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SCHURITY OF S-RINGS OVER A CYCLIC GROUP AND GENERALIZED WREATH PRODUCT OF PERMUTATION GROUPS

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Abstract. With the help of the generalized wreath product of permutation groups introduced in the paper, the automorphism group of an S-ring over a finite cyclic group G is studied. Criteria for the generalized wreath product of two such S-rings to be Schurian or non-Schurian are proved. As a byproduct, it is shown that the group G is a Schur one (i.e., any S-ring over it is Schurian) whenever the total number $\Omega(n)$ of prime factors of the integer $n = |G|$ does not exceed 3. Moreover, the structure of a non-Schurian S-ring over G is described in the case where $\Omega(n) = 4$. In particular, the last result implies that if $n = p^3q$, where p and q are primes, then G is a Schur group.

§1. Introduction

A central problem in the theory of S-rings over a finite group goes back to H. Wielandt and consists in identifying the Schurian S-rings, i.e., those arising from suitable permutation groups (for the background concerning S-rings, see Subsection 2.2). At present this problem is still open even for S-rings over a cyclic group, which are said to be *circulant* below. In [9, 2] it was proved that any circulant S-ring can be constructed from S-rings of rank 2 and normal S-rings via two operations: tensor product and generalized wreath product. It should be noted that the S-rings of rank 2 and the normal S-rings are Schurian, and the tensor product of Schurian S-rings is Schurian. However, the generalized wreath product of Schurian S-rings is not always Schurian: the first examples of non-Schurian generalized wreath products were constructed in [1]. Among these examples, there are non-Schurian S-rings over a cyclic group of order n with $\Omega(n) = 4$. On the other hand, in [10, 8] it was proved that any cyclic group of order n with $\Omega(n) \le 2$ is a *Schur group*, i.e., any S-ring over it is Schurian.

Our main goal in this paper is to lay the foundation of a complete characterization of Schur cyclic groups. As an easy byproduct of the theory developed here, we obtain the following result (which is a reformulation of Theorem 11.1).

Theorem 1.1. *Any cyclic group of order n with $\Omega(n) \le 3$ is a Schur group.*

The identification problem for Schurian S-rings leads naturally to studying the automorphism group of an S-ring $\mathcal{A}$ over a group G. By definition, $\mathrm{Aut}(\mathcal{A})$ is the automorphism group of the Cayley scheme corresponding to $\mathcal{A}$; this group always contains the subgroup G_{right} consisting of all permutations induced by right multiplications. If the group G is cyclic, then the above arguments show that all we need to do is to study the

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group $\mathrm{Aut}(\mathcal{A})$ when $\mathcal{A}$ is a generalized wreath product:

$$\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}, \tag{1}$$

where U and L are $\mathcal{A}$-subgroups of G such that $L \le U$ and $\mathcal{A}_U$ (respectively, $\mathcal{A}_{G/L}$) is the restriction of $\mathcal{A}$ to U (respectively, to G/L); this is equivalent to the fact that any basic set of $\mathcal{A}$ contained in $G \setminus U$ is a union of cosets modulo L. With these notation, the group $\mathrm{Aut}(\mathcal{A})$ coincides with the largest group $\Gamma \le \mathrm{Sym}(G)$ such that

$$\Gamma^U \le \mathrm{Aut}(\mathcal{A}_U) \quad \text{and} \quad \Gamma^{G/L} \le \mathrm{Aut}(\mathcal{A}_{G/L})$$

where Γ^U (respectively, $\Gamma^{G/L}$) is the permutation group induced by the action on U of the setwise stabilizer of U in Γ (respectively, by the action of Γ on G/L). This suggests the following construction, which is central for our paper.

Let V be a finite set and E_0, E_1 equivalence relations on V such that $E_0 \subset E_1$. Suppose we are given two permutation groups $\Delta_U \le \mathrm{Sym}(U)$, where $U \in V/E_1$ and $\Delta_0 \le \mathrm{Sym}(V/E_0)$ such that

$$(\Delta_U)^{U/E_0} = (\Delta_0)^{U/E_0},$$

where U/E_0 is the set of E_0-classes contained in U. Then it is easily seen that all maximal groups $\Gamma \le \mathrm{Sym}(V)$ that leave the relations E_0 and E_1 fixed and satisfy

$$\Gamma^U = \Delta_U \quad \text{and} \quad \Gamma^{V/E_0} = \Delta_0,$$

are permutationally isomorphic. In fact, any such group is uniquely determined by choosing a family of bijections that identify the classes of V/E_1 with the set U. We call it the *generalized wreath product* of the groups Δ_U and Δ_0 (with respect to this family). Obviously, our construction coincides with the usual wreath product whenever $E_0 = E_1$.

An important special case arises when $V = G$ is a group and the classes of the equivalence relations E_0 and E_1 are (respectively) the cosets modulo a normal subgroup L of G and the left cosets modulo a subgroup $U \le G$ that contains L. Then the classes of V/E_1 can be identified with U via permutations from G_{right}, and the generalized wreath product of Δ_U and Δ_0 does not depend on choosing bijections of this type. The permutation group on G obtained in this way is called the *canonical* generalized wreath product of the groups Δ_U and Δ_0 over G, and is denoted by $\Delta_U \wr_{U/L} \Delta_0$ (see Subsection 5.2). It always contains G_{right} as a subgroup.

In the early 1950's, D. K. Faddeev introduced the generalized wreath product of two abstract groups (see [7, p.46]). It can be checked that the generalized wreath product of permutation groups defined in the above paragraph is isomorphic (as an abstract group) to the group obtained by the Faddeev construction. Curiously, the authors came to the generalized wreath product of S-rings independently and only after that got aware of Faddeev′s work. It should be noted that the concept of the generalized wreath product of permutation groups proposed in [4] differs from ours.

A motivation to define the generalized wreath product of permutation groups is the following theorem, which immediately follows from Corollaries 5.5 and 5.7. In particular, this gives a necessary and sufficient condition for a generalized wreath product of two S-rings to be Schurian in terms of groups that are 2-equivalent to their automorphism groups (concerning the concept of 2-equivalence we refer to Wielandt's book [12], see also Notation below).

Theorem 1.2. *Let an* S*-ring* $\mathcal{A}$ *over an Abelian group* G *be the generalized wreath product* (1). *Then* $\mathrm{Aut}(\mathcal{A}) = \Delta_U \wr_{U/L} \Delta_0$ *for some groups* Δ_U *and* Δ_0 *such that*

$$U_{\mathrm{right}} \le \Delta_U \le \mathrm{Aut}(\mathcal{A}_U) \quad \textit{and} \quad (G/L)_{\mathrm{right}} \le \Delta_0 \le \mathrm{Aut}(\mathcal{A}_{G/L}).$$

Moreover, the S*-ring* $\mathcal{A}$ *is Schurian if and only if so are the* S*-rings* $\mathcal{A}_U$ *and* $\mathcal{A}_{G/L}$ *and the groups* Δ_U *and* Δ_0 *can be chosen to be* 2*-equivalent to the groups* $\mathrm{Aut}(\mathcal{A}_U)$ *and* $\mathrm{Aut}(\mathcal{A}_{G/L})$*, respectively.*

To a large extent, the proof of Theorem 1.2 is based on the characterization of the automorphisms of the generalized wreath products of S-rings, as given in [3]. However, the theory of generalized wreath products of circulant S-rings initiated in [3] and developed in the present paper (§§ 3–8), especially the part of it concerning singularities and their resolutions, enables us to reveal a special case of Theorem 1.2 in which the property of an S-ring to be Schurian is checked easily. In fact, this special case, formulated in the theorem below, is the key ingredient in the proofs of Theorem 1.1 and the other results of this paper.

Theorem 1.3. *Let* $\mathcal{A}$ *be an* S*-ring over a cyclic group* G *that is the generalized wreath product* (1)*. Suppose that the* S*-ring* $\mathcal{A}_{U/L}$ *is the tensor product of a normal* S*-ring and* S*-rings of rank* 2*. Then the* S*-ring* $\mathcal{A}$ *is Schurian if and only if so are the* S*-rings* $\mathcal{A}_U$ *and* $\mathcal{A}_{G/L}$*.*

By Theorem 4.1 of this paper, first proved in [2], any circulant S-ring with trivial radical[1] is the tensor product of a normal S-ring and S-rings of rank 2. Thus, the following statement is a special case of Theorem 1.3.

Corollary 1.4. *Let* $\mathcal{A}$ *be an* S*-ring over a cyclic group* G *that is the generalized wreath product* (1)*. Suppose that* $\mathcal{A}_{U/L}$ *is an* S*-ring with trivial radical. Then the* S*-ring* $\mathcal{A}$ *is Schurian if and only if so are the* S*-rings* $\mathcal{A}_U$ *and* $\mathcal{A}_{G/L}$*.*

With the exception of the Burnside–Schur theorem, not so much is known on the structure of a permutation group containing a regular cyclic subgroup. In fact, only the primitive case was studied in detail, but the main results are based on the classification of finite simple groups. On the other hand, as a byproduct of the theory developed below to prove Theorem 1.3, we can get some information on a 2-closed permutation group containing a regular cyclic subgroup (such a group is none other than the automorphism group of a circulant S-ring). Namely, in §9 we prove the following result.

Theorem 1.5. *Let* Γ *be the automorphism group of an* S*-ring over a cyclic group* G*. Then*:

(1) *every non-Abelian composition factor of* Γ *is an alternating group*;
(2) *the group* Γ *is* 2*-equivalent to a solvable group containing* G_{right} *if and only if every alternating composition factor of* Γ *is of prime degree.*

By Theorem 1.1, non-Schurian S-rings over a cyclic group of order n can exist only if $\Omega(n) \geq 4$. In §11 we study the structure of a non-Schurian S-ring $\mathcal{A}$ with $\Omega(n) = 4$. On the basis of Theorem 1.3, we prove that $\mathcal{A}$ is the generalized wreath product of two circulant S-rings such that either the two are generalized wreath products, or exactly one of them is normal. We note that the non-Schurian circulant S-rings constructed in [1] are of the former type. In §12, non-Schurian circulant S-rings of the latter type are presented. The following result is a consequence of Theorem 11.3 and statement (1) of Theorem 11.4.

Theorem 1.6. *Any cyclic group of order* p^3q*, where* p *and* q *are primes, is a Schur group.*

[1]The notion of the radical of a circulant S-ring is different from that generally accepted in the ring theory; see §4.

Concerning permutation groups we refer to [11] and [12]. For the reader's convenience, we collect the basic facts on S-rings and Cayley schemes over an Abelian group in §2. The theory of S-rings over a cyclic group, developed by the authors in [1, 2, 3], is presented in §4.

Notation. As usual, $\mathbb{Z}$ denotes the ring of integers.

For a positive integer n, we denote by $\Omega(n)$ the total number of prime factors of n.

The set of all equivalence relations on a set V is denoted by $\mathcal{E}(V)$.

For $X \subset V$ and $E \in \mathcal{E}(V)$, we set $X/E = X/E_X$, where $E_X = X^2 \cap E$. If the classes of E_X are singletons, we identify X/E with X.

If $R \subset V^2$, $X \subset V$, and $E \in \mathcal{E}(V)$, we set

$$R_{X/E} = \{(Y, Z) \in (X/E)^2 \,:\, R_{Y,Z} \neq \varnothing\},$$

where $R_{Y,Z} = R \cap (Y \times Z)$.

The set of all bijections from V onto V' is denoted by $\mathrm{Bij}(V, V')$.

If $B \subset \mathrm{Bij}(V, V')$, $X \subset V$, $X' \subset V'$, $E \in \mathcal{E}(V)$, and $E' \in \mathcal{E}(V')$, we set

$$B^{X/E,X'/E'} = \{f^{X/E} \,:\, f \in B,\ X^f = X',\ E^f = E'\},$$
$$B^{X/E} = B^{X/E,X/E},$$

where $f^{X/E}$ is the bijection from X/E onto X'/E' induced by f.

The group of all permutations of V is denoted by $\mathrm{Sym}(V)$.

The set of orbits of a group $\Gamma \leq \mathrm{Sym}(V)$ is denoted by $\mathrm{Orb}(\Gamma) = \mathrm{Orb}(\Gamma, V)$.

The setwise stabilizer of a set $U \subset V$ in the group Γ is denoted by $\Gamma_{\{U\}}$.

For an equivalence relation $E \in \mathcal{E}(V)$, we set $\Gamma_E = \bigcap_{X \in V/E} \Gamma_{\{X\}}$.

We write $\Gamma \underset{2}{\approx} \Gamma'$ if groups $\Gamma, \Gamma' \leq \mathrm{Sym}(V)$ are 2-equivalent, i.e., have the same orbits in the coordinatewise action on V^2.

The permutation group defined by the right multiplications of a group G on itself is denoted by G_{right}.

The holomorph $\mathrm{Hol}(G)$ is identified with the permutation group on the set G generated by G_{right} and $\mathrm{Aut}(G)$.

The natural epimorphism from G onto G/H is denoted by $\pi_{G/H}$.

For sections $S = L_1/L_0$ and $T = U_1/U_0$ of G, we write $S \leq T$ if $U_0 \leq L_0$ and $L_1 \leq U_1$.

§2. Schemes and S-rings

2.1. Schemes. Concerning the background on scheme theory presented here, see [5] and the references therein. Let V be a finite set and $\mathcal{R}$ a partition of V^2. Denote by $\mathcal{R}^*$ the set of all unions of the elements of $\mathcal{R}$.

A pair $\mathcal{C} = (V, \mathcal{R})$ is called a *coherent configuration* or a *scheme* on V if the following conditions are satisfied:

- the diagonal of V^2 belongs to $\mathcal{R}^*$;
- the set $\mathcal{R}$ is closed with respect to transposition;
- given $R, S, T \in \mathcal{R}$, the number $|\{v \in V \,:\, (u, v) \in R,\ (v, w) \in S\}|$ does not depend on the choice of $(u, w) \in T$.

The elements of V, $\mathcal{R} = \mathcal{R}(\mathcal{C})$, $\mathcal{R}^* = \mathcal{R}^*(\mathcal{C})$ and the number occurring in the third condition are called (respectively) the *points*, the *basis relations*, the *relations*, and the *intersection number* (associated with R, S, T) of the scheme $\mathcal{C}$. The numbers $|V|$ and $|\mathcal{R}|$ are called the *degree* and *rank* of it. If the diagonal of V^2 belongs to $\mathcal{R}$, the scheme $\mathcal{C}$ is said to be *homogeneous*.

Two schemes are *isomorphic* if there exists a bijection between their point sets preserving the basis relations. Any such bijection is called an *isomorphism* of these schemes. The group of all isomorphisms of a scheme $\mathcal{C}$ contains the normal subgroup

$$\mathrm{Aut}(\mathcal{C}) = \{f \in \mathrm{Sym}(V) \,:\, R^f = R,\ R \in \mathcal{R}\}$$

called the *automorphism group* of this scheme. If V coincides with a group G and $G_{\mathrm{right}} \le \mathrm{Aut}(\mathcal{C})$, then $\mathcal{C}$ is called a *Cayley scheme* over G. By definition, such a scheme is *normal* if the group G_{right} is normal in the group $\mathrm{Aut}(\mathcal{C})$.

Given a permutation group $\Gamma \le \mathrm{Sym}(V)$, set $\mathrm{Orb}_2(\Gamma) = \mathrm{Orb}(\Gamma, V^2)$ to be the set of orbits in the coordinatewise action of Γ on the set V^2. Then the pair

$$\mathrm{Inv}(\Gamma) = (V, \mathrm{Orb}_2(\Gamma))$$

is a scheme; we call it the *scheme of the group* Γ. Any scheme of this type is called *Schurian*. It is easily seen that $\Gamma \le \mathrm{Aut}(\mathcal{C})$, where $\mathcal{C} = \mathrm{Inv}(\Gamma)$.

Suppose $\mathcal{C}$ is a homogeneous scheme and $\mathcal{E}(\mathcal{C}) = \mathcal{R}^*(\mathcal{C}) \cap \mathcal{E}(V)$. Take a class X of an equivalence relation belonging to $\mathcal{E}(\mathcal{C})$. Then for any $E \in \mathcal{E}(\mathcal{C})$, the pair $\mathcal{C}_{X/E} = (X/E, \mathcal{R}_{X/E})$, where $\mathcal{R}_{X/E} = \{R_{X/E} \,:\, R \in \mathcal{R},\ R_{X/E} \ne \varnothing\}$, is a scheme on X/E. It can be checked that

$$\mathrm{Aut}(\mathcal{C})^{X/E} \le \mathrm{Aut}(\mathcal{C}_{X/E}).$$

Moreover, if the scheme $\mathcal{C}$ is Schurian, then these two groups are 2-equivalent. When $\mathcal{C}$ is a Cayley scheme over a group G and $V/E = G/L$ for a normal subgroup L in G, we set $\mathcal{C}_{X/L} = \mathcal{C}_{X/E}$.

Two schemes $\mathcal{C}$ and $\mathcal{C}'$ are said to be *similar* if there exists a bijection

$$\varphi \,:\, \mathcal{R} \to \mathcal{R}',$$

called a *similarity* from $\mathcal{C}$ to $\mathcal{C}'$, such that the intersection number associated with $R, S, T \in \mathcal{R}$ is equal to the intersection number associated with $R^\varphi, S^\varphi, T^\varphi \in \mathcal{R}'$. The set of all isomorphisms from $\mathcal{C}$ to $\mathcal{C}'$ inducing a similarity φ is denoted by $\mathrm{Iso}(\mathcal{C}, \mathcal{C}', \varphi)$.

Let $\mathcal{C}$ be a homogeneous scheme, and let $E \in \mathcal{E}(\mathcal{C})$. Then for any $X, X' \in V/E$ there exists a unique bijection

$$\varphi_{X,X'} \,:\, \mathcal{R}_X \to \mathcal{R}_{X'},\ R_X \mapsto R_{X'}. \tag{2}$$

Moreover, this bijection determines a similarity from $\mathcal{C}_X$ to $\mathcal{C}_{X'}$.

2.2. Schur rings and Cayley schemes. Let G be a finite group. A subring $\mathcal{A}$ of the group ring $\mathbb{Z}G$ is called a *Schur ring* (S-*ring*, for short) over G if it has a (uniquely determined) $\mathbb{Z}$-basis consisting of elements of the form $\sum_{x \in X} x$, where X runs over the classes of a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of G such that

$$\{1\} \in \mathcal{S} \quad \text{and} \quad X \in \mathcal{S} \ \Rightarrow\ X^{-1} \in \mathcal{S}.$$

Let $\mathcal{A}'$ be an S-ring over a group G'. A group isomorphism $f \,:\, G \to G'$ is called a *Cayley isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$.

The elements of the set $\mathcal{S}$ and the number $\mathrm{rk}(\mathcal{A}) = |\mathcal{S}|$ are called (respectively) the *basic sets* and the *rank* of the S-ring $\mathcal{A}$. Any union of basic sets is called an $\mathcal{A}$-*subset of* G or an $\mathcal{A}$-*set*; the set of all of them is denoted by $\mathcal{S}^*(\mathcal{A})$. It is easily seen that the latter set is closed with respect to taking inverse and product. Given $X \in \mathcal{S}^*(\mathcal{A})$, set

$$\mathcal{S}(\mathcal{A})_X = \{Y \in \mathcal{S}(\mathcal{A}) \,:\, Y \subset X\}.$$

The $\mathbb{Z}$-submodule of $\mathcal{A}$ spanned by this set is denoted by $\mathcal{A}_X$. If $\mathcal{A}'$ is an S-ring over G such that $\mathcal{S}^*(\mathcal{A}) \subset \mathcal{S}^*(\mathcal{A}')$, then we write $\mathcal{A} \le \mathcal{A}'$.

Any subgroup of G that is an $\mathcal{A}$-set is called an *$\mathcal{A}$-subgroup* of G or an *$\mathcal{A}$-group*; the set of all of them is denoted by $\mathcal{G}(\mathcal{A})$. The S-ring $\mathcal{A}$ is said to be *dense* if every subgroup of G is an $\mathcal{A}$-group, and *primitive* if the only $\mathcal{A}$-subgroups are 1 and G.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are S-rings over groups G_1 and G_2 (respectively), then the subring $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ of the ring $\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 = \mathbb{Z}G$, where $G = G_1 \times G_2$, is an S-ring over the group G with

$$\mathcal{S}(\mathcal{A}) = \{X_1 \times X_2 \,:\, X_1 \in \mathcal{S}(\mathcal{A}_1),\ X_2 \in \mathcal{S}(\mathcal{A}_2)\}.$$

It is called the *tensor product* of $\mathcal{A}_1$ and $\mathcal{A}_2$. The following statement was proved in [6].

Lemma 2.1. *Let $\mathcal{A}$ be an* S*-ring over the group $G = G_1 \times G_2$, where $G_1, G_2 \in \mathcal{G}(\mathcal{A})$. Then $\pi_i(X) \in \mathcal{S}(\mathcal{A})$ for all $X \in \mathcal{S}(\mathcal{A})$, where π_i is the projection of G on G_i, $i = 1, 2$. In particular, $\mathcal{A} \geq \mathcal{A}_{G_1} \otimes \mathcal{A}_{G_2}$.*

For any group G, there is a one-to-one correspondence between the S-rings over G and the Cayley schemes over G that preserves the natural partial orders on these sets: any basis relation of the scheme $\mathcal{C}$ corresponding to an S-ring $\mathcal{A}$ is of the form

$$\{(g, xg) \,:\, g \in G,\ x \in X\}, \qquad X \in \mathcal{S}(\mathcal{A}).$$

It is easily seen that, given a group $H \leq G$, we have $H \in \mathcal{G}(\mathcal{A})$ if and only if $E_H \in \mathcal{E}(\mathcal{C})$, where E_H is the equivalence relation on the set G for which

$$G/E_H = \{Hg \,:\, g \in G\}.$$

The group $\mathrm{Aut}(\mathcal{A}) := \mathrm{Aut}(\mathcal{C})$ is called the *automorphism group* of the S-ring $\mathcal{A}$. An S-ring is *Schurian* (respectively, *normal*) if so is the corresponding Cayley scheme.

§3. Sections in groups and S-rings

3.1. Sections of a group. Let G be a group. Denote by $\mathcal{F}(G)$ the set of all its sections, i.e., the quotients of subgroups of G. A section U_1/U_0 is called a *multiple* of a section L_1/L_0 if

$$L_0 = U_0 \cap L_1 \quad \text{and} \quad U_1 = U_0 L_1.$$

An equivalence relation on the set $\mathcal{F}(G)$ can be defined as the transitive closure of the relation "to be a multiple". The set of all equivalence classes is denoted by $\mathcal{P}(G)$. Any two sections belonging to one and the same equivalence class are called *projectively equivalent*. The trivial G-sections, i.e., sections of order 1, are obviously projectively equivalent.

Let $S = L_1/L_0$ and $T = U_1/U_0$ be projectively equivalent G-sections. If T is a multiple of S, then, obviously, the groups $S = L_1/U_0 \cap L_1$ and $T = U_0 L_1/U_0$ are isomorphic under the natural isomorphism

$$f_{S,T} \,:\, S \to T, \quad gL_0 \mapsto gU_0. \tag{3}$$

Generally, the sections S and T remain isomorphic, and an isomorphism can be defined by a suitable composition of the above isomorphisms and their inverses. Any such isomorphism will be called a *projective* isomorphism from S onto T.

Let $\Gamma \leq \mathrm{Sym}(G)$ be a group containing G_{right}. We say that a section $U/L \in \mathcal{F}(G)$ is *Γ-invariant* if the equivalence relations E_U and E_L are Γ-invariant. In this case we write $\Gamma^{U/L}$ instead of Γ^{U/E_L}, and $\gamma^{U/L}$ instead of γ^{U/E_L}, $\gamma \in \Gamma$. The following easy lemma is very useful.

Lemma 3.1. *Suppose G is a group and $G_{\mathrm{right}} \leq \Gamma \leq \mathrm{Sym}(G)$. Then for any projectively equivalent Γ-invariant G-sections S and T, any projective isomorphism $f \,:\, S \to T$ induces a permutation isomorphism from Γ^S onto Γ^T. Moreover,*

$$(S_{\mathrm{right}})^f = T_{\mathrm{right}}$$

and $(\gamma^S)^f = \gamma^T$ *for all* $\gamma \in \Gamma$ *leaving the point* 1_G *fixed.*

Proof. Without loss of generality we may assume that the section $T = U_1/U_0$ is a multiple of the section $S = L_1/L_0$ and $f = f_{S,T}$, where $f_{S,T}$ is the bijection defined in (3). Then for any permutation $\gamma \in \Gamma_{\{L_1\}}$ and $g \in L_1$, the block $(gL_0)^\gamma$ contains the element g^γ. It follows that $(gL_0)^\gamma = g^\gamma L_0$, whence by the definition of f we have

$$((gL_0)^\gamma)^f = (g^\gamma L_0)^f = g^\gamma U_0.$$

Thus, $(\Gamma^S)^f \le \Gamma^T$ and $S^f_{\text{right}} = T_{\text{right}}$. Moreover, this also proves the last statement of the lemma. Conversely, let $\gamma \in \Gamma_{\{U_1\}}$. By the above, it suffices to find a permutation $\gamma' \in \Gamma_{\{U_1\}}$ such that $(\gamma')^T = \gamma^T$ and $\gamma' \in \Gamma_{\{L_1\}}$. However, since the equivalence relation E_{L_1} is Γ-invariant, there exists $g \in U_1$ for which $L_1^\gamma = L_1 g$. Since $U_1 = L_1 U_0$, we may assume that $g \in U_0$. Set $\gamma' = \gamma\gamma_1$, where γ_1 is a permutation of G taking x to xg^{-1}. Then

$$(L_1)^{\gamma'} = L_1, \qquad (\gamma')^T = \gamma^T.$$

Obviously, $\gamma' \in \Gamma_{\{U_1\}}$, and we are done. □

3.2. Sections of an S-ring. Let $\mathcal{A}$ be an S-ring over a group G. Denote by $\mathcal{F}(\mathcal{A})$ the set of all $\mathcal{A}$-*sections*, i.e., the G-sections $S = U/L$ for which both U and L are $\mathcal{A}$-groups. Given such an $\mathcal{A}$-section and a set $X \in \mathcal{S}(\mathcal{A})_U$, we put $X_S = \pi_{U/L}(X)$. Then the $\mathbb{Z}$-module

$$\mathcal{A}_S = \operatorname{span}\{X_S \,:\, X \in \mathcal{S}(\mathcal{A})_U\}$$

is an S-ring over the group S the basic sets of which are exactly the sets X_S occurring on the right-hand side of the formula. We say that the section S is of rank 2 (respectively, normal, primitive) if so is the S-ring $\mathcal{A}_S$.

Denote by $\mathcal{P}(\mathcal{A})$ the set of nonempty sets $C \cap \mathcal{F}(\mathcal{A})$, where $C \in \mathcal{P}(G)$. Then $\mathcal{P}(\mathcal{A})$ forms a partition of the set $\mathcal{F}(\mathcal{A})$ into classes of projectively equivalent $\mathcal{A}$-sections. It should be noted that if $\mathcal{A}' \ge \mathcal{A}$, then $\mathcal{F}(\mathcal{A}') \supset \mathcal{F}(\mathcal{A})$, and each class of projectively equivalent $\mathcal{A}$-sections is contained in a unique class of projectively equivalent $\mathcal{A}'$-sections.

Theorem 3.2. *Let* $\mathcal{A}$ *be an* S*-ring over a group* G*. Then for any two projectively equivalent* $\mathcal{A}$*-sections* $S = L_1/L_0$ *and* $T = U_1/U_0$*, any projective isomorphism* $f : S \to T$ *is a Cayley isomorphism from* $\mathcal{A}_S$ *onto* $\mathcal{A}_T$*. Moreover, if* T *is a multiple of* S*, then*

$$(X_S)^f = X_T, \qquad X \in \mathcal{S}(\mathcal{A})_{L_1}.$$

Proof. It suffices to verify the second statement. However, it follows obviously from (3). □

Obviously, any two sections of a class $C \in \mathcal{P}(G)$ have the same order; we call it the *order* of this class. If, in addition, $C \in \mathcal{P}(\mathcal{A})$, then from Theorem 3.2 it follows that all sections in C have the same rank r, and also if C contains a primitive (respectively, normal) section, then all sections in C are primitive (respectively, normal). In these cases we say that the class C is a class of rank r, and is a primitive (respectively, normal) class.

Corollary 3.3. *Let* $\mathcal{A}$ *be an* S*-ring over a group* G*, and* C *a class of projectively equivalent* $\mathcal{A}$*-sections. Suppose that there exists a section* $S \in C$ *such that*

$$\operatorname{Aut}(\mathcal{A})^S \le \operatorname{Hol}(S). \tag{4}$$

Then this inclusion is valid also for all sections belonging to C*.*

Proof. Let $T \in C$. By Lemma 3.1, for $\Gamma = \operatorname{Aut}(\mathcal{A})$ there exists a permutation isomorphism of the group $\operatorname{Aut}(\mathcal{A})^S$ onto the group $\operatorname{Aut}(\mathcal{A})^T$. This isomorphism takes S_{right} to T_{right}, and hence $\operatorname{Hol}(S)$ to $\operatorname{Hol}(T)$. Thus, $\operatorname{Aut}(\mathcal{A})^T \le \operatorname{Hol}(T)$. □

3.3. Generalized wreath product. Let $S = U/L$ be a section of an S-ring $\mathcal{A}$ over a group G. As in [2], we say that $\mathcal{A}$ satisfies the *S-condition* if the group L is normal in G and

$$LX = XL = X, \qquad X \in \mathcal{S}(\mathcal{A})_{G\setminus U}.$$

If, moreover, $L \neq 1$ and $U \neq G$, we say that $\mathcal{A}$ satisfies the S-condition *nontrivially.* The following theorem is a specialization of [1, Theorem 3.1].

Theorem 3.4. *Let $S = U/L$ be a section of an Abelian group G, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be* S*-rings over groups U and G/L, respectively. Suppose that*

$$S \in \mathcal{F}(\mathcal{A}_1) \cap \mathcal{F}(\mathcal{A}_2) \quad \text{and} \quad (\mathcal{A}_1)_S = (\mathcal{A}_2)_S.$$

Then there is a unique S*-ring $\mathcal{A}$ over the group G that satisfies the S-condition and such that $\mathcal{A}_U = \mathcal{A}_1$ and $\mathcal{A}_{G/L} = \mathcal{A}_2$.*

The S-ring $\mathcal{A}$ as in Theorem 3.4 is called the *S-wreath product* of the S-rings $\mathcal{A}_1$ and $\mathcal{A}_2$; we denote it by $\mathcal{A}_1 \wr_S \mathcal{A}_2$ and omit S when $|S| = 1$. Thus, an S-ring $\mathcal{A}$ over G satisfies the S-condition if and only if $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$. We say that $\mathcal{A}$ is a *nontrivial* or *proper* S-wreath product if it satisfies the S-condition nontrivially. When the explicit indication of the section S is not important, we use the term *generalized wreath product.*

§4. S-rings over a cyclic group

4.1. General theory. Let $\mathcal{A}$ be an S-ring over a cyclic group G. It is easily seen that for any $X \subset G$ the set

$$\mathrm{rad}(X) = \{g \in G \,:\, gX = Xg = X\}$$

is an $\mathcal{A}$-group whenever $X \in \mathcal{S}^*(\mathcal{A})$. The well-known Schur theorem on multipliers [11, Theorem 23.9] implies that the group $\mathrm{rad}(X)$ does not depend on the choice of $X \in \mathcal{S}(\mathcal{A})$ such that X contains a generator of G. We call this group the *radical* of $\mathcal{A}$ and denote it by $\mathrm{rad}(\mathcal{A})$.

Theorem 4.1. *Let $\mathcal{A}$ be a circulant* S*-ring. Then*:

(1) $\mathrm{rad}(\mathcal{A}) \neq 1$ *if and only if $\mathcal{A}$ is a proper generalized wreath product*;
(2) $\mathrm{rad}(\mathcal{A}) = 1$ *if and only if $\mathcal{A}$ is the tensor product of a normal* S*-ring with trivial radical and* S*-rings of rank* 2.

Proof. This follows from Corollaries 5.5 and 6.4 in [2]. □

The S-ring $\mathcal{A}$ is said to be *cyclotomic*[2] if $\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(K, G)$ for a group $K \leq \mathrm{Aut}(G)$; in this case we write $\mathcal{A} = \mathrm{Cyc}(K, G)$.

Theorem 4.2. *Let $\mathcal{A}$ be a normal* S*-ring over a cyclic group G. Then*:

(1) *$\mathcal{A}$ is cyclotomic, and in particular, $\mathcal{A}$ is dense and Schurian*;
(2) $|\,\mathrm{rad}(\mathcal{A})|$ *is equal to* 1 *or* 2;
(3) *if* $\mathrm{rad}(\mathcal{A}) \neq 1$, *then $\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}$, where $|G/U| = |L| = 2$ and $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are normal* S*-rings with trivial radicals.*

Proof. This follows from Theorems 6.1 and 5.7 in [2]. □

Corollary 4.3. *Any circulant* S*-ring with trivial radical is Schurian.*

Proof. By statement (2) of Theorem 4.1, it suffices to verify that any normal circulant ring is Schurian. However, this is true by statement (1) of Theorem 4.2. □

[2] In [2], such an S-ring was called an orbit one.

It is well known that

$$\mathrm{Aut}(G) = \prod_{p\in\mathcal{P}} \mathrm{Aut}(G_p), \tag{5}$$

where $\mathcal{P}$ is the set of primes dividing $|G|$ and G_p is the Sylow p-subgroup of G. For any odd prime $p \in \mathcal{P}$, the group $\mathrm{Aut}(G_p)$ is a cyclic group of order $(p-1)|G_p|/p$. In this case it is easily seen that if X is an orbit of a group $K \le \mathrm{Aut}(G_p)$ containing a generator of G_p, then $\mathrm{rad}(X) \ne 1$ if and only if p divides $|K|$. Thus, the cyclotomic S-ring $\mathrm{Cyc}(K, G_p)$ has trivial radical if and only if $|K|$ is coprime to p.

Corollary 4.4. *Suppose $\mathcal{A}$ is a normal* S*-ring over a cyclic p-group, S is an $\mathcal{A}$-section, p is odd, and $|S| \ge p^2$. Then the* S*-ring $\mathcal{A}_S$ is normal.*

Proof. Since $p > 2$, statements (1) and (2) of Theorem 4.2 imply that $\mathcal{A} = \mathrm{Cyc}(K, G)$ where $K \le \mathrm{Aut}(G)$, and $\mathrm{rad}(\mathcal{A}) = 1$. In particular, p does not divide $|K|$. Therefore, $\mathcal{A}_S = \mathrm{Cyc}(K^S, G)$ and $\mathrm{rad}(\mathcal{A}_S) = 1$. Since $\mathrm{rk}(\mathcal{A}_S) > 2$, statement (2) of Theorem 4.1 shows that the S-ring $\mathcal{A}_S$ is normal. □

4.2. Singularities. As in [3], we say that an $\mathcal{A}$-section S is the *smallest* (respectively, *greatest*) if every section projectively equivalent to S is a multiple of it (respectively, if it is a multiple of every section projectively equivalent to it). The following result was proved in Lemma 5.2 of [3].

Theorem 4.5. *Let $\mathcal{A}$ be a circulant* S*-ring. Then any class of projectively equivalent $\mathcal{A}$-sections contains smallest and greatest elements.*

We say that a class of projectively equivalent $\mathcal{A}$-sections of rank 2 is *singular* if its order is greater than 2 and it contains two sections $S = L_1/L_0$ and $T = U_1/U_0$ such that T is a multiple of S and the following conditions are satisfied:

(S1) $\mathcal{A} = \mathcal{A}_{U_0} \wr_{U_0/L_0} \mathcal{A}_{G/L_0} = \mathcal{A}_{U_1} \wr_{U_1/L_1} \mathcal{A}_{G/L_1}$,
(S2) $\mathcal{A}_{U_1/L_0} = \mathcal{A}_{L_1/L_0} \otimes \mathcal{A}_{U_0/L_0}$.

Theorem 4.6. *Let $\mathcal{A}$ be an* S*-ring over a cyclic group G. For any primitive class $C \in \mathcal{P}(\mathcal{A})$, one of the following statements holds true*:

(1) *C is singular, and then $\mathrm{Aut}(\mathcal{A})^S = \mathrm{Sym}(S)$ for all $S \in C$*;
(2) *C is of prime order and $\mathrm{Aut}(\mathcal{A})^S \le \mathrm{Hol}(S)$ for all $S \in C$.*

Proof. First, suppose that the class C contains a *subnormal* section S; by definition, this means that S is a section of a normal $\mathcal{A}$-section. Then $\mathrm{Aut}(\mathcal{A})^S \le \mathrm{Hol}(S)$. By Corollary 3.3, this implies that $\mathrm{Aut}(\mathcal{A})^{S'} \le \mathrm{Hol}(S')$ for all $S' \in C$. Moreover, by statement (1) of Theorem 4.2, the S-ring over the above normal $\mathcal{A}$-section is dense. Therefore, the S-ring $\mathcal{A}_S$ is dense. Since the assumptions of the theorem show that it is also primitive, the number $|S|$ is prime. Thus, in this case statement (2) is true.

To complete the proof, suppose that C contains no subnormal sections (subnormal flags in the sense of [3]). Then, by Proposition 5.3 of [3], the Cayley scheme $\mathcal{C}$ associated with $\mathcal{A}$ has singularity of degree at least 4 in the pair (S, T), where S and T are the smallest and greatest sections of the class C. In our terms, this means that C is a singular class and $|S| = |T| \ge 4$. Then from [3, Lemma 4.3] it follows that $\mathrm{Aut}(\mathcal{C})^S = \mathrm{Sym}(S)$ for all $S \in C$.[3] Thus, since $\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{C})$, statement (1) holds true in this case. □

[3]This result is a special case of statement (1) of Theorem 6.9 proved below.

§5. Generalized wreath product

5.1. Definition. Let E_0 and E_1 be equivalence relations on a set V, and let $E_0 \subset E_1$. Suppose we are given

(a) a group $\Delta_0 \le \mathrm{Sym}(V/E_0)$ such that the equivalence relation $(E_1)_{V/E_0}$ is Δ_0-invariant;

(b) for each $X \in V/E_1$, a group $\Delta_X \le \mathrm{Sym}(X)$ such that the equivalence relation $(E_0)_X$ is Δ_X-invariant;

(c) for each $X, X' \in V/E_1$, a bijection $f_{X,X'} : X \to X'$ taking Δ_X to $\Delta_{X'}$ and X/E_0 to X'/E_0 such that

$$(\Delta_X)^{f_{X,X'} f_{X',X''}} = \Delta_{X''}, \qquad X, X', X'' \in V/E_1.$$

The condition in (c) implies that

$$\Delta_X f_{X,X'} = f_{X,X'} \Delta_{X'} =: \Delta_{X,X'},$$

whence $\Delta_X = \Delta_{X,X}$ and $f_{X,X'} \in \Delta_{X,X'}$ for all X, X'. Thus, the data in (b) and (c) can be recovered by the sets $\Delta_{X,X'}$. Moreover, instead of the families $\{\Delta_X\}$ and $\{f_{X,X'}\}$, we could equivalently be given

(b-c) for each $X, X' \in V/E_1$, a nonempty set $\Delta_{X,X'} \subset \mathrm{Bij}(X, X')$ taking $(E_0)_X$ to $(E_0)_{X'}$ and such that

$$\Delta_{X,X'} \Delta_{X',X''} = \Delta_{X,X''}, \qquad X, X', X'' \in V/E_1.$$

Indeed, set $\Delta_X = \Delta_{X,X}$. Then from the above identity it follows that $\Delta_X \Delta_X = \Delta_X$, whence by the finiteness of V we conclude that Δ_X is a group. Moreover, the same identity shows that $\Delta_X f = \Delta_{X,X'} = f \Delta_{X'}$ for any $f \in \Delta_{X,X'}$.

Set

$$\Gamma = \{\gamma \in \mathrm{Aut}(E_0, E_1) : \gamma^{V/E_0} \in \Delta_0 \text{ and } \gamma^X \in \Delta_{X,X^\gamma},\ X \in V/E_1\}, \tag{6}$$

where $\mathrm{Aut}(E_0, E_1)$ is the subgroup of $\mathrm{Sym}(V)$ leaving the relations E_0 and E_1 fixed. It is easily seen that Γ is a group; if $E_0 = E_1$, then for any $X \in V/E_1$ it is permutationally isomorphic to the wreath product $\Delta_X \wr \Delta_0$ (in imprimitive action). It is also clear that

$$\Gamma_{E_0} = \prod_{X \in V/E_1} \Delta_{X,E_0}, \tag{7}$$

where $\Delta_{X,E_0} = (\Delta_X)_{(E_0)_X}$.

Lemma 5.1. *The equivalence relations E_0 and E_1 are Γ-invariant. Moreover, if*

$$(\Delta_{X,X'})^{X/E_0, X'/E_0} = (\Delta_0)^{X/E_0, X'/E_0}, \quad X, X' \in V/E_1, \tag{8}$$

then $\Gamma^{V/E_0} = \Delta_0$, and $\Gamma^{X,X'} = \Delta_{X,X'}$ for all X, X'. In particular, the group Γ^{V/E_1} is transitive.

Proof. The first statement and the inclusions $\Gamma^{V/E_0} \le \Delta_0$, $\Gamma^{X,X'} \subset \Delta_{X,X'}$ follow from the definition of the group Γ. To prove the reverse inclusions, let $X_0 \in V/E_1$. We claim: *given permutations $\delta_0 \in \Delta_0$ and $\delta_{X_0} \in \Delta_{X_0,X_0'}$, where the class $X_0' \in V/E_1$ is defined by the condition $X'/E_0 = (X/E_0)^{\delta_0}$, such that*

$$(\delta_{X_0})^{X_0/E_0} = (\delta_0)^{X_0/E_0}, \tag{9}$$

we can find a permutation $\delta \in \Gamma$ such that $\delta^{X_0} = \delta_{X_0}$ and $\delta^{V/E_0} = \delta_0$. Then $\Gamma^{V/E_0} \ge \Delta_0$ because by (8) for any $\delta_0 \in \Delta_0$ there exists $\delta_{X_0} \in \Delta_{X_0,X_0'}$ satisfying (9), whereas the inclusion $\Gamma^{X,X'} \supset \Delta_{X,X'}$ follows from the above claim for $X_0 = X$, because, by (8), for any $\delta_X \in \Delta_{X,X'}$ there exists $\delta_0 \in \Delta_0$ satisfying (9).

To prove the claim we observe that, by (8), for each $X \in V/E_1$ other than X_0, there exists $\delta_X \in \Delta_{X,X'}$ with

$$(\delta_X)^{X/E_0} = (\delta_0)^{X/E_0}, \tag{10}$$

where the class $X' \in V/E_1$ is determined by the condition $X'/E_0 = (X/E_0)^{\delta_0}$. Denote by δ the permutation of V such that $\delta^X = \delta_X$ for all $X \in V/E_1$. Then (9) and (10) imply that $\delta^{V/E_0} = \delta_0$. Thus, $\delta \in \Gamma$. □

Definition 5.2. If condition (8) is satisfied, the group Γ defined by formula (6) is called the generalized wreath product of the family of bijections $\{\Delta_{X,X'}\}$ by the group Δ_0; it will be denoted by $\{\Delta_{X,X'}\} \wr \Delta_0$.

From the arguments at the beginning of this section, it follows that for a fixed $X \in V/E_1$ the family $\{\Delta_{X,X'}\}$ is uniquely determined by the group Δ_X and the family of bijections $f_{X,X'} : X \to X'$, $X' \in V/E_1$. Therefore, we can also say that the group Γ is the generalized wreath product of the groups Δ_X and Δ_0 with respect to the above family of bijections.

5.2. Canonical generalized wreath product. An important special case arises when $V = G$ is a group and the classes of the equivalence relations E_0 and E_1 are the left cosets of G modulo subgroups L and U (respectively), where L is a normal subgroup of G contained in U. Suppose that we are also given groups $\Delta_0 \le \mathrm{Sym}(G/L)$ and $\Delta_U \le \mathrm{Sym}(U)$ such that

$$(G/L)_{\mathrm{right}} \le \Delta_0, \quad U_{\mathrm{right}} \le \Delta_U, \quad (\Delta_U)^S = (\Delta_0)^S, \tag{11}$$

where $S = U/L$. Given $X, X' \in V/E_1$, we set

$$\Delta_{X,X'} = (G_{\mathrm{right}})^{X,U}\, \Delta_U\, (G_{\mathrm{right}})^{U,X'}. \tag{12}$$

Then $\Delta_{X,X'} \supset (G_{\mathrm{right}})^{X,X'}$ and all conditions in (a) and (b-c) are satisfied: the equivalence relation $(E_1)_{V/E_0}$ is Δ_0-invariant, $\Delta_{X,X'}$ takes $(E_0)_X$ to $(E_0)_{X'}$, and the identity occurring in (b-c) is true. Moreover, from (11) it follows that

$$\begin{aligned}(\Delta_{X,X'})^{X/E_0,X'/E_0} &= (G_{\mathrm{right}})^{X/E_0,U/E_0}\, (\Delta_U)^{U/L}\, (G_{\mathrm{right}})^{U/E_0,X'/E_0}\\ &= ((G/L)_{\mathrm{right}})^{X/E_0,U/E_0}\, (\Delta_0)^{U/L}\, ((G/L)_{\mathrm{right}})^{U/E_0,X'/E_0}\\ &= (\Delta_0)^{X/E_0,X'/E_0}.\end{aligned}$$

Therefore, relation (8) in Lemma 5.1 is satisfied, and we can form the generalized wreath product $\Gamma = \{\Delta_{X,X'}\} \wr \Delta_0$.

Definition 5.3. The group Γ is called the *canonical generalized wreath product of the group* Δ_U *by the group* Δ_0 *over* G; it is denoted by $\Delta_U \wr_S \Delta_0$.

It is easily seen that any canonical generalized wreath product over G contains the group G_{right}.

5.3. Automorphism groups. Let $\mathcal{C} = (V, \mathcal{R})$ be a homogeneous scheme and let $E_0, E_1 \in \mathcal{E}(\mathcal{C})$ be such that $E_0 \subset E_1$. Suppose that $R_{X,Y} = X \times Y$ for all $R \in \mathcal{R}$ contained in $V^2 \setminus E_1$ and all $X, Y \in V/E_0$ for which $R_{X,Y} \ne \varnothing$. Then we say that the scheme $\mathcal{C}$ satisfies the E_1/E_0-*condition* (this definition is obviously equivalent to the definition given in [3]).

Theorem 5.4. *Let $\mathcal{C}$ be a homogeneous scheme on V, and let $f \in \mathrm{Sym}(V)$. Suppose that $\mathcal{C}$ satisfies the E_1/E_0-condition. Then $f \in \mathrm{Aut}(\mathcal{C})$ if and only if*

$$f^{V/E_0} \in \mathrm{Aut}(\mathcal{C}_{V/E_0}) \quad \textit{and} \quad f^X \in \mathrm{Iso}(\mathcal{C}_X, \mathcal{C}_{X'}, \varphi_{X,X'})$$

for all $X \in V/E_1$, where $X' = X^f$ and $\varphi_{X,X'}$ is the similarity defined in (2).

Proof. In the sense of [3], the family $\{f^X\}$ together with the permutation f^{V/E_0} forms an admissible E_1/E_0-pair compatible with $\mathcal{C}$. Thus, the required statement is a consequence of Theorem 2.5 in [3]. □

Corollary 5.5. *Let $\mathcal{C}$ be a homogeneous scheme on V satisfying the E_1/E_0-condition. Suppose that the group $\Gamma = \mathrm{Aut}(\mathcal{C})$ is transitive on V/E_1 (this is always true when $\mathcal{C}$ is a Cayley scheme). Then $\Gamma = \{\Gamma^{X,X'}\} \wr \Gamma^{V/E_0}$.*

Proof. Set $\Delta_0 = \Gamma^{V/E_0}$ and $\Delta_{X,X'} = \Gamma^{X,X'}$, $X, X' \in V/E_1$. Then, obviously, the condition in (a) and relation (8) in Lemma 5.1 are satisfied. Moreover, from the transitivity of Γ on V/E_1 it follows that the condition in (b-c) is also satisfied. Thus, we can form the generalized wreath product $\Gamma' = \{\Delta_{X,X'}\} \wr \Delta_0$. Clearly, $\Gamma' \geq \Gamma$. The reverse inclusion follows from Theorem 5.4. □

We consider an example illustrating Corollary 5.5. Let $\mathcal{C} = (V, \mathcal{R})$ be a homogeneous scheme satisfying the E_1/E_0-condition. Set

$$\Delta_0 = \mathrm{Aut}(\mathcal{C}_{V/E_0}), \qquad \Delta_{X,X'} = \mathrm{Iso}(\mathcal{C}_X, \mathcal{C}_{X'}, \varphi_{X,X'}), \quad X, X' \in V/E_1.$$

In general, we cannot form the generalized wreath product of $\{\Delta_{X,X'}\}$ by Δ_0, because the condition in (b-c) or identity (8) is not necessarily satisfied. Now suppose that *the scheme $\mathcal{C}_{V/E_0}$ and all schemes $\mathcal{C}_X$ are regular* (i.e., are schemes of regular permutation groups). Then both the condition and the identity follow from the fact that any similarity from a regular scheme to another scheme is induced by an isomorphism. Thus, in this case the above generalized wreath product can be constructed. But then Theorem 5.4 shows that

$$\mathrm{Aut}(\mathcal{C}) = \{\Delta_{X,X'}\} \wr \Delta_0.$$

5.4. Schurian and non-Schurian generalized wreath products. In the rest of the section, we are going to get a necessary and sufficient condition for the generalized wreath product of S-rings to be Schurian. This condition and a criterion for being non-Schurian will be deduced from the following result.

Theorem 5.6. *Let $\mathcal{C}$ be a homogeneous scheme on V satisfying the E_1/E_0-condition, and let $\Gamma = \{\Delta_{X,X'}\} \wr \Delta_0$ be a generalized wreath product such that*

$$\Delta_0 \leq \mathrm{Aut}(\mathcal{C}_{V/E_0}) \quad \textit{and} \quad \Delta_{X,X'} \subset \mathrm{Iso}(\mathcal{C}_X, \mathcal{C}_{X'}, \varphi_{X,X'})$$

for all $X, X' \in V/E_1$. Suppose that

$$\mathrm{Orb}(\Delta_{X,E_0}) = X/E_0, \qquad X \in V/E_1. \tag{13}$$

Then $\mathcal{C} = \mathrm{Inv}(\Gamma)$ if and only if $\mathcal{C}_{V/E_0} = \mathrm{Inv}(\Delta_0)$ and $\mathcal{C}_X = \mathrm{Inv}(\Delta_X)$ for all $X \in V/E_1$.

Proof. The "only if" part follows from Lemma 5.1 because

$$\mathrm{Inv}(\Gamma)^{V/E_0} = \mathrm{Inv}(\Gamma^{V/E_0}) \quad \text{and} \quad \mathrm{Inv}(\Gamma)^X = \mathrm{Inv}(\Gamma^X)$$

for all $X \in V/E_1$. To prove the "if" part, we check that each relation $R \in \mathcal{R}(\mathcal{C})$ is an orbit of the group Γ. We note that, since $\Gamma^{V/E_0} = \Delta_0$ (Lemma 5.1) and $\mathcal{C}_{V/E_0} = \mathrm{Inv}(\Delta_0)$, the group Γ^{V/E_0} acts transitively on R_{V/E_0}. On the other hand, since $\Delta_{X,X'} \subset \mathrm{Iso}(\mathcal{C}_X, \mathcal{C}_{X'}, \varphi_{X,X'})$ and the scheme $\mathcal{C}$ satisfies the E_1/E_0-condition, the group Γ acts on the nonempty sets $R_{X,X'}$, $X, X' \in V/E_1$. Thus, Γ acts on R. To prove that this action is transitive, first we suppose that $R \subset E_1$. Then the relation R is a disjoint union of the relations R_X, $X \in V/E_1$. Since the group Γ^{V/E_1} is transitive (Lemma 5.1), it suffices to verify that the group Γ^X acts transitively on each R_X. However, this is true because $\Gamma^X = \Delta_X$ (Lemma 5.1) and $\mathcal{C}_X = \mathrm{Inv}(\Delta_X)$.

Now, let $R \subset V^2 \setminus E_1$. Then, given $Y, Y' \in V/E_0$, we have either $R_{Y,Y'} = \varnothing$ or $R_{Y,Y'} = Y \times Y'$. So it suffices to verify that in the latter case the group $\Gamma_{Y,Y'} = \Gamma_{\{Y\}} \cap \Gamma_{\{Y'\}}$ acts transitively on the set $R_{Y,Y'}$. However, by (7),

$$\Delta_{X,E_0} \times \Delta_{X',E_0} \times \{\mathrm{id}_{V\setminus(X\cup X')}\} \le \Gamma_{E_0} \le \Gamma_{Y,Y'},$$

where X and X' are the classes of the equivalence relation E_1 that contain Y and Y', respectively. Thus, the required statement follows from (13). □

Corollary 5.7. *Let $\mathcal{A}$ be an* S*-ring over an Abelian group G. Suppose $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ for some $\mathcal{A}$-section $S = U/L$. Then $\mathcal{A}$ is Schurian if and only if so are the* S*-rings $\mathcal{A}_{G/L}$ and $\mathcal{A}_U$ and there exist groups $\Delta_0 \le \mathrm{Sym}(G/L)$ and $\Delta_U \le \mathrm{Sym}(U)$ satisfying* (11) *and such that*

$$\Delta_0 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{G/L}) \quad \textit{and} \quad \Delta_U \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_U). \tag{14}$$

Moreover, in this case $\mathrm{Aut}(\mathcal{A}) \underset{2}{\approx} \Delta_U \wr_S \Delta_0$.

Proof. It is easily seen that the S-ring $\mathcal{A}$ satisfies the U/L-condition if and only if the Cayley scheme $\mathcal{C}$ associated with $\mathcal{A}$ satisfies the E_1/E_0-condition, where $E_0 = E_L$ and $E_1 = E_U$. Thus, the "only if" part follows immediately for $\Delta_0 = \mathrm{Aut}(\mathcal{A})^{G/L}$ and $\Delta_U = \mathrm{Aut}(\mathcal{A})^U$; the second part of the statement is a consequence of Corollary 5.5.

To prove the "if" part, suppose we are given groups $\Delta_0 \le \mathrm{Sym}(G/L)$ and $\Delta_U \le \mathrm{Sym}(U)$ satisfying (11) and (14). Setting $\Gamma = \Delta_U \wr_S \Delta_0$ to be the canonical generalized wreath product, we verify that the assumptions of Theorem 5.6 are satisfied. Indeed, the inclusion $\Delta_0 \le \mathrm{Aut}(\mathcal{C}_{G/L})$ is clear, and the inclusion $\Delta_{X,X'} \subset \mathrm{Iso}(\mathcal{C}_X, \mathcal{C}_{X'}, \varphi_{X,X'})$, where $X, X' \in V/E_1$, is true by (12), because $(G_{\mathrm{right}})^{Y,Y'} \subset \mathrm{Iso}(\mathcal{C}_Y, \mathcal{C}_{Y'}, \varphi_{Y,Y'})$ for all $Y, Y' \in V/E_1$. Moreover, from the definition of the set $\Delta_{X,X'}$ it follows that $\Delta_X = (\Delta_U)^f$, where $f \in (G_{\mathrm{right}})^{U,X}$. Therefore, by the second inclusion in (11), this implies that

$$\mathrm{Orb}(\Delta_{X,E_0}) = \mathrm{Orb}((\Delta_{U,E_0})^f) = \mathrm{Orb}(\Delta_{U,E_0})^f = (U/L)^f = X/E_0.$$

So, condition (13) of Theorem 5.6 is also satisfied. Using that theorem, we conclude that $\mathcal{C} = \mathrm{Inv}(\Gamma)$, whence it follows that the S-ring $\mathcal{A}$ is Schurian. □

The following statement gives a criterion for an S-ring over an Abelian group to be non-Schurian.

Corollary 5.8. *Let $\mathcal{A}$ be an* S*-ring over an Abelian group G. Suppose $\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}$ for some $\mathcal{A}$-section $S = U/L$. Then the* S*-ring $\mathcal{A}$ is non-Schurian whenever*

$$\mathrm{Aut}(\mathcal{A}_U)^S \cap \mathrm{Aut}(\mathcal{A}_{G/L})^S \underset{2}{\not\approx} \mathrm{Aut}(\mathcal{A}_S). \tag{15}$$

Proof. Suppose, to the contrary, that the S-ring $\mathcal{A}$ is Schurian. Then by Corollary 5.7, the S-rings $\mathcal{A}_{G/L}$ and $\mathcal{A}_U$ are Schurian and there exist groups $\Delta_0 \le \mathrm{Sym}(G/L)$ and $\Delta_U \le \mathrm{Sym}(U)$ satisfying conditions (11) and (14). Therefore,

$$(\Delta_0)^S \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{G/L})^S \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_S).$$

Thus, the intersection on the left-hand side of (15) contains the subgroup $(\Delta_U)^S = (\Delta_0)^S$, which is 2-equivalent to the group $\mathrm{Aut}(\mathcal{A}_S)$, a contradiction. □

§6. Isolated classes

6.1. Isolated pairs. Let $S = L_1/L_0$ and $T = U_1/U_0$ be nontrivial sections of an S-ring $\mathcal{A}$ over an Abelian group G.

Definition 6.1. We say that S and T form an *isolated pair* in $\mathcal{A}$ if T is a multiple of S and conditions (S1) and (S2) are satisfied.

This definition shows immediately that U_0, $U_1 \setminus U_0$, and $G \setminus U_1$ are $\mathcal{A}$-subsets of the group G. Moreover, the set $\mathcal{S} = \mathcal{S}(\mathcal{A})$ is uniquely determined by the sets $\mathcal{S}_{U_0} = \mathcal{S}(\mathcal{A}_{U_0})$, $\mathcal{S}_{L_1/L_0} = \mathcal{S}(\mathcal{A}_{L_1/L_0})$, and $\mathcal{S}_{G/L_1} = \mathcal{S}(\mathcal{A}_{G/L_1})$ as follows:

$$\mathcal{S} = \mathcal{S}_{U_0} \cup (\pi_0^{-1}(\mathcal{S}_{L_1/L_0}))_{L_1\setminus L_0}\mathcal{S}_{U_0} \cup (\pi_1^{-1}(\mathcal{S}_{G/L_1}))_{G\setminus U_1}, \tag{16}$$

where $\pi_0 = \pi_{L_1/L_0}$ and $\pi_1 = \pi_{G/L_1}$ are natural epimorphisms. Moreover, the three sets on the right-hand side are pairwise disjoint and are equal to $\mathcal{S}(\mathcal{A})_{U_0}$, $\mathcal{S}(\mathcal{A})_{U_1\setminus U_0}$, and $\mathcal{S}(\mathcal{A})_{G\setminus U_1}$ respectively.

Obviously, the $\mathcal{A}$-sections forming an isolated pair are projectively equivalent. The projective equivalence class containing them will be called *isolated*; we also say that it contains this pair.

Lemma 6.2. *Let $\mathcal{A}$ be a circulant* S*-ring. Then any isolated class $C \in \mathcal{P}(\mathcal{A})$ contains exactly one isolated pair. This pair consists of the smallest and the largest elements of C.*

Proof. By Theorem 4.5, the class C has the smallest and greatest sections; we denote them by L_1/L_0 and U_1/U_0. Clearly, $\pi(L_1)$ and $\pi(U_0)$ are $\mathcal{A}_{\pi(U_1)}$-subgroups with $\pi = \pi_{U_1/L_0}$. Moreover, since U_1/U_0 is a multiple of L_1/L_0, we also have

$$\pi(U_1) = \pi(L_1) \times \pi(U_0). \tag{17}$$

Let L_1'/L_0' and U_1'/U_0' be sections in C forming an isolated pair in $\mathcal{A}$. Then it suffices to verify that $U_1' = U_1$ and $L_0' = L_0$ (then, obviously, $U_0' = U_0$ and $L_1' = L_1$). For this, we observe that, by the definition of the smallest and largest sections, we have

$$L_1 \le L_1' \le U_1' \le U_1 \quad \text{and} \quad L_0 \le L_0' \le U_0' \le U_0. \tag{18}$$

First, suppose that $U_1' \ne U_1$. Then there exists $X \in \mathcal{S}(\mathcal{A})_{U_1\setminus U_1'}$. From the second identity in (S1) with $U_1 = U_1'$ and $L_1 = L_1'$ it follows that $L_1'X = X$; hence, by the left-hand side of (18), we also have $L_1X = X$. This implies that

$$\pi(L_1)\pi(X) = \pi(X). \tag{19}$$

On the other hand, by (17), Lemma 2.1 implies that $\mathrm{pr}_{\pi(L_1)}(\pi(X))$ is a basic set of the S-ring $\mathcal{A}_{\pi(L_1)}$. However, (19) shows that this set coincides with $\pi(L_1)$, which is impossible because $\pi(L_1) = L_1/L_0 \ne 1$. Thus, $U_1' = U_1$.

To complete the proof, suppose that $L_0 \ne L_0'$. Then there exists $X' \in \mathcal{S}(\mathcal{A})_{L_0'\setminus L_0}$. By (18), we have $X' \subset U_0$, whence $\pi(X') \subset \pi(U_0)$. Due to (17) and the inequality $L_1 \ne L_0$, the full π-preimage of $\pi(X')$ does not coincide with X'. Therefore, we can find (in this preimage) a basic set $X \in \mathcal{S}(\mathcal{A})_{U_1\setminus U_0}$ such that $\mathrm{pr}_{\pi(U_0)}(\pi(X)) = \pi(X')$. On the other hand, from the first identity in (S1) with $U_0 = U_0'$ and $L_0 = L_0'$ it follows that $L_0'X = X$. This implies that $\pi(L_0') = \mathrm{pr}_{\pi(U_0)}(\pi(X)) = \pi(X')$ is a basic set of the S-ring $\mathcal{A}_{\pi(U_0)}$. However, this is impossible because $\pi(X') \ne \{1\}$. Thus, $L_0' = L_0$. □

In what follows, for a circulant S-ring $\mathcal{A}$ the smallest and the largest elements of a class $C \in \mathcal{P}(\mathcal{A})$ (the existence of which follows from Theorem 4.5) are denoted by $S_{\min}(C) = L_1(C)/L_0(C)$ and $S_{\max}(C) = U_1(C)/U_0(C)$, respectively.

Corollary 6.3. *Let $C \in \mathcal{P}(\mathcal{A})$ be a primitive isolated class of a circulant* S*-ring $\mathcal{A}$. Suppose that an $\mathcal{A}$-section $S = U/L$ has no subsection belonging to this class. Then either $L \geq L_1$ or $U \leq U_0$, where $L_1 = L_1(C)$ and $U_0 = U_0(C)$.*

Proof. By Lemma 6.2, the sections $S_{\min} = L_1/L_0$ and $S_{\max} = U_1/U_0$, where $L_0 = L_0(C)$ and $U_1 = U_1(C)$, form an isolated pair in $\mathcal{A}$. Suppose that the section S is such that $L \not\geq L_1$ and $U \not\leq U_0$. Then

$$U \geq L_1 \quad \text{and} \quad L \leq U_0. \tag{20}$$

Indeed, since the S-ring $\mathcal{A}$ satisfies the U_1/L_1-condition, it follows that either $U \geq L_1$ or $U \leq U_1$. In the latter case the first relation in (20) is obvious, whereas the second follows from the assumption $U \not\leq U_0$, the primitivity of the class C, and the relation $L_1 \cap U_0 = L_0$. The former case is proved in a similar way.

Now, the first relation in (20) yields

$$S = U/L \geq U \cap U_1/LL_0 \geq LL_1/LL_0.$$

Furthermore, the latter section is projectively equivalent to $S_{\min}$ because the second relation in (20) shows that $L_1 \cap LL_0 = L_0$ and $L_1\, LL_0 = LL_1$. Thus, S has a subsection from C, a contradiction. □

6.2. Extension. Let $\mathcal{A}$ be an S-ring over a cyclic group G and $C \in \mathcal{P}(\mathcal{A})$ an isolated class. By Definition 6.1 and Lemma 6.2, the S-ring $\mathcal{A}$ satisfies the U_i/L_i-condition, where $U_i = U_i(C)$ and $L_i = L_i(C)$, $i = 0, 1$, and $\mathcal{A}_{U_1/L_0} = \mathcal{A}_{L_1/L_0} \otimes \mathcal{A}_{U_0/L_0}$.

Suppose we are additionally given an S-ring $\mathcal{B}$ over the group $S = L_1/L_0$ such that $\mathcal{B} \geq \mathcal{A}_S$. Then by Theorem 3.4, there are uniquely determined S-rings

$$\mathcal{A}_1 = \mathcal{A}_{U_0} \wr_{U_0/L_0} (\mathcal{B} \otimes \mathcal{A}_{U_0/L_0}) \quad \text{and} \quad \mathcal{A}_2 = (\mathcal{B} \otimes \mathcal{A}_{U_0/L_0}) \wr_{U_1/L_1} \mathcal{A}_{G/L_1} \tag{21}$$

over the groups U_1 and G/L_0, respectively. Obviously, the restrictions of these S-rings to U_1/L_0 coincide with $\mathcal{B} \otimes \mathcal{A}_{U_0/L_0}$. Therefore, by Theorem 3.4, there is a unique S-ring

$$\mathrm{Ext}_C(\mathcal{A}, \mathcal{B}) = \mathcal{A}_1 \wr_{U_1/L_0} \mathcal{A}_2 \tag{22}$$

over the group G.

Definition 6.4. The S-ring $\mathrm{Ext}_C(\mathcal{A}, \mathcal{B})$ is called the *extension* of the S-ring $\mathcal{A}$ by means of the S-ring $\mathcal{B}$ with respect to the class C.

The following statement is straightforward.

Lemma 6.5. *In the above notation, set $\mathcal{A}' = \mathrm{Ext}_C(\mathcal{A}, \mathcal{B})$. Then:*

(1) *$\mathcal{A}' \geq \mathcal{A}$; moreover, $\mathcal{A} = \mathcal{A}'$ if and only if $\mathcal{B} = \mathcal{A}_S$;*
(2) *$\mathcal{A}'$ is both a U_0/L_0- and a U_1/L_1-wreath product;*
(3) *$\mathcal{A}'_{U_0} = \mathcal{A}_{U_0}$, $\mathcal{A}'_{G\setminus L_1} = \mathcal{A}_{G\setminus L_1}$, and $\mathcal{A}'_{U_1/L_0} = \mathcal{B} \otimes \mathcal{A}_{U_0/L_0}$.*

From statements (2) and (3) of Lemma 6.5 it follows that the $\mathcal{A}'$-sections $S_{\min}(C)$ and $S_{\max}(C)$ form an isolated pair in the S-ring $\mathcal{A}'$. Therefore, Lemma 6.2 implies the following statement.

Theorem 6.6. *Let $C \in \mathcal{P}(\mathcal{A})$ be an isolated class of a circulant* S*-ring $\mathcal{A}$, and let $\mathcal{A}' = \mathrm{Ext}_C(\mathcal{A}, \mathcal{B})$ as above. Then the class $C' \in \mathcal{P}(\mathcal{A}')$ containing the sections $S = S_{\min}(C)$ and $T = S_{\max}(C)$ is isolated in $\mathcal{A}'$, and $S_{\min}(C') = S$, $S_{\max}(C') = T$.*

The following statement gives a necessary and sufficient condition for the extension (22) to be Schurian.

Theorem 6.7. *Let $C \in \mathcal{P}(\mathcal{A})$ be an isolated class of a Schurian circulant* S*-ring $\mathcal{A}$. Then the* S*-ring $\mathcal{A}' = \mathrm{Ext}_C(\mathcal{A}, \mathcal{B})$ is Schurian if and only if the* S*-ring $\mathcal{B}$ is Schurian.*

Proof. The only "if"part is obvious. To prove the "if" part, suppose that the S-ring $\mathcal{B}$ is Schurian. Then, since $\mathcal{A}$ is Schurian, the S-rings $\mathcal{B} \otimes \mathcal{A}_{U_0/L_0}$ and $\mathcal{A}_{U_0}$ are also Schurian. Therefore, by Corollary 5.7 applied to the S-ring $\mathcal{A}_1$ defined by formula (21) and to the groups $U = U_0$, $\Delta_0 = \mathrm{Aut}(\mathcal{B}) \times \mathrm{Aut}(\mathcal{A})^{U_0/L_0}$ and $\Delta_U = \mathrm{Aut}(\mathcal{A})^{U_0}$, this S-ring is Schurian and

$$\mathrm{Aut}(\mathcal{A}_1)^{U_1/L_0} = \mathrm{Aut}(\mathcal{B}) \times \mathrm{Aut}(\mathcal{A})^{U_0/L_0}. \tag{23}$$

Similarly, the S-ring $\mathcal{A}_2$ defined by formula (21) is Schurian and

$$\mathrm{Aut}(\mathcal{A}_2)^{U_1/L_0} = \mathrm{Aut}(\mathcal{B}) \times \mathrm{Aut}(\mathcal{A})^{U_0/L_0}. \tag{24}$$

Thus, the S-ring $\mathcal{A}'$ is the generalized wreath product of two Schurian S-rings $\mathcal{A}_1$ and $\mathcal{A}_2$ (see (22)). Therefore, by (23), (24), and Corollary 5.7 with $\Delta_0 = \mathrm{Aut}(\mathcal{A}_2)$ and $\Delta_U = \mathrm{Aut}(\mathcal{A}_1)$, we conclude that $\mathcal{A}'$ is Schurian. □

6.3. Automorphisms. Let $S = L_1/L_0$ and $T = U_1/U_0$ be sections of the group G such that T is a multiple of S. For a transitive group $M \le \mathrm{Sym}(S)$ and cosets $X, X' \in T$, put

$$\Delta_0 = M \times \{\mathrm{id}_{U_0/L_0}\} \quad \text{and} \quad \Delta_{X,X'} = (((U_1)_{\mathrm{right}})_{E_{L_1}})^{X,X'}. \tag{25}$$

Then, obviously, the equivalence relation $(E_1)_{U_1/E_0}$, where $E_1 = E_{U_0}$ and $E_0 = E_{L_0}$, is Δ_0-invariant. Moreover, all conditions in (a) and (b-c) in Subsection 5.1 are satisfied for $V = U_1$. Finally, for any $X, X' \in U_1/E_1$ there exist $a, a' \in L_1$ such that $X = aU_0$ and $X' = a'U_0$ (we note that a and a' are determined uniquely modulo L_0). Then from the transitivity of M and the definition of Δ_0 it follows that

$$(\Delta_{X,X'})^{X/E_0, X'/E_0} = \{\gamma_{a,a'}\} = (\Delta_0)^{X/E_0, X'/E_0},$$

where $\gamma_{a,a'}$ is the bijection from X/E_0 onto X'/E_0 taking axL_0 to $a'xL_0$, $x \in U_0$. Thus, condition (8) of Lemma 5.1 is also satisfied and we can consider the *generalized wreath product* $\{\Delta_{X,X'}\} \wr \Delta_0$.

Definition 6.8. We define a permutation group $\Delta \le \mathrm{Sym}(G)$ as follows:

$$\Delta^{G \setminus U_1} = \{\mathrm{id}_{G \setminus U_1}\} \quad \text{and} \quad \Delta^{U_1} = \{\Delta_{X,X'}\} \wr \Delta_0.$$

Below we denote this group by $\mathrm{Gwr}(S, T, M)$.

It is easily seen that, given a group $H \le G$ such that $(L_1 \cap L_0 H)/L_0$ is a block of M, we have

$$\mathrm{Gwr}(S, T, M)^{\pi(G)} = \mathrm{Gwr}(\pi(S), \pi(T), M^{\pi(S)})$$

where $\pi = \pi_{G/H}$.

If $\mathcal{A}$ is a circulant S-ring, $C \in \mathcal{P}(\mathcal{A})$, and $S = S_{\min}(C)$, $T = S_{\max}(C)$, then we set $\mathrm{Gwr}_{\mathcal{A}}(C, M) = \mathrm{Gwr}(S, T, M)$. The first statement of the following theorem generalizes [3, Lemma 4.3].

Theorem 6.9. *Let $\mathcal{A}$ be a circulant* S*-ring, $C \in \mathcal{P}(\mathcal{A})$ an isolated class, and $\Delta = \mathrm{Gwr}_{\mathcal{A}}(C, M)$, where $M = \mathrm{Aut}(\mathcal{A}_S)$ with $S = S_{\min}(C)$. Then*:

(1) $\Delta \le \mathrm{Aut}(\mathcal{A})$,
(2) $\Delta^{S'} = \mathrm{Aut}(\mathcal{A}_{S'})$ *for all* $S' \in C$,
(3) $\Delta^{S'} \le (S')_{\mathrm{right}}$ *whenever C is primitive and a section $S' \in \mathcal{F}(\mathcal{A}) \setminus C$ is either primitive or of order coprime to $|S|$.*

Proof. To prove statement (1), let $f \in \Delta$. Then $f^{G/U_1} = \mathrm{id}_{G/U_1}$ and $f^X = \mathrm{id}_X$ for all cosets $X \in G/U_1$ other than U_1 (we keep the notation of Subsection 6.1). Since $\mathcal{A}$ is the U_1/L_1-wreath product, Theorem 5.4 for the Cayley scheme $\mathcal{C}$ associated with $\mathcal{A}$ shows

that it suffices to verify that $g := f^{U_1}$ is an automorphism of the S-ring $\mathcal{A}_{U_1}$. However, the definition of the group Δ implies that

$$g^X \in ((U_1)_{\mathrm{right}})^{X,X'} \subset \mathrm{Iso}(\mathcal{C}_X, \mathcal{C}_{X'}, \varphi_{X,X'})$$

for all $X \in U_1/U_0$, where $X' = X^g$, and

$$g^{U_1/L_0} \in \Delta_0 = M \times \mathrm{id}_{U_0/L_0} \leq \mathrm{Aut}(\mathcal{C}_{U_1/L_0})$$

because $\mathcal{A}_{U_1/L_0} = \mathcal{A}_{L_1/L_0} \otimes \mathcal{A}_{U_0/L_0}$. Since the S-ring $\mathcal{A}_{U_1}$ satisfies the U_0/L_0-condition, Theorem 5.4 shows that $g \in \mathrm{Aut}(\mathcal{A}_{U_1})$.

To prove statement (2), let $S' \in C$. Denote by Δ' the subgroup of $\mathrm{Sym}(U_1)$ generated by Δ^{U_1} and $(U_1)_{\mathrm{right}}$. Then by statement (1) we have

$$\Delta' \leq \mathrm{Aut}(\mathcal{A})^{U_1}. \tag{26}$$

So, $(\Delta')^{S'} \leq \mathrm{Aut}(\mathcal{A})^{S'}$. Moreover, by Lemma 3.1, the groups $(\Delta')^{S'}$ and $(\Delta')^S$ are isomorphic. On the other hand, since $(U_1)_{\mathrm{right}}$ normalizes the group Δ^{U_1} and $\Delta^S \geq S_{\mathrm{right}}$, we have $(\Delta')^S = \Delta^S = \mathrm{Aut}(\mathcal{A}_S)$. Thus, $(\Delta')^{S'} = \mathrm{Aut}(\mathcal{A}_{S'})$ by Theorem 3.2.

To prove statement (3), let $S' = U'/L'$ be either a primitive section not in C, or a section of order coprime to $|S|$. Obviously, this section has no subsection from C. By Corollary 6.3 with $S = S'$, we have either $L' \geq L_1$ or $U' \leq U_0$. To complete the proof, it suffices to note that in the former case $\Delta^{S'} = \mathrm{id}_{S'}$, whereas in the latter case $\Delta^{S'} = (S')_{\mathrm{right}}$. □

§7. Extension of a singular S-ring

Any singular class is obviously isolated (see Subsection 4.2). The following result shows how the set $P_{\mathrm{sgl}}(\mathcal{A})$ of all singular classes in the S-ring $\mathcal{A}$ varies when we pass to the special case of extension of the form (22).

Theorem 7.1. *Let $\mathcal{A}$ be a circulant* S*-ring, $C \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A})$ a singular class of prime order, and $\mathcal{A}' = \mathrm{Ext}_C(\mathcal{A}, \mathbb{Z}S)$, where $S = S_{\min}(C)$. Then:*

(1) *$\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{A}')$, and in particular, $\mathcal{F}(\mathcal{A}) = \mathcal{F}(\mathcal{A}')$ and $\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathcal{A}')$;*

(2) *$\mathcal{P}_{\mathrm{sgl}}(\mathcal{A}') = \mathcal{P}_{\mathrm{sgl}}(\mathcal{A}) \setminus \{C\}$.*

Proof. We keep the notation of Subsection 6.1. To prove statement (1), it suffices to verify that any $\mathcal{A}'$-subgroup H belongs to the set $\mathcal{G}(\mathcal{A})$. However, if $H \not\subset U_1$, then H is generated by an element in $G \setminus U_1$. Denote by X the basic set of $\mathcal{A}'_H$ containing this element. Then $H = \langle X \rangle$. On the other hand, $X \in \mathcal{S}(\mathcal{A})$ by statements (2) and (3) of Lemma 6.5. Thus, $H \in \mathcal{G}(\mathcal{A})$. Let $H \subset U_1$. By statement (3) of the same lemma, we have $\mathcal{G}(\mathcal{A}_{U_0}) = \mathcal{G}(\mathcal{A}'_{U_0})$. So, we may assume that $H \not\subset U_0$. Then H contains a subset belonging to the set $\mathcal{S}(\mathcal{A}')_{U_1 \setminus U_0}$. Since by statement (2) of the same lemma the S-ring $\mathcal{A}'$ is a U_0/L_0-wreath product, the group generated by this set contains L_0. Thus, $H \geq L_0$. Since $H \not\subset U_0$ and the group $\pi(U_1)$, where $\pi = \pi_{G/L_0}$, is the direct product of the group L_1/L_0 of prime order and the group $\pi(U_0)$, it follows that $H = L_1 H'$, where H' is the full π-preimage of the group $\pi(H) \cap \pi(U_0)$. Since L_1 and H' are $\mathcal{A}$-subgroups, we are done.

To prove statement (2), we observe that $\mathrm{rk}(\mathcal{A}'_S) > 2$ because the order of the class C is at least 3. Therefore, the class C is not singular in $\mathcal{A}'$. In the remaining part of the proof we shall need the following auxiliary lemma. We set $p = |S|$.

Lemma 7.2. *Let $X \in \mathcal{S}(\mathcal{A})$ and $X' \in \mathcal{S}(\mathcal{A}')$ be such that $X' \subset X$. Then* $\mathrm{rad}(X) = \mathrm{rad}(X')$*. Moreover, if $X \subset U_1 \setminus U_0$, then*

$$X = \bigcup_{\sigma \in T_p} (X')^\sigma \quad \textit{and} \quad |X| = (p-1)|X'|, \tag{27}$$

where T_p is the subgroup of $\mathrm{Aut}(G)$ *of order $p-1$ that acts trivially on each Sylow q-subgroup of G, $q \neq p$.*

Proof. If $X \subset U_0$ or $X \subset G \setminus U_1$, then the required statement immediately follows from statement (3) of Lemma 6.5. Suppose that $X \subset U_1 \setminus U_0$. Then, by (16), there exists $Y \in \mathcal{S}(\mathcal{A})_{U_0}$ and $x' \in L_1 \setminus L_0$ such that

$$X = (L_1 \setminus L_0)Y, \qquad X' = (x'L_0)Y. \tag{28}$$

Since $Y \subset U_0$ and $U_0 \cap L_1 = L_0$, we have

$$\mathrm{rad}(X) = \mathrm{rad}(L_1 \setminus L_0)\,\mathrm{rad}(Y) = L_0\,\mathrm{rad}(Y).$$

On the other hand, obviously, $\mathrm{rad}(X') = L_0\,\mathrm{rad}(Y)$. Thus, $\mathrm{rad}(X) = \mathrm{rad}(X')$. Since $\pi(U_1) = \pi(L_1) \times \pi(U_0)$, where $\pi = \pi_{G/L_0}$, the group T_p leaves the set L_0Y fixed. Therefore, (27) follows from (28). □

Let $\widetilde{C} \neq C$ be a class belonging to the set $\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathcal{A}')$. We prove that it is singular in the S-ring $\mathcal{A}$ if and only if it is singular in the S-ring $\mathcal{A}'$. Set

$$\widetilde{L}_i = L_i(\widetilde{C}) \quad \text{and} \quad \widetilde{U}_i = U_i(\widetilde{C}), \qquad i = 0, 1$$

(see Subsection 6.1). By Theorem 6.2, it suffices to verify that conditions (S1) and (S2) for $L_i = \widetilde{L}_i$ and $U_i = \widetilde{U}_i$ are satisfied or are not satisfied in $\mathcal{A}$ and $\mathcal{A}'$ simultaneously, and that $\mathrm{rk}(\mathcal{A}_{H_1}) = 2$ if and only if $\mathrm{rk}(\mathcal{A}'_{H_1}) = 2$, where $H_1 = \widetilde{L}_1/\widetilde{L}_0$ and $H_2 = \widetilde{U}_0/\widetilde{L}_0$. However, from the first statement of Lemma 7.2 it follows that the S-rings $\mathcal{A}$ and $\mathcal{A}'$ are or are not the $\widetilde{U}_0/\widetilde{L}_0$-wreath products (as well as $\widetilde{U}_1/\widetilde{L}_1$-wreath products) simultaneously. Thus, it suffices to verify that

$$\mathcal{A}_H = \mathcal{A}_{H_1} \otimes \mathcal{A}_{H_2},\ \mathrm{rk}(\mathcal{A}_{H_1}) = 2 \quad \Leftrightarrow \quad \mathcal{A}'_H = \mathcal{A}'_{H_1} \otimes \mathcal{A}'_{H_2},\ \mathrm{rk}(\mathcal{A}'_{H_1}) = 2$$

where $H = \widetilde{U}_1/\widetilde{L}_0$. For this, we note that $H = H_1 \times H_2$, and H_i is both an $\mathcal{A}_H$- and $\mathcal{A}'_H$-subgroup, $i = 1, 2$. By Lemma 2.1, this implies that for any $X \in \mathcal{S}(\mathcal{A})$ (respectively, $X' \in \mathcal{S}(\mathcal{A}')$) the set $\widetilde{X}_i = \mathrm{pr}_{H_i}\,\widetilde{\pi}(X)$ (respectively, $\widetilde{X}'_i = \mathrm{pr}_{H_i}\,\widetilde{\pi}(X')$) is an $\mathcal{A}$-set (respectively, $\mathcal{A}'$-set), where $\widetilde{\pi} = \pi_{\widetilde{U}_1/\widetilde{L}_0}$. Thus, the required statement is a consequence of the following lemma.

Lemma 7.3. *Suppose that* $\mathrm{rk}(\mathcal{A}_{H_1}) = 2$. *If $X \in \mathcal{S}(\mathcal{A})$ and $X' \in \mathcal{S}(\mathcal{A}')$ are such that $X' \subset X \subset \widetilde{U}_1$, then*

$$\widetilde{X} = \widetilde{X}_1 \times \widetilde{X}_2 \quad \Leftrightarrow \quad \widetilde{X}' = \widetilde{X}'_1 \times \widetilde{X}'_2, \tag{29}$$

where $\widetilde{X} = \widetilde{\pi}(X)$ and $\widetilde{X}' = \widetilde{\pi}(X')$. Moreover, $\widetilde{X}_1 = \widetilde{X}'_1$.

Proof. Without loss of generality we may assume that $\widetilde{X} \neq \widetilde{X}'$. Then, obviously, $X \neq X'$, whence $X \subset U_1 \setminus U_0$ by statement (3) of Lemma 6.5. By (28), this implies that $L_1 \subset \langle X \rangle \subset \widetilde{U}_1$. We claim that

$$\widetilde{\pi}(L_1) \subset H_2. \tag{30}$$

Suppose, to the contrary, that this is not true. Denote by M_i (respectively, N_i) the projection of the group $\widetilde{\pi}(L_i)$ to H_1 (respectively, to H_2), $i = 0, 1$. Then

$$\widetilde{\pi}(L_i) = M_i \times N_i.$$

Moreover, since $\widetilde{\pi}(L_i)$ is an $\mathcal{A}$-group, Lemma 2.1 shows that M_i and N_i are $\mathcal{A}$-groups. Since also $\mathrm{rk}(\mathcal{A}_{H_1}) = 2$, the group M_i equals 1 or H_1; consequently, either $M_1 = M_0$, or $N_1 = N_0$. Moreover, our supposition implies that $M_1 = H_1$.

Next, the S-ring $\mathcal{A}$ is the U_0/L_0-wreath product and $\mathrm{rk}(\mathcal{A}_{L_1/L_0}) = 2$. Therefore, $L_1 \setminus L_0 \in \mathcal{S}(\mathcal{A})$, and hence the set $\widetilde{Z} = \widetilde{\pi}(L_1 \setminus L_0)$ is basic. Also,

$$\widetilde{Z} = \begin{cases} \widetilde{\pi}(L_1) & \text{if } \widetilde{\pi}(L_1) = \widetilde{\pi}(L_0), \\ \widetilde{\pi}(L_1) \setminus \widetilde{\pi}(L_0) & \text{otherwise.} \end{cases} \tag{31}$$

(Indeed, in the first case this is true because the group $\widetilde{\pi}(L_1)$ is the union of the sets $\widetilde{\pi}(xL_0) = \widetilde{\pi}(x)\widetilde{\pi}(L_0)$, $x \in L_1$; in the second case the basic set $\widetilde{Z}$ obviously contains the $\mathcal{A}$-set $\widetilde{\pi}(L_1) \setminus \widetilde{\pi}(L_0)$ and hence coincides with it.) Now, the relation $\widetilde{Z} = \widetilde{\pi}(L_1)$ is impossible because otherwise the set $H_1 = \mathrm{pr}_{H_1}(\widetilde{Z})$ is basic, which is possible only if $H_1 = 1$. Thus, by (31) we have

$$\widetilde{Z} = \widetilde{\pi}(L_1) \setminus \widetilde{\pi}(L_0)$$

and so either $\mathrm{pr}_{H_1}(\widetilde{Z}) = H_1$ or $\mathrm{pr}_{H_2}(\widetilde{Z}) = N_1$. As before, the former case is impossible because $\widetilde{Z}$ is a basic set, and the latter case is possible only if $N_1 = 1$. Therefore,

$$\widetilde{\pi}(L_0) = 1 \quad \text{and} \quad \widetilde{\pi}(L_1) = H_1.$$

It follows that $\widetilde{L}_1 = L_1\widetilde{L}_0$ and $\widetilde{L}_0 \geq L_0$, whence $L_0 \leq L_1 \cap \widetilde{L}_0 < L_1$. Since the number $|L_1/L_0|$ is prime, this implies that $L_0 = L_1 \cap \widetilde{L}_0$. Thus, the section $\widetilde{L}_1/\widetilde{L}_0 \notin C$ is a multiple of the section $L_1/L_0 \in C$, a contradiction. This proves (30).

Now we complete the proof of (29). First, since $X \subset U_1 \setminus U_0$ (see the beginning of the proof), from (28) it follows that $\widetilde{X} = \widetilde{\pi}(L_1 \setminus L_0)\widetilde{Y}$ and $\widetilde{X}' = \widetilde{\pi}(x'L_0)\widetilde{Y}$, where $x' \in X'$ and $\widetilde{Y} = \widetilde{\pi}(Y)$. By (30), this implies

$$\widetilde{X}_1 = \widetilde{X}'_1 = \widetilde{Y}_1, \tag{32}$$

where $\widetilde{Y}_1 = \mathrm{pr}_{H_1}\widetilde{Y}$. Next, since the radicals of the sets $\widetilde{X}$ and $\widetilde{X}'$ contain the group $\widetilde{\pi}(L_0)$, without loss of generality we may assume that this group is trivial. Therefore, by Lemma 7.2 we have

$$\widetilde{X}_2 = \bigcup_{\sigma \in T_p} \widetilde{\pi}(x')^\sigma \widetilde{Y}_2, \quad \widetilde{X}'_2 = \widetilde{\pi}(x')\widetilde{Y}_2. \tag{33}$$

If the group $A = \widetilde{\pi}(L_1)$ is trivial, then $\widetilde{X}_2 = \widetilde{X}'_2 = \widetilde{Y}_2$ and the required statement immediately follows from (32). Suppose that A is nontrivial. Then it is an $\mathcal{A}$-group of order p and the set $A \setminus \{1\}$ is the orbit of the group T_p that contains the element $\widetilde{\pi}(x')$. Moreover, by condition (S2) we have $A\langle\widetilde{Y}\rangle = A \times \langle\widetilde{Y}\rangle$. Therefore, the union in (33) is a disjoint one. Now, to complete the proof of the equivalence (29), it suffices to observe that, by (32) and (33), any of the relations $\widetilde{X} = \widetilde{X}_1 \times \widetilde{X}_2$ and $\widetilde{X}' = \widetilde{X}'_1 \times \widetilde{X}'_2$ is equivalent to $\widetilde{Y} = \widetilde{Y}_1 \times \widetilde{Y}_2$. □

Thus, statements (1) and (2) of Theorem 7.1 are completely proved. □

§8. Resolving singularities

Let $\mathcal{A}$ be a circulant S-ring, $C \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A})$ a singular class, and $S = S_{\min}(C)$. Set

$$\mathrm{Gwr}_{\mathcal{A}}(C) = \begin{cases} \mathrm{Gwr}_{\mathcal{A}}(C, \mathrm{Sym}(S)) & \text{if } C \text{ is of composite order,} \\ \mathrm{Gwr}_{\mathcal{A}}(C, \mathrm{Hol}(S)) & \text{otherwise} \end{cases} \tag{34}$$

(see Subsection 6.3). Statement (1) of Theorem 8.1 below implies that this group is contained in $\mathrm{Aut}(\mathcal{A})$.

Theorem 8.1. *Let $\mathcal{A}$ be a Schurian* S*-ring over a cyclic group G. Then the group* $\mathrm{Aut}(\mathcal{A})$ *contains a subgroup $\Gamma \geq G_{\mathrm{right}}$ 2-equivalent to* $\mathrm{Aut}(\mathcal{A})$ *and such that for any class $C \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A})$ the following statements hold true*:

(1) $\mathrm{Gwr}_{\mathcal{A}}(C) \leq \Gamma$;
(2) *for any $S \in C$ we have $\Gamma^S = \mathrm{Hol}(S)$ if $|S|$ is prime, and $\Gamma^S = \mathrm{Sym}(S)$ otherwise.*

Proof. We use induction on the number $m = m(\mathcal{A})$ of singular classes of prime order. If $m = 0$, then we are done with $\Gamma = \mathrm{Aut}(\mathcal{A})$ by statements (1) and (2) of Theorem 6.9.

Suppose $m > 0$ and C is a singular class of $\mathcal{A}$ of prime order. Set

$$\mathcal{A}' = \mathrm{Ext}_C(\mathcal{A}, \mathbb{Z}S), \tag{35}$$

where $S = S_{\min}(C)$. Then $m(\mathcal{A}') = m(\mathcal{A}) - 1$ by statement (2) of Theorem 7.1. So, by the inductive hypothesis applied to the S-ring $\mathcal{A}'$, the group $\mathrm{Aut}(\mathcal{A}')$ is 2-equivalent to a group $\Gamma' \geq G_{\mathrm{right}}$ such that for any $C' \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A}')$ statements (1) and (2) are valid with $\mathcal{A}$ and C replaced by $\mathcal{A}'$ and C', respectively. Set

$$\Gamma = \langle \Gamma', \Delta \rangle, \tag{36}$$

where $\Delta = \mathrm{Gwr}_{\mathcal{A}}(C)$ is the group defined in (34). Obviously, $G_{\mathrm{right}} \leq \Gamma$. To complete the proof, it suffices to verify that the groups Γ and $\mathrm{Aut}(\mathcal{A})$ are 2-equivalent and statements (1) and (2) are true. Below we set $L_i = L_i(C)$ and $U_i = U_i(C)$, $i = 0, 1$.

To prove the above 2-equivalence, it suffices to verify that the group Γ_u with $u = 1_G$ acts transitively on each set $X \in \mathcal{S}(\mathcal{A})$. But the S-ring $\mathcal{A}'$ is Schurian by Theorem 6.7. Therefore, by statement (3) of Lemma 6.5, transitivity occurs when $X \subset U_0$ or $X \subset G \setminus U_1$, because in these cases $X \in \mathcal{S}(\mathcal{A}')$, so that $X \in \mathrm{Orb}(\Gamma'_u)$. Suppose that $X \subset U_1 \setminus U_0$. Since $\mathrm{rk}(\mathcal{A}_S) = 2$ and $\mathcal{A}'_S = \mathbb{Z}S$, statements (2) and (3) of Lemma 6.5 imply that there exists a set $Y \in \mathcal{S}(\mathcal{A})_{U_0}$ such that

$$X = (L_1 \setminus L_0)Y = \bigcup_x xL_0Y,$$

where x runs over a full set of representatives of L_0-cosets in L_1 other than L_0. Moreover, the set xL_0Y is basic in $\mathcal{A}'$ for all x. Therefore, the group Γ'_u acts transitively on each of such sets. On the other hand, the group Δ_u acts on S as $\mathrm{Hol}(S)_u = \mathrm{Aut}(S)$, whence it is transitive on the set $S \setminus \{1\}$. Thus, the group Γ_u acts transitively on X and we are done.

To prove statements (1) and (2), we take $\widetilde{C} \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A})$. If $\widetilde{C} = C$, then statement (1) is obvious because $\mathrm{Gwr}_{\mathcal{A}}(\widetilde{C}) = \Delta \leq \Gamma$. If $\widetilde{C} \neq C$, then $\widetilde{C} \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A}')$ by statement (2) of Theorem 7.1. However, for the group Γ' statement (1) holds true, whence $\mathrm{Gwr}_{\mathcal{A}'}(\widetilde{C}) \leq \Gamma'$. Since $\mathrm{Gwr}_{\mathcal{A}'}(\widetilde{C}) = \mathrm{Gwr}_{\mathcal{A}}(\widetilde{C})$ and $\Gamma' \leq \Gamma$, statement (1) is proved.

Now we prove statement (2). First, suppose that the class $\widetilde{C}$ is of composite order. Then $\widetilde{C} \neq C$. By Theorem 7.1, this implies that $\widetilde{C} \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A}')$. Since for the group Γ' statement (2) holds true, for any section $\widetilde{S} \in \widetilde{C}$ we have

$$\Gamma^{\widetilde{S}} \geq (\Gamma')^{\widetilde{S}} = \mathrm{Sym}(\widetilde{S}),$$

and we are done. Now suppose that the class $\widetilde{C}$ is of prime order. Since the groups Γ' and $\mathrm{Aut}(\mathcal{A}')$ are 2-equivalent by the inductive hypothesis, so are the groups $(\Gamma')^{\widetilde{S}}$ and $\mathrm{Aut}(\mathcal{A}')^{\widetilde{S}}$, where $\widetilde{S} \in \widetilde{C}$. On the other hand, if $\widetilde{C} = C$, then, by the definition of $\mathcal{A}'$ and Theorem 3.2, the latter group coincides with $\widetilde{S}_{\mathrm{right}}$. Since statement (2) is true for Γ', we conclude that

$$(\Gamma')^{\widetilde{S}} = \begin{cases} \widetilde{S}_{\mathrm{right}} & \text{if } \widetilde{C} = C, \\ \mathrm{Hol}(\widetilde{S}) & \text{if } \widetilde{C} \neq C. \end{cases} \tag{37}$$

By Lemma 3.1, there is no loss of generality in assuming that $\widetilde{S} = \widetilde{U}_1/\widetilde{U}_0$, where $\widetilde{U}_i = U_i(\widetilde{C})$, $i = 0, 1$. Also, we may assume that $\widetilde{U}_0 = 1$, because the natural epimorphism $\Gamma \to \Gamma^{G/\widetilde{U}_0}$ induces an isomorphism from the group $\Gamma^{\widetilde{S}}$ onto the group $(\Gamma^{G/\widetilde{U}_0})^{\widetilde{S}}$. Thus, $\widetilde{S} = \widetilde{U}_1$ is a cyclic group of prime order and $\widetilde{C} = \{\widetilde{S}\}$.

To formulate the next auxiliary lemma, we need some notation. It is easily seen that, for any $\widetilde{U}_1$-coset $\widetilde{X}$, the group $\mathrm{Hol}(\widetilde{U}_1)^\gamma \le \mathrm{Sym}(\widetilde{X})$ does not depend on the choice of $\gamma \in (G_{\mathrm{right}})^{\widetilde{U}_1,\widetilde{X}}$. We denote this group by $K(\widetilde{X})$. Clearly, $K(\widetilde{U}_1) = \mathrm{Hol}(\widetilde{U}_1)$.

Lemma 8.2. *Let $K \le \mathrm{Sym}(G)$ be the intransitive direct product of the groups $K(\widetilde{X}) \le \mathrm{Sym}(\widetilde{X})$, where $\widetilde{X}$ runs over the set $G/\widetilde{U}_1$. Then the group Δ normalizes the group K.*

Proof. Since $\widetilde{U}_1$ is a block of the group $\Gamma \ge \Delta$, it suffices to verify that for any permutation $\gamma \in \Delta$ we have

$$K(\widetilde{X})^{\gamma^{\widetilde{X}}} = K(\widetilde{X}^\gamma) \tag{38}$$

for all $\widetilde{X} \in G/\widetilde{U}_1$. For this, first we suppose that $\widetilde{X} \not\subset U_1$. Then, since $\widetilde{X}$ and U_1 are blocks of the group Γ, and the number $|\widetilde{X}|$ is prime, it follows that $|\widetilde{X} \cap U_1| \le 1$. However, the group $\Delta = \mathrm{Gwr}_{\mathcal{A}}(C)$ acts trivially outside the set U_1. Therefore, the permutation γ fixes all but possibly one point, and hence all points of $\widetilde{X}$. Thus, in this case (38) is obvious.

Let $\widetilde{X} \subset U_1$. Then, obviously, $\widetilde{U}_1 \subset U_1$. First, suppose that $\widetilde{U}_1 \subset U_0$. Then $\widetilde{X}$ is contained in some set $X \in U_1/U_0$. By the definition of the group $\mathrm{Gwr}_{\mathcal{A}}(C)$, this implies that $\gamma^{U_1} \in ((U_1)_{\mathrm{right}})^{X,X'}$, where $X' = X^\gamma$ (see (25)). This implies that $\gamma^{\widetilde{X}} \in (G_{\mathrm{right}})^{\widetilde{X},\widetilde{X}'}$, where $\widetilde{X}' = \widetilde{X}^\gamma$, whence (38) follows. Finally, let $\widetilde{U}_1 \not\subset U_0$. Since the index of the subgroup U_0 in the group U_1 is prime, this implies that $\widetilde{U}_1 U_0 = U_1$. On the other hand, since the number $|\widetilde{U}_1|$ is prime, we have $\widetilde{U}_1 \cap U_0 = \{1\}$. Thus, the section $S = U_1/U_0$ is a multiple of the section $\widetilde{U}_1/1 = \widetilde{S}$, so that $\widetilde{C} = C$. This implies that $U_0 = L_0 = 1$, whence $\widetilde{X} = U_1 = L_1$. Thus,

$$K(\widetilde{X}) = \mathrm{Hol}(U_1) = \mathrm{Hol}(L_1) = \mathrm{Gwr}_{\mathcal{A}}(C)^{U_1} = \Delta^{U_1}$$

and (38) is proved because, obviously, $(\Delta^{U_1})^{\gamma^{U_1}} = \Delta^{U_1}$. □

To complete the proof of the theorem, we observe that the $\widetilde{U}_1$-cosets form an imprimitivity system for the group $\Gamma' \ge G_{\mathrm{right}}$. From (37) it follows that Γ' normalizes K. Therefore, by Lemma 8.2 and formula (36), the group Γ, and hence the group $\Gamma_{\{\widetilde{S}\}}$, normalizes K. Consequently, $\Gamma^{\widetilde{S}}$ normalizes $K^{\widetilde{S}}$ in $\mathrm{Sym}(\widetilde{S})$. Recalling (37) and the fact that $\Delta^{\widetilde{S}} = \mathrm{Hol}(\widetilde{S})$ for $\widetilde{C} = C$, we have

$$\mathrm{Hol}(\widetilde{S}) \le \Gamma^{\widetilde{S}} \le N(K^{\widetilde{S}}) = N(\mathrm{Hol}(\widetilde{S})), \tag{39}$$

where $N(\cdot)$ denotes the normalizer in $\mathrm{Sym}(\widetilde{S})$. However, since $\widetilde{S}$ is a cyclic group of prime order, it is a characteristic subgroup of its holomorph. It follows that the group on the right-hand side of (39) coincides with $N(\widetilde{S}_{\mathrm{right}}) = \mathrm{Hol}(\widetilde{S})$. Thus, $\Gamma^{\widetilde{S}} = \mathrm{Hol}(\widetilde{S})$, and we are done. □

§9. Proof of Theorem 1.5

In what follows, under a *section* of a transitive group

$$\Gamma \le \mathrm{Sym}(V)$$

we mean a permutation group $\Gamma^{X/E}$, where X is a block of Γ and E is a Γ-invariant equivalence relation. It is easily seen that this section is transitive; it is primitive whenever each block of Γ properly contained in X is contained in a class of E.

Now, let Γ be as in Theorem 1.5. Then any section of Γ is permutationally isomorphic to a section Γ^S with $S \in \mathcal{F}(G)$. Therefore, statement (1) immediately follows from Theorem 4.6 and the next lemma.

Lemma 9.1. *Any composition factor of a transitive group is isomorphic to a composition factor of some of its primitive sections.*

Proof. Let $\Gamma \leq \mathrm{Sym}(V)$ be a transitive group. We take a minimal Γ-invariant equivalence relation E. By the Jordan–Hölder theorem, any composition factor of Γ is isomorphic either to a composition factor of the transitive group $\Gamma^{V/E}$, or a composition factor of the kernel Γ_E of the epimorphism from Γ onto $\Gamma^{V/E}$. Since any primitive section of $\Gamma^{V/E}$ is permutationally isomorphic to a primitive section of Γ, by induction it suffices to verify that any composition factor of the group Γ_E is isomorphic to a composition factor of the group Γ^X with $X \in V/E$. However, the group Γ_E is a subdirect product of the groups $(\Gamma_E)^X$, $X \in V/E$. Therefore, any composition factor of the group Γ_E is isomorphic to a composition factor of some of the groups $(\Gamma_E)^X$. Since the latter group is normal in Γ^X, we are done. $\square$

To prove the "if" part in statement (2), suppose that every alternating composition factor of the group Γ is of prime degree. Denote by $\mathcal{A}$ the S-ring over the group G associated with Γ. Then $\Gamma = \mathrm{Aut}(\mathcal{A})$. We claim that any class $C \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A})$ of composite degree is of degree 4. To prove this, let $S \in C$. Then $\Gamma^S = \mathrm{Sym}(S)$ by Theorem 4.6. Moreover, if $S = U/L$, then the group Γ_E with $E = E_U$ is obviously normal in Γ. Therefore, $\Gamma_E \trianglelefteq \Gamma^S$ and

$$(\Gamma_E)^S \trianglelefteq \mathrm{Sym}(S).$$

Since $S_{\mathrm{right}} \leq (\Gamma_E)^S$, this implies that $|S| = 4$ or $(\Gamma_E)^S \in \{\mathrm{Alt}(S), \mathrm{Sym}(S)\}$. By the Jordan–Hölder theorem, any composition factor of the group $(\Gamma_E)^S$ is isomorphic to a composition factor of Γ. Thus, our supposition implies that $|S| = 4$, and the claim is proved.

By the claim and Theorem 8.1, in Γ one can find a subgroup Γ' that is 2-equivalent to it and satisfies $G_{\mathrm{right}} \leq \Gamma' \leq \mathrm{Sym}(G)$ and $(\Gamma')^S = \mathrm{Hol}(S)$ for any section S belonging to a singular class $C \in \mathcal{P}_{\mathrm{sgl}}(\mathcal{A})$ of degree other than 4. Since the group $\mathrm{Sym}(4)$ is solvable, Theorem 4.6 shows that any primitive section of Γ' is solvable. By Lemma 9.1, this implies that the last group is solvable. This proves the "if" part in statement (2).

To prove the "only if" part of statement (2), suppose that the group Γ is 2-equivalent to a solvable group Γ' containing G_{right}. Let Γ have a composition factor isomorphic to the group $\mathrm{Alt}(m)$, where m is a composite number. Obviously, $m > 4$. By Lemma 9.1, there exists a primitive section $S \in \mathcal{F}(\mathcal{A})$ such that the group Γ^S has a composition factor isomorphic to $\mathrm{Alt}(m)$. Since $m > 4$, the last group is not solvable. Therefore, by Theorem 4.6 we have

$$\Gamma^S = \mathrm{Sym}(S) \quad \text{and} \quad m = |S|.$$

On the other hand, the groups Γ^S and $(\Gamma')^S$ are 2-equivalent. Therefore, the latter group is primitive. Since it is also solvable, its degree is a prime power by [11, Theorem 11.5]. Since this group contains the regular cyclic subgroup S_{right}, we can apply [11, Theorem 27.3] to conclude that either $|S|$ is prime, or $|S| = 4$. However, $m = |S|$ is a composite number greater than 4, a contradiction.

§10. Proof of Theorem 1.3

We deduce Theorem 1.3 from Theorem 10.2 to be proved below with the help of the following auxiliary lemma.

Lemma 10.1. *Let $G = \prod_{i\in I} G_i$ and $\Delta = \prod_{i\in I} \Delta_i$, where G_i is a cyclic group and $(G_i)_{\mathrm{right}} \le \Delta_i \le \mathrm{Hol}(G_i)$, $i \in I$. Then $\Delta' = \Delta$ for any subgroup Δ' of the group Δ that is 2-equivalent to Δ.*

Proof. Let Δ' be a subgroup of Δ 2-equivalent to Δ. Then the 2-orbits of Δ and Δ' are the same. On the other hand, denote by X_i the 2-orbit of Δ_i that contains the pair $(1_{G_i}, g_i)$, where g_i is a generator of the group G_i. Then the group Δ_i acts regularly on X_i because $\Delta_i \le \mathrm{Hol}(G_i)$. Therefore the set $X = \prod_i X_i$ is a regular 2-orbit of the group Δ. Thus,

$$|\Delta| = |X| \le |\Delta'|.$$

Since $\Delta' \le \Delta$, this implies that $\Delta' = \Delta$. □

Turning to the proof of Theorem 1.3, we set $S = U/L$. Then, by assumption, there exists an integer $k \ge 0$ such that

$$\mathcal{A}_S = \bigotimes_{i=0}^{k} \mathcal{A}_{S_i}, \tag{40}$$

where $S = \prod_{i=0}^{k} S_i$ with $S_i \in \mathcal{G}(\mathcal{A})_S$, and the S-ring $\mathcal{A}_{S_i}$ is normal for $i = 0$ and is primitive of rank 2 and of degree at least 3 for $i \ge 1$. Denote by I (respectively, by J) the set of all $i \in \{1, \dots, k\}$ for which the number $|S_i|$ is prime (respectively, composite).

Theorem 10.2. *With the above notation, suppose that the* S*-ring $\mathcal{A}$ is Schurian and $\Gamma \le \mathrm{Sym}(G)$ is the group the existence of which was stated in Theorem* 8.1*. Then*

$$\Gamma^S = \mathrm{Aut}(\mathcal{A}_{S_0}) \times \prod_{i\in I} \mathrm{Hol}(S_i) \times \prod_{i\in J} \mathrm{Sym}(S_i). \tag{41}$$

Proof. Let $j \in J$. By Theorem 4.6, the class C_j of projectively equivalent sections that contains S_j is singular. Set $\Delta_j = \mathrm{Gwr}_{\mathcal{A}}(C_j)$. Then $\Delta_j \le \Gamma$ by the choice of Γ (see statement (1) of Theorem 8.1). Moreover,

$$(\Delta_j)^{S_j} = \mathrm{Sym}(S_j) \quad \text{and} \quad (\Delta_j)^{S_i} \le (S_i)_{\mathrm{right}}, \quad i \ne j$$

(the first relation follows from the definition of the group $\mathrm{Gwr}_{\mathcal{A}}(C_j)$ and statement (2) of Theorem 6.9, whereas the inclusion is a consequence of statement (3) of the same theorem). Since obviously $\Gamma^S \ge \langle \Delta_j^S, S_{\mathrm{right}} \rangle$ and S_{right} is the direct product of the groups $(S_i)_{\mathrm{right}}$ over all i, it follows that

$$\Gamma^S \ge \mathrm{Sym}(S_j) \times \{\mathrm{id}_{S_{j'}}\},$$

where $S_{j'}$ is the direct product of the groups S_i, $i \ne j$. Thus,

$$\Gamma^S = \Gamma^{S_{I^*}} \times \prod_{j\in J} \mathrm{Sym}(S_j), \tag{42}$$

where $I^* = I \cup \{0\}$ and S_{I^*} is the product of all S_i with $i \in I^*$.

By assumption, the groups Γ and $\mathrm{Aut}(\mathcal{A})$ are 2-equivalent. Therefore, for any section $S' \in \mathcal{F}(\mathcal{A})$ the groups $\Gamma^{S'}$ and $\mathrm{Aut}(\mathcal{A})^{S'}$ are also 2-equivalent. Since the S-ring $\mathcal{A}$ is Schurian, by (40) it follows that

$$\Gamma^{S_{I^*}} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A})^{S_{I^*}} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{S_{I^*}}) = \prod_{i\in I^*} \mathrm{Aut}(\mathcal{A}_{S_i}) \underset{2}{\approx} \prod_{i\in I^*} \mathrm{Aut}(\mathcal{A})^{S_i} \underset{2}{\approx} \prod_{i\in I^*} \Gamma^{S_i}.$$

Thus, by (42) and Lemma 10.1, to complete the proof of the theorem it suffices to verify that, given $i \in I^*$, we have

$$\Gamma^{S_i} = \begin{cases} \mathrm{Hol}(S_i) & \text{if } i \neq 0, \\ \mathrm{Aut}(\mathcal{A}_{S_0}) & \text{if } i = 0. \end{cases} \tag{43}$$

For this, let $i \in I^*$. First, suppose that $i = 0$. Since the S-ring $\mathcal{A}_{S_0}$ is normal, we see that

$$\Gamma^{S_0} \leq \mathrm{Aut}(\mathcal{A})^{S_0} \leq \mathrm{Aut}(\mathcal{A}_{S_0}) \leq \mathrm{Hol}(S_0).$$

On the other hand, we have

$$\Gamma^{S_0} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A})^{S_0} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{S_0}).$$

Thus, $\Gamma^{S_0} = \mathrm{Aut}(\mathcal{A}_{S_0})$ by Lemma 10.1 for $I = \{0\}$ and $\Delta = \mathrm{Aut}(\mathcal{A}_{S_0})$.

Suppose that $i > 0$. Then $i \in I$ and the number $|S_i|$ is prime. Therefore, without loss of generality we may assume that the section S_i is not singular (see statement (2) of Theorem 8.1). Then by Theorem 4.6 we have

$$\Gamma^{S_i} \leq \mathrm{Aut}(\mathcal{A})^{S_i} \leq \mathrm{Hol}(S_i).$$

On the other hand, since $\mathcal{A}$ is Schurian and $\mathrm{rk}(\mathcal{A}_{S_i}) = 2$, we obtain

$$\Gamma^{S_i} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A})^{S_i} \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{S_i}) \underset{2}{\approx} \mathrm{Hol}(S_i).$$

Thus, $\Gamma^{S_i} = \mathrm{Hol}(S_i)$ by Lemma 10.1 for $I = \{i\}$ and $\Delta = \mathrm{Hol}(S_i)$. □

We turn to the proof of Theorem 1.3. Since the property to be Schurian is preserved under taking restrictions and factors, the "only if" part is obvious. To prove the "if" part, suppose that the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are Schurian. Denote by Γ_0 and Γ_U the groups the existence of which is provided by Theorem 8.1 applied to the S-rings $\mathcal{A}_{G/L}$ and $\mathcal{A}_U$, respectively. Then for the groups $\Delta_0 = \Gamma_0$ and $\Delta_U = \Gamma_U$ the first two inclusions in (11) and condition (14) are fulfilled. By Corollary 5.7, to complete the proof it suffices to verify that the third relation in (11) is also fulfilled for them. However, this is true by Theorem 10.2 applied both to the S-ring $\mathcal{A}_{G/L}$ with $\Gamma = \Delta_0$ and $S = U/L$, and to the S-ring $\mathcal{A}_U$ with $\Gamma = \Delta_U$ and $S = U/L$.

§11. Circulant S-rings with $\Omega(n) \leq 4$

Our main goal in the section is to study non-Schurian S-rings over a cyclic group of order n with $\Omega(n) \leq 4$.

Theorem 11.1. *Any* S*-ring over a cyclic group of order n with $\Omega(n) \leq 3$ is Schurian.*

Proof. This claim is trivial for $n = 1$. Let $\Omega(n) > 0$, and let $\mathcal{A}$ be an S-ring over a cyclic group G of order n. If the radical of this S-ring is trivial, then we are done by Corollary 4.3. Suppose the radical is not trivial. Then by statement (1) of Theorem 4.1, there exists an $\mathcal{A}$-section $S = U/L$ such that $\mathcal{A}$ is a proper S-wreath product. By the inductive hypothesis, the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are Schurian. Moreover, since $\Omega(n) \leq 3$, we have $\Omega(|S|) \leq 1$. Therefore, $\mathrm{rad}(\mathcal{A}_S) = 1$. Thus, $\mathcal{A}$ is Schurian by Theorem 1.3. □

The following auxiliary lemma will be used in the analysis of the case of four primes. We fix a cyclic group G of order pqr, where p, q, and r are primes. The subgroups of G of orders pq and r are denoted by M and N, respectively. Also, we fix a section S of G of order pq; obviously, $S = M/1$ or $S = G/N$.

Lemma 11.2. *Let $\mathcal{A}$ be an* S*-ring over the group G; suppose $\mathcal{A}$ is not a proper wreath product. Also, suppose that $S \in \mathcal{F}(\mathcal{A})$, $|S| \neq 4$, and $\mathcal{A}_S = \mathcal{A}_{S_p} \wr \mathcal{A}_{S/S_p}$, where S_p is the $\mathcal{A}_S$-group of order p. Then the set of orders of the proper $\mathcal{A}$-subgroups equals $\{p, r, pq, pr\}$, and*

(1) *if $S = M/1$, then $q \neq r$;*
(2) *if $S = G/N$, then $p \neq r$.*

Moreover, either the S*-ring $\mathcal{A}$ is normal and then $p = q$, or it is an M'/N'-wreath product, where M' and N' are the subgroups of G of orders pr and p, respectively.*

Proof. First, we find the set $\mathcal{G}(\mathcal{A})$ and prove that statements (1) and (2) are valid under the assumption that $G = M \times N$ with $M, N \in \mathcal{G}(\mathcal{A})$. Indeed, in this case obviously $r \notin \{p, q\}$ and the section S is projectively equivalent both to $M/1$ and to G/N. Therefore, statements (1) and (2) hold true. Next, since by Theorem 3.2 the S-ring $\mathcal{A}_M$ is Cayley isomorphic to $\mathcal{A}_S$, the assumption of the lemma implies that $\mathcal{G}(\mathcal{A}_M) = \{1, N', M\}$. Moreover, $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{A}_M)\mathcal{G}(\mathcal{A}_N)$ by Lemma 2.1. Thus, $\mathcal{G}(\mathcal{A}) = \{1, N, N', M', M, G\}$, and we are done.

Suppose that the S-ring $\mathcal{A}$ is normal. Then it is dense by statement (1) of Theorem 4.2. This implies that $p = q$, because otherwise $|\mathcal{G}(\mathcal{A}_S)| = 3$, which contradicts the assumptions of the lemma. Thus, if G is not a p-group, then $G = M \times N$ with $M, N \in \mathcal{G}(\mathcal{A})$, and we are done by the preceding paragraph. However, if G is a p-group, then $p \neq 2$ because $|S| \neq 4$ by assumption. By Corollary 4.4, this implies that the S-ring $\mathcal{A}_S$ is normal, which is impossible; see statement (3) of Theorem 4.2.

Finally, suppose that the S-ring $\mathcal{A}$ is not normal. If $\mathrm{rad}(\mathcal{A}) = 1$, then statement (2) of Theorem 4.1 shows that

$$\mathcal{A} = \mathcal{A}_{\widetilde{M}} \otimes \mathcal{A}_{\widetilde{N}} \tag{44}$$

for some $\mathcal{A}$-groups $\widetilde{M}$ and $\widetilde{N}$ with $\Omega(|\widetilde{M}|) = 2$ and $\Omega(|\widetilde{N}|) = 1$ (we have used the fact that $\mathrm{rk}(\mathcal{A}) \geq 3$). However, if $|\widetilde{N}| \in \{p, q\}$, then it is easily seen that the S-ring $\mathcal{A}_S$ is a nontrivial tensor product, which is impossible. Thus, $\widetilde{M} = M$ and $\widetilde{N} = N$, whence it follows that the section $M/1$ is projectively equivalent to S, so that $N' \in \mathcal{G}(\mathcal{A})$ and $\mathcal{A}_M = \mathcal{A}_{N'} \wr \mathcal{A}_{M/N'}$. By (44), the latter relation implies that $\mathcal{A}$ is an M'/N'-wreath product; by statement (1) of Theorem 4.1, this contradicts the assumption $\mathrm{rad}(\mathcal{A}) = 1$.

Let $\mathrm{rad}(\mathcal{A}) \neq 1$. Then statement (1) of Theorem 4.1 shows that there exist $\mathcal{A}$-groups $\widetilde{M}$ and $\widetilde{N}$ such that $1 < \widetilde{N} \leq \widetilde{M} < G$ and the S-ring $\mathcal{A}$ is an $\widetilde{M}/\widetilde{N}$-wreath product. Since this S-ring is not a proper wreath product, we also have $\widetilde{M} \neq \widetilde{N}$. For the same reason, both $\mathcal{A}_{\widetilde{M}}$ and $\mathcal{A}_{G/\widetilde{N}}$ are not proper wreath products (because otherwise $\mathcal{A} = \mathcal{A}_{\widetilde{N}} \wr \mathcal{A}_{G/\widetilde{N}}$ in the former case, and $\mathcal{A} = \mathcal{A}_{\widetilde{M}} \wr \mathcal{A}_{G/\widetilde{M}}$ in the latter case).

Assume that $S = M/1$ (the case where $S = G/N$ can be treated in a similar way). Obviously, then M is an $\mathcal{A}$-group. Moreover,

$$\widetilde{N} \leq M. \tag{45}$$

Indeed, otherwise $\widetilde{N} \cap M = 1$, whence $G = M \times \widetilde{N}$. Since $\mathcal{A}_{G/\widetilde{N}}$ is Cayley isomorphic to $\mathcal{A}_M$, it is a proper wreath product, which is impossible by the above. Next, since $\mathcal{A}_{\widetilde{M}}$ is also not a proper wreath product, we have

$$\widetilde{M} \neq M. \tag{46}$$

Now, since $\mathcal{G}(\mathcal{A}_M) = \{1, N', M\}$, from (45) we deduce that $\widetilde{N} = N'$. Therefore, $\widetilde{M} \in \{M, M'\}$, and (46) implies $\widetilde{M} = M'$. Therefore, $M' \neq M$, whence $q \neq r$. Finally, to find the set $\mathcal{G}(\mathcal{A})$ we observe that, since the S-ring $\mathcal{A}_{M'} = \mathcal{A}_{\widetilde{M}}$ is of rank at least 3 and not a proper wreath product, Theorem 4.1 implies that this S-ring is either a normal

one or a nontrivial tensor product. So, it is dense (in the former case this follows from statement (1) of Theorem 4.2), and hence $\mathcal{G}(\mathcal{A}_{M'}) \supset \{N', N\}$. Therefore, the set $\mathcal{G}(\mathcal{A})$ is as required: if $N = N'$, then this is obvious, and if $N \neq N'$, then $G = M \times N$ and we can use the statement proved in the first paragraph. □

Theorem 11.3. *Let $\mathcal{A}$ be a non-Schurian* S*-ring over a cyclic group G of order n with $\Omega(n) = 4$. Then $\mathcal{A}$ is a proper S-wreath product for some $\mathcal{A}$-section S. Moreover, if in this case $S = U/L$, then*:

(1) *the numbers $|L|$ and $|G/U|$ are prime and $|S| \neq 4$*;
(2) *the* S*-ring $\mathcal{A}_S$ is a proper wreath product*;
(3) *the* S*-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are not proper wreath products and cannot be normal simultaneously.*

Proof. By Lemma 4.3, the radical of the S-ring $\mathcal{A}$ is nontrivial. By statement (1) of Theorem 4.1, there exists an $\mathcal{A}$-section $S = U/L$ of the group G such that the S-ring $\mathcal{A}$ is a proper S-wreath product. It follows that $L \neq 1$ and $U \neq G$. Therefore, by Theorem 11.1, the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are Schurian, and hence, by Theorem 1.3, the S-ring $\mathcal{A}_S$ can be neither normal nor of rank 2. In particular, $\Omega(|S|) = 2$. Since, obviously, any S-ring over a cyclic group of order 4 is of rank 2 or normal, statement (1) follows. Moreover, by Theorem 4.1, the S-ring $\mathcal{A}_S$ is a proper generalized wreath product, and hence a proper wreath product. Statement (2) is proved.

Next, since the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are Schurian, we see that the groups $\mathrm{Aut}(\mathcal{A}_U)^S$ and $\mathrm{Aut}(\mathcal{A}_{G/L})^S$ are 2-equivalent. If, moreover, these S-rings are normal, then the above groups are between S_{right} and $\mathrm{Hol}(S)$. Thus, they are equal. By Corollary 5.7, this implies that the S-ring $\mathcal{A}$ is Schurian, a contradiction. This proves the second part of statement (3). To prove the first, suppose that

$$\mathcal{A}_U = \mathcal{A}_H \wr \mathcal{A}_{U/H}$$

for some proper $\mathcal{A}$-subgroup H of U (the case where $\mathcal{A}_{G/L}$ is a proper wreath product can be treated in a similar way). Then any $\mathcal{A}$-subgroup of the group U either contains H or is a subgroup of H. Since $|L|$ is prime, this implies that $L \leq H$. However, then the S-ring $\mathcal{A}$ is an H/L-wreath product, which is impossible by statement (1). □

Now we fix a cyclic group G of order $n = p_1p_2p_3p_4$, where the numbers p_i are primes. Then for any divisor m of n there is a unique subgroup of G of order m. Denote by U, V, G_1, H, G_2, K, L subgroups of G such that

$$|U| = p_1p_2p_3, \quad |V| = p_1p_3p_4,$$
$$|G_1| = p_1p_2, \quad |H| = p_1p_3, \quad |G_2| = p_3p_4,$$
$$|K| = p_1, \quad |L| = p_3.$$

It is easily seen that if $p_1 \neq p_3$ and $p_2 \neq p_4$, then $K \neq L$, $U \neq V$ and these subgroups form a sublattice of the lattice of all subgroups of G, as in Figure 1 (if $p_2 = p_3$, then $G_1 = H$, whereas if $p_1 = p_4$, then $G_2 = H$).

Theorem 11.4. *Let $\mathcal{A}$ be a non-Schurian* S*-ring over the group G as above. Suppose that $\mathcal{A}$ is a U/L-wreath product. Then $|U/L| \neq 4$ and the following is true*:

(1) *$p_1 \neq p_3$, $p_2 \neq p_4$ and the lattice of $\mathcal{A}$-subgroups of G is as in Figure* 1;
(2) *$\mathcal{A}_{U/L} = \mathcal{A}_{H/L} \wr \mathcal{A}_{U/H}$*;
(3) *the* S*-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ cannot be normal simultaneously, and moreover, if one of them is normal, then $p_1 = p_2$,*
(4) *if the* S*-ring $\mathcal{A}_U$ is not normal, then $\mathcal{A}_U = \mathcal{A}_H \wr_{H/K} \mathcal{A}_{U/K}$, and if the* S*-ring $\mathcal{A}_{G/L}$ is not normal, then $\mathcal{A}_{G/L} = \mathcal{A}_{V/L} \wr_{V/H} \mathcal{A}_{G/H}$*;

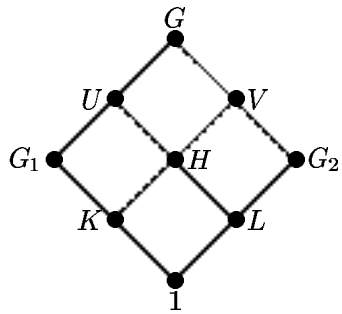

FIGURE 1

(5) *if each of the* S*-rings* $\mathcal{A}_U$ *and* $\mathcal{A}_{G/L}$ *is not normal, then so are the* S*-rings* $\mathcal{A}_V$ *and* $\mathcal{A}_{G/K}$*, and* $\mathcal{A} = \mathcal{A}_V \wr_{V/K} \mathcal{A}_{G/K}$.

Proof. Let $S = U/L$. Then by Theorem 11.3 we have $|S| \neq 4$, and statement (2) and the first part of statement (3) hold true. Moreover, by the same theorem the assumptions of Lemma 11.2 are satisfied for the S-ring $\mathcal{A}_U$ with $(p, q, r) = (p_1, p_2, p_3)$, and for the S-ring $\mathcal{A}_{G/L}$ with $(p, q, r) = (p_1, p_2, p_4)$. Thus, by that lemma, all the remaining statements follow except for statement (5). To prove (5), suppose that the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are not normal. Then by statement (4) we have

$$\mathcal{A}_U = \mathcal{A}_H \wr_{H/K} \mathcal{A}_{U/K} \quad \text{and} \quad \mathcal{A}_{G/L} = \mathcal{A}_{V/L} \wr_{V/H} \mathcal{A}_{G/H}. \tag{47}$$

Take $X \in \mathcal{S}(\mathcal{A})_{G\setminus V}$. If $X \subset U \setminus V$, then obviously $X \subset U \setminus H$. By the first relation in (47), this implies that $K \leq \mathrm{rad}(X)$. Now, let $X \subset G \setminus U$. Then $L \leq \mathrm{rad}(X)$ because by assumption the S-ring $\mathcal{A}$ is a U/L-wreath product. However, $\pi_L(X) \subset \pi_L(G) \setminus \pi_L(V)$. By the second relation in (47), this implies that $\pi_L(H) \leq \mathrm{rad}(\pi_L(X))$. Therefore, $K \leq H \leq \mathrm{rad}(X)$ and

$$\mathcal{A} = \mathcal{A}_V \wr_{V/K} \mathcal{A}_{G/K}.$$

Next, statement (1) of Theorem 11.3 shows that $|V/K| \neq 4$. Moreover, the S-ring $\mathcal{A}_V$ is an H/L-wreath product, because the S-ring $\mathcal{A}$ is a U/L-wreath product, and $U \cap V = H$ due to the relation $p_2 \neq p_4$ proved above. Therefore, by statement (3) of Theorem 4.2, the S-ring $\mathcal{A}_V$ can be normal only if $|V/H| = |L| = 2$. But this is not the case because $4 \neq |V/K| = p_3 p_4 = |L|\,|V/H|$. Thus, the S-ring $\mathcal{A}_V$ is not normal. The fact that the S-ring $\mathcal{A}_{G/K}$ is not normal is proved in a similar way. □

§12. EXAMPLE

Throughout this section, $\mathbb{Z}_n$ is the additive group of integers modulo a positive integer n.

For any divisor m of n, we denote by $i_{m,n} : \mathbb{Z}_m \to \mathbb{Z}_n$ and $\pi_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ the group homomorphisms taking 1 to n/m and to 1, respectively. Using them, we identify the groups $i_{m,n}(\mathbb{Z}_m)$ and $\mathbb{Z}_n/\ker(\pi_{n,m})$ with $\mathbb{Z}_m$. Thus, every section of the group $\mathbb{Z}_n$ of order m is identified with $\mathbb{Z}_m$. Moreover, the permutation in $\mathrm{Aut}(\mathbb{Z}_n)$ afforded by multiplication by an integer induces the permutation in $\mathrm{Aut}(\mathbb{Z}_m)$ afforded by multiplication by the same integer.

If $\mathcal{A}$ is an S-ring over $G = \mathbb{Z}_n$ and H is the $\mathcal{A}$-group of order m, then $\mathcal{A}_H$ and $\mathcal{A}_{G/H}$ are denoted (respectively) by $\mathcal{A}_m$ and $\mathcal{A}^{n/m}$. Finally, let $\mathcal{A}_i$ be an S-ring over $\mathbb{Z}_{n_i}$ $(i = 1, 2)$ and $(\mathcal{A}_1)^m = (\mathcal{A}_2)_m$ for some m dividing both n_1 and n_2. Then a unique S-ring $\mathcal{A}$ over $\mathbb{Z}_{n_1 n_2/m}$ defined in Theorem 3.4 is denoted by $\mathcal{A}_1 \wr_m \mathcal{A}_2$. We omit m if $m = 1$. Given a

group $K \le \mathrm{Aut}(\mathbb{Z}_n)$ and a prime p dividing n, we denote by K_p the p-projection of K in the sense of the decomposition (5), and write $\mathrm{Cyc}(K,n)$ instead of $\mathrm{Cyc}(K,\mathbb{Z}_n)$.

In what follows, we shall construct a family of non-Schurian S-rings over a cyclic group of order $n = p_1p_2p_3p_4$, where the p_i are primes with $p_1 = p_2 = p \ne p_4$, and p divides $p_3 - 1$. We fix a positive integer d dividing $p-1$.

For an arbitrary cyclic group $M \le \mathrm{Aut}(\mathbb{Z}_{p^2p_3})$ of order pd such that $|M_p| = pd$ and $|M_{p_3}| = p$, set $\mathcal{A}_1 = \mathrm{Cyc}(M, p^2p_3)$. We claim that

$$(\mathcal{A}_1)^{p^2} = \mathrm{Cyc}(d,p) \wr \mathrm{Cyc}(d,p), \tag{48}$$

where $\mathrm{Cyc}(d,p) = \mathrm{Cyc}(K,p)$ and K is the subgroup of $\mathrm{Aut}(\mathbb{Z}_p)$ of order d. Indeed, it is easily seen that $(\mathcal{A}_1)^{p^2} = \mathrm{Cyc}(M_p, p^2)$. On the other hand, since p divides $|M_p|$, the group $\mathrm{rad}((\mathcal{A}_1)^{p^2})$ is nontrivial. By statement (1) of Theorem 4.1, this implies that the S-ring $(\mathcal{A}_1)^{p^2}$ is a nontrivial generalized wreath product. So, this S-ring is the wreath product of $(\mathcal{A}_1)^p$ by $(\mathcal{A}_1)_p$. Since obviously $(\mathcal{A}_1)^p = (\mathcal{A}_1)_p = \mathrm{Cyc}(d,p)$, the claim is proved.

Next, suppose in addition that d divides $p_4 - 1$. Given two cyclic groups $M_1, M_2 \le \mathrm{Aut}(\mathbb{Z}_{pp_4})$ of order d such that $|(M_i)_p| = |(M_i)_{p_4}| = d$, $i = 1,2$, we set

$$\mathcal{A}_2 = \mathrm{Cyc}(M_1, pp_4) \wr_{p_4} \mathrm{Cyc}(M_2, pp_4).$$

Then $\mathrm{rad}(\mathcal{A}_2)$ contains the subgroup of order p. Moreover, since $p \ne p_4$, Lemma 2.1 implies that $\mathcal{A}_2 \ge (\mathcal{A}_2)_{p^2} \otimes (\mathcal{A}_2)_{p_4}$. Thus, the group $\mathrm{rad}((\mathcal{A}_2)_{p^2})$ is nontrivial. By statement (1) of Theorem 4.1, this implies that $(\mathcal{A}_2)_{p^2}$ is a nontrivial generalized wreath product. So, this S-ring is the wreath product of $(\mathcal{A}_2)^p$ by $(\mathcal{A}_2)_p$. Since $(\mathcal{A}_2)^p = (\mathcal{A}_2)_p = \mathrm{Cyc}(d,p)$, we conclude that

$$(\mathcal{A}_2)_{p^2} = \mathrm{Cyc}(d,p) \wr \mathrm{Cyc}(d,p). \tag{49}$$

From (48) and (49) it follows that $(\mathcal{A}_1)^{p^2} = (\mathcal{A}_2)_{p^2}$, and we set

$$\mathcal{A} = \mathcal{A}_1 \wr_{p^2} \mathcal{A}_2. \tag{50}$$

Observe that $\mathcal{A}$ is an S-ring over the group $\mathbb{Z}_n$, and the S-rings $\mathcal{A}_1$, $\mathrm{Cyc}(M_1, pp_4)$, and $\mathrm{Cyc}(M_2, pp_4)$ are cyclotomic, and hence dense. The density of the last two S-rings implies that the S-ring $\mathcal{A}_2$ is also dense. Thus, the set $\mathcal{G}(\mathcal{A})$ contains the subgroups K, L, G_1, H, G_2, U, and V of the group $G = \mathbb{Z}_n$ of orders p, p_3, p^2, pp_3, p_3p_4, p^2p_3, and pp_3p_4, respectively (see Figure 1).

Theorem 12.1. *The* S*-ring* $\mathcal{A}$ *is non-Schurian whenever* $M_1 \ne M_2$.

Proof. Suppose that $M_1 \ne M_2$. It is easily seen that each of the S-rings $\mathcal{A}_1$, $\mathrm{Cyc}(M_1, pp_4)$, and $\mathrm{Cyc}(M_2, pp_4)$ has trivial radical, and is not a proper tensor product (because otherwise $d = 1$ whence $M_1 = M_2$). By Theorem 4.1, this implies that all these S-rings are normal. To prove that the S-ring $\mathcal{A}$ is not Schurian, it suffices to find an element

$$\gamma \in (\Gamma^{U/L})_u \setminus (\Delta^{U/L})_u, \tag{51}$$

where $\Gamma = \mathrm{Aut}(\mathcal{A}_1)$, $\Delta = \mathrm{Aut}(\mathcal{A}_2)$, and $u = 0$ (in the group U/L). Indeed, in this case the group $\Gamma' = \Gamma^{U/L} \cap \Delta^{U/L}$ is a proper subgroup of $\Gamma^{U/L}$. Also, since the S-ring $\mathcal{A}_1$ is normal, we have

$$\Gamma^{U/L} \le \mathrm{Hol}(U/L). \tag{52}$$

Thus, the groups Γ' and $\Gamma^{U/L}$ are not 2-equivalent by Lemma 10.1. Since, obviously, the latter group is 2-equivalent to the group $\mathrm{Aut}(\mathcal{A}_{U/L})$, the S-ring $\mathcal{A}$ is not Schurian by Corollary 5.8.

To prove the existence of an element γ satisfying (51), we note that, by definition, the group M_1 is a subgroup of the group

$$\mathrm{Aut}(V/L) = \mathrm{Aut}(H/L) \times \mathrm{Aut}(G_2/L) = \mathrm{Aut}(\mathbb{Z}_p) \times \mathrm{Aut}(\mathbb{Z}_{p_4}).$$

Moreover, if $Q \leq \mathrm{Aut}(\mathbb{Z}_p)$ and $R \leq \mathrm{Aut}(\mathbb{Z}_{p_4})$ are groups of order d, then

$$M_1 = \{(x, y) \in Q \times R \,:\, \varphi_1(x) = y\} \tag{53}$$

where φ_1 is an isomorphism from Q onto R. Similarly, M_2 is a subgroup of the group

$$\mathrm{Aut}(G/H) = \mathrm{Aut}(U/H) \times \mathrm{Aut}(V/H) = \mathrm{Aut}(\mathbb{Z}_p) \times \mathrm{Aut}(\mathbb{Z}_{p_4}),$$

and

$$M_2 = \{(x, y) \in Q \times R \,:\, \varphi_2(x) = y\}, \tag{54}$$

where φ_2 is an isomorphism from Q onto R.

To find γ, we take $\delta \in \Delta_u$. Then, since the S-rings $(\mathcal{A}_2)_U = \mathrm{Cyc}(M_1, pp_4)$ and $(\mathcal{A}_2)_{G/H} = \mathrm{Cyc}(M_2, pp_4)$ are normal (see above), we conclude that $(\delta)^{V/L} \in M_1$ and $(\delta)^{G/H} \in M_2$. By (53) and (54), this implies that

$$\varphi_1(\delta^{H/L}) = \delta^{V/H} \quad \text{and} \quad \varphi_2(\delta^{U/H}) = \delta^{V/H}. \tag{55}$$

Next, since $M_1 \neq M_2$, from (53) and (54) it follows that there exist distinct elements $x_1, x_2 \in Q$ such that

$$\varphi_1(x_1) = \varphi_2(x_2). \tag{56}$$

Now, the inclusion (52) shows that there exists an element γ in the group $\Gamma_u = M$ for which $\gamma^{H/L} = \gamma^{U/H} = x_1$.

To complete the proof, we need to check that γ satisfies (51). Suppose to the contrary that $\gamma^{U/L} \in (\Delta_u)^{U/L}$. Then there exists $\delta \in \Delta_u$ such that $\delta^{U/L} = \gamma^{U/L}$. So, from (55), (56), and the definition of γ it follows that

$$\varphi_2(\delta^{U/H}) = \varphi_1(\delta^{H/L}) = \varphi_1(\gamma^{H/L}) = \varphi_1(x_1) = \varphi_2(x_2).$$

This implies that $x_2 = \delta^{U/H} = \gamma^{U/H} = x_1$, which is impossible because $x_1 \neq x_2$. □

Theorem 12.2. *Let $n = p_1p_2p_3p_4$, where the p_i are odd primes. Suppose that $\{p_1, p_2\} \cap \{p_3, p_4\} = \varnothing$ and that the numbers $p_1 - 1$ and $p_4 - 1$ have a common divisor $d \geq 3$. Then a cyclic group of order n is not Schur whenever either d divides both $p_2 - 1$ and $p_3 - 1$, or $p_1 = p_2$ and p_1 divides $p_3 - 1$.*

Proof. If d divides both $p_2 - 1$ and $p_3 - 1$, then d divides $p_i - 1$ for $i = 1, 2, 3, 4$. By the results of [1], this implies that there is a non-Schurian S-ring over the group $\mathbb{Z}_n$. Thus, this group is not Schur. To complete the proof, suppose that $p_1 = p_2 = p$ and p divides $p_3 - 1$. Using (5), we set

$$M = \{(x, y) \in \mathrm{Aut}(\mathbb{Z}_{p^2p_3}) \,:\, \varphi(x) = y\}$$

and, for $i = 1, 2$,

$$M_i = \{(x, y) \in \mathrm{Aut}(\mathbb{Z}_{pp_4}) \,:\, \varphi_i(x) = y\}$$

where φ is an epimorphism from the subgroup of $\mathrm{Aut}(\mathbb{Z}_{p_3})$ of order pd to the subgroup of $\mathrm{Aut}(\mathbb{Z}_{p^2})$ of order p, and φ_i is an isomorphism from the subgroup of $\mathrm{Aut}(\mathbb{Z}_p)$ of order d to the subgroup of $\mathrm{Aut}(\mathbb{Z}_{p_4})$ of order d. Then, obviously, M is a cyclic group of order pd such that $|M_p| = pd$ and $|M_{p_3}| = p$, and M_i is a cyclic group of order d such that $|(M_i)_p| = |(M_i)_{p_4}| = d$. Since $d \geq 3$, the isomorphisms φ_1 and φ_2, and hence the groups M_1 and M_2, can be chosen to be distinct. Thus, by Theorem 12.1, the S-ring $\mathcal{A}$ is non-Schurian and we are done. □

The minimal integer n for which the hypotheses of Theorem 12.2 are satisfied is equal to $3575 = 5 \cdot 5 \cdot 11 \cdot 13$.

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