \in \mathbb{R}^n, \quad u \in \mathcal{S}_n. \tag{3.15}$$

Of course, if $u \in \mathcal{S}_n$ is continuous, then "ess sup" in (3.15) can be replaced by "sup". From (3.12)–(3.15) we see that the polarization P is an open and compact rearrangement continuous from the inside and from the outside.

For simplicity, we shall often use the subscript "$_H$" to denote any one of the polarized objects, i.e., we write u_H and M_H for Pu and $P(M)$, respectively.

It should be mentioned that the polarization of a connected set is not necessarily connected and may contain one multiply connected component.

There are three major approaches to polarization. The first, initiated in [33, 2] and [7], uses convolution type inequalities. In the most powerful and general form, this approach culminated in Baernstein's fundamental work described in [6].

The second approach was introduced by Dubinin [15] who used the following representation of polarized functions. Let $v(x) = u(\overline{x})$, $w(x) = u_H(\overline{x})$, with $x \in H$. Then

$$u_H(x) = (u(x) - v(x))_+ \quad \text{and} \quad w(x) = u(x) - (u(x) - v(x))_+.$$

Under certain conditions, this leads to

$$\nabla u_H(x) = \begin{cases} \nabla u(x) & \text{a.e. on } \ \{u > v\} \cap H, \\ \nabla v(x) & \text{a.e. on } \ \{u \le v\} \cap H, \end{cases}$$

$$\nabla w(x) = \begin{cases} \nabla v(x) & \text{a.e. on } \ \{u > v\} \cap H, \\ \nabla u(x) & \text{a.e. on } \ \{u \le v\} \cap H, \end{cases}$$

which readily results in several Dirichlet type inequalities, see [14, Lemma 5.3].

The third approach suggested by Solynin [28] and developed further in [14] rests on a direct application of the maximum principle to prove comparison theorems for solutions of two related boundary-value problems for certain partial differential equations defined in a given domain Ω and in the polarized domain Ω_H. This approach will be used in §11.

§4. Continuous $(1, n)$-Steiner symmetrization

First, we define a continuous transformation on $\mathbb{R}$. For $M \in \mathcal{M}_1$ and $-\infty < t \le \infty$, the *measuring function of* M is defined by

$$m_M(t) = \mathcal{L}^1\big((-\infty, t) \cap M\big). \tag{4.1}$$

This definition shows that $m_M(t)$ is monotone nondecreasing and Lipschitz continuous with constant 1:

$$0 \le m_M(t_2) - m_M(t_1) \le t_2 - t_1 \quad \text{for all } \ t_1 \le t_2. \tag{4.2}$$

This implies that, for every $t \in \mathbb{R}$, the equation

$$y - (1/2)m_M(y) = t \tag{4.3}$$

has a unique solution $y = y_M(t) \in [t, \infty)$. The function $y = y_M(t)$, called *the separating function of* M will play an important role in this study. The two basic properties of $y_M(t)$ given by the following lemma are immediate consequences of the above definitions and inequalities (4.2).

Lemma 4.1. **(a)** *If $M \subset N$, then $y_M(t) \le y_N(t)$ for all $t \in \mathbb{R}$.*
(b) *If $t_1 < t_2$, then*

$$t_2 - t_1 \le y_M(t_2) - y_M(t_1) \le 2(t_2 - t_1). \tag{4.4}$$

For $t \in \mathbb{R}$, the T^t-transformation of M is defined as follows:

$$M^t = (y_M(t) - m_M(t), y_M(t)) \cup (M \cap [y_M(t), \infty)) \quad \text{if } M \text{ is open,} \tag{4.5}$$

$$M^t = [y_M(t) - m_M(t), y_M(t)] \cup (M \cap [y_M(t), \infty)) \quad \text{if } M \text{ is compact.} \tag{4.6}$$

If $M \in \mathcal{M}_n$ but is neither open or compact, then M^t is defined in the a.e. sense by either one of the equations (4.5) or (4.6).

Definition 4.1. The family of mappings $T^t : \mathcal{M}_1 \to \mathcal{M}_1$, $t \in \mathbb{R}$, defined by $T^t(M) = M^t$ is called the continuous symmetrization on $\mathbb{R}$.

Now we turn to $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. For $\Omega \in \mathcal{M}_n$ and $x' \in \mathbb{R}^{n-1}$, let $\Omega(x') = \{y \in \mathbb{R} : (x', y) \in \Omega\}$ be the 1-slice of Ω at x'. If $\Omega \in \mathcal{M}_n$ is open or compact, then Ω' will denote the orthogonal projection of Ω to $\mathbb{R}^{n-1}$. Otherwise we put $\Omega' = \{x' \in \mathbb{R}^{n-1} : \mathcal{L}^1(\Omega(x')) \neq 0\}$. The measuring function of Ω is defined by

$$m_\Omega(x', t) = \begin{cases} m_{\Omega(x')}(t) & \text{if } x' \in \Omega', \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

One can easily show that $m_\Omega(x', t)$ is lower semicontinuous in x' if Ω is open and it is upper semicontinuous in x' if Ω is compact.

Definition 4.2. The function $y_\Omega : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ defined by

$$y_\Omega(x', t) = \begin{cases} y_{\Omega(x')}(t) & \text{if } x' \in \Omega', \\ t & \text{otherwise,} \end{cases} \tag{4.8}$$

is called the separating function, and the graph $F_\Omega(t) = \{(x', y_\Omega(x', t)) : x' \in \mathbb{R}^{n-1}\}$ is called the frontier of symmetrization of Ω at t.

To simplify notation, we shall skip the symbol of the set if its meaning is clear from the context. Thus, we often write $m(x', t)$, $y(x', t)$, etc. instead of $m_\Omega(x', t)$, $y_\Omega(x', t)$, etc.

Lemma 4.2. *For a fixed $t \in \mathbb{R}$ and $\Omega \in \mathcal{M}_n$, the separating function $y(x', t)$ is lower semicontinuous on $\mathbb{R}^{n-1}$ if Ω is open and it is upper semicontinuous on $\mathbb{R}^{n-1}$ if Ω is compact.*

Proof. Suppose Ω is open and $x'_0 \in \mathbb{R}^{n-1}$. If $m(x'_0, t) = 0$, then $y(x', t) \ge t = y(x'_0, t)$ for all x', and lower semicontinuity follows.

Assume that $y(x', t)$ is not lower semicontinuous at x'_0 such that $m(x'_0, t) > 0$. Then for some $\delta > 0$ and some sequence $x'_k \to x'_0$,

$$y(x'_k, t) \le y(x'_0, t) - \delta, \quad k = 1, 2, \ldots. \tag{4.9}$$

From (4.3), (4.2), and (4.9) we obtain

$$\begin{aligned} &m(x'_k, y(x'_0,t)-\delta) - 2(y(x'_k,t)-t) \\ &\qquad = m(x'_k, y(x'_0,t)-\delta) - m(x'_k, y(x'_k,t)) \le y(x'_0,t) - y(x'_k,t) - \delta. \end{aligned} \tag{4.10}$$

Combined with (4.9), this gives

$$\limsup_{k\to\infty} m(x'_k, y(x'_0,t)-\delta) \le 2(y(x'_0,t)-t-\delta). \tag{4.11}$$

Since Ω is open, the function $m(x',t)$ is lower semicontinuous in x'. Hence,

$$\liminf_{k\to\infty} m(x'_k, y(x'_0,t)-\delta) \ge m(x'_0, y(x'_0,t)-\delta). \tag{4.12}$$

Using (4.2) and (4.3), we obtain

$$m(x'_0, y(x'_0,t)-\delta) \ge m(x'_0, y(x'_0,t)) - \delta = 2(y(x'_0,t)-t) - \delta. \tag{4.13}$$

Now (4.12) and (4.13) yield the inequality

$$\liminf_{k\to\infty} m(x'_k, y(x'_0,t)-\delta) \ge 2(y(x'_0,t)-t) - \delta,$$

which contradicts (4.11). This proves Lemma 4.2 for the case of open sets. If Ω is compact the proof is similar and is left to the reader. □

Now we are ready to define what the 1-dimensional continuous symmetrization is.

Definition 4.3. A family of set transformations $T^t : \mathcal{M}_n \to \mathcal{M}_n$, $t \in \mathbb{R}$, defined by

$$T^t(\Omega) = \Omega^t := \{(x',y) : x' \in \Omega',\ y \in \Omega^t(x')\} \tag{4.14}$$

will be called a continuous 1-symmetrization or SC 1-symmetrization. Any single transformation of this family will be called a T^t-transformation. For $t = -\infty$, we define $T^{-\infty}$ to be the identity transformation.

The set Ω^t itself will be called the *T^t-transformation of* Ω or the *partial symmetrization of* Ω with respect to the plane $\{y = t\}$. Sometimes we shall refer to this plane $\{y = t\}$ as the *moving plane of symmetrization*. Accordingly, the parameter t will be called the *height* of the moving plane.

The frontier of symmetrization $F_\Omega(t)$ divides $\mathbb{R}^n$ into two parts

$$H_+(t) = \big\{(x',y) : y > y(x',t)\big\} \quad \text{and} \quad H_- = \big\{(x',y) : y < y(x',t)\big\} \tag{4.15}$$

called the upper and the lower subspace of symmetrization, respectively. Let

$$\Omega_-(t) = \Omega \cap H_-(t), \quad \Omega_+(t) = \Omega \setminus H_-(t) \quad \text{if } \Omega \text{ is open}, \tag{4.16}$$

and let

$$\Omega_+(t) = \Omega \cap H_+(t), \quad \Omega_-(t) = \Omega \setminus H_+(t) \quad \text{if } \Omega \text{ is compact}. \tag{4.17}$$

If Ω is neither open nor compact, then $\Omega_-(t)$ and $\Omega_+(t)$ are defined a.e. by (4.17).

In Lemma 4.3 below we list some useful properties of SC symmetrization, which follow directly from the above definitions. By $\Omega^*_-(t)$ we denote the $(1,n)$-Steiner symmetrization of $\Omega_-(t)$ with respect to $\{y = t\}$.

Lemma 4.3.

(a) *$\Omega^t = \Omega^*_-(t) \cup \Omega_+(t)$ for all $t \in \mathbb{R}$ and every $\Omega \in \mathcal{M}_n$.*
(b) *$\mathcal{L}^1(\Omega^t(x')) = \mathcal{L}^1\big(\Omega(x')\big)$ for all $x' \in \mathbb{R}^{n-1}$, and $\mathcal{L}^n(\Omega^t) = \mathcal{L}^n(\Omega)$.*
(c) *If Ω is open, then $\Omega_-(t)$, $\Omega^*_-(t)$, and Ω^t are open sets, and $\Omega^*_-(t) \cap \{y \ge t\} = H_-(t) \cap \{y \ge t\}$.*
(d) *If Ω is compact, then $\Omega_-(t)$, $\Omega^*_-(t)$, and Ω^t are compact sets, and $\Omega^*_-(t) \cap \{y \ge t\} = \{y \ge t\} \setminus H_+(t)$.*

We note that Lemma 4.3(a) gives an equivalent definition of Ω^t and Lemma 4.3(b) shows that the T^t-transformation preserves measures.

Now we show that SC symmetrization possesses all basic geometric properties of classical symmetrizations.

Lemma 4.4. *For a fixed $t \in \mathbb{R}$, the T^t-transformation is an open, compact, and smoothing rearrangement, which is continuous from the inside and from the outside.*

Proof. Lemma 4.1(a) and Definition 4.3 imply that $T^t(M) \subset T^t(N)$ if $M \subset N$ and therefore the T^t-transformation is monotone. Since, by Lemma 4.3(b), the T^t-transformation preserves the measure of sets, it follows that the T^t-transformation is a rearrangement.

Using Definition 4.3 and Lemma 4.3(c) and (d), one can easily verify that the T^t-transformation is continuous from the inside and from the outside.

To prove that the T^t-transformation is smoothing, we fix a compact set K and $\varepsilon > 0$. Since, by Lemma 4.3(a), $K^t = K^*_-(t) \cup \overline{K_+(t)}$, we have

$$K^t + \varepsilon\overline{B^{(n)}} = \big(K^*_-(t) + \varepsilon\overline{B^{(n)}}\big) \cup \big(\overline{K_+(t)} + \varepsilon\overline{B^{(n)}}\big). \tag{4.18}$$

Lemma 4.3(d) and the smoothing property of the Steiner symmetrization show that

$$T^t\big(K + \varepsilon\overline{B^{(n)}}\big) \supset T^t\big(K_-(t) + \varepsilon\overline{B^{(n)}}\big) = \big(K_-(t) + \varepsilon\overline{B^{(n)}}\big)^* \supset K^*_-(t) + \varepsilon\overline{B^{(n)}}, \tag{4.19}$$

where $(\cdot)^*$ denotes the Steiner symmetrization with respect to $\{y = t\}$. Since $K_+(t)$ remains unchanged under the T^t-transformation of K, one can easily see that

$$T^t\big(K + \varepsilon\overline{B^{(n)}}\big) \supset T^t\big(K_+(t) + \varepsilon\overline{B^{(n)}}\big) = \overline{K_+(t)} + \varepsilon\overline{B^{(n)}}. \tag{4.20}$$

Now (4.18)–(4.20) yield

$$T^t\big(K + \varepsilon\overline{B^{(n)}}\big) \supset T^t(K) + \varepsilon\overline{B^{(n)}},$$

which shows that the T^t-transformation is smoothing.

It was shown by Sarvas (see Lemmas 3.2 and 3.3 in [14]) that the properties established above imply that the T^t-transformation is open and compact. The proof is complete. □

Since the T^t-transformation is a rearrangement of sets for any fixed $t \in \mathbb{R}$, there is a standard way to define a corresponding transformation of functions.

Definition 4.4 (SC 1-symmetrization of functions)**.** Let $u \in \mathcal{S}_n$. Then the family of functions u^t, $t \in \mathbb{R}$, defined for $x \in \mathbb{R}^n$ by

$$u^t(x) := \begin{cases} \operatorname{ess\,sup}\{c > \inf u \,:\, x \in T^t(\{u > c\})\} & \text{if } x \in \bigcup_{c>\inf u} T^t(\{u > c\}), \\ \inf u & \text{if } x \notin \bigcup_{c>\inf u} T^t(\{u > c\}), \end{cases}$$

is called the SC 1-symmetrization of u.

§5. Continuity properties of SC symmetrization

Now we prove that the T^t-transformation depends continuously on the parameter t, which justifies the use of the word "continuous" in the name of this transformation.

Lemma 5.1. *Let t_k, $k = 1, 2, \dots$, be an increasing sequence such that $t_k \to t' < \infty$, and let $\Omega' = \Omega^{t'}$ and $\Omega_k = \Omega^{t_k}$. Then*

$$d(\partial\Omega_k, \partial\Omega') \to 0 \quad \text{as} \quad k \to \infty \tag{5.1}$$

if Ω is an open bounded set, and

$$d(\Omega_k, \Omega') \to 0 \quad \text{as} \quad k \to \infty \tag{5.2}$$

if Ω is a compact set.

Proof. We prove the lemma for open sets. The proof for compact sets follows the same lines, it is easier and is left to the reader.

If (5.1) fails, then there exist subsequences of the sequences t_k and Ω_k, which we still call t_k and Ω_k, such that

$$d(\partial\Omega_k, \partial\Omega') \geq \varepsilon, \quad k = 1, 2, \dots, \tag{5.3}$$

for some $\varepsilon > 0$. Since Ω is bounded, equation (5.3) implies that there are subsequences (again denoted by t_k and Ω_k) and a sequence of points $x_k \in \mathbb{R}^n$ with $x_k \to x_0 \in \mathbb{R}^n$ such that one of the following two conditions is satisfied:

(a) $x_k \in \partial\Omega'$ and $\operatorname{dist}(x_k, \partial\Omega_k) \geq \varepsilon$ for all $k = 0, 1, 2, \dots$,

(b) $x_k \in \partial\Omega_k$ for $k = 1, 2, \dots$ and $\operatorname{dist}(x_k, \partial\Omega') \geq \varepsilon$ for $k = 0, 1, 2, \dots$

Case (a) contradicts Lemma 5.2 below, which characterizes the stability property of the boundary of Ω^t.

Now we consider case (b). Let

$$x_k = (x'_k, y_k) \to (x'_0, y_0) = x_0.$$

Then

$$x_k \in \partial(\Omega^*_-(t_k)) \cup \partial(\Omega_+(t_k)).$$

Taking a subsequence if necessary, we may restrict ourselves to two cases.

1) In the first case we assume that $x_k \in \partial\Omega_+(t_k)$ for all $k = 1, 2, \dots$. Since $x_k \notin \Omega_k$, we have $x_k \notin \Omega_+(t_k)$. Now if $x_k \in \Omega$, then $x_k \in \Omega_-(t_k)$. Since $\Omega_-(t_k)$ is open and $\Omega_-(t_k) \cap \Omega_+(t_k) = \varnothing$, we conclude that $x_k \notin \partial\Omega_+(t_k)$, which contradicts the above assumption. Thus in the case under consideration we have $x_k \notin \Omega$, whence $x_k \in \partial\Omega$ for all $k = 1, 2, \dots$. Taking the limit, we get

$$x_0 \in \partial\Omega. \tag{5.4}$$

Since $x_k \in \partial\Omega_+(t_k)$, we can use (4.4) to obtain

$$y_k \geq y(x'_k, t_k) \geq y(x'_k, t') - 2(t' - t_k).$$

Passing to the limit, we get

$$y_0 \geq y(x'_0, t'), \tag{5.5}$$

because $y(x', t)$ is lower semicontinuous in x'. Equations (5.4) and (5.5) imply

$$x_0 \notin \Omega'. \tag{5.6}$$

Now we investigate possible dispositions of y_0, $y(x'_0, t')$, and t'.

(i) If $y_0 = y(x'_0, t') > t'$, then $x_0 \in \partial\Omega'$, contradicting assumption (b).

(ii) The second possibility is

$$y_0 > y(x'_0, t') \geq t'.$$

From assumption (b) and (5.6) it follows that there exists a ball $B^{(n)}_{\varepsilon_1}(x_0)$ with a small radius $\varepsilon_1 > 0$ such that

$$\overline{B^{(n)}_{\varepsilon_1}}(x_0) \subset (\mathbb{R}^n \setminus \Omega') \cap H_+(t'). \tag{5.7}$$

By (5.4), there exists a point $\widehat{x} \in \Omega \cap B^{(n)}_{\varepsilon_1}(x_0)$. Since $B^{(n)}_{\varepsilon_1}(x_0) \subset H_+(t')$, we must have $\widehat{x} \in \Omega'$. But this contradicts (5.7) and, therefore, contradicts (b).

(iii) Let $y_0 = t'$. Assumption (b) and (5.6) imply that there is a ball $B^{(n)}_{\varepsilon_1}(x_0)$ such that $B^{(n)}_{\varepsilon_1}(x_0) \subset \mathbb{R}^n \setminus \Omega'$. Then the definition of $\Omega^{t'}$ shows that $B^{(n)}_{\varepsilon}(x_0) \subset \mathbb{R}^n \setminus \Omega$, which contradicts (5.4). This finishes the proof of the lemma in case 1) under assumption (b).

2) In the second case we assume that

$$x_k \in \partial(\Omega^*_-(t_k)) \setminus \partial\Omega_+(t_k) \quad \text{for all} \quad k = 1, 2, \dots \tag{5.8}$$

As in the case 1), we consider possible dispositions of y_0, $y(x_0', t')$, and t'.

(i) If $y_0 < t'$, then $y_k < t_k$ for all sufficiently large k. Since $x_k \in \partial(\Omega_-^*(t_k))$ and the open interval between the points $(x_k', 2t_k - y(x_k', t_k))$ and $(x_k', y(x_k', t_k))$ is in $\Omega_-^*(t_k)$, we have

$$y_k \le 2t_k - y(x_k', t_k) \le 2t' - y(x_k', t') \tag{5.9}$$

for all sufficiently large k. The second inequality here follows from (4.4). The latter inequality implies that $x_k \notin \Omega'$, so that x_0 is not an inner point of Ω'.

We show that x_0 cannot be an outer point of Ω'. Since $x_k \in \partial\Omega_-^*(t_k)$ and $x_k \to x_0$, there is a sequence of points $\widehat{x}_k = (\widehat{x}_k', \widehat{y}_k) \in \Omega_-^*(t_k)$ such that $\widehat{x}_k \to x_0$. Then we have $2t_k - y(\widehat{x}_k', t_k) < \widehat{y}_k$. Therefore, using (4.4), we obtain

$$2t' - y(\widehat{x}_k', t') \le 2t_k - y(\widehat{x}_k', t_k) + (t' - t_k) < (t' - t_k) + \widehat{y}_k.$$

This yields $(\widehat{x}_k', (t' - t_k) + \widehat{y}_k) \in \Omega'$. Since $x_0 \notin \Omega'$, taking the limit we get $x_0 \in \partial\Omega'$, which contradicts assumption (b).

(ii) Let $y_0 > t'$. Then $y(x_k', t_k) \le y_k$ for all sufficiently large k. Therefore, by (4.4),

$$y(x_k', t') \le 2(t' - t_k) + y(x_k', t_k) \le 2(t' - t_k) + y_k.$$

Passing to the limit and using the semicontinuity property of the separating function, we obtain

$$y(x_0', t') \le y_0. \tag{5.10}$$

The case of equality $y(x_0', t') = y_0 (> t')$ leads to the fact that $x_0 \in \partial\Omega'$, which contradicts assumption (b). Indeed, if $x_0 \notin \partial\Omega'$, then $x_0 \in \Omega'$. Therefore, $x_k \in \Omega$ for all sufficiently large k. Since $x_k \in \partial\Omega_-^*(t_k)$ and $x_k \in \Omega$, it follows that $x_k \in \Omega_+(t_k)$. Therefore, $x_k \in \partial\Omega_+(t_k)$ if k is sufficiently large, which contradicts (5.8).

Now, let $y(x_0', t') < y_0$. If $x_0 \notin \Omega'$, then assumption (b) shows that there is a ball $B_{\varepsilon_2}^{(n)}(x_0)$ with some small radius $\varepsilon_2 > 0$ such that

$$\overline{B_{\varepsilon_2}^{(n)}}(x_0) \subset \big(\mathbb{R}^n \setminus \Omega'\big) \cap H_+(t').$$

Then $\overline{B_{\varepsilon_2}^{(n)}}(x_0) \cap \Omega = \varnothing$. Since $\overline{B_{\varepsilon_2}^{(n)}}(x_0) \subset H_+(t_k)$, we have $\overline{B_{\varepsilon_2}^{(n)}}(x_0) \cap \Omega_k = \varnothing$ for all $k = 1, 2, \ldots$. The latter is impossible because $x_k \in \partial\Omega_k$ and $x_k \to x_0$.

If $x_0 \in \Omega'$ and $y(x_0', t') < y_0$, then $x_0 \in \Omega$. Hence, $x_k \in \Omega$ for all sufficiently large k. Since $x_k \in \partial\Omega_-^*(t_k)$, this implies that $x_k \in \Omega_k$, contradicting assumption (b) if k is sufficiently large. This completes the proof of the lemma in the case where $y_0 > t'$.

(iii) Let $y_0 = t'$. If $x_0 \notin \Omega'$, then $\overline{B_\varepsilon^{(n)}}(x_0) \cap \Omega' = \varnothing$ by assumption (b). This implies that $\overline{B_\varepsilon^{(n)}}(x_0) \cap \Omega_-^*(t_k) = \varnothing$ for all $k = 1, 2, \ldots$, which contradicts the assumptions $x_k \in \partial\Omega_-^*(t_k)$ and $x_k \to x_0$.

Let $x_0 \in \Omega'$. Then, by assumption (b), $B_\varepsilon^{(n)}(x_0) \subset \Omega_-^*(t')$. Together with inequality (4.4), this implies that $B_{\varepsilon/2}^{(n)}(x_0) \subset \Omega_-^*(t_k)$ for all sufficiently large k, again contradicting assumption (b). Now the proof of the lemma for open sets is complete. □

We pass to a lemma characterizing a certain stability property of the boundary $\partial\Omega^t$.

Lemma 5.2. *Let $-\infty < s < t < \infty$. Then*

$$\partial\Omega^t \subset \partial\Omega^s + (t - s)\overline{B^{(n)}} \tag{5.11}$$

if Ω is open, and

$$\Omega^t \subset \Omega^s + (t - s)\overline{B^{(n)}} \tag{5.12}$$

if Ω is a compact.

Proof. Once again we give a proof for open sets. The easier case of compact sets is left to the reader.

Suppose that Ω is open and (5.11) fails. Then there exists a point $x_0 \in \partial\Omega^t$ such that

$$\operatorname{dist}(x_0, \partial\Omega^s) > (t-s). \tag{5.13}$$

By Lemma 4.3(a), we have $\Omega^t = \Omega^*_-(t) \cup \Omega_+(t)$. Hence,

$$\partial\Omega^t \subset \partial\Omega^*_-(t) \cup \partial\Omega_+(t).$$

As in the proof of Lemma 5.1, we consider two cases.

1) In the first case we assume that

$$x_0 \in \partial\Omega_+(t). \tag{5.14}$$

By Lemma 4.3(c), the set $\Omega_-(t)$ is open. By the definitions of $\Omega_-(t)$ and $\Omega_+(t)$, we have $\Omega_-(t) \cap \Omega_+(t) = \varnothing$. Therefore, (5.14) implies that $x_0 \notin \Omega_-(t)$.

Since $\Omega_+(t) \subset \Omega$, we must have $x_0 \in \bar{\Omega}$. If $x_0 \in \Omega$, we conclude that $x_0 \in \Omega_+(t)$, whence $x_0 \in \Omega^t$. Since Ω^t is open by Lemma 4.3(c), this contradicts the assumption $x_0 \in \partial\Omega^t$. Therefore,

$$x_0 \in \partial\Omega. \tag{5.15}$$

Since $\Omega_+(t) \subset \Omega_+(s) \subset \Omega$, relations (5.14) and (5.15) imply

$$x_0 \in \partial\Omega_+(s). \tag{5.16}$$

Hence, $x_0 \in \overline{\Omega^s}$. This implies that $x_0 \in \partial\Omega^s$. Indeed, let $x_0 \in \Omega^s$. Since $x_0 \notin \Omega$, we conclude that $x_0 \notin \Omega_+(s)$. Therefore, in the case under consideration we must have

$$x_0 \in \Omega^*_-(s). \tag{5.17}$$

By Lemma 4.3(c), the set $\Omega^*_-(s)$ is open, and the definitions show that $\Omega^*_-(s) \cap \Omega_+(s) = \varnothing$. Therefore, (5.17) contradicts (5.16), and we must have $x_0 \in \partial\Omega^s$, contradicting (5.13).

2) In the second case we assume that $x_0 = (x_0', y_0) \in \partial\Omega^*_-(t) \setminus \partial\Omega_+(t)$. Now we consider two subcases.

(i) Suppose that $y_0 \le t$. Since $x_0 \in \Omega^t$, we have

$$y_0 \le 2t - y(x_0', t).$$

Together with (4.4), this implies

$$y_0 - (t-s) \le 2s - y(x_0', s),$$

which yields

$$\widetilde{x} := (x_0', y_0 - (t-s)) \notin \Omega^s. \tag{5.18}$$

Since $x_0 \in \partial\Omega^*_-(t)$, for each $\varepsilon > 0$ there exists a point $\widehat{x} = (\widehat{x}', \widehat{y}) \in \Omega^*_-(t)$ such that

$$|\widehat{x} - x_0| < \varepsilon. \tag{5.19}$$

Let $y_0 < s$. Then $m(\widehat{x}', \widehat{y}) > 0$; therefore, $m(\widehat{x}', s) > 0$ if $\widehat{x}$ is sufficiently close to x_0. Therefore, the interval

$$((\widehat{x}', y) \,:\, 2s - y(\widehat{x}', s) < y < y(\widehat{x}', s))$$

is nonempty and is included in Ω^s.

Using (4.4) and the relation $\widehat{x} \in \Omega^*_-(t)$, we conclude that

$$2s - y(\widehat{x}', s) \le 2t - y(\widehat{x}', t) < \widehat{y} < s$$

if $\varepsilon > 0$ in (5.19) is sufficiently small. Then we have $\widehat{x} \in \Omega^s$. Therefore, the closed interval $[\widehat{x}, \widetilde{x}]$ contains a point $z \in \partial\Omega^s$. Hence,

$$|z - x_0| \le \max\{|x_0 - \widetilde{x}|,\ |x_0 - \widehat{x}|\} = (t-s)$$

if $\varepsilon > 0$ in (5.19) does not exceed $t - s$, which contradicts inequality (5.13).

Let $y_0 = s$, and let $\widehat{y} \le s$. Then $m(\widehat{x}', s) > 0$ and $(\widehat{x}', s) \in \Omega^s$. The closed interval $[x_0, (\widehat{x}', s)]$ contains a point $z \in \partial\Omega^s$. Hence, $|x_0 - z| \le \varepsilon$, which again contradicts (5.13) if $\varepsilon < t - s$.

Let $y_0 = s$, $\widehat{y} > s$. If $m(\widehat{x}', s) > 0$, then we argue as in the previous case.

Let $m(\widehat{x}', s) = 0$. Since $m(\widehat{x}', \widehat{y}) > 0$, there exists a point $z = (\widehat{x}', \zeta) \in \Omega$ such that $s < \zeta \le \widehat{y}$. But in this case we have $z \in \Omega^s$ and $|z - x_0| \le |\widehat{x} - x_0| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrarily small, this leads to a contradiction with (5.13).

Let $s < y_0 \le t$. If $m(\widehat{x}', s) > 0$, then $(\widehat{x}', s) \in \Omega^s$ and

$$|x_0 - (\widehat{x}', s)| \le |x_0 - (x_0', s)| + |(x_0', s) - (\widehat{x}', s)| \le (t - s) + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this inequality contradicts (5.13).

Let $m(\widehat{x}', s) = 0$. Since $m(\widehat{x}', \widehat{y}) > 0$, there exists a point $z = (\widehat{x}', \zeta) \in \Omega$ such that $s < \zeta < \widehat{y}$. Then $z \in \Omega^s$. Moreover,

$$|z - x_0| \le (t - s) + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the latter inequality again contradicts (5.13).

(ii) Let $y_0 > t$. Since $x_0 \notin \Omega^t$, we have $y_0 \ge y(x_0', t) \ge y(x_0', s)$. This implies that $x_0 \notin \Omega^s$. Since $x_0 \in \partial\Omega^*_-(t)$, we conclude that for every $\varepsilon > 0$ there exists a point $\widehat{x} = (\widehat{x}', \widehat{y}) \in \Omega^*_-(t)$ such that

$$|\widehat{x} - x_0| < \varepsilon. \tag{5.20}$$

If $\varepsilon > 0$ is sufficiently small, we have $t < \widehat{y} < y(\widehat{x}', t)$.

If $y(\widehat{x}', s) \ge \widehat{y}$, then $\widehat{x} \in \overline{\Omega^s}$. Therefore, (5.20) contradicts (5.13).

Now we consider the case where $y(\widehat{x}', s) < \widehat{y}$. Denoting $I := \big((\widehat{x'}, y(\widehat{x'}, t)), (\widehat{x'}, y(\widehat{x'}, s))\big)$, we put $\Omega_I = I \cap \Omega$. From (4.3), it follows that the linear measure of Ω_I equals $2(y(\widehat{x'}, t) - y(\widehat{x'}, s)) - 2(t - s)$. Since $\widehat{x} \in I$, we have

$$\operatorname{dist}(\widehat{x}, \Omega_I) \le \mathcal{L}^1(I) - \mathcal{L}^1(\Omega_I).$$

This inequality together with (4.4) implies

$$\operatorname{dist}(\widehat{x}, \Omega_I) \le 2(t - s) - (y(\widehat{x'}, t) - y(\widehat{x'}, s)) \le t - s.$$

Since $\Omega_I \subset \Omega^s$, there exists a point $z \in \overline{\Omega^s}$ such that $|\widehat{x} - z| \le t - s$. Finally, we have

$$|x_0 - z| \le |x_0 - \widehat{x}| + |\widehat{x} - z| \le (t - s) + \varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this inequality contradicts (5.13). The proof of the lemma for open sets is complete. □

§6. Stability and semigroup properties of the T^t-transformation

First, we prove a lemma characterizing stability with respect to polarization, cf. [27, Theorem 3].

Lemma 6.1. *For $s \in \mathbb{R}$, let P_s denote polarization with the polarizer $H_s = \{(x', y) \in \mathbb{R}^n : y > s\}$. If $t_1 < t_2$ and $s \le (1/2)(t_1 + t_2)$, then*

$$T^{t_2} \circ P_s \circ T^{t_1} = T^{t_2} \quad \text{on } \mathcal{M}_n. \tag{6.1}$$

Proof. Let $\Omega \in \mathcal{M}$ be an open set, and let $D = P_s(\Omega^{t_1})$. We must show that

$$D^{t_2} = \Omega^{t_2}. \tag{6.2}$$

By Lemma 4.3(a),

$$D^{t_2} = D^*_-(t_2) \cup D_+(t_2), \quad \Omega^{t_2} = \Omega^*_-(t_2) \cup \Omega_+(t_2). \tag{6.3}$$

Since the T^t-transformation and polarization both preserve measure in 1-slices, from (6.3) it follows that (6.2) is true if and only if

$$D_+(t_2) = \Omega_+(t_2). \tag{6.4}$$

Since $\Omega^{t_1} \cap \{(x', y) : y < 2t_1 - y_\Omega(x', t_1)\} = \varnothing$, the definition of polarization and T^t-transformation shows that

$$\begin{aligned} &P_s(\Omega^{t_1}) \cap \{(x', y) : y \geq 2(s - t_1) + y_\Omega(x', t_1)\} \\ &\quad = \Omega^{t_1} \cap \{(x', y) : y \geq 2(s - t_1) + y_\Omega(x', t_1)\} \\ &\quad = \Omega \cap \{(x', y) : y \geq 2(s - t_1) + y_\Omega(x', t_1)\}. \end{aligned} \tag{6.5}$$

Since $s \leq (1/2)(t_1 + t_2)$, (4.4) implies

$$y_\Omega(x', t_2) \geq y_\Omega(x', t_1) + 2(s - t_1). \tag{6.6}$$

Now from (6.5) and (6.6) we conclude that

$$P_s(\Omega^{t_1}) \cap \{(x', y) : y \geq y_\Omega(x', t_2)\} = \Omega \cap \{(x', y) : y \geq y_\Omega(x', t_2)\},$$

which implies (6.4).

For open sets the proof is complete. If Ω is not open, the proof follows the same lines. □

Lemma 6.1 and Definition 4.4 immediately imply the next statement.

Corollary 6.1. *Let $u(x) \in \mathcal{S}_n$, $t_1 < t_2$, and let $s \leq (1/2)(t_1 + t_2)$. Then*

$$\big((u^{t_1})_{H_s}\big)^{t_2} = u^{t_2}.$$

In the particular case where $s = t_1$, Lemma 6.1 and its corollary show that our T^t-transformation possesses the following "presemigroup property", which was first mentioned in [27]. A similar property was used in [13] to define the BC symmetrization.

Corollary 6.2. *Let T^t denote a T^t-transformation on $\mathcal{M}_n$ or on $\mathcal{S}_n$. Then*

$$T^{t_2} \circ T^{t_1} = T^{t_2} \quad \textit{for all} \;\; t_1 < t_2. \tag{6.7}$$

To justify the name "presemigroup property", we consider the transformation $\widehat{T}^t = St^t \circ T^t$, where $t \geq 0$ and St^t denotes a continuous shift of t units in the direction of the negative y-axis. Then it is easily seen that (6.7) is equivalent to the following familiar semigroup property of $\widehat{T}^t$:

$$\widehat{T}^{t+s} = \widehat{T}^t \circ \widehat{T}^s \quad \text{for all} \;\; t \geq 0,\, s \geq 0.$$

In §8, a similar composite transformation will be used to replace the parameter t, the range of which is $-\infty < t < \infty$, by a parameter τ with the standard range $0 \leq \tau \leq 1$.

Now we prove a criterion for Ω and u to possess a partial symmetry.

Lemma 6.2.

(a) *If $\Omega \in \mathcal{M}_n$ is open or compact, then $\Omega = \Omega^t$ if and only if $P_s(\Omega) = \Omega$ for all $s \leq t$.*

(b) *If $u \in \mathcal{S}_n$, then $u \equiv u^t$ if and only if $u_{H_s} \equiv u$ for all $s \leq t$.*

Proof. (a) We work with an open set Ω. The "only if" part is obvious: if $\Omega = \Omega^t$, then Lemma 4.3(a) shows that $P_s(\Omega) = \Omega$ for all $s \le t$.

To prove the "if" part, we assume that $P_s(\Omega) = \Omega$ for all $s \le t$ but $\Omega \ne \Omega^t$. Since $\Omega_+(t) = (\Omega^t)_+(t)$, we conclude that $\Omega_-(t) \ne \Omega^*_-(t)$. Then, by Lemma 6.3 in [14], there is a polarizer $H_{s_0} = \{y > s_0\}$ with $s_0 < t$ such that

$$P_{s_0}(\Omega_-(t)) \ne \Omega_-(t). \tag{6.8}$$

Then there is $x_0 = (x_0', y_0)$ with $y_0 < s_0$ such that

$$x_0 \in \Omega_-(t) \quad \text{and} \quad \widehat{x} := (x_0', 2s_0 - y_0) \notin \Omega_-(t). \tag{6.9}$$

If $2s_0 - y_0 < y(x_0', t)$, then (6.9) contradicts the assumption $P_s(\Omega) = \Omega$ for $s = s_0 < t$.

Let $2s_0 - y_0 \ge y(x_0', t)$. Since $P_s(\Omega) = \Omega$ for all $s \le t$ and $x_0 \in \Omega$, it follows that the closed interval $\{(x_0', y) : y_0 \le y \le 2t - s_0\}$ is in Ω. Then

$$m_\Omega(x_0', y(x_0', t)) > y(x_0', t) - y_0 \ge 2(y(x_0', t) - s_0) \ge 2(y(x_0', t) - t),$$

contradicting (4.3). The proof of Lemma 6.2(a) for open sets is complete. If Ω is not open the proof is left to the reader.

The proof of Lemma 6.2(b) easily follows from Lemma 6.2(a) and Definition 4.4. □

Our next result will be used in the proof of Theorem 11.3 in §11.

Corollary 6.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f \in L^2_+(\Omega)$, $f > 0$ on Ω. If for some $T \in \mathbb{R}$, f^T is not a translate of f in the direction of the y-axis, then we can find a polarizer $H_{t_0} = \{y > t_0\}$ with $t_0 < T$ such that $\left(f_{H_{t_0}}\right)^T = f^T$, $f \ne f_{H_{t_0}}$, and $\sigma_{H_{t_0}}(f) \ne f_{H_{t_0}}$.*

In particular, if Ω^T is not a translate of Ω in the direction of the y-axis, then we can find a polarizer $H_{t_0} = \{y > t_0\}$ with $t_0 < T$ such that $\left(\Omega_{H_{t_0}}\right)^T = \Omega^T$, $\Omega \ne \Omega_{H_{t_0}}$, and $\sigma_{H_{t_0}}(\Omega) \ne \Omega_{H_{t_0}}$.

Proof. Let $t_1 = \inf t$, where the infimum is taken over all $t \in \mathbb{R}$ such that $\Omega_+(t) = \varnothing$. Then f^t is a translate of f^{t_1} for all $t \ge t_1$. Therefore, when proving this corollary, we may assume that $T \le t_1$. In this case, f^T is not a translation of f if and only if $f^T \ne f$.

Let $t_\Omega = \sup t$, where the supremum is taken over all $t \in \mathbb{R}$ such that $P_\tau(f) = f$ for all $\tau \le t$. Then $f^{t_\Omega} = f$ by Lemma 6.2. Since $f^T \ne f$, the same lemma shows that $t_\Omega < T$. Combining this with Corollary 6.1, we obtain

$$(f_{H_s})^T = \left((f^{t_\Omega})_{H_s}\right)^T = f^T \quad \text{for all } s \le (1/2)(t_\Omega + T).$$

Since $t_\Omega < T$, $f^{t_\Omega} = f$, and $\Omega_+(t_\Omega) \ne \varnothing$, we have $\sigma_{H_s}(\Omega) \ne \Omega_{H_s}$, whence $\sigma_{H_s}(f) \ne f_{H_s}$ for all s sufficiently close to t_Ω. Therefore, every polarizer H_s with $t_\Omega < s < (1/2)(t_\Omega + T)$ such that $f_{H_s} \ne f$ satisfies the requirements of the corollary.

By the definition of t_Ω, there is a sequence $\{s_k\}_{k=1}^\infty$ converging to t_Ω from the right and such that $f_{H_{s_k}} \ne f$ for all k. Choosing k sufficiently large and such that $s_k \le (1/2)(t_\Omega + T)$, we obtain the desired polarizer $H_{t_0} = H_{s_k}$. This proves the first assertion of the corollary. Since $\Omega = \{f > 0\}$, the second assertion is a particular case of the first. □

Lemma 6.3. *Let $\Omega \in \mathcal{M}_n$, and let $(\cdot)^*$ denote the $(1,n)$-Steiner symmetrization with respect to the plane $\{y = t\}$. If $T^t(\Omega) = \Omega^*$ and $\Omega \in \mathcal{F} \cup \mathcal{G}_b$, then $T^t(\Omega_\varepsilon) = (\Omega_\varepsilon)^*$ for every $\varepsilon > 0$.*

Proof. Assume that $\Omega \in \mathcal{G}_b$. For a fixed $\varepsilon > 0$, let $y_\varepsilon(x', t)$ and $m_\varepsilon(x', y)$ denote, respectively, the separating function and the measuring function of Ω_ε. For $x' \in (\Omega_\varepsilon)'$, let

$$y_\varepsilon(x') = \sup\left\{y : (x', y) \in T^t(\Omega_\varepsilon)\right\}.$$

Then $y_\varepsilon(x') \geq y_\varepsilon(x', t)$. If $y_\varepsilon(x') = y_\varepsilon(x', t)$ for all $x' \in (\Omega_\varepsilon)'$, the claim of the lemma is immediate.

Suppose that

$$y_\varepsilon(x_0') > y_\varepsilon(x_0', t)$$

for some $x_0' \in (\Omega_\varepsilon)'$. Then (4.3) implies

$$m_\varepsilon(x_0', y_\varepsilon(x_0')) < 2(y_\varepsilon(x_0') - t). \tag{6.10}$$

Now we fix $\delta > 0$ sufficiently small. Then we can find two points $\widehat{x} = (x_0', \widehat{y})$ and $\widetilde{x} = (\widetilde{x}', \widetilde{y})$ satisfying the following conditions:

$$\widehat{x} \in T^t(\Omega_\varepsilon) \quad \text{and} \quad y_\varepsilon(x_0') - \delta < \widehat{y} < y_\varepsilon(x_0'), \tag{6.11}$$

$$\widetilde{x} \in \Omega \quad \text{and} \quad |\widehat{x} - \widetilde{x}| < \varepsilon. \tag{6.12}$$

Since $T^t(\Omega) = \Omega^*$ and $\widetilde{x} \in \Omega$, we have

$$m(\widetilde{x}', \widetilde{y}) \geq 2(\widetilde{y} - t). \tag{6.13}$$

Using (6.12) and (6.13), after an elementary geometric argument we obtain the inequality

$$m_\varepsilon(x_0', \widehat{y}) \geq 2(\widehat{y} - t);$$

together with (6.11), this implies

$$m_\varepsilon(x_0', y_\varepsilon(x_0')) \geq m_\varepsilon(x_0', \widehat{y}) \geq 2(y_\varepsilon(x_0') - t) - 2\delta.$$

Since $\delta > 0$ can be chosen arbitrarily small, the resulting inequality contradicts (6.10). This proves the lemma for open sets. If Ω is a compact set, the proof follows the same lines. □

§7. Approximation by polarizations

A compact set $K \subset \mathbb{R}^n$ is said to be *simple* if it can be decomposed into a finite number of blocks $\bar{R}_i = \bar{R}(a^i, b^i)$ such that $R_i \cap R_j = \varnothing$ if $i \neq j$. Here $\bar{R}(a, b)$ denotes the closure of an open block

$$R(a, b) := \{x \in \mathbb{R}^n \,:\, b_k < x_k < b_k + a_k, 1 \leq k \leq n\},$$

where $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$.

If K is a simple compact set in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, $n \geq 2$, then its projection K' is a simple compact set in $\mathbb{R}^{n-1}$. Let $K' = \bigcup_{i=1}^N \bar{R}_i'$ be a block decomposition of K'.

A collection of open sets $Cu(K) := \{C_i = R_k' \times \mathbb{R}\}_1^N$ will be called a *ubicle structure of* K. If C_i is a cubicle in $Cu(K)$, then $C_i \cap K = R_i' \times K^i$, where K^i is a simple compact set in $\mathbb{R}$. Therefore $C_i \cap K$ consists of a finite number of disjoint blocks $K_{i,j} = R_i' \times [\alpha_{ij}, \beta_{ij}]$, $1 \leq j \leq j_i$. We always enumerate these blocks in such a way that

$$\alpha_{i1} < \beta_{i1} < \alpha_{i2} < \cdots < \alpha_{ij_i} < \beta_{ij_i}, \quad 1 \leq i \leq N.$$

Thus, every simple compact set K admits a *block decomposition* of the form

$$K = \bigcup_{i=1}^N \overline{R'}_i \times K^i = \bigcup_{i,j} \bar{K}_{ij}.$$

In this section we show that the T^t-transformation of sets and functions can be approximated by sequences of polarizations. For the first time, this approach was used by Wolontis in [33]. The method was developed further by Dubinin [16, 17]. So, we shall call this method the *Wolontis–Dubinin approach*. In the context of continuous symmetrization, the Wolontis–Dubinin approach was first used in [27].

Lemma 7.1. *For every $t \in \mathbb{R}$ and every collection of simple compact sets $K_1, \dots, K_m$ in $\mathbb{R}^n$, there exist a finite number of polarizations P_k, $1 \le k \le N$, with polarizers $\{y > y_k\}$ such that*

$$y_1 < y_2 < \dots < y_N < t \tag{7.1}$$

and

$$\bigcup_{k=1}^{N} P_k(K_i) = K_i^t \quad \textit{for all } 1 \le i \le m. \tag{7.2}$$

Proof. 1) Let $t \in \mathbb{R}$ be fixed. First, we consider the case where $n = 1$. Then the block decomposition of K_i consists of a finite number of disjoint segments:

$$K_i = \bigcup_{j=1}^{j_i} [a_{ij}, b_{ij}], \tag{7.3}$$

where

$$a_{i1} < b_{i1} < a_{i2} < b_{i2} < \dots < a_{ij_i} < b_{ij_i}, \quad j_i \ge 1, \quad 1 \le i \le m. \tag{7.4}$$

For every index i, we define two numbers:

$$\alpha_i = (1/2)\big(t + (1/2)(a_{i1} + b_{i1})\big), \tag{7.5}$$

and

$$\beta_i = \begin{cases} (1/2)(a_{i1} + a_{i2}) & \text{if } \ j_i > 1, \\ t & \text{otherwise.} \end{cases} \tag{7.6}$$

Let $\mathcal{A}_t^{(0)}$ denote the set of all numbers α_i and all numbers β_i that are less than t. It is easy to verify that $T^t\big((K_i)\big) = K_i$ for some i if and only if $\alpha_i \ge t$. Therefore, if $\mathcal{A}_t^{(0)} = \varnothing$, then $\alpha_i \ge t$ for all i and there is nothing to prove.

If $\mathcal{A}_t^{(0)} \ne \varnothing$, then we set

$$y_1 := \min\{\gamma \,:\, \gamma \in \mathcal{A}_t^{(0)}\}.$$

Now we consider a polarization P_1 with the polarizer $\{y > y_1\}$. Let $K_i^{(1)} = P_1(K_i)$, $1 \le i \le m$. For each i, the compact set $K_i^{(1)}$ satisfies the following conditions:

(a) $K_i^{(1)}$ is a simple compact set whose block decomposition $K_i^{(1)} = \bigcup_{j=1}^{j_i^{(1)}} [a_{ij}^{(1)}, b_{ij}^{(1)}]$ consists of $j_i^{(1)}$ disjoint segments, where $j_i^{(1)} \le j_i$.

(b) $\big(K_i^{(1)}\big)_+(t) = (K_i)_+(t)$, and therefore $T^t\big(K_i^{(1)}\big) = T^t(K_i)$ for all $1 \le i \le m$.

Now, starting with simple compact sets $K_i^{(1)}$, $1 \le i \le m$, we define a set $\mathcal{A}_t^{(1)}$ of elements $\alpha_i^{(1)} < t$ and $\beta_i^{(1)} < t$ in the same way as $\mathcal{A}_t^{(0)}$ was defined for K_i, see formulas (7.3)–(7.6). As has been observed above, if $\mathcal{A}_t^{(1)} = \varnothing$, then $K_i^{(1)} = T^t(K_i^{(1)}) = T^t(K_i)$, $1 \le i \le m$, where the second relation follows from condition (b). In this case our construction is finished.

If $\mathcal{A}_t^{(1)} \ne \varnothing$, we continue our construction further to get simple compact sets $K_i^{(2)}$, $K_i^{(3)}, \dots, 1 \le i \le m$. Let $\mathcal{A}_t^{(k)}, \alpha_i^{(k)}, \beta_i^{(k)}$, and y_{k+1} correspond $K_i^{(k)}$ in the process of this construction. It is easily seen that at each step of our construction, the $K_i^{(k)}$, $\mathcal{A}_t^{(k)}$, and y_{k+1} satisfy the following conditions.

(i) If $\mathcal{A}_t^{(k)} \ne \varnothing$ for some $k \ge 1$, then $y_1 < y_2 < \dots < y_k < t$.

(ii) If for some $k \geq 1$ and some s, $1 \leq s \leq m$, we have $y_k = \alpha_s^{(k-1)}$, then $\#\mathcal{A}_t^{(k)} \leq \#\mathcal{A}_t^{(k-1)} - 1$, where $\#$ denotes the cardinality of the corresponding set and $\mathcal{A}_t^{(0)} = \mathcal{A}_t$.

(iii) If $y_k = \beta_s^{(k-1)}$ for some s, $1 \leq s \leq m$, then $j_s^{(k)} \leq j_s^{(k-1)} - 1$, where $j_i^{(l)}$ denotes the number of disjoint segments in $K_i^{(l)}$ if $l \geq 1$ and $j_i^{(0)} = j_i$.

Since the total number of segments constituting $K_1, \dots, K_m$ is finite and $\#\mathcal{A}_t < \infty$, conditions (ii) and (iii) show that the above construction of compact sets $K_i^{(k)}$ terminates after a finite number of steps. Thus, there is an integer $N \geq 0$ such that $\mathcal{A}_t^{(N)} = \varnothing$. Since at each step of our construction conditions (a) and (b) are satisfied, we have

$$K_i^{(N)} = T^t\big(K_i^{(N)}\big) = T^t\big(K_i^{(N-1)}\big) = \cdots = T^t(K_i), \quad 1 \leq i \leq m,$$

which proves the lemma in the case under consideration.

(2) Now we consider the case where $n \geq 2$. Since K_i is a simple compact set in $\mathbb{R}^n$, it can be decomposed as follows:

$$K_i = \bigcup_{j=1}^{j_i} \overline{R'}_{ij} \times K_i^j,$$

where the $\overline{R'}_{ij}$ are closed blocks in $\mathbb{R}^{n-1}$ such that $R'_{ij} \cap R'_{il} = \varnothing$ if $j \neq l$ and K_i^j are simple compact sets in $\mathbb{R}$.

By part 1) of this proof, for every $t \in \mathbb{R}$ there are polarizations P_k, $1 \leq k \leq N$, in $\mathbb{R}$ with the polarizers $\{y > y_k\}$ such that

$$y_1 < y_2 < \cdots < y_N < t$$

and

$$\bigcirc_{k=1}^{N} P_k(K_i^j) = T^t(K_i^j) \quad \text{for all } 1 \leq j \leq j_i \text{ and } 1 \leq i \leq m.$$

Viewing the P_k, $1 \leq k \leq N$, as polarizations in $\mathbb{R}^n$ with the corresponding polarizers $\{y > y_k\}$, we obtain

$$\bigcirc_{k=1}^{N} P_k(K_i) = \bigcup_{j=1}^{j_i} \overline{R'}_{ij} \times \big(T^t(K_i^j)\big) = T^t(K_i), \quad 1 \leq i \leq m.$$

The proof of the lemma is complete. □

Corollary 7.1. *Let $t \in \mathbb{R}$, and let D, Ω be bounded open sets such that $\bar{D} \subset \Omega$. Then there is a finite number of polarizations P_k, $1 \leq k \leq N$, with polarizers $\{y > y_k\}$, where $y_1 < y_2 < \cdots < y_N \leq t$, such that*

$$\bar{D}_N \subset \Omega^t, \quad \textit{where } \bar{D}_N = \bigcirc_{k=1}^{N} P_k(\bar{D}).$$

Moreover,

$$\operatorname{dist}(\partial D, \partial\Omega) \leq \operatorname{dist}(\partial D_N, \partial\Omega^t), \quad \textit{where } D_N = \bigcirc_{k=1}^{N} P_k(D).$$

Proof. Let $\rho_0 = \operatorname{dist}(\partial D, \partial\Omega)$. Then $\rho_0 > 0$, and for every ρ with $0 \leq \rho < \rho_0$ there exists a simple compact set $K = K(\rho)$ such that

$$(\bar{D})_\rho \subset K \subset \Omega.$$

By Lemma 7.1, we can find a finite number of polarizations P_k, $1 \le k \le N$, such that

$$K^t = K_N := \bigcirc_{k=1}^{N} P_k(K).$$

Since the T^t-transformation and polarization are monotone and smoothing transformations, we have

$$\Omega^t \supset K^t = K_N \supset \bigcirc_{k=1}^{N} P_k((\bar{D})_\rho) \supset (\bar{D}_N)\rho. \tag{7.7}$$

Since (7.7) is true for every ρ, $0 \le \rho < \rho_0$, the corollary follows. □

Definition 7.1. A function $f \in \mathcal{S}_n$ is said to be *simple* if f can be represented as

$$f = \varepsilon\left(-n_0 + \sum_{i=1}^{m} \chi(K_i)\right) \tag{7.8}$$

with some $\varepsilon > 0$, $n_0 \in \mathbb{N}$, and some simple compact sets K_i, $1 \le i \le m$, such that $K_1 \supset \cdots \supset K_m$.

We note that simple functions are dense in $L^p_+(\mathbb{R}^n)$ for every $1 \le p < \infty$. The following lemma shows that the T^t-transformation of a simple function reduces to a finite sequence of polarizations.

Lemma 7.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a simple function. Then for every $t \in \mathbb{R}$ there are polarizations P_k with polarizers $\{y > y_k\}$, $1 \le k \le N$, such that*

$$y_1 < y_2 < \cdots < y_N < t, \tag{7.9}$$

and

$$\bigcirc_{k=1}^{N} P_k(f) = f^t. \tag{7.10}$$

Proof. Since f is simple, it can be represented in the form (7.8). By Lemma 7.1, we can find polarizations P_k, $1 \le k \le N$, with polarizers $\{y > y_k\}$ satisfying (7.9) and such that

$$\bigcirc_{k=1}^{N} P_k(K_i) = K_i^t, \quad 1 \le i \le m.$$

Now (7.10) follows from the relations

$$\bigcirc_{k=1}^{N} P_k(f) = \varepsilon\left(-n_0 + \sum_{i=1}^{m} \chi\left(\bigcirc_{k=1}^{N} P_k(K_i)\right)\right)$$

and

$$f^t = \varepsilon\left(-n_0 + \sum_{i=1}^{m} \chi(K_i^t)\right),$$

which in their turn follow from Definition 4.4 and Definition 3.2. □

The following lemma shows that, for functions $u \in L^p_+(\mathbb{R}^n)$ such that $u \ne u^t$, one polarization suffices for finding a better approximation of u^t. Its proof, which is left to the reader, follows the ideas of [7, p. 252] and repeats the proof of Lemma 6.4 in [14].

Lemma 7.3. *Let $u \in L^p_+(\mathbb{R}^n)$, $1 \le p < \infty$. If $u = u^{t_1}$ and $u \ne u^t$ for some $t_1, t \in \mathbb{R}$ such that $t_1 < t$, then there is a polarizer $H_s = \{y > s\}$ with $t_1 < s < t$ such that*

$$(u_{H_s})^t = u^t$$

and

$$\|u_{H_s} - u^t\|_p < \|u - u^t\|_p.$$

Lemma 7.4. *Let $u \in L^p_+(\mathbb{R}^n)$, $1 \le p < \infty$. Then for every $t \in \mathbb{R}$ there is a sequence of polarizations P_k with polarizers $\{y > y_k\}$, where $y_k \le t$ for all $k = 1, 2, \dots,$ such that the sequence of functions $u_m := \bigcirc_{i=1}^m P_i u$ satisfies the minimality condition*

$$\|u_{m+1} - u^t\|_p = \min\{\|(u_m)_H - u^t\|_p\}, \quad m = 1, 2, \dots, \ u_0 = u, \tag{7.11}$$

where the minimum is taken over all polarizations with polarizers $H = \{y > s\}$ such that $-\infty < s \le t$.

Moreover,

$$u_m \to u^t \quad \text{in} \ \ L^p(\mathbb{R}^n).$$

Proof. Lemma 5.2 in [14] shows that the minimum in (7.11) is attained for some polarizer $\{y > y_{k+1}\}$, $-\infty < y_{k+1} \le t$. Then by [14, Lemma 6.1], there is a function $v \in L^p_+(\mathbb{R}^n)$ and a subsequence $u_{m'}$ such that

$$u_{m'} \to v \quad \text{in } L^p(\mathbb{R}^n).$$

Using the nonexpansivity Lemma 3.1, we obtain $v^t = u^t$.

Now assume that $v \ne u^t$. By Lemma 7.3, we can choose a polarizer H such that

$$\|v_H - u^t\|_p < \|v - u^t\|_p.$$

It follows that

$$\begin{aligned} &\|(u_{m'})_H - u^t\|_p - \|u_{m'} - u^t\|_p \\ &\quad \le \|(u_{m'})_H - v_H\|_p + \|v_H - u^t\|_p + \|u_{m'} - v\|_p - \|v - u^t\|_p \\ &\quad \le 2\|u_{m'} - v\|_p + \|v_H - u^t\|_p - \|v - u^t\|_p \to \|v_H - u^t\|_p - \|v - u^t\|_p < 0 \end{aligned} \tag{7.12}$$

as $m' \to \infty$.

On the other hand, the sequence $\|u_m - u^t\|_p$ is monotone decreasing. Hence,

$$\|u_{m+1} - u^t\|_p - \|u_m - u^t\|_p \to 0 \quad \text{as } m \to \infty.$$

Together with (7.12), this contradicts the minimality property (7.11). □

Lemma 7.5. *Let $u \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and let u_m be defined as in Lemma 7.4, so that condition (7.11) is satisfied with $p = 1$. Then*

$$u_m \to u^t \quad \text{in} \ \ C(\mathbb{R}^n).$$

Proof. By Lemma 7.4, $u_m \to u^t$ in $L^1(\mathbb{R}^n)$, and the functions u_m are equicontinuous by [14, Lemma 5.1]. Lemma 6.2 in [14] shows that we also have

$$u_{m'} \to v \quad \text{in} \ \ C(\mathbb{R}^n)$$

for a subsequence $u_{m'}$ and $\omega_v \le \omega_u$. Thus $v = u^t$ and the claim follows. □

§8. Rescaling and limit cases

The T^t-transformation of sets and functions defined in §4 depends on the height t of the moving plane of symmetrization $\{y = t\}$, the range of which is $-\infty < t < \infty$. We recall that, in its original setting as described in the Introduction, the continuous symmetrization problem requires the standard range $0 \le t \le 1$ with the "boundary conditions"

$$T^0(\Omega) = \Omega, \quad T^1(\Omega) = \Omega^*, \tag{8.1}$$

where Ω^* denotes the $(1,n)$-Steiner symmetrization of Ω with respect to a fixed plane $\{y = t_0\}$. If $t_0 = 1$, then the T^t-transformation perfectly matches these conditions when restricted to the family $\mathcal{M}_n[0,1]$, where

$$\mathcal{M}_n[a,b] := \{\Omega \in \mathcal{M}_n \,:\, \Omega \subset \Pi(a,b)\}$$

and

$$\Pi(a,b) := \{(x,y) \in \mathbb{R}^n \,:\, a \le y \le b\}, \quad -\infty \le a < b \le \infty.$$

If $\Omega \in \mathcal{M}_n[a,b]$ with $-\infty < a < b < \infty$, then the linear change of variables $\tau = (t-a)/(b-a)$ gives the desired range $0 \le \tau \le 1$ with $\widetilde{T}^0(\Omega) = \Omega$ and $\widetilde{T}^1(\Omega) = \Omega^*$, where $\widetilde{T}^\tau = T^t$ and the asterisk denotes the $(1,n)$-Steiner symmetrization with respect to $\{y = b\}$. The case of $\Omega \in \mathcal{M}_n[-\infty, b]$ with $b < \infty$ can be handled in a similar way with a nonlinear change of variables, for instance, $\tau = (1 + b - t)^{-1}$.

The remaining case where $\Omega \in \mathcal{M}_n[a,\infty]$ with $-\infty \le a < \infty$ requires some additional consideration, because the limit set Ω^∞ is not defined yet. A possible way to "correct" this defect is to use a post-composition $\widehat{T}^t = Sh^t \circ T^t$ of the T^t-transformation with a continuous shift Sh^t by $(t - t_0)^+$ units in the direction of the negative y-axis. Here $(t-t_0)^+ = \max\{0, t - t_0\}$. In this case, the limit set $\Omega^\infty = \widehat{T}^\infty(\Omega)$ may be identified with the Steiner symmetrization Ω^* of Ω with respect to the plane $\{y = t_0\}$. Of course, $\widehat{T}^t$ still possesses all basic properties of the T^t-transformation. In particular, Lemmas 4.4 and 5.1 are still valid for $\widehat{T}^t$ for all $-\infty < t \le +\infty$. The case where $t = \infty$ is included here because $\widehat{T}^\infty$ is simply the Steiner symmetrization with respect to $\{y = t_0\}$. Changing the variables via $\tau = (1/\pi)\arctan t + 1/2$, we arrive at a family of transformations $\widetilde{T}^\tau$ defined on the standard interval $I = \{\tau \,:\, 0 \le \tau \le 1\}$. As we have seen in previous sections, the T^t-transformations is continuous from the left for $-\infty < t < \infty$. Therefore, the $\widetilde{T}^t$-transformation is continuous from the left on the open interval $0 < \tau < 1$. For this reason, the $\widetilde{T}^t$-transformation will be called the continuous symmetrization defined on the standard interval.

Being restricted to the class of bounded sets and functions with bounded supports, the $\widetilde{T}^t$-transformation is continuous at $\tau = 1$ as well. For unbounded sets and functions with unbounded supports the situation is different: some functionals of domains or functions are continuous at $\tau = 1$ while some other are not. For instance, Lemma 5.1 is obviously not valid if Ω is an unbounded set, while the relevant Steiner symmetrization of Ω is bounded.

In order to define a one-dimensional T^t-transformation for an arbitrary half-space $H = H(a,n)$, we may combine the T^t-transformation of Definition 4.3 with appropriate affine transformations of $\mathbb{R}^n$. To be more precise, let $k > 0$ and let A be an orthogonal $(n \times n)$-matrix such that the operator $L := k(A - Aa)$ maps $H(a,n)$ onto $\{y > 0\}$. Then $T^t_H := L^{-1} \circ T^t \circ L$ is the desired continuous symmetrization with respect to H. If $M \in \mathcal{M}_n$ is bounded, then the scaling factor k can be chosen so that $L(M) \subset B^{(n)}$; therefore, when working with bounded sets M, we may assume without loss of generality

that $M \subset B^{(n)}$. Similarly, if $u \in \mathcal{S}_n$ has bounded support, we often assume without loss of generality that $\operatorname{supp} u \subset B^{(n)}$.

We emphasize once again that *any T_H^t-transformation can be rescaled to get a $\widetilde{T}_H^t$-transformation defined on the standard interval I and that all basic properties of continuous symmetrization discussed in §4–7 still remain true for any $\widetilde{T}_H^t$-transformation.*

§9. Continuous (k,n)-Steiner symmetrizations

To define a one-parameter family of rearrangements T_k^t left continuous in $t \in [0,1]$ and transforming sets and functions to their (k,n)-Steiner symmetrization for any $2 \le k \le n$, we suggest the following inductive algorithm.

Let $2 \le k \le n$. Suppose that for any $(n-k+1)$-dimensional plane L' we have defined a continuous family of rearrangements that transform sets, functions, etc. into their $(k-1,n)$-Steiner symmetrizations with respect to L'. Then, let S be a (k,n)-Steiner symmetrization with respect to an $(n-k)$-dimensional plane L. By Lemma 3.1, we can choose two intersecting $(n-k+1)$-dimensional planes L_1 and L_2 giving rise to $(k-1,n)$-Steiner symmetrizations S_1 and S_2 such that

$$S = \lim_{i\to\infty} (S_2 \circ S_1)^i = \lim_{i\to\infty} S_1 \circ (S_2 \circ S_1)^i, \tag{9.1}$$

where $(S_2 \circ S_1)^0$ is the identity transformation.

By the inductive assumption, there are continuous $(k-1,n)$-Steiner symmetrizations $T^{1,t}$ and $T^{2,t}$, $t \in [0,1]$, into S_1 and S_2, respectively. Let $t(j) = (j-1)/j$, and let $I_j = \{t : t(j) \le t < t(j+1)\}$, $j = 1,2,\dots$ Then $\left(\cup_{j=1}^{\infty} I_j\right) \cup \{1\}$ is a disjoint decomposition of the standard interval I. Next, for every $j = 1,2,\dots$ and every $t \in I_j$ we define the transformation T_k^t by

$$T_k^t = \begin{cases} T^{1,\tau} \circ (S_2 \circ S_1)^{i-1} & \text{with } \tau = \frac{t-t(2i-1)}{t(2i)-t(2i-1)} \text{ if } j = 2i-1, \\ T^{2,\tau} \circ S_1 \circ (S_2 \circ S_1)^{i-1} & \text{with } \tau = \frac{t-t(2i)}{t(2i+1)-t(2i)} \text{ if } j = 2i, \end{cases} \tag{9.2}$$

where $i = 1,2,\dots$.

The inductive algorithm (9.2) determines a continuous transformation into a (k,n)-Steiner symmetrization S as follows. First, we choose a complete binary tree $\mathcal{T}$ of planes $L_{0i_1\dots i_l}$, $1 \le l \le k-1$, $i_s \in \{1,2\}$, rooted at $L = L_0$ and such that, for every $1 \le l \le k-2$ and every multiindex $i_1 \dots i_l$, the planes $L_{0i_1\dots i_l 1}$ and $L_{0i_1\dots i_l 2}$ are $(n-k+l+1)$-dimensional and the corresponding $(k-l-1,n)$-Steiner symmetrizations $S_{0i_1\dots i_l 1}$ and $S_{0i_1\dots i_l 2}$ approximate $S_{0i_1\dots i_l}$ as in (9.1).

Definition 9.1. By an SC (k,n)-Steiner symmetrization corresponding to a tree $\mathcal{T}$ we mean a transformation T_k^t of sets and functions defined inductively by formulas (9.2), where the initial 2^{k-1} continuous 1-dimensional symmetrizations are selected to be SC 1-symmetrizations with respect to the corresponding hyperplanes $L_{0i_1\dots i_{k-1}}$, where $i_1 \dots i_{k-1} \in \{1,2\}^{k-1}$.

Three remarks are in order now.

Remark 9.1. The Sarvas approximation pattern used in the algorithm (9.2) can be replaced by any other approximation pattern. For example, we can use a sequence S_i, $i = 1,2,\dots$, of $(1,n)$-Steiner symmetrizations to approximate a given (k,n)-Steiner symmetrization: $S = \lim_{n\to\infty} \bigcirc_{i=1}^n S_i$. In this case a T^t-transformation can be defined as $T^t = T_n^\tau \circ (\bigcirc_{i=1}^n S_i)$ with $\tau = (t - t(i))/(t(i+1) - t(i))$.

Remark 9.2. One can define a continuous (k,n)-symmetrization by replacing the S-continuous one-dimensional symmetrization with a B-continuous one-dimensional symmetrization. Alternatively, one may want to define a continuous (k,n)-symmetrization

by "mixing" the SC symmetrizations with BC symmetrizations (or with other types of continuous $(1,n)$-symmetrization if those will be discovered) in various stages of the algorithm (9.2).

Remark 9.3. Any continuous deformation defined by the algorithm (9.2) is generated by two kinds of transformations, continuous one-dimensional symmetrizations and (k,n)-Steiner symmetrizations. Such a "hybrid" will inherit the properties that both its "parents" have; the proof of a particular property of the transformation T_k^t, for any *fixed* t, can often be given via what we call the "standard inductive argument". More precisely, this means that if a certain statement is true for any (k,n)-Steiner symmetrization, as well as for the corresponding continuous one-dimensional symmetrization, and if the property in question is invariant under translations and scaling, then the corresponding result will remain true for any continuous k-dimensional symmetrization defined by the algorithm (9.2). We shall demonstrate how this "standard inductive argument" works in the proof of Theorem 9.1 below. Here we want to emphasize that if the assumptions of some theorem or lemma deal with the *varying* parameter t, then the standard inductive argument is *not valid*, and, usually, the proof of such statement is not so straightforward, cf. the proof of Theorem 9.2.

Theorem 9.1. *Let T_k^t be an SC (k,n)-Steiner symmetrization into a (k,n)-Steiner symmetrization corresponding to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$, and let $T_{k,x_0'}^t$ be the restriction of T_k^t to the slice $\mathbb{R}(x_0') := \{x = (x_0', y) : y \in \mathbb{R}^k\}$. Then T_k^t (respectively, $T_{k,x_0'}^t$) acts on $\mathbb{R}^n$ (respectively, on $\mathbb{R}(x_0')$) as an open, compact, and smoothing rearrangement, which is continuous from the inside and from the outside.*

Proof. The result is already known for the one-dimensional symmetrization, see Lemma 4.4.

Assume now that the theorem is true for any integer $n \geq 3$ and some $k-1$ such that $2 \leq k < n$. For a fixed t, $0 \leq t \leq 1$, let T_k^t be a continuous symmetrization as described in the formulation of the theorem. If $t = 1$ or $t = t(j)$ for some $j \geq 2$, then this T_k^t-transformation coincides with the (k,n)-Steiner symmetrization or it coincides with a finite composition of $(k-1,n)$-Steiner symmetrizations, respectively. In both cases the theorem is known to be true.

Let $t(j) < t < t(j+1)$ for some integer $j \geq 2$. Then, in accordance with the algorithm (9.2), the T_k^t-transformation can be represented as a finite composition of $(k-1,n)$-Steiner symmetrizations postcomposed by a continuous $(k-1,n)$-Steiner symmetrization. Then, using the known result for $(k-1,n)$-Steiner symmetrizations and the inductive assumption on the continuous $(k-1,n)$-Steiner symmetrizations, we conclude that the theorem is true for any k, $1 \leq k \leq n$. □

The following lemma shows that continuous symmetrizations are monotone in measure on slices. Its proof easily follows from Theorem 9.1 and the non-expansivity property of Lemma 3.1.

Lemma 9.1. *Let T_k^t be a continuous symmetrization as in Theorem* 9.1*, and let $\Omega \in \mathcal{G}_b \cup \mathcal{F}$. If $0 \leq t_1 < t_2 < t_3 \leq 1$, then*

$$\mathcal{L}^k(\Omega^{t_2}(x') \triangle \Omega^{t_3}(x')) \leq \mathcal{L}^k(\Omega^{t_1}(x') \triangle \Omega^{t_3}(x')) \tag{9.3}$$

for every $x' \in \mathbb{R}^{n-k}$, whence

$$\mathcal{L}^n(\Omega^{t_2} \triangle \Omega^{t_3}) \leq \mathcal{L}^n(\Omega^{t_1} \triangle \Omega^{t_3}). \tag{9.4}$$

Theorem 9.2. *Let T_k^t be a continuous symmetrization as in Theorem 9.1 and let t_s, $s = 1, 2, \dots$, be an increasing sequence such that $t_s \in [0, 1]$ and $t_s \to t_0$. Then*

$$d(\partial\Omega^{t_s}, \partial\Omega^{t_0}) \to 0 \quad \text{as} \quad s \to \infty \tag{9.5}$$

if Ω is a bounded open set, and

$$d(\Omega^{t_s}, \Omega^{t_0}) \to 0 \quad \text{as} \quad s \to \infty \tag{9.6}$$

if Ω is a compact set.

Proof. To be specific, we assume that Ω is a bounded open set. In the case of compact sets the proof is easier and is left to the reader.

We proceed by induction. For $k = 1$, the statement is true by Lemma 5.1 modulo some remarks in §8. Suppose that for every SC (j, n)-Steiner symmetrization with $1 \le j \le k-1$ the conclusion of the theorem is true. Our goal now is to prove that the claim is true for $j = k$.

If $t_0 < 1$, then $t(m) < t_0 \le t(m + 1)$ for some positive integer m. By Definition 9.1 and the algorithm (9.2), the set $\Omega^{t(m+1)}$ is the image of $\Omega^{t(m)}$ under some $(k - 1, n)$-Steiner symmetrization S_1, and moreover, there exists a continuous $(k - 1, n)$-Steiner symmetrization $T^{1,\tau}$ into S_1 such that

$$\Omega^t = T^{1,\tau}(\Omega^{t(m)}) \quad \text{for all} \quad t(m) < t \le t(m + 1), \tag{9.7}$$

where $\tau = \tau(t) = (t - t(m))/(t(m + 1) - t(m))$. Since $\tau(t_s) \to \tau(t_0)$ as $s \to \infty$, the theorem follows from (9.7) and the inductive assumption.

In the case where $t_0 = 1$, the proof is by contradiction. If (9.5) fails, then, as in the proof of Lemma 5.1, we can find $\varepsilon > 0$ and, if necessary, subsequences (for which we keep the previous notation) t_s and $\Omega_s = \Omega^{t_s}$, and a sequence of points $x_s \in \mathbb{R}^n$ with $x_s \to x_0 \in \mathbb{R}^n$ such that one of the following two conditions is satisfied:

(a) $x_s \in \partial\Omega^*$ and $\operatorname{dist}(x_s, \partial\Omega_s) \ge \varepsilon$ for all $s = 0, 1, 2, \dots$;

(b) $x_s \in \partial\Omega_s$ for $s = 1, 2, \dots$ and $\operatorname{dist}(x_s, \partial\Omega^*) \ge \varepsilon$ for all $s = 0, 1, 2, \dots$.

In case **(a)** we consider two subcases:

(i) $B_\varepsilon^{(n)}(x_0) \cap \Omega^{t_s} = \varnothing$ for all $s = 1, 2, \dots$;

(ii) $B_\varepsilon^{(n)}(x_0) \subset \Omega^{t_s}$ for all $s = 1, 2, \dots$.

Case (i). Let $t(m_s) < t_s \le t(m_s + 1)$. In the case under consideration there is $\delta_0 > 0$ such that

$$\mathcal{L}^n\big(\Omega^{t_s} \triangle \Omega^*\big) \ge \delta_0 \tag{9.8}$$

for all $s = 1, 2, \dots$. Since

$$\Omega^{t(m_s)} \triangle \Omega^{t(m_s+1)} \subset (\Omega^{t(m_s)} \triangle \Omega^*) \cup (\Omega^{t(m_s+1)} \triangle \Omega^*),$$

relation (3.11) and Theorem 3.1 show that

$$\mathcal{L}^n\big(\Omega^{t(m_s)} \triangle \Omega^{t(m_s+1)}\big) \le \mathcal{L}^n\big(\Omega^{t(m_s)} \triangle \Omega^*\big) + \mathcal{L}^n\big(\Omega^{t(m_s+1)} \triangle \Omega^*\big) \to 0 \tag{9.9}$$

as $m \to \infty$. Next, by Lemma 9.1,

$$\mathcal{L}^n\big(\Omega^{t_s} \triangle \Omega^{t(m_s+1)}\big) \le \mathcal{L}^n\big(\Omega^{t(m_s)} \triangle \Omega^{t(m_s+1)}\big).$$

Combined with (9.9), this contradicts (9.8).

Case (ii). Let $x_0 = (x_0', y_0)$, where $x_0' \in \mathbb{R}^{n-k}$. Let $\Omega^*(x_0')$ and $\Omega^t(x_0')$ denote the restrictions of Ω^* and Ω^t (respectively) to the slice $\mathbb{R}_{x_0'}$, each of which is not empty by condition (ii). Since $\Omega^*(x_0')$ is a (k, k)-Steiner symmetrization of $\Omega(x_0')$, it is a nonempty k-dimensional ball. By assumption **(a)**, $x_0 \in \partial\Omega^*$ and, therefore,

$$\mathcal{L}^k\big(B_\varepsilon^{(k)}(y_0) \setminus \Omega^*(x_0')\big) \ge \delta_1$$

for some $\delta_1 > 0$. Hence,

$$\mathcal{L}^k\big(\Omega^*(x_0') \triangle \Omega^{t_s}(x_0')\big) \geq \delta_1, \quad s = 1, 2, \ldots. \tag{9.10}$$

By inequality (9.3) of Lemma 9.1 we have

$$\mathcal{L}^k\big(\Omega^{t_s}(x_0') \triangle \Omega^{t(m_s+1)}(x_0')\big) \leq \mathcal{L}^k\big(\Omega^{t(m_s)}(x_0') \triangle \Omega^{t(m_s+1)}(x_0')\big).$$

By equation (3.11),

$$\mathcal{L}^k\big(\Omega^{t(m)}(x_0') \triangle \Omega^*(x_0')\big) \to 0 \quad \text{as} \quad m \to \infty.$$

As in case (i), the latter two relations contradict (9.10). This completes the proof in case **(a)**.

Now we turn to condition **(b)**. In this case relation (3.10) in Theorem 3.1 implies that there is a neighborhood $\mathcal{U}$ of x_0 such that one of the following two conditions is satisfied:

(i) $\mathcal{U} \cap \Omega^* = \varnothing$ and $\mathcal{U} \cap \Omega^{t(m)} = \varnothing$ for all m sufficiently large;

(ii) $\mathcal{U} \subset \Omega^*$ and $\mathcal{U} \subset \Omega^{t(m)}$ for all m sufficiently large.

For $\delta > 0$, $\tau > 0$, and $x_0 = (x_0', y_0)$ with $y_0 \in \mathbb{R}^k$, let

$$Q_\delta(x_0) = B_\delta^{(n-k)}(x_0') \times \mathbb{R}^k, \quad Q(x_0, \delta, \tau) = B_\delta^{(n-k)}(x_0') \times B_\tau^{(k)}.$$

Considering case (i), first we assume that $|y_0| > 0$. Since for every $x' \in \mathbb{R}^{n-k}$ the section $\Omega^*(x')$ is a ball in $\mathbb{R}^k$ centered at the origin, condition (i) implies that we can find $\delta_0 > 0$ and $\tau_0 > 0$ sufficiently small such that

$$\Omega^* \cap Q_{\delta_0}(x_0) \subset Q(x_0, \delta_0, |y_0| - \tau_0) \quad \text{and} \quad \Omega_m \cap Q_{\delta_0}(x_0) \subset Q(x_0, \delta_0, |y_0| - \tau_0) \tag{9.11}$$

for all sufficiently large m. By (9.2), the set Ω^{t_s} is obtained as a result of the action of a certain continuous $(k-1, n)$-Steiner symmetrization of the set $\Omega^{t(m_k)}$ into the set $\Omega^{t(m_s+1)}$. Since the intersections $\Omega^{t(m_k)} \cap Q_{\delta_0}(x_0)$ and $\Omega^{t(m_k+1)} \cap Q_{\delta_0}(x_0)$ both belong to $Q(x_0, \delta_0, |y_0| - \tau_0)$, and since the set $Q(x_0, \delta_0, |y_0| - \tau_0)$ is invariant under the continuous $(k-1, n)$-Steiner symmetrization mentioned above, from the property of monotonicity in slices of the continuous symmetrization it follows that $\Omega^{t_s} \cap Q_{\delta_0}(x_0) \subset Q(x_0, \delta_0, |y_0| - \tau_0)$ for all sufficiently large s. This contradicts the assumptions $x_s \in \partial\Omega^{t_s}$ and $x_s \to x_0$.

If $y_0 = 0$ then, in case (i), we have $\Omega^* \cap Q_{\delta_0}(x_0) = \varnothing$ for some $\delta_0 > 0$. Hence, the intersection $\Omega \cap Q_{\delta_0}(x_0)$ is also empty, so that $\Omega^{t_s} \cap Q_{\delta_0}(x_0) = \varnothing$ for all s, which again contradicts the assumptions $x_s \in \partial\Omega^{t_s}$ and $x_s \to x_0$. This finishes the proof in case (i).

In case (ii), we argue as above to conclude that there are numbers $\delta_0 > 0$ and $\tau_0 > 0$ such that

$$Q(x_0, \delta_0, |y_0| + \tau_0) \subset \Omega^* \cap Q_{\delta_0}(x_0) \quad \text{and} \quad Q(x_0, \delta_0, |y_0| + \tau_0) \subset \Omega_m \cap Q_{\delta_0}(x_0) \tag{9.12}$$

for all sufficiently large m. As above, the monotonicity property of the continuous $(k-1, n)$-Steiner symmetrization from $\Omega^{t(m_s)}$ into $\Omega^{t(m_s+1)}$ shows that $Q(x_0, \delta_0, \tau_0) \subset \Omega^{t_s}$ if s is sufficiently large. This contradicts the assumptions $x_s \in \partial\Omega^{t_s}$ and $x_s \to x_0$. The proof of the theorem is complete. □

Lemma 9.2. *Let $\Omega, \Omega' \in \mathcal{G}_{n,b}$ be such that $\overline{\Omega'} \subset \Omega$, and let $(\cdot)^t$ denote an SC symmetrization into the (k, n)-Steiner symmetrization Sym corresponding to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$. Then for every t, there exist finitely many transformations $P_1, \ldots, P_N$ such that*

$$\Omega_N' := P_1 \circ \cdots \circ P_N(\Omega') \subset \Omega^t$$

and

$$\operatorname{dist}(\partial\Omega', \partial\Omega) \leq \operatorname{dist}(\partial\Omega_N', \partial\Omega^t),$$

where each P_j is either a polarization with the polarizer H_j such that $\Sigma_j := \partial H_j \supset \mathbb{R}^{n-k}$, or a shift in a direction $\vec{l}$ such that $\vec{l} \perp \mathbb{R}^{n-k}$.

Proof. If $k = 1$, then the symmetrization in question is one-dimensional, and the lemma follows from Corollary 7.1.

Let $k \geq 2$. If $t = 1$ or $t = t(j)$ for some $j \geq 2$, then the T_k^t-transformation under consideration coincides with the (k, n)-Steiner symmetrization or with a finite composition of $(k-1, n)$-Steiner symmetrizations. In either case, the result is known to be true, cf. Lemmas 7.1 and 7.2 in [14]. Therefore, for any fixed t, the desired conclusion follows via the standard inductive argument, as was explained in Remark 9.3 and demonstrated in the proof of Theorem 9.1. □

§10. Integral inequalities

Many integral inequalities well-known in the theory of symmetrizations remain valid for the continuous symmetrization as well. We start with the following L^p-continuity lemma.

Lemma 10.1. *Let $u \in L^p_{0+}(\mathbb{R}^n)$, $1 \leq p < \infty$, and let $u^t = T^t(u)$, $0 \leq t \leq 1$, denote a continuous (k, n)-Steiner symmetrization. Then the mapping $t \mapsto u^t$ is continuous in the L^p-norm, i.e.,*

$$\lim_{t \to 0} \|u^{s+t} - u^s\|_p = 0 \quad \textit{for all} \;\; 0 \leq s \leq 1. \tag{10.1}$$

Proof. First we assume that u is a step function. Then

$$u = \delta \sum_{i=1}^{m} \chi(E_i)$$

with some $\delta > 0$ and some measurable sets $E_1 \supseteq \cdots \supseteq E_m$. Let $E_i^\tau = T^\tau(E_i)$, $0 \leq \tau \leq 1$. By Lemma 9.1,

$$\|u^{s+t} - u^s\|_p \leq \delta \sum_{i=1}^{m} \mathcal{L}^n(E_i^{t+s} \Delta E_i^s) \to 0 \quad \text{as} \;\; t \to 0.$$

In general, u can be approximated in the L^p-norm by step functions u_m. Then, using the inequality

$$\|u^{s+t} - u^s\| \leq \|u^{s+t} - u_m^{s+t}\| + \|u_m^{s+t} - u_m^s\| + \|u_m^s - u^s\|$$

and the nonexpansivity Lemma 3.1, we obtain (10.1). □

Our next lemma shows that a similar continuity property remains valid for the space of continuous functions.

Lemma 10.2. *Let $u \in C_0(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$. Then for every $0 \leq s \leq 1$ we have $\|u^{s+t} - u^s\|_\infty \to 0$ as $t \to 0$.*

Proof. Fix $0 \leq s \leq 1$ and $\varepsilon > 0$. Since $u \in C_0(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$, we can find $v \in W_0^{1,\infty}(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$ such that $\|v - u\|_\infty < \varepsilon/3$. Then, using the non-expansivity property of Lemma 3.1, for all sufficiently small t we have

$$\begin{aligned} \|u^s - u^{s+t}\|_\infty &\leq \|u^s - v^s\|_\infty + \|v^s - v^{s+t}\|_\infty + \|v^{s+t} - u^{s+t}\|_\infty \\ &\leq 2\|u - v\|_\infty + \|v^s - v^{s+t}\|_\infty < \varepsilon, \end{aligned}$$

which proves the lemma. □

The rest of this section contains seven theorems. The first six of them are monotonicity statements that correspond to inequalities that are well-known in the theory of Steiner symmetrization. Their proofs in all cases follow the same approach. First, we use approximation by polarizations to prove the required monotonicity for the continuous

SC 1-symmetrization. Then we apply the standard inductive argument to show that the same kind of monotonicity holds true for the continuous (k,n)-symmetrization for any k.

Theorem 10.1 below is related to the well-known convolution type inequalities, cf. [14, Lemmas 8.1 and 8.2], [8], and [6, Corollary 2]. This theorem shows that, in fact, the convolutions are monotone functions of the parameter of the corresponding continuous symmetrization.

Theorem 10.1. *Let $u, v, w \in \mathcal{S}_+$ with $w = w^*$, where $(\ )^*$ denotes a (k,n)-Steiner symmetrization, and let j be a Young function. Then the integral*

$$\iint_{\mathbb{R}^{2n}} j(|u^t(x) - v^t(y)|)w(x-y)\,dx\,dy$$

decreases in $0 \le t \le 1$ provided that the integral converges for $t = 0$.

Proof. First, we prove the claim for the case of SC 1-symmetrization. By the semigroup property (6.7), we need to prove the inequality

$$\iint_{\mathbb{R}^{2n}} j(|u^t(x) - v^t(x)|)w(x-y)\,dx\,dy \le \iint_{\mathbb{R}^{2n}} j(|u(x) - v(x)|)w(x-y)\,dx\,dy \tag{10.2}$$

for $-\infty < t < \infty$.

First, observe that, by the nonexpansivity Lemma 3.1, we can restrict ourselves to the case where u and v are continuous functions with bounded support. Then we define inductively two sequences u_m and v_m of polarizations of u and v as in Lemma 7.4, where the corresponding half-spaces H_m are chosen in such a way that the minimality property (7.11) is satisfied. By Lemma 7.5, the sequences u_m and v_m converge in $C(\mathbb{R}^n)$ to u^t and v^t, respectively. Then (10.2) follows by applying [14, Lemma 8.1] inductively.

To make the values of t vary in the standard range $0 \le t \le 1$, we may use scaling and translation as was explained in §8. Of course, these two operations do not change the integrals in (10.2).

Finally, it is well known that the desired monotonicity result is valid for the (k,n)-Steiner symmetrization for any k, $1 \le k \le n$, see [14, Lemma 8.2]. Therefore, for $k \ge 2$ the proof of Theorem 10.1 follows via our standard inductive argument. □

The Dirichlet-type inequalities for functions and their (k,n)-Steiner symmetrizations also admit continuous counterparts.

Theorem 10.2. *Let $u \in W^{1,p}_+(\mathbb{R}^n) \cap \mathcal{S}_+$, $1 \le p \le +\infty$. Then $u^t \in W^{1,p}_+(\mathbb{R}^n) \cap \mathcal{S}_+$ for all $0 \le t \le 1$, and $\|\nabla u^t\|_p$ decreases in $0 \le t \le 1$.*

Furthermore, if V is some linear subspace such that it either contains all "y-directions" $x_{n-k+1}, \dots, x_n$, or is orthogonal to each of these directions, then $\|\nabla_V u^t\|_p$ decreases in $0 \le t \le 1$.

Proof. For the (k,n)-Steiner symmetrization, this result is well known, see [14, Theorem 8.2]. So, we give the proof for the case of the SC 1-symmetrization. Then for $k \ge 2$, Theorem 10.2 will follow via the standard inductive argument.

Let u^t denote the SC 1-symmetrization of u. The semigroup property (6.7) shows that it suffices to prove the inequalities

$$\|\nabla u^t\|_p \le \|\nabla\|_p \quad \text{and} \quad \|\nabla_V u^t\|_p \le \|\nabla_V u\|_p \tag{10.3}$$

for $-\infty < t < \infty$.

For a fixed t, let u_m be the sequence of polarizations of u defined by Lemma 7.4, which converges to u^t in $L^p(\mathbb{R}^n)$. We consider two cases.

(i). Let $1 < p < \infty$. Since $\|\nabla u_m\|_p = \|\nabla u\|_p$, by Lemma 5.3 in [14] we can find a function $v \in W^{1,p}(\mathbb{R}^n)$ and a subsequence $u_{m'}$ such that

$$u_{m'} \rightharpoonup v \quad \text{weakly in } W^{1,p}(\mathbb{R}^n).$$

This means that, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $i = 1, \dots, n$,

$$\int_{\mathbb{R}^n} \varphi v_{x_i}\, dx \leftarrow \int_{\mathbb{R}^n} \varphi \frac{\partial(u_{m'})}{\partial x_i}\, dx = -\int_{\mathbb{R}^n} \varphi_{x_i} u_{m'}\, dx \to -\int_{\mathbb{R}^n} \varphi_{x_i} u^t\, dx,$$

that is, $v = u^t$. In view of the lower semicontinuity of the norm, it follows that

$$\|\nabla u^t\|_p \le \liminf \|\nabla(u_m)\|_p = \|\nabla u\|_p,$$

yielding the first inequality in (10.3). By using relation (5.10) in [14, Lemma 5.3], the second inequality in (10.3) can be proved similarly.

(ii). Let $p = 1$. By Lemma 5.3 in [14], the functions $|\nabla u_m|$ and $|\nabla u|$ are rearrangements of each other. This means that for every $\delta > 0$,

$$\begin{aligned}\sup\left\{\int_E |(u_m)_{x_i}|\, dx \,:\, L^n(E) \le \delta\right\} &\le \sup\left\{\int_E |\nabla u_m|\, dx \,:\, L^n(E) \le \delta\right\} \\ &= \sup\left\{\int_E |\nabla v|\, dx \,:\, L^n(E) \le \delta\right\}.\end{aligned}$$

Hence, if E_k is any sequence of measurable sets with $\lim(L^n(E_k)) = 0$, we infer that

$$\sup\left\{\int_{E_k} |(u_m)_{x_i}|\, dx \,:\, m \in \mathbb{N}\right\} \to 0 \quad \text{as} \quad k \to \infty.$$

Applying a well-known weak compactness criterion in $L^1(\mathbb{R}^n)$, see [3, p. 199], again we can extract a subsequence $u_{m'}$ converging weakly in $W^{1,1}(\mathbb{R}^n)$. Finally, proceeding as in case **(i)**, we obtain the assertion also in the case of $p = 1$. □

The following theorem gives an analog of the previous theorem for the spaces of continuous functions.

Theorem 10.3. *Let $u \in C(\mathbb{R}^n) \cap \mathcal{S}_+$. Then ω_{u^t} decreases in $0 \le t \le 1$.*

Proof. Let $u \in L^1_+(\mathbb{R}^n)$. For SC 1-symmetrizations the result follows from the proof of Lemma 7.5. In the general case, we choose a sequence u_m of functions in $C(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$ converging to u in $C(\mathbb{R}^n)$. Then, applying the nonexpansivity Lemma 3.1, we obtain

$$\|(u_m)^t - u^t\|_\infty \le \|u_m - u\|_\infty, \quad m = 1, 2, \dots,$$

and the claim for the SC 1-symmetrization follows. For $k \ge 2$, we apply the standard inductive argument. □

It is also easy to prove the monotonicity property of convex functionals.

Theorem 10.4. *Let $u \in W^{1,1}_+(\mathbb{R}^n)$, and let j be a Young function such that*

$$\int_{\mathbb{R}^n} j(|\nabla u|)\, dx < \infty.$$

Then the integral mean $\int_{\mathbb{R}^n} j(|\nabla u^t|)\, dx$ decreases in $0 \le t \le 1$.

Furthermore, if V is a linear subspace such that it either contains all "y- directions" $x_{n-k+1}, \dots, x_n$, or is orthogonal to each of these directions, then $\int_{\mathbb{R}^n} j(|\nabla_V u^t|)\, dx$ decreases in $0 \le t \le 1$.

Proof. The standard inductive argument still works. Therefore, it suffices to prove the theorem for the SC 1-symmetrization.

Fix t, $-\infty < t < \infty$. By the semigroup property (6.7), we need to prove the inequalities

$$\int_{\mathbb{R}^n} j(|\nabla u^t|)\,dx \le \int_{\mathbb{R}^n} j(|\nabla u|)\,dx, \quad \int_{\mathbb{R}^n} j(|\nabla_V u^t|)\,dx \le \int_{\mathbb{R}^n} j(|\nabla_V u|)\,dx. \tag{10.4}$$

First, assume that $u \in C^{0,1}_{0+}(\mathbb{R}^n)$. If we choose a sequence of polarizations of u converging to u^t in $L^1(\mathbb{R}^n)$ and apply [14, Lemma 5.3], we obtain

$$\int_{\mathbb{R}^n} j(|\nabla u_m|)\,dx = \int_{\mathbb{R}^n} j(|\nabla u|)\,dx.$$

By the weak lower semicontinuity of the integral functional, this leads to

$$\int_{\mathbb{R}^n} j(|\nabla u|)\,dx = \liminf \int_{\mathbb{R}^n} j(|\nabla u_m|)\,dx \ge \int_{\mathbb{R}^n} j(|\nabla u^t|)\,dx.$$

If $u \in W^{1,1}_+(\mathbb{R}^n)$, we choose a sequence $v_m \in C^{0,1}_{0+}(\mathbb{R}^n)$ such that

$$v_m \to u \quad \text{in } W^{1,1}(\mathbb{R}^n)$$

and

$$\int_{\mathbb{R}^n} j(|\nabla v_m|)\,dx \to \int_{\mathbb{R}^n} j(|\nabla u|)\,dx.$$

This means that for a subsequence $v_{m'}$ we have

$$(v_{m'})^t \rightharpoonup u^t \quad \text{weakly in } W^{1,1}(\mathbb{R}^n),$$

and the weak lower semicontinuity of the functionals shows that the first inequality in (10.4) holds true.

The second inequality in (10.4) is proved similarly. □

An analog of Theorem 10.2 is also valid for BV-functions.

Theorem 10.5. *If $u \in BV(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$, then $u^t \in BV(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$ and $\|Du^t\|_{BV}$ decreases in $0 \le t \le 1$.*

Proof. As in the previous theorems, we need to prove the result for the SC 1-symmetrization. By the semigroup property (6.7), we must prove that

$$\|Du^t\|_{BV} \le \|Du\|_{BV} \tag{10.5}$$

for any fixed t, $-\infty < t < \infty$.

We choose a sequence of functions $u_m \in W^{1,1}_+(\mathbb{R}^n)$ that converges to u in $BV(\mathbb{R}^n)$. By Lemma 5.3 in [14], the functions $(u_m)^t$ are equibounded in $W^{1,1}(\mathbb{R}^n)$. Therefore, there is a function $v \in BV(\mathbb{R})^n$ and a sequence $u_{m'}$ such that

$$(u_{m'})^t \rightharpoonup v \quad \text{weakly in } BV(\mathbb{R}^n).$$

On the other hand, the inequalities

$$\|(u_m)^t - u^t\|_1 \le \|u_m - u\|_1$$

show that

$$(u_m)^t \to u^t \quad \text{in } L^1(\mathbb{R}^n).$$

Now let μ_i denote the Radon measure associated with the weak partial derivative v_{x_i}, $i = 1, \dots, n$. Then for every $\varphi \in C^\infty_0(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \varphi\,d\mu_i \leftarrow \int_{\mathbb{R}^n} \varphi \frac{\partial((u_{m'})^t)}{\partial x_i}\,dx = -\int_{\mathbb{R}^n} \varphi_{x_i}(u_{m'})^t\,dx \to -\int_{\mathbb{R}^n} \varphi_{x_i} u^t\,dx,$$

which means that $v = u^t$.

Finally, the weak lower semi-continuity of the norm gives

$$\|Du^t\|_{BV} \le \liminf \|\nabla((U_m)^t)\|_1 = \lim \|\nabla u_m\|_1 = \|Du\|_{BV}. \qquad \square$$

Choosing the characteristic function of a set of finite perimeter for the role of u in Theorem 10.5, we derive the following "monotonic isoperimetric inequality" in $\mathbb{R}^n$.

Theorem 10.6. *Let E be a Caccioppoli set in $\mathbb{R}^n$. Then the perimeter*

$$\|D\chi_{E^t}\|_{BV}$$

of E^t is a monotone decreasing function of $0 \le t \le 1$.

Now we prove that the mapping $t \mapsto u^t$ is continuous from the left in the Sobolev spaces $W^{1,p}_+(\mathbb{R}^n)$. It should be mentioned here that an analog of this result in the space $BV(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$ fails. The characteristic function $u = \chi_I$ of a single interval $I \subset \mathbb{R}$ provides a simple counterexample in dimension $n = 1$.

Theorem 10.7. *Let $u \in W^{1,p}_+(\mathbb{R}^n)$, $1 \le p < \infty$, and let t_m, $m = 1, 2, \dots,$ be an increasing sequence in $(0, 1)$ such that $t_m \to t$ as $m \to \infty$. Then*

$$u^{t_m} \longrightarrow u^t \quad \text{in } W^{1,p}(\mathbb{R}^n). \tag{10.6}$$

Proof. First we consider the case where $p > 1$. By Lemma 10.1, we have $u^{t_m} \to u^t$ in $L^p(\mathbb{R}^n)$. From this we conclude that

$$u^{t_m} \rightharpoonup u^t \quad \text{weakly in } W^{1,p}(\mathbb{R}^n). \tag{10.7}$$

Moreover, Theorem 10.2 combined with the weak lower semicontinuity of the L^p-norm implies

$$\lim_{m\to\infty} \|\nabla u^{t_m}\|_p = \|\nabla u^t\|_p. \tag{10.8}$$

Since for $p > 1$ the spaces $W^{1,p}(\mathbb{R}^n)$ are uniformly convex, (10.6) follows from (10.7) and (10.8).

Now, let $p = 1$. We note that (10.7) and (10.8) remain true for $p = 1$. Then we fix an index $i \in \{1, \dots, n\}$ and put $v_m := (u^{t_m})_{x_i}$, $v := (u^t)_{x_i}$. Let $j(z) := \sqrt{1+z^2} - 1$. Then $j(z)$ is a Young function satisfying the inequality $j(z) \le z$. Therefore, Theorem 10.4 and the weak lower semicontinuity property of the integral imply that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \big(\sqrt{1+v_m^2} - 1\big)\, dx = \int_{\mathbb{R}^n} \big(\sqrt{1+v^2} - 1\big)\, dx. \tag{10.9}$$

Applying Taylor's formula, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} &\big(\sqrt{1+v_m^2} - 1\big)\, dx \\ &\ge \int_{\mathbb{R}^n} \big(\sqrt{1+v^2} - 1\big)\, dx + \int_{\mathbb{R}^n} \frac{v}{\sqrt{1+v^2}}(v_m - v)\, dx + \frac{1}{2}\int_{\mathbb{R}^n} \frac{(v_m - v)^2}{(1+c_m^2)^{3/2}}\, dx,\end{aligned}$$

where $c_m^2 = \max\{v^2, v_m^2\}$. The limit passage as $m \to \infty$ yields

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \frac{(v_m - v)^2}{(1+c_m^2)^{3/2}}\, dx = 0.$$

This implies that for every positive integer k we have

$$\lim_{m\to\infty} \int_{F_{k,m}} |v_m - v|\, dx = 0, \tag{10.10}$$

where $F_{k,m} = \{x \in \mathbb{R}^n : |v_m| \le k, |v| \le k\}$. Since $v_m \rightharpoonup v$ weakly in $L^p(\mathbb{R}^n)$, it follows that

$$\lim_{k\to\infty} \int_{G_{k,m}} |v_m|\, dx = 0 \tag{10.11}$$

uniformly over all m. Here $G_{k,m} = \{x \in \mathbb{R}^n : |v_m| > k\}$.

Now (10.6) follows from (10.9) and (10.11) via the inequality

$$\|v_m - v\|_1 \le \int_{F_{k,m}} |v_m - v|\, dx + 2\int_{G_{k,m}} |v_m|\, dx + 2\int_{\{|v|>k\}} |v|\, dx,$$

which is true for all positive integers m and k. □

§11. Comparison theorems

First we introduce a partial order $\prec^t$ related to the continuous (k,n)-Steiner symmetrization T^t.

Definition 11.1. For functions $f, g \in L^1_+(\mathbb{R}^n)$ and $0 \le t \le 1$, we write

$$f \prec^t g \text{ if and only if } \int_{\mathbb{R}^n} fh\, dx \le \int_{\mathbb{R}^n} g^t h^t\, dx \quad \text{for all } h \in L^\infty_+(\mathbb{R}^n). \tag{11.1}$$

The following theorem generalizes the well-known equivalences in the theory of symmetrization (see [14, Remark 10.1], [4]) to the case of SC 1-symmetrization.

Theorem 11.1. *Let $f, g \in L^1_+(\mathbb{R}^n)$, and let $(\cdot)^t$ denote the SC 1-symmetrization, $-\infty < t < \infty$. Then the following relations are equivalent*:

$$f \prec^t g; \tag{11.2}$$

$$\int_{\mathbb{R}^n} f^t h^t\, dx \le \int_{\mathbb{R}^n} g^t h^t\, dx \quad \textit{for all } h \in L^\infty_+(\mathbb{R}^n); \tag{11.3}$$

$$j(f) \prec^t j(g) \quad \textit{for all Lipschitz continuous Young functions } j; \tag{11.4}$$

$$\int_{2t-y}^{y} f^t(x', s)\, ds \le \int_{2t-y}^{y} g^t(x', s)\, ds \quad \textit{for all } x' \in \mathbb{R}^{n-1} \textit{ and every } y \ge t; \tag{11.5}$$

$$f^t(x', y) \le g^t(x', y) \quad \textit{for all } x' \in \mathbb{R}^{n-1} \textit{ and every } y \ge t. \tag{11.6}$$

Proof. **(a)** To prove that (11.2) implies (11.5), we fix $c > 0$ and set $M_c := \{f > c\}$. Then for fixed $x'_0 \in \mathbb{R}^{n-1}$ and positive ε, we introduce the function

$$h_\varepsilon(x', y) := \kappa(x', y)\varphi_\varepsilon(x', y)$$

with $x = (x', y) \in \mathbb{R}^n$ and put

$$\varphi_\varepsilon = \varphi_\varepsilon(x', y) := \chi(\{(x', y) : |x' - x'_0| < \varepsilon\}),$$

$$\kappa(x', y) := \begin{cases} 1 & \text{if } f(x', y) > c \text{ and } y < y(x', t), \\ 0 & \text{otherwise,} \end{cases}$$

where $y(x'_0, t) = y_{M_c}(x'_0, t)$ is the separating function defined in §4.

Now (11.2) implies

$$\int_{\{|x'-x'_0|<\varepsilon\}} \int_{2t-y(x',t)}^{y(x',t)} f^t(x', s)\, ds\, dx' = \int_{\mathbb{R}^n} f^t (h_\varepsilon)^t\, dx$$
$$= \int_{\mathbb{R}^n} f h_\varepsilon\, dx \le \int_{\mathbb{R}^n} g^t (h_\varepsilon)^t\, dx = \int_{\{|x'-x'_0|<\varepsilon\}} \int_{2t-y(x',t)}^{y(x',t)} g^t(x', s)\, ds\, dx'.$$

Taking the limit here as $\varepsilon \to 0^+$, we obtain

$$\int_{2t-y(x_0',t)}^{y(x_0',t)} f^t(x_0',s)\,ds \le \int_{2t-y(x_0',t)}^{y(x_0',t)} g^t(x_0',s)\,ds,$$

where $y(x_0',t) = y_{M_c}(x_0',t)$ depends on c. Since $c > 0$ can be chosen arbitrarily small, the latter inequality implies (11.5).

(b) To show that (11.2) implies (11.6), we fix $(x_0', y_0) \in \mathbb{R}^n$ with $y_0 \ge t$. Then, using the notation of part **(a)** of this proof, we note that the separating function $y(x_0',t) = y_{M_c}(x_0',t)$ depends monotonously on the polarization height c. Therefore, we can choose $c > 0$ so small that $y_0 \ge y(x_0',t)$. Then $f^t(x_0',y) = f(x_0',y)$ for all $y \ge y_0$. Now we choose the function h in (11.1) to be the Dirac δ-function at $x_0 = (x_0', y_0)$, i.e., $h := \delta(x_0)$. Then $h = h^t$. Finally, applying (11.1), we derive

$$f^t(x_0',y_0) = f(x_0',y_0) = \int_{\mathbb{R}^n} fh\,dx \le \int_{\mathbb{R}^n} g^t h^t\,dx = g^t(x_0',y_0),$$

which is (11.6).

(c) Now we show that (11.5) and (11.6) together imply (11.3). First, we assume that $h = \chi(M)$ is a characteristic function of $M \in \mathcal{M}(\mathbb{R}^n)$. Let $y(x',t) = y_M(x',t)$, and let $M(x',t) = M(x') \cap \{s : s > y(x',t)\}$. Then (11.5) and (11.6) imply

$$\begin{aligned}\int_{M^t} f^t\,dx &= \int_{\mathbb{R}^{n-1}} \left\{ \int_{2t-y(x',t)}^{y(x',t)} f^t(x',s)\,ds + \int_{M(x',t)} f^t(x',s)\,ds \right\} dx' \\ &\le \int_{\mathbb{R}^{n-1}} \left\{ \int_{2t-y(x',t)}^{y(x',t)} g^t(x',s)\,ds + \int_{M(x',t)} g^t(x',s)\,ds \right\} dx' \\ &= \int_{M^t} g^t\,dx.\end{aligned} \tag{11.7}$$

Next, let h be a step function, i.e.,

$$h := \varepsilon \sum_{i=1}^m \chi(M_i)$$

with some $\varepsilon > 0$ and some sets $M_i \in \mathcal{M}(\mathbb{R}^n)$ such that $M_1 \supset \cdots \supset M_m$. Applying inequality (11.7) to the functions $\chi(M_i)$, we obtain the desired inequality:

$$\int_{\mathbb{R}^n} f^t g^t\,dx = \varepsilon \sum_{i=1}^m \int_{M_i^t} f^t\,dx \le \varepsilon \sum_{i=1}^m \int_{M_i^t} g^t\,dx = \int_{\mathbb{R}^n} g^t h^t\,dx.$$

Finally, every $h \in L_+^\infty(\mathbb{R}^n)$ can be approximated by step functions. Therefore, in the general case (11.3) follows from the previous inequality.

(d) To prove that (11.3) implies (11.4), we may assume without loss of generality that $j \in C^1$. For $p \in L_+^\infty(\mathbb{R}^n)$, we set $h := j'(f)p$. Since j' is a monotone increasing function, we have $h^t = j'(f^t)p^t$. Combined with (11.3), this implies

$$\int_{\mathbb{R}^n} f^t j'(f^t) p^t\,dx \le \int_{\mathbb{R}^n} g^t j'(f^t) p^t\,dx. \tag{11.8}$$

Since j is a convex function, we have

$$j(g^t) - j(f^t) \ge j'(f^t)(g^t - f^t),$$

which together with (11.8) leads to (11.4).

(e) Finally, (11.2) is a special case of (11.4). □

The following corollary shows that the equivalencies (11.2)–(11.4) remain valid for the continuous (k,n)-Steiner symmetrizations for all k, $1 \le k \le n$. Its proof follows from Theorem 11.1 via the standard inductive argument.

Corollary 11.1. *Let $f, g \in L^1_+(\mathbb{R}^n)$, and let $(\cdot)^t$ (where $0 \le t \le 1$) denote the continuous (k,n)-Steiner symmetrization. Then the equivalencies* (11.2)–(11.4) *remain valid for all $0 \le t \le 1$.*

Corollary 11.2. *Let $f, g \in L^1_+(\mathbb{R}^n)$, and let $(\cdot)^t$ denote the continuous (k,n)-Steiner symmetrization into a given (k,n)-Steiner symmetrization $(\cdot)^*$. Then for all s and t such that $0 \le s < t \le 1$, the following is true.*

If $f \prec^s g$, then $f \prec^t g$.

In particular, if $f \prec^t g$, then $f \prec^* g$.

For the SC 1-symmetrization, Corollary 11.2 follows immediately from the semigroup property (6.7). Then, for any $k \ge 2$, its proof follows via the standard inductive argument.

Now we prove two comparison lemmas concerning partial symmetry of solutions of certain elliptic and parabolic PDE's. In fact, these results and their proofs show that the approach to comparison theorems in partially symmetric domains based on continuous symmetrization is a close relative of Alexandrov's moving plane method, see [26] and [18]. In the context of comparison theorems, the approach based on continuous symmetrization was used for the first time in [27].

Lemma 11.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, $c \ge 0$, $f, g \in L^2_+(\Omega)$, and u, v are solutions of the following boundary-value problems*:

$$u, v \in W^{1,2}_0(\Omega), \quad -\Delta u + cu = f, \quad -\Delta v + cv = g \quad \text{in } \Omega. \tag{11.9}$$

For $0 \le t \le 1$, let $(\cdot)^t$ denote a continuous (k,n)-Steiner symmetrization. If for some $0 \le t \le 1$ we have $\Omega = \Omega^t$, $f = f^t$, $g = g^t$, and $f \prec^t g$, then

$$u = u^t, \quad v = v^t \tag{11.10}$$

and

$$u \prec^t v. \tag{11.11}$$

Proof. First, we prove the lemma for the SC 1-symmetrization. Then, of course, $-\infty < t < \infty$. Relation (11.10) follows easily from the maximum principle.

To prove (11.11), we recall that $f \prec^t g$ if and only if

$$\int_{\mathbb{R}^n} fh\,dx \le \int_{\mathbb{R}^n} g^t h^t\,dx \quad \text{for all} \quad h \in L^\infty_+(\mathbb{R}^n).$$

Since $g = g^t$, the latter inequality with the test function $h = (u-v)_+ = (u^t - v^t)_+$ gives

$$\int_{\mathbb{R}^n} f(u-v)_+\,dx \le \int_{\mathbb{R}^n} g(u-v)_+\,dx,$$

which easily leads to the desired conclusion (11.11) in the case where $k = 1$.

Now for $k \ge 2$, the lemma follows via the standard inductive argument. □

Similar lemma holds true also for parabolic problems. Its proof follows along the lines of the proof of Lemma 11.1. Therefore, the details are left to the reader.

Lemma 11.2. *Let $(\cdot)^s$ denote a continuous (k,n)-Steiner symmetrization, $0 \le s \le 1$. Let $c \ge 0$, $T > 0$, and let Ω be a bounded domain in $\mathbb{R}^n$ such that $\Omega = \Omega^s$. Suppose that functions $u_0, v_0 \in L^2_+(\Omega)$ and $f, g \in L^2_+(\Omega \times (0,T))$ satisfy the following conditions*:

$$u_0 = u_0^s, \quad v_0 = v_0^s, \quad u_0 \prec^s v_0,$$

and

$$f(\cdot,t) = f^s(\cdot,t), \quad g = g(\cdot,t) = g^s(\cdot,t), \quad f(\cdot,t) \prec^s g(\cdot,t) \quad \textit{for all} \;\; t \in (0,T).$$

Next, let $u, v \in L^2(0,T; W_0^{1,2}(\Omega)) \cap C([0,T]; L^2(\Omega))$ *be solutions of the following initial boundary-value problems*:

$$\begin{aligned} &u_t - \Delta u + cu = f, \quad v_t - \Delta v + cv = g, \quad \textit{in} \;\; \Omega \times (0,T), \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \textit{in} \;\; \Omega. \end{aligned} \tag{11.12}$$

Then

$$u(\cdot,t) = u^s(\cdot,t), \quad v(\cdot,t) = v^s(\cdot,t) \quad \textit{for all} \;\; t \in (0,T) \tag{11.13}$$

and

$$u(\cdot,t) \prec^s v(\cdot,t) \quad \textit{for all} \;\; t \in (0,T). \tag{11.14}$$

To simplify the notation in our formulations and proofs, we shall use the following definitions from [14].

Definition 11.2. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open set, $c \geq 0$, and $f \in L^2_+(\Omega)$. We say that u solves the problem $\mathbb{B}_1(\Omega, c, f)$ if u is the solution of the following boundary-value problem:

$$u \in W_0^{1,2}(\Omega), \quad -\Delta u + cu = f \quad \text{in} \;\; \Omega. \tag{11.15}$$

Definition 11.3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open set, $c \geq 0$, $f \in L^2_+(\Omega)$, and $\gamma : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is a continuous and monotone nondecreasing function. We say that $\underline{u}$ *is a solution of the problem* $\mathbb{B}_2(\Omega, c, \gamma, f)$ if $\underline{u}$ is the nonnegative minimal solution of the following boundary-value problem:

$$u \in W_0^{1,2}(\Omega), \quad u \geq 0, \quad -\Delta u + cu = \gamma(u) + f \quad \text{in} \;\; \Omega, \tag{11.16}$$

that is,

(i) $\underline{u}$ is a solution of (11.16), and
(ii) $0 \leq \underline{u} \leq u$ for all other solutions u of (11.16).

For a brief discussion of important properties of solutions of problems in Definitions 11.2 and 11.3, we refer to [14, §9].

Theorem 11.2. *Let* $(\cdot)^t$ *denote a continuous* (k,n)*-Steiner symmetrization,* $0 \leq t \leq 1$. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a bounded open set,* $c \geq 0$, $f \in L^2_+(\Omega)$, *and* γ *is a Young function. For a fixed* t, $0 \leq t \leq 1$, *let* $g \in L^2_+(\Omega^t)$ *be such that* $g^t = g$ *and* $f \prec^t g$. *Let* u *and* v *be solutions of the problems* $B_2(\Omega, c, \gamma, f)$ *and* $B_2(\Omega^t, c, \gamma, g)$, *respectively. Then*

$$v = v^t \tag{11.17}$$

and

$$u \prec^t v. \tag{11.18}$$

Proof. As is well known, the claim of this theorem holds true for the (k,n)-Steiner symmetrization for any k, $1 \leq k \leq n$; see, for example, Theorem 10.1 in [14]. Thus, in view of the standard inductive argument we need to prove it for the SC 1-symmetrization only. Then, of course, $-\infty < t < \infty$.

(1) First, we assume that $\gamma \equiv 0$ and that f is a simple function with compact support in Ω, see Definition 7.1. For a fixed $t \in \mathbb{R}$, let $\widetilde{v}$ denote the solution of the problem $\mathbb{B}_1(\Omega^t, c, f^t)$. The maximum principle tells us that $v = v^t$ and $\widetilde{v} = \widetilde{v}^t$. Furthermore, let h be an arbitrary function in $L^2_+(\Omega^t)$ satisfying $h = h^t$, and let w be the solution of

the problem $\mathbb{B}_1(\Omega^t, c, h)$. Since again we have $w = w^t$, $w \geq 0$, and $f \prec^t g$, after partial integration we get

$$\int_{\Omega^t} \widetilde{v} h^t \, dx = \int_{\Omega^t} w f^t \, dx \leq \int_{\Omega^t} w g \, dx = \int_{\Omega^t} v h^t \, dx,$$

which means that

$$\widetilde{v} \prec^t v. \tag{11.19}$$

Next, let Ω' be an open set such that $\operatorname{supp}(f) \subset \Omega' \subset \overline{\Omega'} \subset \Omega$. By Corollary 7.1 and Lemma 7.2, we can find a finite number of polarizations P_i with polarizers $H_i = \{y > y_i\}$, $i = 1, \dots, N$, where

$$y_1 < y_2 < \dots < y_N \leq t,$$

such that the closure of $\bigcirc_{i=1}^N P_i(\Omega')$ is in Ω^t and $\left(\bigcirc_{i=1}^N P_i f\right)^t = f^t$.

Let $\Omega_m = \bigcirc_{i=1}^m P_i \Omega'$ and let $f_m = \bigcirc_{i=1}^m P_i f$, $m = 1, \dots, N$. Let u' and u'_m be solutions of the problems $\mathbb{B}_1(\Omega', c, f)$ and $\mathbb{B}_1(\Omega'_m, c, f_m)$, respectively, $m = 1, \dots, N$. Applying [14, Theorem 9.1], we conclude that

$$u' \prec_{H_1} u'_1 \prec_{H_2} u'_2 \prec_{H_3} \dots \prec_{H_N} u'_N, \tag{11.20}$$

where "$\prec_H$" denotes the partial order related to polarization with the polarizer H, see [14, p. 1783]. Since $\Omega'_N \subset \Omega^t$ and $f_N = f^t$, the solutions u'_N and $\widetilde{v}$ satisfy the inequality

$$0 \leq u'_N \leq \widetilde{v} \quad \text{a.e. in } \Omega_N.$$

Together with (11.19) and (11.20), this implies $u' \prec^t v$.

Now we choose open bounded sets Ω^k such that $\overline{\Omega^k} \subset \Omega^{k+1}$, $k = 1, 2, \dots$, and $\cup_k \Omega^k = \Omega$. Let u^t denote the solution of the problem $\mathbb{B}_1(\Omega^k, c, f)$, $k = 1, 2, \dots$ By the above consideration we have $u^k \prec^t v$, $k = 1, 2, \dots$ By the convergence property of elliptic boundary-value problems in varying domains, see Lemma A in [14], the sequence u^k converges to u in $L^2(\Omega)$. This proves the claim in the case under consideration.

(2) Next we assume that $\gamma \equiv 0$ but f is an arbitrary function in $L^2_+(\Omega)$. Since simple functions are dense in $L^2_+(\Omega)$, there is a sequence of simple functions f^k, $k = 1, 2, \dots$, with compact supports in Ω and such that $f^k \to f$ in $L^2(\Omega)$. Then we can find open sets Ω^k, $k = 1, 2, \dots$, such that $\operatorname{supp}(f^k) \subset \Omega^k \subset \overline{\Omega^k} \subset \Omega^{k+1} \subset \Omega$ and $\cup_k \Omega^k = \Omega$.

Let u^k and v_k be solutions of the problems $\mathbb{B}_1(\Omega^k, c, f^k)$ and $\mathbb{B}_1((\Omega^k)^t, c, (f^k)^t)$, respectively, $k = 1, 2, \dots$. By part **(1)**, we have

$$u_k \prec^t v_k, \quad k = 1, 2, \dots. \tag{11.21}$$

By Lemma 7.4, $(f^k)^t \to f^t$ in $L^2(\Omega^t)$. Also by the monotonicity property of rearrangements,

$$\operatorname{closure}((\Omega^k)^t) \subset (\Omega^{k+1})^t \subset \Omega^t \text{ and } \cup_k (\Omega^k)^t = \Omega^t.$$

Therefore, by the convergence property of elliptic boundary-value problems in varying domains, see Lemma A in [14], we have

$$u_k \to u \text{ in } L^2(\Omega) \text{ and } v_k \to v \text{ in } L^2(\Omega^t).$$

Combined with (11.21), this proves the theorem in the case where $\gamma \equiv 0$.

(3) Next, let $\gamma \not\equiv 0$. In accordance with Remark 9.2 in [14], we approximate u and v by solutions of the problems $\mathbb{B}_1(\Omega, c, \gamma(u_{m-1}) + f)$, $\mathbb{B}_1(\Omega^t, c, \gamma(v_{m-1}) + g)$, respectively, $m = 1, 2, \dots$ Here $u_0 \equiv v_0 \equiv 0$. Assume we have proved that $u_m \prec^t v_m$ for some m. Notice that for $m = 0$ this is trivial. Then, by relation (11.4) in Theorem 11.1, we see that $\gamma(u_m) + f \prec^t \gamma(v_m) + g$. By parts **(1)** and **(2)** above, this means that also $u_{m+1} \prec^t v_{m+1}$, and the proof can be finished by induction. $\square$

Theorem 11.2 and the equivalence relations in Theorem 11.1 lead to the following corollary.

Corollary 11.3. *Let u and v be the solutions defined in Theorem* 11.2*. Then for every Young function j, we have*

$$\int_\Omega j(u)\,dx \le \int_{\Omega^t} j(v)\,dx \tag{11.22}$$

if the above integrals converge. In particular,

$$\|u\|_p \le \|v\|_p \quad \text{for all} \quad 1 \le p \le \infty. \tag{11.23}$$

One might ask under what conditions equality occurs in inequalities (11.22) and (11.23), and believe that equality is only possible — roughly speaking — if Ω possesses a partial symmetry. For Steiner symmetrizations this result was proved in [14]. To prove this uniqueness result for the continuous (k,n)-Steiner symmetrization, we restrict ourselves to the case where Ω is a domain.

Theorem 11.3. *Let $(\cdot)^t$, $0 \le t \le 1$, denote the continuous (k,n)-Steiner symmetrization into some (k,n)-Steiner symmetrization with respect to a plane Σ. Let Ω, Ω^t, c, f, g, γ, u, and v be as in Theorem* 11.2 *and assume that Ω is a bounded domain and $f > 0$ on Ω. Also, assume that there is a Lipschitz continuous Young function j that satisfies*

$$\int_\Omega j(u)\,dx = \int_{\Omega^t} j(v)\,dx > 0 \tag{11.24}$$

for some t, $0 \le t \le 1$. Then $\Omega = \Omega^t$ and $f = g$ modulo some translation in a direction orthogonal to Σ.

Proof. For (k,n)-Steiner symmetrizations, this result was proved in Theorem 10.3 in [14]. Thus, thanks to the standard inductive argument, we need to prove the theorem for the SC 1-symmetrization only. Then $-\infty < t < \infty$ and we may assume that $\Sigma = \{y = 0\}$. Here $x = (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Assume that, for some fixed $t \in \mathbb{R}$, Ω^t is not a translate of Ω in the direction of the y-axis. Corollary 6.3 shows that we can find a polarizer $H = \{y > t_0\}$ with $t_0 < t$ such that either $(\Omega_H)^t = \Omega^t$, $(f_H)^t = f^t$, $\Omega \ne \Omega_H$, and $\sigma_H(\Omega) \ne \Omega_H$, or $(\Omega_H)^t = \Omega^t$, $(f_H)^t = f^t$, $\Omega = \Omega_H$, $f \ne f_H$, and $\sigma_H(f) \ne f_H$. Then, if w is the solution of the problem $\mathbb{B}_2(\Omega_H, c, \gamma, f_H)$, we can use Theorem 9.3 in [14] to conclude that

$$\int_\Omega j(u)\,dx < \int_{\Omega_H} j(w)\,dx. \tag{11.25}$$

Next, since $(\Omega_H)^t = \Omega^t$ and $(f_H)^t = f^t$, we also have $w \prec^t v$. By Corollary 11.3, this means that

$$\int_{\Omega_H} j(w)\,dx \le \int_{\Omega^t} j(v)\,dx,$$

which together with (11.25) contradicts (11.24).

Now we assume that, for some $t \in \mathbb{R}$, $\Omega = \Omega^t$ modulo translation in the direction of the y-axis. Without loss of generality we may assume that $\Omega = \Omega^t$. Assume also that $f \ne f^t$. By Corollary 6.3, there is a polarizer $H = \{y > t_1\}$ with $t_1 < t$ such that $\Omega_H = \Omega$ and $f \ne f_H$, and we can argue as before to derive a contradiction to (11.24).

Thus, we have $f = f^t$ and it remains to show that $f^t = g$.

Assume that $f^t \ne g$. We set $\widetilde{f} = \gamma(u) + f$ and $\widetilde{g} = \gamma(v) + g$. Since $u = u^t$, $v = v^t$, and $u \prec^t v$, from Theorem 11.1 it follows that

$$\widetilde{f} \prec^t \widetilde{g}, \quad \widetilde{f} = \widetilde{f}^t \ne \widetilde{g}^t = \widetilde{g},$$

and also

$$\int_{2t-y}^{y} \big(\widetilde{f}(x',s) - \widetilde{g}(x',s)\big)\, ds \le 0 \tag{11.26}$$

for all $x' \in \mathbb{R}^{n-1}$ and every $y \ge t$, where the inequality in (11.26) must be strict on a subset of Ω of positive measure. Now, let h be an arbitrary function in $L^2_+(\Omega)$ satisfying $h = h^t \not\equiv 0$. Then, if w is the solution of the problem $\mathbb{B}_1(\Omega, c, h)$, we conclude that $w = w^t$. Moreover, the strong maximum principle yields

$$\left|\frac{\partial}{\partial y} w(x)\right| > 0 \quad \text{a.e. in } \Omega. \tag{11.27}$$

After partial integration and using (11.26) and (11.27), we obtain

$$\begin{aligned}\int_\Omega (u-v)h\, dx &= \int_\Omega w(\widetilde{f} - \widetilde{g})\, dx \\ &= \frac{1}{2}\int_\Omega \left|\frac{\partial}{\partial y} w(x',y)\right| \left(\int_{2t-y}^{y} (\widetilde{f}(x',\tau) - \widetilde{g}(x',\tau))\, d\tau\right) dx'\, dy < 0.\end{aligned}$$

Since j' is monotone nondecreasing and $u = u^t$, we have $j'(u) = (j'(u))^t$ (see relation (3.6) in [14]), and by (11.24) it follows that $j'(u) \ne 0$. Therefore, we may take $h = j'(u)$ in the equation above. Since j is convex, we get

$$\int_\Omega (j(u) - j(v))\, dx \le \int_\Omega j'(u)(u-v)\, dx < 0,$$

a contradiction. The theorem is proved. □

Next we prove that solutions of the problems considered above are continuous from the left with respect to the parameter t of symmetrization.

Theorem 11.4. *Let $(\cdot)^t$, $0 \le t \le 1$, denote the continuous (k,n)-Steiner symmetrization into some (k,n)-Steiner symmetrization. Let Ω, f, c, γ be defined as in Theorem 11.2, and let t_m, $m = 1, 2, \dots$, be a monotone increasing sequence in the standard interval $I = [0,1]$ such that $t_m \to t_0 \in I$. Next, let v and v_m be the positive minimal solutions of the problems $\mathbb{B}_2(\Omega^{t_0}, c, \gamma, f^{t_0})$ and $\mathbb{B}_2(\Omega^{t_m}, c, \gamma, f^{t_m})$, respectively. Then*

$$v_m \longrightarrow v \quad \text{in } W^{1,2}(\mathbb{R}^n). \tag{11.28}$$

Proof. **(a)** First we prove the theorem for the SC 1-symmetrization. Then $-\infty < t < \infty$. Let t_m be a monotone increasing sequence such that $t_m \to t_0 \in \mathbb{R}$. Since $(f^{t_m})^{t_0} = f^{t_0}$ for $m = 1, 2, \dots$, we may apply Theorem 10.4 to conclude that the functions v_m, $m = 1, 2, \dots$, are equibounded in $W^{1,2}(\mathbb{R}^n)$. Therefore, we can find a subsequence $v_{m'}$ and a function $w \in W^{1,2}(\mathbb{R}^n)$ such that

$$v_{m'} \rightharpoonup w \quad \text{weakly in } W^{1,2}(\mathbb{R}^n) \tag{11.29}$$

and

$$v_{m'} \longrightarrow w \quad \text{in } L^2(\mathbb{R}^n) \text{ and a.e.} \tag{11.30}$$

By Lemma 5.2, we have $\partial\Omega^{t_0} \subset \partial\Omega^{t_m} + r_m \bar{B}_1$, where $r_m = t_0 - t_m$. Then $r_{m+1} \le r_m$ for $m = 1, 2, \dots$ and $r_m \to 0$. Since $v_m \equiv 0$ in $\mathbb{R}^n \setminus \Omega^{t_m}$, we see that $w \in W^{1,2}_0(\Omega^{t_0})$. Thus, we can argue as in the proof of Theorem 11.2 to deduce that $w = v$.

Next, since $(f^{t_m})^{t_{m+1}} = f^{t_{m+1}}$, Theorem 11.2 shows that

$$v_1 \prec^{t_2} v_2 \prec^{t_3} \cdots \prec^{t_0} v. \tag{11.31}$$

Together with (11.30) and (11.23), this implies that the sequence of norms $\|v_m\|_2^2$ decreases and

$$\|v_m\|_2^2 \to \|v\|_2^2.$$

By the uniform convexity of $L^2(\mathbb{R}^n)$, this means that

$$v_m \longrightarrow v \quad \text{in } L^2(\mathbb{R}^n). \tag{11.32}$$

Now, combining (11.32) and the fact that the functions v_m, $m = 1, 2, \dots$, are equibounded in $W^{1,2}(\mathbb{R}^n)$, we conclude that

$$v_m \rightharpoonup v \quad \text{weakly in } W^{1,2}(\mathbb{R}^n).$$

Finally, we have

$$\|\nabla v_m\|_2^2 + c\|v_m\|_2^2 = \int_{\Omega^{t_m}} \big(\gamma(v_m) + f\big) v_m \, dx \to \int_{\Omega^{t_0}} \big(\gamma(v) + f\big) v \, dx = \|\nabla v\|_2^2 + c\|v\|_2^2.$$

In view of the uniform convexity of $W^{1,2}(\mathbb{R}^n)$, this yields (11.28).

(b) Now we prove the theorem for the continuous (k, n)-Steiner symmetrization and $t_0 = 1$. In this case, $\Omega^{t_0} = \Omega^*$ and $f^{t_0} = f^*$, i.e., Ω^{t_0} and f^{t_0} are the corresponding (k, n)-Steiner symmetrizations of Ω and f, respectively. Similarly, we have $(\Omega^{t_m})^* = \Omega^*$ and $(f^{t_m})^* = f^*$ for $m = 1, 2, \dots$. Applying Theorem 10.1 in [14], we conclude as before that the functions v_m, $m = 1, 2, \dots$, are equibounded in $W^{1,2}(\mathbb{R}^n)$. Therefore, we can find a subsequence $v_{m'}$ and a function $w \in W^{1,2}(\mathbb{R}^n)$ such that (11.29) and (11.30) remain valid. By (9.5) of Theorem 9.2, we have $\partial\Omega^{t_0} \subset \partial\Omega^{t_m} + \varepsilon_m \bar{B}_1$ with some $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ as $m \to \infty$. Since $v_m \equiv 0$ in $\mathbb{R}^n \setminus \Omega^{t_m}$, we conclude that $w \in W_0^{1,2}(\Omega^{t_0})$. As in part **(a)**, this implies that $w \equiv v$.

By Definition 9.1, $\Omega^{t_{m+1}}$ and $f^{t_{m+1}}$ are obtained from Ω^{t_m} and f^{t_m}, respectively, after a finite number of $(k-1, n)$-Steiner symmetrizations followed by an appropriate SC $(k-1, n)$-Steiner symmetrization. Therefore, Theorem 10.1 in [14] and Theorem 11.2 in this section show that relations (11.31) remain valid in this case as well. As in part **(a)** of this proof, the last statement yields (11.28).

(c) The theorem is proved for the SC one-dimensional symmetrization and for $t_0 = 1$, which corresponds to the (k, n)-Steiner symmetrization. Now, the general case follows from these two cases via the standard inductive argument. □

Since the proofs of Theorems 11.2 and 11.4 depend only on the maximum principle, we can derive similar results for parabolic problems. The proofs of Theorems 11.5 and 11.6 below are based on an approximation argument involving solutions of some elliptic problems. This idea was used in [5] for the Schwarz symmetrization and then in [14] for the (k, n)-Steiner symmetrization. As we shall see, this method works also for continuous symmetrizations. First, following [14], we shall define solutions of parabolic problems.

Definition 11.4. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open set, $c \geq 0$, $T > 0$, $f \in L^2_+(\Omega \times (0, T))$, $\varphi \in L^2_+(\Omega)$, and let $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a globally Lipschitzian function. We say that u *solves the problem* $\mathbb{I}(\Omega, T, c, \gamma, f, \varphi)$ if u is a solution of the following initial boundary-value problem:

$$\begin{aligned} &u \in L^2\big(0, T; W_0^{1,2}(\Omega)\big) \cap C\big([0, T]; L^2(\Omega)\big), \quad \frac{\partial u}{\partial t} \in L_2\big([0, T]; L^2(\Omega)\big), \\ &u_t - \Delta u + cu = \gamma(u) + f \quad \text{in } \Omega \times (0, T), \\ &u(x, 0) = \varphi(x) \quad \text{in } \Omega. \end{aligned} \tag{11.33}$$

Under the assumptions of Definition 11.4, the problem $\mathbb{I}(\Omega, T, c, \gamma, f, \varphi)$ has a unique nonnegative solution that can be approximated with the help of the so-called *time discretization method*, see [19] or [14, Section 10]. To define this approximation, we choose

$N \in \mathbb{N}$. Then we divide the interval $(0,T)$ into N subintervals $[t_{i-1}, t_i]$, where $t_i = iT/N$, and set

$$f_i(x) = \frac{T}{N} \int_{t_{i-1}}^{t_i} f(x,s)\, ds, \quad i = 1, \dots, N. \tag{11.34}$$

Putting $u_0 = \varphi$, we let u_i be the solution of the problem

$$\mathbb{B}_1\big(\Omega, c + (N/T), \gamma(u_{i-1}) + f_i + (N/T)u_{i-1}\big), \tag{11.35}$$

defined inductively for all $i = 1, \dots, N$.

Let $u^N(x,t)$ denote the function of $x \in \Omega$ and $t \in [0,T]$ defined for $t_{i-1} \le t \le t_i$, $i = 1, \dots, N$, by

$$u^N(x,t) = u_{i-1}(x) + (t - t_{i-1})(N/T)(u_i(x) - u_{i-1}(x)). \tag{11.36}$$

This gives the desired approximation to the solution of problem (11.33). Namely, we have (see [19, Theorem 2.2.4, p. 42]):

$$u^N(\cdot,t) \rightharpoonup u(\cdot,t) \quad \text{weakly in } W_0^{1,2}(\Omega) \text{ for all } \ t \in (0,T),$$

$$\frac{\partial u^N}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } \ L^2([0,T]; L^2(\Omega)),$$

and

$$u^N \to u \quad \text{in } C(0,T; L^2(\Omega)).$$

Theorem 11.5. *Let $(\cdot)^s$, $0 \le s \le 1$, denote a continuous (k,n)-Steiner symmetrization. Let Ω, c, T, f, γ, φ, and u be as in Definition 11.4, and let $g \in L^2_+(\Omega^s \times (0,T))$ and $\psi \in L^2_+(\Omega^s)$, with $f(\cdot,t) \prec^s g(\cdot,t)$ and $g(\cdot,t) = (g(\cdot,t))^s$ for all $t \in (0,T)$, and $\varphi \prec^t \psi$, $\psi = \psi^t$. Let v be the solution of the problem $\mathbb{I}(\Omega^t, T, c, \gamma, g, \psi)$. Then*

$$u(\cdot,t) \prec^s v(\cdot,t) \quad \textit{for all} \ \ t \in (0,T) \tag{11.37}$$

and

$$v(\cdot,t) = (v(\cdot,t))^s \quad \textit{for all} \ \ t \in (0,T). \tag{11.38}$$

Proof. Fix $0 < s \le 1$. For any $N \in \mathbb{N}$ and $1 \le i \le N$, let u_i be the solution of problem (11.35), and let v_i be the solution of problem (11.35) with Ω, f, and φ replaced by Ω^s, g, and ψ, respectively. By assumption, we have $u_0 \prec^s v_0$. If N is sufficiently large, then $\gamma(\tau) + N\tau/T$ is monotone increasing and convex in τ. Since $f \prec^s g$, by (11.34) we also have $f_i \prec^s g_i$ for $i = 1, \dots, N$.

Applying Theorem 11.2, we conclude that $u_i \prec^s v_i$ for $i = 1, \dots, N$. By (11.36), we see that $u^N \prec^s v^N$. Then the limit passage as $N \to \infty$ yields (11.37).

Since $v_i(\cdot,t) = (v_i(\cdot,t))^s$ for all $t \in (0,T)$ and all $i = 1, \dots, N$, we have $v^N(\cdot,t) = (v^N(\cdot,t))^s$. Passing to the limit as $N \to \infty$, we get (11.38). □

The solutions of the above parabolic problems are continuous from the left with respect to the parameter of symmetrization s.

Theorem 11.6. *Let Ω, c, T, f,γ, φ, and u be as in Theorem 11.5, and let s_m, $m = 1, 2, \dots$, be a monotone increasing sequence in $I = [0,1]$ such that $s_m \to s_0 \in I$. Next, let v and v_m be the positive solutions of the problems $\mathbb{I}(\Omega^{s_0}, T, c, \gamma, g^{s_0}, \psi)$ and $\mathbb{I}(\Omega^{s_m}, T, c, \gamma, g^{s_m}, \psi))$, respectively. Then*

$$v_m \longrightarrow v \quad \textit{in} \ \ W^{1,2}(\mathbb{R}^n) \times (0,T). \tag{11.39}$$

Proof. Using time discretization as in the proof of Theorem 11.5, we approximate v and v_m, $m = 1, 2, \dots$, with solutions $v^{(i)}$ and $v_m^{(i)}$ of problem (11.35) for an appropriate initial data. By Theorem 11.4, $v_m^{(i)} \to v^{(i)}$ in $W^{1,2}(\mathbb{R}^n)$. Taking the limit as $N \to \infty$ and using Theorem 11.5, we obtain (11.39). □

Remark 11.1. The major results in §10 and 11 remain valid if we replace the operator $(-\Delta + c)$ by any uniformly elliptic operator that is invariant under the transformations considered. In the case of the continuous (k,n)-Steiner symmetrization, this is true, e.g., for operators of the type

$$-\sum_{i=1}^{n-k}\frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n-k}a_{ij}(x')\frac{\partial}{\partial x_j}+b_i(x')\Big)-\sum_{i=n-k+1}^{n}\frac{\partial^2}{\partial x_i^2}+c(x'),$$

where the coefficients a_{ij}, b_i, and c are bounded and independent of $y=(x_{n-k+1},\dots,x_n)$, c nonnegative, and

$$\sum_{i,j=1}^{n-k}a_{ij}(x')\xi_i\xi_j \ge \lambda\sum_{i=1}^{n-k}\xi_i^2, \quad \lambda>0.$$

§12. Appendix

In this appendix we prove Theorem 3.1. For compact sets Ω and even $j=2s$, relation (3.9) is a part of Theorem 4.32 in [25]. If $j=2s-1$ is odd, we apply an even number of symmetrizations to the set Ω_1 and again obtain (3.9).

To prove (3.11) for $\Omega\in\mathcal{F}_n$, we consider a nonempty slice $\Omega(x')$ that is compact in $\mathbb{R}^k$. Applying (3.9) to symmetrizations in slices, we obtain

$$\lim_{j\to\infty} d(\Omega_j(x'),\Omega^*(x'))=0. \tag{12.1}$$

Since $\Omega^*(x')$ is a k-dimensional ball and $\mathcal{L}^k(\Omega_j(x'))=\mathcal{L}^k(\Omega^*(x'))$, relation (12.1) implies (3.11) in the case of compact sets.

In the rest of this section, we work with open bounded sets Ω. When proving (3.10), we may assume without loss of generality that all open sets under consideration are included in the unit ball $B^{(n)}$.

First, we prove three technical lemmas. In all these lemmas, by S we denote the (k,k)-Steiner symmetrization in $\mathbb{R}^k$ with respect to the origin $x=(0,\dots,0)$. Let S_1 and S_2 be $(k-1,k)$-Steiner symmetrizations that approximate S in the sense of Theorem 3.1, and let Σ_1 and Σ_2 be the symmetry planes (one-dimensional) of the symmetrizations S_1 and S_2, respectively. For notational convenience, we assume without loss of generality that $\Sigma_1=\{x=(t\cos\gamma\pi,t\sin\gamma\pi,0,\dots,0)\in\mathbb{R}^k : -\infty<t<\infty\}$ for some irrational $\gamma\in(0,1/2)$ and that Σ_2 is the x_1-axis.

For given R, x_1, and ρ such that $R>0$, $-R\le x_1\le R$, and $0<\rho<R$, let

$$Z(x_1,R,\rho)=\overline{B_R^{(k)}}\cap\Big(\bigcup_{\zeta}B_\rho^{(k)}(\zeta)\Big),$$

where the union is taken over all balls $B_\rho^{(k-1)}(\zeta)$ centered at $\zeta=(\zeta_1,\dots,\zeta_k)\in\mathbb{R}^k$ such that $|\zeta|=R$ and $\zeta_1=x_1$.

Lemma 12.1. *Let $0<r<R_1<1$. Then there exists $\tau=\tau(r,R_1)$, $0<\tau<1$, such that for every compact set K satisfying the conditions $K\subset\overline{B_R^{(k)}}$ for some $R\in[R_1,1]$ and $\mathcal{L}^k(K)\le\mathcal{L}^k(B_r^{(k)})$ there is a real number $x_1=x_1(K)$ such that $-r\le x_1\le r$ and*

$$\overline{B_R^{(k)}}\setminus K_2\supset Z(x_1,R,\tau R).$$

Proof. Since K_2 is Steiner symmetric with respect to Σ_2 and $\mathcal{L}^k(K_2)\le\mathcal{L}^k(B_r^{(k)})$, there is a point $x_0=(x_1^0,x_2^0,0,\dots,0)\in S_r^{(k)}\setminus K_2$ such that $-r\le x_1^0\le r$ and $x_2^0=\sqrt{r^2-(x_1^0)^2}\ge 0$.

Since the slice $K_2(x_1^0)$ is a $(k-1)$-dimensional ball, we have $K_2(x_1^0) \subset B_{x_2^0}^{(k-1)}$. Since at the same time $K_2(x_1^0)$ is the result of the symmetrization S_2 of $K_1(x_1^0)$, we have

$$\begin{aligned}\mathcal{L}^{k-1}\big(B_R^{(k)}(x_1^0) \setminus K_1\big) &= \mathcal{L}^{k-1}\big(B_R^{(k)}(x_1^0) \setminus K_2\big) \\ &\geq \mathcal{L}^{k-1}\big(B_{R_1}^{(k)}(x_1^0) \setminus B_{x_2^0}^{(k-1)}\big) \geq c(r, R_1),\end{aligned} \tag{12.2}$$

where

$$c(r, R_1) = \min_{|x_1^0| \leq r} \mathcal{L}^{k-1}\big(B_{R_1}^{(k)}(x_1^0) \setminus B_{x_2^0}^{(k-1)}\big) > 0.$$

For $x \in B_R^{(k)} \setminus \Sigma_1$, let $\Gamma_1(x)$ denote the ray through x that is orthogonal to Σ_1 and has its origin at some point $\widetilde{x} \in \Sigma_1 \cap B_R^{(k)}$. Let $I(x)$ be a closed segment of $\Gamma_1(x)$ joining x and $\partial B_R^{(k)}$. We note that $I(x) \cap K_1 = \varnothing$ if $x \notin K_1$. Let

$$J(x_1^0) = \bigcup_x I(x),$$

where the union is taken over all $x \in B_R^{(k)}(x_1^0) \setminus K_1(x_1^0)$. Let $A = A(K, x_1^0)$ be the maximal segment of the x_1-axis such that $x_1^0 \in A$ and if $x_1 \in A$, then

$$\mathcal{L}^{k-1}\big(\mathbb{R}^k(x_1) \cap J(x_1^0)\big) \geq (1/2)c(r, R_1). \tag{12.3}$$

Using (12.2), one can show that there is a constant $t = t(r, R_1) > 0$ such that

$$\mathcal{L}^1\big(A(K, x_1^0)\big) \geq t(r, R_1) \tag{12.4}$$

for every compact set K satisfying the assumptions of the lemma.

Finally, it is not difficult to see that (12.3) and (12.4) imply the lemma. □

The following lemma allows us to control how fast the approximation process is for compact sets.

Lemma 12.2. *Let $0 < r < R_1 < 1$. There exists a positive integer $N = N(r, R_1)$ and a real number $\beta = \beta(r, R_1)$, $0 < \beta < 1$, such that for all $j \geq N$ and all R with $R_1 \leq R \leq 1$, we have*

$$K_j \subset \overline{B_{\beta R}^{(k)}}$$

for every compact set $K \subset \overline{B_R^{(k)}}$ such that $\mathcal{L}^k(K) \leq \mathcal{L}^k\big(B_r^{(k)}\big)$.

Proof. We may assume without loss of generality that

$$K \subset \overline{B_R^{(k)}} \setminus Z(x_1, R, \tau R) \tag{12.5}$$

with $\tau = \tau(r, R_1) > 0$ defined as in Lemma 12.1 and with some $x_1 = x_1(K)$ such that $-r \leq x_1 \leq r$. Indeed, if K does not satisfy (12.5), then we replace K with its second symmetrization K_2, which by Lemma 12.1 satisfies the required condition.

Let C_R be the circle of radius R in the plane

$$\mathcal{E} = \{x \in \mathbb{R}^k \,:\, x = (x_1, x_2, 0, \ldots, 0)\}.$$

Using complex notation, we shall write $Re^{i\theta}$ for

$$x = (R\cos\theta, R\sin\theta, 0, \ldots, 0) \in C_R.$$

From (12.5) it follows that there is a sufficiently small constant $a_0 = a_0(r, R_1) > 0$ such that for every compact set K satisfying the conditions of the lemma there is a point $Re^{i\theta_0}$ with $\theta_0 = \theta_0(K)$ such that the intersection $C_R \cap Z(x_1, R, \tau R)$ contains an arc α centered at $Re^{i\theta_0}$ with angular measure of at least a_0.

Moreover, it is not difficult to check that there exists a constant $\nu = \nu(r, R_1)$, $0 < \nu < 1$, such that the set $C_R \cap Z(x_1, R, \nu\tau R)$ contains a subarc $\alpha' \subset \alpha$ that is again centered at $Re^{i\theta_0}$ and has angular measure of at least $(2/3)a_0$.

Let Ref_1 and Ref_2 denote the reflections in the plane $\mathcal{E}$ with respect to the lines $l_1 = \{z = te^{i\gamma\pi} : -\infty < t < \infty\}$ and $l_2 = \{z = t : -\infty < t < \infty\}$, respectively. Let $\mathcal{R}^s = (\mathrm{Ref}_1 \circ \mathrm{Ref}_2)^s$, $s = 2, 3, \dots$ Then $\mathcal{R}^s(Re^{i\theta_0}) = Re^{i(2s\gamma\pi+\theta_0)}$, $s = 2, 3, \dots$

Since γ is irrational, the set $\{Re^{2is\gamma\pi}\}_{s=2}^{\infty}$ is dense in C_R (see, e.g., [25, Lemma 3.25]). Therefore, there are indices $s_1, \dots, s_{N_1}$ such that for any θ_0, the points $Re^{i(2s_j\gamma\pi+\theta_0)}$, $j = 1, \dots, N_1$, divide C_R into N_1 arcs each of which has angular measure of at most $a_0/4$. Let $N = \max\{s_1, \dots, s_{N_1}\}$. Let

$$\alpha_s = \mathcal{R}^s(\alpha), \quad \alpha'_s = \mathcal{R}^s(\alpha'), \quad s = 2, 3, \dots, N,$$

where α and α' are the arcs defined above. By our choice of the points $Re^{i(2s_j\gamma\pi+\theta_0)}$, we have

$$C_R = \bigcup_{s=1}^{N} \alpha'_s = \bigcup_{s=1}^{N} \alpha_s.$$

Let

$$W(x_1) = \overline{B_R^{(k)}} \setminus Z(x_1, R, \tau R), \quad W'(x_1) = \overline{B_R^{(k)}} \setminus Z(x_1, R, \nu\tau R),$$

where $x_1 = x_1(K)$ and τ and ν are positive constants defined at the beginning of this proof. For $s = 1, 2, \dots$ and $x_1 = x_1(K)$, let

$$W_s(x_1) = (S_2 \circ S_1)^s(W(x_1)), \quad W'_s(x_1) = (S_2 \circ S_1)^s(W'(x_1)).$$

Relation (12.5) and our construction show that

$$K_{2N} \subset W_N(x_1) \subset W'_N(x_1)$$

and

$$W'_N(x_1) \cap S_R^{(k)} = \varnothing. \tag{12.6}$$

Let

$$\beta(x_1) = \inf\left\{\beta > 0 : W_N(x_1) \subset \overline{B^{(k)}}_{\beta R}\right\}.$$

By (12.6), $0 < \beta(x_1) < 1$ for all x_1 such that $-r \le x_1 \le r$. Let $\widetilde{\beta} = \sup_{|x_1| \le r} \beta(x_1)$. Our construction above works for every compact set K satisfying the assumptions of the lemma. Choosing $K = \overline{B_r^{(k)}}$, we see that $\widetilde{\beta} \ge r/R \ge r > 0$.

Thus, to complete the proof, we need to show that $\widetilde{\beta} < 1$. If not, we can find a sequence $x_1^{(s)} \in [-r, r]$, $s = 1, 2, \dots$, such that $x_1^{(s)} \to x_1^{(0)}$ and $\beta(x_1^{(s)}) \to 1$ as $s \to \infty$. For all sufficiently large s we have

$$Z(x_1^{(0)}, R, \nu\tau R) \subset Z(x_1^{(s)}, R, \tau R).$$

Therefore, for all such s we have

$$W'_N(x_1^{(0)}) \supset W_N(x_1^{(s)}).$$

By (12.6), $W'_N(x_1^{(0)}) \subset \overline{B_{\beta' R}^{(k)}}$ for some $0 < \beta' < 1$. Hence, for all sufficiently large s we have

$$W_N(x_1^{(s)}) \subset \overline{B_{\beta' R}^{(k)}},$$

which contradicts the assumption that $\beta(x^{(s)}) \to 1$ as $s \to \infty$. The proof of the lemma is complete. $\square$

Lemma 12.3. *Let S, S_1, and S_2 be the symmetrizations as in Lemmas* 12.1 *and* 12.2. *Let $0 < r < R$ and $0 < \rho < R$. Then there is a constant $c_1 = c_1(r, \rho, R) > 0$ such that*

$$\mathcal{L}^k\big(\Omega_1 \setminus B_r^{(k)}\big) \geq c_1 \tag{12.7}$$

for every open set $\Omega \subset \mathbb{R}^k$ with the property that Ω_2 contains some point $x_0 = (x_0', y_0)$ with $x_0' \in \mathbb{R}$, $y_0 \in \mathbb{R}^{k-1}$ satisfying $|x_0| \geq R$ and $|y_0| \geq \rho$.

Proof. Let Ω satisfy the assumptions of the lemma, and let

$$c_2 = c_2(r, \rho, R) = \min_{x_0'} \mathcal{L}^{k-1}\big((\{x_0'\} \times B_\rho^{(k-1)}) \setminus B_r^{(k)}\big),$$

where the minimum is taken over all x_0' such that $(x_0')^2 + \rho^2 \geq R^2$. It is clear that $c_2 > 0$. Since $\Omega_2(x_0')$ is a $(k-1)$-dimensional ball with radius of at least ρ, we have $\mathcal{L}^{k-1}\big(\Omega_2(x_0') \setminus B_r^{(k)}\big) \geq c_2$.

For $x \notin \Sigma_1$, let $\widehat{I}(x)$ denote the segment orthogonal to Σ_1 that joins x and Σ_1. For $x \in \mathbb{R}^k \setminus (B_r^{(k)} \cup \Sigma_1)$, we set $I(x) = \widehat{I}(x) \setminus B_r^{(k)}$. We note that $I(x) \subset \Omega_1$ whenever $x \in \Omega_1$. For $x_0' \in \mathbb{R}$ defined above, let

$$J_1(x_0') = \bigcup_x I(x),$$

where the union is taken over all $x = (x_0', y) \in \mathbb{R}^k \setminus B_r^{(k)}$ such that $y \in \Omega_1(x_0')$. Let $A_1(\Omega, x_0')$ be the maximal closed segment of the x_1-axis such that $x_0' \in A_1(\Omega, x_0')$ and

$$\mathcal{L}^{k-1}\big((J_1(x_0'))(x_1)\big) \geq c_2/2, \tag{12.8}$$

where $(J_1(x_0'))(x_1)$ denotes the $(k-1)$-slice of $J_1(x_0')$ at x_1. It is not difficult to check that there is a constant $t_1 = t_1(r, \rho, R) > 0$ such that

$$\mathcal{L}^1\big(A_1(\Omega, x_0')\big) \geq t_1 \tag{12.9}$$

for every domain Ω satisfying the assumptions of the lemma.

Since $J_1(x_0') \subset \Omega_1$, equations (12.8) and (12.9) imply (12.7). □

Proof of Theorem 3.1 *for open sets.* The proof is by contradiction. Suppose that (3.10) is not true for some open set $\Omega \subset B^{(k)}$. Then we consider the following two cases.

(i) There is a real number $\varepsilon_0 > 0$ and a sequence $x_s \to x_0$ such that $x_s \in \partial\Omega_{m_s}$ for some subsequence of indices m_s and

$$d(x_s, \partial\Omega^*) \geq \varepsilon_0 \quad \text{for} \quad s = 0, 1, \ldots \tag{12.10}$$

(ii) There is a real number $\varepsilon_0 > 0$ and a sequence $x_s \to x_0$ such that $x_s \in \partial\Omega^*$ for all s and there is a subsequence of indices m_s such that

$$d(x_s, \partial\Omega_{m_s}) \geq \varepsilon_0 \quad \text{for} \quad s = 1, 2, \ldots \tag{12.11}$$

In each of these cases, we may assume without loss of generality that all indices m_s are odd and $m_s \geq 3$. Then $\Omega_{m_s} = S_1(\Omega_{m_s - 1})$.

To prove **(i)**, we consider two subcases.

(1) First, we assume that $B_{\varepsilon_0}^{(n)}(x_0) \subset \Omega^*$. Let $x_s = (x_s', y_s)$, $s = 0, 1, \ldots$ Then there are constants $\delta_0 > 0$ and $R > |y_0|$ such that

$$\Omega^*(x') \supset B_R^{(k)} \quad \text{for all } x' \in B_{\delta_0}^{(n-k)}(x_0'). \tag{12.12}$$

Next we estimate how much the slice $\Omega_{m_s - 1}(x_s')$ differs from the ball $B_R^{(k)}$. Let $L_1(x', y)$ and $L_2(x', y)$ be the symmetrizing planes through $(x', y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ of S_1 and S_2, respectively. Then $L_j(x', y)$ can be represented in the form $L_j(x', y) = \big\{(x', t) : t \in L_j(y)\big\}$, where $L_j(y)$ is a $(k-1)$-dimensional plane in $\mathbb{R}^k$, which does not depend on x'.

Since $y_s \to y_0$ and $R > |y_0|$, we have $|y_s| < R_1 = (R + |y_0|)/2$ for all sufficiently large s. Since Ω_{m_s} is the $(k-1)$-dimensional Steiner symmetrization of Ω_{m_s-1} with respect to S_1, the set $\Omega_{m_s}(x'_s) \cap L_1(y_s)$ is a $(k-1)$-dimensional ball in the corresponding $(k-1)$-dimensional plane. Since $(x'_s, y_s) \in \partial\Omega_{m_s}$, the latter implies that

$$\Omega_{m_s}(x'_s) \cap L_1(y_s) \subset B^{(k)}_{R_1} \cap L_1(y_s) \subset B^{(k)}_R \cap L_1(y_s).$$

Therefore, there is $\delta_1 = \delta_1(R, |y_0|) > 0$ such that

$$\mathcal{L}^{k-1}\big((B^{(k)}_R \cap L_1(y_s)) \setminus (\Omega_{m_s}(x'_s) \cap L_1(y_s))\big) \geq \delta_1$$

for all sufficiently large s. Since $\Omega_{m_s} = S_1(\Omega_{m_s-1})$, this inequality implies that

$$\mathcal{L}^{k-1}\big((B^{(k)}_R \cap L_1(y_s)) \setminus (\Omega_{m_s-1}(x'_s) \cap L_1(y_s))\big) \geq \delta_1 \tag{12.13}$$

for all sufficiently large s.

Now, since $\Omega_{m_s-1}(x'_s)$ omits a set of positive $(k-1)$-dimensional measure in $B^{(k)}_R \cap L_1(y_s)$ and at the same time $\Omega_{m_s-1}(x'_s) \cap L_2(y_s)$ is a $(k-1)$-dimensional ball in $L_2(y_s)$, we can argue as in the proof of Lemma 12.3 (cf. the deduction of (12.7) from (12.8) and (12.9)) to show that there is $\delta_2 = \delta_2(R, |y_0|) > 0$ such that

$$\mathcal{L}^k\big(B^{(k)}_R \setminus \Omega_{m_s-1}(x'_s)\big) \geq \delta_2 \tag{12.14}$$

for all sufficiently large s.

Next we show that (12.14) leads to a contradiction. The slice $\Omega(x'_0)$ is an open set in $B^{(k)}$. Therefore, for any small $\varepsilon > 0$ there is a compact set $F \subset \Omega(x'_0)$ such that

$$\mathcal{L}^k\big(\Omega(x'_0) \setminus F\big) < \varepsilon. \tag{12.15}$$

Then, for sufficiently small $\varepsilon_1 > 0$ we have $B^{(n-k)}_{\varepsilon_1}(x'_0) \times F \subset \Omega$. By the monotonicity property of symmetrizations, we have

$$\big((\{x'_0\} \times F)_{m_s-1}\big)(x'_0) \subset \Omega_{m_s-1}(x'_s),$$

where $((\{x'_0\} \times F)_{m_s-1})(x'_0)$ denotes the slice of $(\{x'_0\} \times F)_{m_s-1}$ at x'_0. Applying relation (3.11) to the compact sets $\{x'\} \times F$ and $\{x'\} \times S(F)$, we obtain the following limit relation for measure in slices:

$$\mathcal{L}^k\big(((\{x'\} \times F)_j)(x')\big) \triangle \big(((\{x'\} \times S(F)))(x')\big)\big) \to 0 \quad \text{as } j \to \infty. \tag{12.16}$$

Now, (12.12), (12.15), and (12.16) imply the inequality

$$\mathcal{L}^k\big(B^{(k)}_R \setminus ((\{x'_s\} \times F)_{m_s-1})(x'_s)\big) \leq 2\varepsilon$$

for all sufficiently large s. Since $\big((\{x'_s\} \times F)_{m_s-1}\big)(x'_s) \subset \Omega_{m_s-1}(x'_s)$, this contradicts (12.14). This completes the proof of the theorem in the case under consideration.

(2) In the second case, we assume that $B^{(n)}_{\varepsilon_0}(x_0) \cap \Omega^* = \varnothing$. We recall that $x_s = (x'_s, y_s) \to x_0 = (x'_0, y_0)$ as $s \to \infty$.

If $y_0 = (0, \dots, 0)$, then $\Omega^*(x') = \varnothing$ for all $x' \in \mathbb{R}^{n-k}$ sufficiently close to x'_0. Hence, $\Omega(x') = \varnothing$ and therefore $\Omega_j(x') = \varnothing$ for all such x' and all $j = 1, 2, \dots$ It is easily seen that this contradicts our assumptions that $x_s \in \partial\Omega_{m_s}$ and $x_s \to x_0$ as $s \to \infty$.

Assume now that $|y_0| = R_0 > 0$. Since $B^{(n)}_{\varepsilon_0}(x_0) \cap \Omega^* = \varnothing$, there is $\delta_0 > 0$ such that for every $x' \in \overline{B^{(n-k)}_{\delta_0}}(x'_0)$ the slice $\Omega^*(x')$ either is empty, or is an open k-dimensional ball $B^{(k)}_{R(x')}$ with the radius $R(x')$ such that

$$0 < R(x') \leq \rho < R_0 \tag{12.17}$$

with some ρ independent of $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$. In particular, (12.17) shows that

$$0 < \mathcal{L}^k(\Omega(x')) = \mathcal{L}^k(\Omega^*(x')) \le \mathcal{L}^k(B_\rho^{(k)}) \tag{12.18}$$

for all $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$.

Let $\varepsilon_0 > 0$ be fixed and sufficiently small. For every $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$, we choose a k-dimensional compact set $K(x')$ such that $K(x') \subset \Omega(x')$ and

$$\mathcal{L}^k\big(\Omega(x') \setminus K(x')\big) \le \varepsilon_0. \tag{12.19}$$

By (12.18), we have

$$\mathcal{L}^k\big(K(x')\big) \le \mathcal{L}^k\big(B_\rho^{(k)}\big) \tag{12.20}$$

for all $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$.

Let $\varepsilon_2 > 0$ be sufficiently small, and let $\varepsilon_1 > 0$ be such that $\rho + \varepsilon_1 < R_0$ and

$$\mathcal{L}^k\big(B_{\rho+\varepsilon_1}^{(k)} \setminus B_\rho^{(k)}\big) < \varepsilon_2 \tag{12.21}$$

for all $\rho \le R_0$.

For $j = 1, 2, \dots$ and $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$, let $K^j(x')$ denote the slice at x' of the jth successive symmetrization of the set $\{x'\} \times K(x')$ defined by formulas (3.7) and (3.8). Alternatively, $K^j(x')$ can be obtained by applying appropriate $(k-1)$-dimensional symmetrizations to the set $K(x')$ in $\mathbb{R}^k$.

Since $K(x') \subset B^{(k)}$ for all x' and $K(x')$ satisfies (12.20) for all $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$, we can apply Lemma 12.2 with $r = \rho$ and $R_1 = \rho + \varepsilon_1$ to the compact sets $K(x')$. Therefore, there exists a positive integer $N = N(\rho, \varepsilon_1)$ such that

$$K^j(x') \subset B_{\rho+\varepsilon_1}^{(k)} \tag{12.22}$$

for all $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$ and all $j \ge N$.

Now, combining (12.19), (12.21), and (12.22), we obtain

$$\mathcal{L}^k\big(\Omega_j(x') \setminus B_\rho^{(k)}\big) \le \varepsilon_0 + \varepsilon_2 \tag{12.23}$$

for all $x' \in \overline{B_{\delta_0}^{(n-k)}}(x_0')$ and all $j \ge N$.

Now we return to the sequence $x_s = (x_s', y_s) \to x_0 = (x_0', y_0)$. Since $x_s \in \partial\Omega_{m_s}$, for every s we can find a point $\widehat{x}_s = (\widehat{x}_s', \widehat{y}_s) \in \Omega_{m_s}$ such that $\widehat{x}_s \to x_0$ as $s \to \infty$. Then, of course, $\Omega(\widehat{x}_s') \ne \varnothing$, whence $\Omega^*(\widehat{x}_s') \ne \varnothing$.

First, suppose that $x_0 \notin \Sigma_2$. Then $d = d(x_0, \Sigma_2) > 0$. Now we can apply Lemma 12.3 with $r = \rho$, $R = (R_0 + \rho)/2$, and $\rho = d$ and with the domains Ω_1 and Ω_2 in that lemma replaced by the domains $\Omega_{m_s-1}(\widehat{x}_s')$ and $\Omega_{m_s}(\widehat{x}_s')$, respectively. By Lemma 12.3, there exists a constant $c_1 = c_1(\rho, d, R_0) > 0$ such that

$$\mathcal{L}^k\big(\Omega_{m_s-1}(x_s') \setminus B_\rho^{(k)}\big) \ge c_1$$

for all sufficiently large s, which obviously contradicts (12.23) if ε_0 and ε_2 are chosen sufficiently small. This proves the theorem in the case under consideration if $x_0 \notin \Sigma_2$.

Suppose now that $x_0 \in \Sigma_2$. For every $s = 1, 2, \dots$, the symmetrizing plane $L_2(\widehat{y}_s)$ contains some point $\widetilde{x}_s = (\widehat{x}_s', \widetilde{y}_s) \in \partial\Omega_{m_s-1}$. Selecting a subsequence if necessary, we may assume that $\widetilde{x}_s \to \widetilde{x}^{(1)} = (x_0', \widetilde{y}^{(1)})$.

Since $B_{\varepsilon_0}^{(n)}(x_0) \cap \Omega^* = \varnothing$ and $x_0 \in \Sigma_2$, the definition of (k,n)-Steiner symmetrization shows that

$$B_{\varepsilon_0}^{(n)}(\widetilde{x}^{(1)}) \cap \Omega^* = \varnothing.$$

Now, if $\widetilde{x}^{(1)} \notin \Sigma_1$, then, to complete the proof, we can apply our argument above replacing the plane Σ_2 and the sequence of points $x_s \in \partial\Omega_{m_s}$ with the plane Σ_1 and the sequence $\widetilde{x}_s \in \partial\Omega_{m_s-1}$, respectively.

In the case where $\widetilde{x}^{(1)} \in \Sigma_1$, we continue our construction to find points $\widetilde{x}^{(2)} \in \Sigma_2$, $\widetilde{x}^{(3)} \in \Sigma_1, \ldots$ The sequence of points $\widetilde{x}^{(j)}$ will be finite if $\widetilde{x}^{(2m)} \notin \Sigma_2$ or $\widetilde{x}^{(2m-1)} \notin \Sigma_1$ for some $m \geq 1$. Otherwise, the sequence $\widetilde{x}^{(j)}$ will contain infinite number of terms. If it is finite, say $j = 1, \ldots, N$, then we apply our previous argument to the point $\widetilde{x}^{(N)}$ and to the plane Σ_i, where $i = 1$ if N is odd and $i = 2$ if N is even.

Assume now that the constructed sequence of points $\widetilde{x}^{(j)} = (x_0', \widetilde{y}^{(j)})$, $j = 1, 2, \ldots$, is infinite. By our construction we have $|\widetilde{y}^{(j+1)}| \geq |\widetilde{y}^{(j)}| \sec(\gamma\pi)$ for all $j = 1, 2, \ldots$ Therefore,

$$|\widetilde{x}^{(j)}| \to \infty \quad \text{as } j \to \infty.$$

Since for every j there is an index $m(j)$ and a point $z^{(j)} \in \Omega_{m(j)}$ such that $|z^{(j)} - \widetilde{x}^{(j)}| \leq 1$, the above limit relation contradicts our assumption that $\Omega \subset B^{(n)}$.

This completes the proof of the theorem in case **(i)**.

In case **(ii)** the proof is simpler. As in case **(i)**, we consider two subcases.

(1) First, we suppose that $B^{(n)}_{\varepsilon_0}(x_0) \cap \Omega_{m_s} = \varnothing$ for some sufficiently small $\varepsilon_0 > 0$ and some infinite subsequence of indices m_s, $s = 1, 2, \ldots$.

Fix $\delta_0 > 0$ sufficiently small, and let K be a compact subset of Ω such that

$$\mathcal{L}^n\big(\Omega \setminus K\big) < \delta_0. \tag{12.24}$$

Let $K^* = S(K)$, and let K_j, $j = 1, 2, \ldots$, be the successive symmetrizations of K defined by (3.7) and (3.8). Since $K_{m_s} \subset \Omega_{m_s}$, we have

$$B^{(n)}_{\varepsilon_0}(x_0) \cap K_{m_s} = \varnothing. \tag{12.25}$$

Since $x_0 \in \partial\Omega^*$, there is $\delta_1 > 0$ such that

$$\mathcal{L}^n\big(B^{(n)}_{\varepsilon_0}(x_0) \cap \Omega^*\big) \geq \delta_1.$$

Together with (12.24), this implies that

$$\mathcal{L}^n\big(B^{(n)}_{\varepsilon_0}(x_0) \cap K^*\big) \geq \delta_2 \tag{12.26}$$

for some $\delta_2 > 0$ if $\delta_0 > 0$ in (12.24) is sufficiently small.

Applying (3.11) to the compact set K, we obtain

$$\mathcal{L}^n\big(K_{m_s} \triangle K^*\big) \to 0 \quad \text{as} \quad s \to \infty. \tag{12.27}$$

It is easily seen that (12.26) and (12.27) contradict (12.25). This proves the theorem in the case under consideration.

(2) Now we suppose that $\varepsilon_0 > 0$ and $x_0 = (x_0', y_0)$ are such that

$$B^{(n)}_{\varepsilon_0}(x_0) \subset \Omega_{m_s} \tag{12.28}$$

for some infinite subsequence m_s, $s = 1, 2, \ldots$. This implies $\Omega_{m_s}(x_0') \neq \varnothing$, so that $\Omega(x_0') \neq \varnothing$ and $\Omega^*(x_0') \neq \varnothing$. Since $\Omega^*(x_0')$ is a k-dimensional ball, there is $r > 0$ such that $\Omega^*(x_0') = B^{(k)}_r$. Since $x_0 \notin \Omega^*$, we have $r \leq |y_0|$. This implies the relation

$$\mathcal{L}^k\big((B^{(n)}_{\varepsilon_0}(x_0) \setminus \Omega^*)(x_0')\big) \geq \delta_1 \tag{12.29}$$

for some $\delta_1 > 0$. Here $(B^{(n)}_{\varepsilon_0}(x_0) \setminus \Omega^*)(x_0')$ denotes the slice of $B^{(n)}_{\varepsilon_0}(x_0) \setminus \Omega^*$ at x_0'. For every small $\delta_2 > 0$, there exists a k-dimensional compact set $K \subset \Omega(x_0')$ such that

$$\mathcal{L}^k\big(\Omega(x_0') \setminus K\big) = \mathcal{L}^k\big(\Omega^*(x_0') \setminus K^*\big) < \delta_2, \tag{12.30}$$

where K^* denotes the (k, k)-Steiner symmetrization of K.

Applying (3.11) to K, we see that

$$\mathcal{L}^k\big(K_s \triangle K^*\big) \to 0 \quad \text{as } s \to \infty, \tag{12.31}$$

where $K_s = \big((\{x_0'\} \times K)_s\big)(x_0')$ denotes the slice at x_0' of the symmetrized set $(\{x_0'\} \times K)_s$ defined by formulas (3.7) and (3.8).

By the monotonicity property of symmetrization, we have $K_{m_s} \subset \Omega_{m_s}(x_0')$. Combining this with (12.30), we get

$$\mathcal{L}^k\big(\Omega_{m_s}(x_0') \setminus K_{m_s}\big) < \delta_2. \tag{12.32}$$

Finally, (12.28), (12.29), and (12.32) imply that there is a constant $\delta_3 > 0$ such that

$$\mathcal{L}^k\big(K_{m_s} \setminus \Omega^*(x_0')\big) \geq \delta_3 \tag{12.33}$$

for all sufficiently large s. Now it is easily seen that (12.33) contradicts (12.30) and (12.31) if $\delta_2 > 0$ in (12.30) is sufficiently small.

The proof of relation (3.10) in Theorem 3.1 is finished.

To prove (3.11) for $\Omega \in \mathcal{G}_{n,b}$, we fix $x' \in \mathbb{R}^{n-k}$ such that the slice $\Omega(x')$ is not empty. Considering the restrictions of the symmetrizations S, S_1, and S_2 to the slice $\mathbb{R}^n(x')$, we obtain

$$\lim_{j\to\infty} d(\partial\Omega_j(x'), \partial\Omega^*(x')) = 0. \tag{12.34}$$

Since $\Omega^*(x')$ is an open k-dimensional ball and $\mathcal{L}^k\big(\Omega^*(x')\big) = \mathcal{L}^k\big(\Omega_j(x')\big)$, (12.34) implies (3.11) in the case of bounded open sets.

Now the proof of Theorem 3.1 is complete. □

Remark 12.1. One can easily show that (3.11) remains valid even for unbounded sets Ω if the measure of the corresponding slice is finite, i.e., if $\mathcal{L}^k\big(\Omega(x')\big) < \infty$.

In contrast, simple examples of unbounded domains Ω with a finite measure, $\mathcal{L}^n(\Omega) < \infty$, show that, in general, (3.10) is not true for unbounded open sets.

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