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AN OPERATOR EQUATION CHARACTERIZING THE LAPLACIAN

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Abstract. The Laplace operator on $\mathbb{R}^n$ satisfies the equation

$$\Delta(fg)(x) = (\Delta f)(x)g(x) + f(x)(\Delta g)(x) + 2\,\langle f'(x), g'(x)\rangle$$

for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. In the paper, an operator equation generalizing this product formula is considered. Suppose $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ are operators satisfying the equation

$$T(fg)(x) = (Tf)(x)g(x) + f(x)(Tg)(x) + \langle (Af)(x), (Ag)(x)\rangle \tag{1}$$

for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. Assume, in addition, that T is $O(n)$-invariant and annihilates the affine functions, and that A is nondegenerate. Then T is a multiple of the Laplacian on $\mathbb{R}^n$, and A a multiple of the derivative,

$$(Tf)(x) = \frac{d(\|x\|)^2}{2}(\Delta f)(x), \quad (Af)(x) = d(\|x\|)f'(x),$$

where $d \in C(\mathbb{R}_+, \mathbb{R})$ is a continuous function. The solutions are also described if T is not $O(n)$-invariant or does not annihilate the affine functions. For this, all operators (T, A) satisfying (1) for scalar operators $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ are determined. The map A, both in the vector and the scalar case, is closely related to T and there are precisely three different types of solution operators (T, A).

No continuity or linearity requirement is imposed on T or A.

§1. Statement of the main results

Recently, it has become clear that some fundamental operations in analysis and geometry like derivatives or duality maps are characterized by simple properties or operator functional equations: the derivative is characterized by the chain or the Leibniz rule, cf. [AAM, AKM, KM1, KM2], the duality of convex bodies and the Legendre transform as a duality map for convex functions are characterized by order-reversing idempotent operations, cf. [AM1, AM2, AM3]. In this paper, we characterize the Laplace operator for C^2-functions on $\mathbb{R}^n$ by the second order Leibniz rule, orthogonal invariance, and simple initial conditions. For $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$, we know by applying the Leibniz rule that

$$\Delta(f \cdot g)(x) = (\Delta f)(x) \cdot g(x) + f(x) \cdot (\Delta g)(x) + 2\langle f'(x), g'(x)\rangle, \quad x \in \mathbb{R}^n.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. We study a general operator equation motivated by this equation, namely,

$$T(fg)(x) = (Tf)(x) \cdot g(x) + f(x) \cdot (Tg)(x) + \langle (Af)(x), (Ag)(x)\rangle \tag{1}$$

for $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$, $x \in \mathbb{R}^n$. Here $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ are unknown operators. It is our aim to find all operators (T, A) satisfying (1)

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and, in particular, to find characterizations of the Laplace operator. The solutions of (1) include all derivations on $C^2(\mathbb{R}^n,\mathbb{R})$ for $A=0$, and these solutions of the "homogeneous equation" may be added to any solution of the "inhomogeneous equation" (1) when $A\neq 0$. It will turn out that A is "of lower order" not involving the second derivative, and thus may be extended to $C^1(\mathbb{R}^n,\mathbb{R}^n)$. No continuity or linearity assumption is imposed on T or A.

Second partial directional derivatives $T=\frac{\partial^2}{\partial v^2}$ satisfy the scalar version of (1) with A being $\sqrt{2}$ times the first directional derivative. To characterize the Laplacian, in addition to (1), we need the invariance of T with respect to the orthogonal group $O(n)$, just as the Laplacian is $O(n)$-invariant.

Definition. A map $T : C^2(\mathbb{R}^n,\mathbb{R}) \to C(\mathbb{R}^n,\mathbb{R})$ is $O(n)$-*invariant* if for all $f,g\in C^2(\mathbb{R}^n,\mathbb{R})$ and all orthogonal maps $u\in O(n)$ we have $T(f\circ u)=(Tf)\circ u$.

We impose a mild nondegeneracy assumption on A in the sense that A should not be close to a multiple of the identity.

Definition. A map $A : C^2(\mathbb{R}^n,\mathbb{R})\to C(\mathbb{R}^n,\mathbb{R}^n)$ is *nondegenerate* if for all open subsets $J\subset\mathbb{R}^n$ and all $x\in J$ there exist $n+1$ functions $g_i\in C^2(\mathbb{R}^n,\mathbb{R})$ with support in J such that the $n+1$ vectors $(g_i(x),Ag_i(x))\in\mathbb{R}^{n+1}$, $i\in\{1,\dots,n+1\}$, are linearly independent in $\mathbb{R}^{n+1}$. If A is a scalar map from $C^2(\mathbb{R}^n,\mathbb{R})$ into $C(\mathbb{R}^n,\mathbb{R})$, we require this for two functions with $(g_i(x),Ag_i(x))\in\mathbb{R}^2$, $i\in\{1,2\}$.

The affine functions on $\mathbb{R}^n$ are given by $c+\langle b,x\rangle$ for $c\in\mathbb{R}$, $b\in\mathbb{R}^n$. The main result of this paper is as follows.

Theorem 1. *Let $n\in\mathbb{N}$. Assume that $T : C^2(\mathbb{R}^n,\mathbb{R})\to C(\mathbb{R}^n,\mathbb{R})$ and $A : C^2(\mathbb{R}^n,\mathbb{R})\to C(\mathbb{R}^n,\mathbb{R}^n)$ are operators satisfying the "second order Leibniz rule" equation*

$$T(f\cdot g)(x)=(Tf)(x)\cdot g(x)+f(x)\cdot(Tg)(x)+\langle (Af)(x),(Ag)(x)\rangle \tag{1}$$

for all $f,g\in C^2(\mathbb{R}^n,\mathbb{R})$, $x\in\mathbb{R}^n$. Suppose also that T is $O(n)$-invariant and vanishes on the affine functions and that A is nondegenerate. Then T is a multiple of the Laplace operator and A a multiple of the derivative: there is a continuous function $d\in C(\mathbb{R}_+,\mathbb{R})$ such that

$$(Tf)(x)=\frac{1}{2}d(\|x\|)^2(\Delta f)(x)\ \text{ and }\ (Af)(x)=d(\|x\|)f'(x);$$
$$f,g\in C^2(\mathbb{R}^n,\mathbb{R}),\quad x\in\mathbb{R}^n.$$

If T is zero on the affine functions but no longer $O(n)$-invariant, T and A must be of the form

$$(Tf)(x)=\frac{1}{2}\sum_{i=1}^n\langle f''(x)c_i(x),c_i(x)\rangle,\quad (Af)(x)=(\langle f'(x),c_i(x)\rangle)_{i=1}^n.$$

This means that T is a sum of multiples of changing second directional derivatives and A a vector of the corresponding directional derivatives.

Remarks.

(1) If T is not assumed to vanish on the affine functions but is still $O(n)$-invariant, there are different operators T and A satisfying (1). In this case, there are continuous functions $a : [0,\infty)\to\mathbb{R}$, $p : [0,\infty)\to[-1,\infty)$, $d,e : [0,\infty)\to\mathbb{R}^n$, and $b : [0,\infty)\to L(\mathbb{R}^n,\mathbb{R}^n)$

such that T and A are given by one of the formulas

$$\begin{aligned}
Tf(x) &= \frac{1}{2}\langle d(\|x\|), d(\|x\|)\rangle f(x)(\log|f(x)|)^2,\\
Af(x) &= d(\|x\|)f(x)(\log|f(x)|),\\
Tf(x) &= \langle e(\|x\|), e(\|x\|)\rangle f(x)(\{\operatorname{sgn} f(x)\}\ |f(x)|^{p(\|x\|)} - 1),\\
Af(x) &= e(\|x\|)f(x)(\{\operatorname{sgn} f(x)\}\ |f(x)|^{p(\|x\|)} - 1),\\
Tf(x) &= b(\|x\|)\cdot f'(x) + d(\|x\|)f(x)\log|f(x)|, \quad Af(x) = 0.
\end{aligned}$$

The brackets $\{\operatorname{sgn} f(x)\}$ mean that this term may appear in both formulas for T and A or in neither one of them, yielding two different solutions. Note here that $Af(x)$ is again $\mathbb{R}^n$-valued: $f(x)$ is scalar-valued, but the coefficient function values $d(x)$, $e(x)$ are in $\mathbb{R}^n$. The first case is an iteration of the entropy solution given in the third case when $A = 0$, which is a derivation in $C(\mathbb{R}^n, \mathbb{R})$. So it may be viewed as an analog of the Laplacian in the space $C(\mathbb{R}^n, \mathbb{R})$. In the second case, T is a multiple of A, which we consider to be less interesting.

(2) If T or A are isotropic, i.e., commute with shifts S_c, $TS_c = S_cT$, where $S_cf(x) := f(x+c)$, then the functions c_i are constant, that is do not depend on $x \in \mathbb{R}^n$.

Let $A_i : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ be the "component operators" of the operator $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ in (1), i.e., $(Af)(x) = ((A_if)(x))_{i=1}^n$. In the proof of Theorem 1 in §5 it will be shown that equation (1) decomposes into n "scalar" equations

$$T_i(f\cdot g) = (T_if)\cdot g + f\cdot(T_ig) + A_if\cdot A_ig, \quad i = 1, \dots, n, \tag{2}$$

where the $T_i : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ are operators with $Tf = \sum_{i=1}^n T_if$. To prove Theorem 1, we shall find all operators (T_i, A_i) satisfying (2) for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$. The following is a generalization of Theorem 1 of [KM2] from $n = 1$ to general $n \in \mathbb{N}$, where an additional continuity assumption was imposed on A.

Theorem 2. *Let $n \in \mathbb{N}$. Assume that $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ are operators such that the equation*

$$T(f\cdot g)(x) = (Tf)(x)\cdot g(x) + f(x)\cdot(Tg)(x) + (Af)(x)\cdot(Ag)(x) \tag{2}$$

is satisfied for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. Let A be nondegenerate. Then there are continuous functions $a, d, p \in C(\mathbb{R}^n, \mathbb{R})$ and $b, c \in C(\mathbb{R}^n, \mathbb{R}^n)$ such that any operators T and A verifying (2) have the form

$$(Tf)(x) = (T_1f)(x) + (Sf)(x),$$

with

$$(Sf)(x) = \langle f'(x), b(x)\rangle + a(x)f(x)\ln|f(x)|$$

solving the "homogeneous equation" and T_1 and A are of one of the following three types:

$$(T_1f)(x) = \frac{1}{2}\langle f''(x)c(x), c(x)\rangle, \quad (Af)(x) = \langle f'(x), c(x)\rangle,$$

or

$$(T_1f)(x) = \frac{1}{2}d(x)^2f(x)(\ln|f(x)|)^2, \quad (Af)(x) = d(x)\ln|f(x)|,$$

or

$$\begin{aligned}
(T_1f)(x) &= d(x)^2f(x)(\{\operatorname{sgn} f(x)\}|f(x)|^{p(x)} - 1),\\
(Af)(x) &= d(x)f(x)(\{\operatorname{sgn} f(x)\}|f(x)|^{p(x)} - 1).
\end{aligned}$$

Here $p(x) \geq -1$ and the term $\{\operatorname{sgn} f(x)\}$ may be present or not in both T_1 and A, giving two independent formulas. In the last two cases, the operators extend to $C^1(\mathbb{R}^n, \mathbb{R})$. In all three cases, the operators (T, A) satisfy (2).

Remarks.

(1) The formula for $(Sf)(x)$ gives the general solution of the homogeneous (derivation) equation
$$S(f \cdot g) = Sf \cdot g + f \cdot Sg$$
on $C^2(\mathbb{R}^n, \mathbb{R})$ or on $C^1(\mathbb{R}^n, \mathbb{R})$. On $C(\mathbb{R}^n, \mathbb{R})$, these operators (with $b(x) = 0$) were determined by Goldmann and Šemrl [GS].

(2) In the last case, T_1 and A are proportional and we consider this a less interesting case.

(3) In the first case, $(T_1 f)(x)$ is a second directional derivative of f in the direction $v(x) = c(x)/\|c(x)\|_2$ multiplied by $\frac{1}{2}\|c(x)\|_2^2$, $(T_1 f)(x) = \frac{1}{2}\|c(x)\|_2^2 \frac{\partial^2 f}{\partial v^2}(x)$ and $(Af)(x) = \|c(x)\|_2 \frac{\partial f}{\partial v}(x)$. However, the directions $v(x)$ change continuously with $x \in \mathbb{R}^n$. Similarly, in $(Sf)(x)$, there are directional derivatives in the direction $w(x) = b(x)/\|b(x)\|_2$.

(4) The formulas for A in Theorems 1 and 2 can be extended to $C^1(\mathbb{R}^n, \mathbb{R})$ and in some cases to $C(\mathbb{R}^n, \mathbb{R})$.

§2. Localization

The first step in the proof of Theorems 1 and 2 is to show that T and A are local operators, i.e., are defined pointwise, and that $(Tf)(x)$ and $(Af)(x)$ only depend on x, $f(x)$, $f'(x)$ and $f''(x)$. The case where $n = 1$ was treated in [KM2]. Now we give a modified proof for $n \geq 2$, starting with localization on open sets.

Proposition 3. *Let $n, m \in \mathbb{N}$. Assume that $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}^m) \to C(\mathbb{R}^n, \mathbb{R}^m)$ are operators such that*

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x) \cdot (Tg)(x) + \langle Af(x), Ag(x) \rangle \tag{1}$$

for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. Assume that A is nondegenerate. Let $J \subset \mathbb{R}^n$ be an open subset and let $f_1, f_2 \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfy $f_1|_J = f_2|_J$. Then $(Tf_1)|_J = (Tf_2)|_J$ and $(Af_1)|_J = (Af_2)|_J$.

Proof. Let $J \subset \mathbb{R}^n$ be open, and let $f_1, f_2 \in C^2(\mathbb{R}^n, \mathbb{R})$ with $f_1|_J = f_2|_J$. For any $g \in C^2(\mathbb{R}^n, \mathbb{R})$ with support in J, we have $f_1 \cdot g = f_2 \cdot g$ and hence by (1)

$$\begin{aligned}(Tf_1) \cdot g + f_1 \cdot (Tg) + \langle Af_1, Ag \rangle &= T(f_1 \cdot g) = T(f_2 \cdot g) \\ &= (Tf_2) \cdot g + f_2 \cdot (Tg) + \langle Af_2, Ag \rangle.\end{aligned}$$

Therefore, for any $x \in J$, with $f_1(x) = f_2(x)$,

$$\big((Tf_1)(x) - (Tf_2)(x)\big) \cdot g(x) + \langle (Af_1)(x) - (Af_2)(x), (Ag)(x) \rangle = 0. \tag{3}$$

Since A is nondegenerate, we can find $n+1$ functions g_i with support in J such that the $n+1$ vectors $(g_i(x), Ag_i(x)) \in \mathbb{R}^{n+1}$, $i \in \{1, \ldots, n+1\}$, are linearly independent. Applying (3) to the $n+1$ functions g_i instead of g, we conclude that $(Af_1)(x) = (Af_2)(x)$ and $Tf_1(x) = (Tf_2)(x)$. Hence, $Tf_1|_J = Tf_2|_J$ and $Af_1|_J = Af_2|_J$. It is easily seen that (1) and the nondegeneracy condition also imply that $T(\nVdash) = 0$ and $A(\nVdash) = 0$. $\square$

As a consequence, we show that T and A are defined by pointwise data. In the following, the action of $f''(x)$ as a bilinear map or an operator is written $f''(x)(u, v) = \langle f''(u), v \rangle$ for $u, v \in \mathbb{R}^n$. Since $f''(x)$ is symmetric, it may be represented by the $\binom{n+1}{2}$ independent partial derivatives $\frac{\partial^2 g}{\partial x_k \partial x_l}(x)$ for $1 \leq k \leq \ell \leq n$, and hence can be regarded as an element of $\mathbb{R}^{\binom{n+1}{2}}$, but of course also as an element of $\mathbb{R}^{n^2}$ if we use all partial derivatives.

Proposition 4. *For any given $n, m \in \mathbb{N}$, assume that $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^m)$ satisfy the operator equation*

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x) \cdot Tg(x) + \langle Af(x),\, Ag(x)\rangle \tag{1}$$

for all $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$. Assume further that A is nondegenerate. Then there are functions

$$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}, \quad B : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^m$$

such that for all f in $C^2(\mathbb{R}^n, \mathbb{R})$ and all $x \in \mathbb{R}^n$ we have

$$(Tf)(x) = F_x\big(f(x),\, f'(x),\, f''(x)\big), \quad (Af)(x) = B_x\big(f(x),\, f'(x),\, f''(x)\big),$$

where the dependence on the first variable $x \in \mathbb{R}^n$ is written as an index.

Proof. Let $x_0 = (x_{01}, \dots, x_{0n}) \in \mathbb{R}^n$ and $f \in C^2(\mathbb{R}^n, \mathbb{R})$. For $k \in \{0, \dots, n\}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define functions $h_k \in C^2(\mathbb{R}^n, \mathbb{R})$ by

$$\begin{aligned} h_k(x) := {} & f(x_{01}, \dots, x_{0k}, x_{k+1}, \dots, x_n) \\ & + \sum_{j=1}^{k} \frac{\partial f}{\partial x_j}(x_{01}, \dots, x_{0k}, x_{k+1}, \dots, x_n)(x_j - x_{0j}) \\ & + \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_{01}, \dots, x_{0k}, x_{k+1}, \dots, x_n)(x_i - x_{0i})(x_j - x_{0j}) \end{aligned}$$

with $f = h_0$ and

$$h(x) := h_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}\langle f''(x_0)(x - x_0), (x - x_0)\rangle$$

being the Taylor polynomial of the second order approximating f at x_0. Let

$$g_k(x) := \begin{Bmatrix} h_{k-1}(x), & x_k < x_{0k} \\ h_k(x), & x_k \geq x_{0k} \end{Bmatrix}$$

for $k = 1, \dots, n$. Then $f \in C^2(\mathbb{R}^n, \mathbb{R})$ implies that $g_k \in C^2(\mathbb{R}^n, \mathbb{R})$. Put $J_{k,-} := \{x \in \mathbb{R}^n \mid x_k < x_{0k}\}$ and $J_{k,+} := \{x \in \mathbb{R}^n \mid x_k > x_{0k}\}$. On these open half-spaces we have $h_{k-1}\big|_{J_{k,-}} = g_k\big|_{J_{k,-}}$, $g_k\big|_{J_{k,+}} = h_k\big|_{J_{k,+}}$. By Proposition 3,

$$Th_{k-1}\big|_{J_{k,-}} = Tg_k\big|_{J_{k,+}}, \quad Tg_k\big|_{J_{k,+}} = Th_k\big|_{J_{k,+}}.$$

Since the functions Th_{k-1}, Tg_k and Th_k are continuous, these identities extend to $\overline{J_{k,-}}$ and $\overline{J_{k,+}}$. Hence, for $x_0 \in \overline{J_{k,-}} \cap \overline{J_{k,+}}$,

$$\begin{aligned} Th_{k-1}(x_0) = Tg_k(x_0) &= Th_k(x_0), \\ Tf(x_0) = Th_1(x_0) &= \cdots = Th_n(x_0) = Th(x_0). \end{aligned}$$

However, h only depends on x_0, $f(x_0)$, $f'(x_0)$, and $f''(x_0)$, the latter identified with a bilinear form, a matrix, or an element of $\mathbb{R}^{\binom{n+1}{2}}$. Therefore, there is a function $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}$ such that for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x_0 \in \mathbb{R}^n$ we have

$$Tf(x_0) = F_{x_0}\big(f(x_0),\, f'(x_0),\, f''(x_0)\big).$$

A similar argument shows that there is a function $B : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^m$ such that for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $x_0 \in \mathbb{R}^n$, we have

$$Af(x_0) = B_{x_0}\big(f(x_0),\, f'(x_0),\, f''(x_0)\big).$$

Note, however, that now B is $\mathbb{R}^m$-valued. $\square$

Remark. If A does not satisfy the nondegeneracy condition, T and A may fail to be pointwise localized. Take $n = m = 1$ in Proposition 4 and define $T, A : C^2(\mathbb{R}) \to C(\mathbb{R})$ by $Tf(x) := -f(x) + f(x+1)$, $Af(x) := f(x) - f(x+1)$. Then T and A satisfy (1) but are not localized at only one point x. A is degenerate and the operator equation (1) by itself does not guarantee pointwise localization.

§3. Functional equations

A function $d : \mathbb{R}^m \to \mathbb{R}$ is additive if

$$d(\alpha + \beta) = d(\alpha) + d(\beta)$$

for all $\alpha, \beta \in \mathbb{R}^m$. For $i \in \{1, \dots, m\}$, let $d_i(\alpha_i) := d(0, \dots, 0, \underbrace{\alpha_i}_{i\text{th position}}, 0, \dots, 0)$. Then, obviously, $d_i : \mathbb{R} \to \mathbb{R}$ is additive and $d(\alpha) = \sum_{i=1}^n d_i(\alpha_i)$. Additive measurable functions are linear by a result of Banach and Sierpinski, cf. Aczél [A, 2.1]. In general, additive functions are not linear. In the following, we write $d[\alpha]$ for additive functions of $\alpha \in \mathbb{R}^m$ that have not yet been shown to be linear. The following is a generalization of Lemma 6 in [KM2] to the case where $m > 1$.

Lemma 5. *Let $m \in \mathbb{N}$. Assume that $G, H : \mathbb{R}^m \to \mathbb{R}$ are functions such that*

$$G(\alpha + \beta) = G(\alpha) + G(\beta) + H(\alpha)H(\beta) \tag{4}$$

for any $\alpha, \beta \in \mathbb{R}^m$. Then there are additive functions $b, d : \mathbb{R}^m \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that G and H are of one of the following forms:

(a) $G(\alpha) = -\gamma^2 + b[\alpha]$, $H(\alpha) = \gamma$;
(b) $G(\alpha) = \frac{1}{2}(d[\alpha])^2 + b[\alpha]$, $H(\alpha) = d[\alpha]$;
(c) $G(\alpha) = \gamma^2(e^{d[\alpha]} - 1) + b[\alpha]$, $H(\alpha) = \gamma(e^{d[\alpha]} - 1)$.

The proof is an extension of that of Lemma 6 in [KM2] for $m = 1$, which itself was a generalization of Proposition 3.1.3 in Aczél [A]. For completeness, we give the proof of Lemma 5 in the Appendix. We also restate Lemma 7 from [KM2] for later use ($m = 1$); it is a multiplicative analog of the previous lemma and follows from it.

Lemma 6. *Assume $F, B : \mathbb{R} \to \mathbb{R}$ are functions such that*

$$F(\alpha\beta) = F(\alpha)\beta + F(\beta)\alpha + B(\alpha)\, B(\beta)$$

for any $\alpha, \beta \in \mathbb{R}$. Then there are additive functions $a, d : \mathbb{R} \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that F and B are of one of the following forms:

(a) $F(\alpha) = \alpha(-\gamma^2 + a[\ln|\alpha|])$, $B(\alpha) = \alpha\gamma$;
(b) $F(\alpha) = \alpha\left(\frac{1}{2}(d[\ln|\alpha|])^2 + a[\ln|\alpha|]\right)$, $B(\alpha) = \alpha d[\ln|\alpha|]$;
(c) $F(\alpha) = \alpha\left(\gamma^2(\{\operatorname{sgn}\alpha\}e^{d[\ln|\alpha|]} - 1) + a[\ln|\alpha|]\right)$, $B(\alpha) = \alpha\gamma(\{\operatorname{sgn}\alpha\}e^{d[\ln|\alpha|]} - 1)$.

In (c) *there are two possibilities, one with the term* $\operatorname{sgn}\alpha$ *and one without. This is indicated by the bracket* $\{\dots\}$.

The following proposition of Faifman was proved in [KM1] for $n = 1$; the extension to $n > 1$ is immediate by the same arguments.

Proposition 7. *Let $n \in \mathbb{N}$, $d \in \mathbb{N}$, and let $H_j : \mathbb{R}^n \times \mathbb{R}^{\binom{n+j-1}{j}} \to \mathbb{R}$ be a family of functions additive in the second variables,*

$$H_j(x, \alpha + \beta) = H_j(x, \alpha) + H_j(x, \beta) \ \textit{for all}\ \alpha, \beta \in \mathbb{R}^{\binom{n+j-1}{j}},$$

for $j = 0, \dots, d-1$. Assume that for any function $g \in C^{d-1}(\mathbb{R}^n, \mathbb{R})$

$$H_0(x, g(x)) + H_1(x, g'(x)) + \cdots + H_{d-1}(x, g^{(d-1)}(x))$$

is continuous in $x \in \mathbb{R}^n$. *The choice of the dimension* $\binom{n+j-1}{j}$ *reflects the fact that, because of symmetry,* $g^{(j)}(x)$ *is determined by* $\binom{n+j-1}{j}$ *partial derivatives whose values might be chosen independently. Then for all* $j \in \{0, \dots, d-1\}$, $H_j(x, \alpha) = \langle c_j(x), \alpha \rangle$ *is linear with respect to the second variable* $\alpha \in \mathbb{R}^{\binom{n+j-1}{j}}$ *with continuous coefficients* $c_j \in C\big(\mathbb{R}^n, \mathbb{R}^{\binom{n+j-1}{j}}\big)$.

§4. Proof of Theorem 2

Theorem 2 is a generalization of Theorem 1 of [KM2], where the case of $m = 1$ was considered. We give a modified and simplified proof for its extension to $m > 1$, which does not require the weak continuity of A assumed there. Then Theorem 2 will be used to prove Theorem 1.

Proof of Theorem 2.

(i) We study the operator solutions of the equation

$$T(fg) = Tf \cdot g + f \cdot Tg + Af \cdot Ag; \quad f, g \in C^2(\mathbb{R}^n, \mathbb{R}), \tag{2}$$

where T and A are maps $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$, $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and A is nondegenerate. By Proposition 4, T and A have the form

$$Tf(x) = F_x(f(x), f'(x), f''(x)), \quad Af(x) = B_x(f(x), f'(x), f''(x)), \tag{5}$$

where F and B are unknown functions.

Suppose $f \in C^2(\mathbb{R}^n, \mathbb{R})$ is nonzero at $x \in \mathbb{R}^n$. Then

$$\begin{aligned} f(x) &= \operatorname{sgn} f(x) \exp(\ln|f|)(x), \\ f'(x) &= f(x)(\ln|f|)'(x), \\ f''(x) &= f(x)\big((\ln|f|)''(x) + \big((\ln|f|)' \otimes (\ln|f|)'\big)(x)\big). \end{aligned}$$

Here we view $f''(x)$ and $(\ln|f|)''(x)$ as bilinear maps from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$, and $\big((\ln|f|)' \otimes (\ln|f|)'\big)(x)$ denotes the bilinear map

$$(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle u, (\ln|f|)'(x) \rangle \langle v, (\ln|f|)'(x) \rangle,$$

i.e., the product of the directional derivatives in the directions u and v. Hence, the $f^{(i)}(x)$, $i = 0, 1, 2$, are functions of $\operatorname{sgn} f(x)$ and $(\ln|f|)^{(j)}(x)$ for $j = 0, 1, 2$. Since by (5), $Tf(x)/f(x)$ and $Af(x)/f(x)$ are functions of x and $f^{(i)}(x)$ for $i = 0, 1, 2$, we conclude that there are functions $\widetilde{G} : \{\pm 1\} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}$ and $\widetilde{H} : \{\pm 1\} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}$ such that

$$\begin{aligned} Tf(x)/f(x) &= \widetilde{G}_x(\operatorname{sgn} f(x),\ (\ln|f|)(x),\ (\ln|f|)'(x),\ (\ln|f|)''(x)), \\ Af(x)/f(x) &= \widetilde{H}_x(\operatorname{sgn} f(x),\ (\ln|f|)(x),\ (\ln|f|)'(x),\ (\ln|f|)''(x)) \end{aligned} \tag{6}$$

for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $f(x) \neq 0$.

Let $g \in C^2(\mathbb{R}^n, \mathbb{R})$ be arbitrary and put $f := \exp(g)$. Then $f \in C^2$, $f > 0$ and, omitting $\operatorname{sgn} f(x) = +1$ and the tilde above G_x and H_x in (6), we see that

$$\begin{aligned} Tf(x)/f(x) &= G_x\big(g(x), g'(x), g''(x)\big), \\ Af(x)/f(x) &= H_x\big(g(x), g'(x), g''(x)\big) \end{aligned} \tag{7}$$

are continuous for all $g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$, by using the fact that $\operatorname{Im}(T), \operatorname{Im}(A) \subset C(\mathbb{R}^n, \mathbb{R})$.

For $g_1, g_2 \in C^2(\mathbb{R}^n, \mathbb{R})$, let $f_1 = \exp(g_1)$, $f_2 = \exp(g_2)$. Then, by (2),

$$\frac{T(f_1 f_2)}{f_1 f_2} = \frac{Tf_1}{f_1} + \frac{Tf_2}{f_2} + \frac{Af_1}{f_1}\,\frac{Af_2}{f_2},$$

which means by (7) that

$$\begin{aligned} &G_x\big((g_1+g_2)(x),(g_1+g_2)'(x),(g_1+g_2)''(x)\big)\\ &\quad= G_x\big(g_1(x),\, g_1'(x),\, g_1''(x)\big) + G_x\big(g_2(x),\, g_2'(x),\, g_2''(x)\big)\\ &\qquad\qquad + H_x\big(g_1(x),\, g_1'(x),\, g_1''(x)\big)\cdot H_x\big(g_2(x),\, g_2'(x),\, g_2''(x)\big). \end{aligned}$$

For arbitrary $\alpha_0,\beta_0 \in \mathbb{R}$, $\alpha_1,\beta_1 \in \mathbb{R}^n$, and $\alpha_2,\beta_2 \in \mathbb{R}^{\binom{n+1}{2}}$ and $x \in \mathbb{R}^n$, there are functions $g_1, g_2 \in C^2(\mathbb{R}^n,\mathbb{R})$ such that

$$\alpha_0 = g_1(x),\;\; \beta_0 = g_2(x),\;\; \alpha_1 = g_1'(x),\;\; \beta_1 = g_2'(x),\;\; a_2 = g_1''(x),\;\; \beta_2 = g_2''(x),$$

with a suitable fixed ordering of the $\binom{n+1}{2}$ different second partial derivatives. Therefore, we know that for all α_i, β_i as above

$$\begin{aligned} G_x(\alpha_0+\beta_0,\, \alpha_1+\beta_1,\, \alpha_2+\beta_2) &= G_x(\alpha_0,\alpha_1,\alpha_2) + G_x(\beta_0,\beta_1,\beta_2)\\ &\qquad + H_x(\alpha_0,\alpha_1,\alpha_2)\, H_x(\beta_0,\beta_1,\beta_2). \end{aligned} \tag{8}$$

(ii) Equation (8) has the form of equation (4) considered in Lemma 5, with $m = 1+n+\binom{n+1}{2}$. The additive functions in Lemma 5, e.g., $b : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}$ split into a sum of three additive functions $b_0 : \mathbb{R}\to\mathbb{R}$, $b_1 : \mathbb{R}^n \to \mathbb{R}$, $b_2 : \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}$, i.e., $b[\alpha_0,\alpha_1,\alpha_2] = b_0[\alpha_0] + b_1[\alpha_1] + b_2[\alpha_2]$.

Therefore, Lemma 5 implies that for any $x \in \mathbb{R}^n$ there are additive functions $a_x, d_x : \mathbb{R}\to\mathbb{R}$, $b_x, c_x : \mathbb{R}^n \to \mathbb{R}$ and $j_x, k_x : \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}$ and a constant $\gamma_x \in \mathbb{R}$ such that G and H are of one of the following forms:

$$\begin{aligned} &\text{(a)}\quad G_x(\alpha_0,\alpha_1,\alpha_2) = \frac{1}{2}(d_x[\alpha_0]+c_x[\alpha_1]+k_x[\alpha_2])^2 + a_x[\alpha_0] + b_x[\alpha_1] + j_x[\alpha_2],\\ &\qquad\;\; H_x(\alpha_0,\alpha_1,\alpha_2) = d_x[\alpha_0]+c_x[\alpha_1]+k_x[\alpha_2];\\ &\text{(b)}\quad G_x(\alpha_0,\alpha_1,\alpha_2) = \gamma_x^2\big(e^{d_x[\alpha_0]+c_x[\alpha_1]+k_x[\alpha_2]}-1\big) + a_x[\alpha_0] + b_x[\alpha_1] + j_x[\alpha_2],\\ &\qquad\;\; H_x(\alpha_0,\alpha_1,\alpha_2) = \gamma_x\big(e^{d_x[\alpha_0]+c_x[\alpha_1]+k_x[\alpha_2]}-1\big);\\ &\text{(c)}\quad G_x(\alpha_0,\alpha_1,\alpha_2) = -\gamma_x^2 + a_x[a_0] + b_x[a_1] + j_x[a_2],\\ &\qquad\;\; H_x(\alpha_0,\alpha_1,\alpha_2) = \gamma_x. \end{aligned} \tag{9}$$

By (7), in case (c) we have $Af(x) = \gamma_x f(x)$, i.e., A is a multiple of the identity and thus is not nondegenerate, as required. Hence, this case will not be considered any further. By (7), we also know that for all $g \in C^2(\mathbb{R}^n,\mathbb{R})$ the function $x \mapsto H_x(g(x),g'(x),g''(x))$ is continuous. In (b), choose g to be an appropriate constant, linear, or quadratic function to have $H_x(g(x),g'(x),g''(x)) = \gamma_x$ (or zero), and hence $e(x) := \gamma_x$ is continuous in x as well. Dividing by $e(x)$ if nonzero and adding 1 shows in (b) that $x \mapsto d_x[g(x)] + c_x[g'(x)] + k_x[g''(x)]$ is continuous for all $g \in C^2(\mathbb{R}^n,\mathbb{R})$; in (a) the same is implied directly by (7) and (9). Therefore, Proposition 7 of Faifman shows that $d_x[\alpha_0] = d(x)\alpha_0$, $c_x[\alpha_1] = \langle c(x),\alpha_1\rangle$, and $k_x[\alpha_2] = \langle k(x),\alpha_2\rangle$, where $d : \mathbb{R}^n \to \mathbb{R}$, $c : \mathbb{R}^n \to \mathbb{R}^n$, and $k : \mathbb{R}^n \to \mathbb{R}^{\binom{n+1}{2}}$ are continuous functions. In particular,

$$k_x[g''(x)] = \langle k(x), g''(x)\rangle = \sum_{1\le i\le j\le n} \frac{\partial^2 g}{\partial x_i \partial x_j}(x)\, k_{ij}(x)$$

with $k_{ij} \in C(\mathbb{R}^n,\mathbb{R})$. Symmetrizing, we may also extend the double sum over all $1 \le i,j \le n$.

We know as well that in cases (a) and (b) $x \mapsto G_x(g(x),g'(x),g''(x))$ is continuous for all $g \in C^2(\mathbb{R}^n,\mathbb{R})$. Subtracting the terms involving d_x, c_x, k_x, and γ_x, which we know already to depend continuously on x, in both remaining cases we see that $x \mapsto a_x[g(x)] + b_x[g'(x)] + j_x[g''(x)]$ is continuous, and hence, again by Proposition 7, that

$a_x[\alpha_0] = a(x)\alpha_0$, $b_x[\alpha_1] = \langle b(x), \alpha_1\rangle$ and $j_x[\alpha_2] = \langle j(x), \alpha_2\rangle$ for suitable continuous functions $a : \mathbb{R}^n \to \mathbb{R}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $j : \mathbb{R}^n \to \mathbb{R}^{\binom{n+1}{2}}$.

(iii) We have identified the various functions appearing in (9) to be linear in the α_j's and continuous in x. Let

$$K_x(\alpha_0, \alpha_1, \alpha_2) := G_x(\alpha_0, \alpha_1, \alpha_2) - L_x(\alpha_0, \alpha_1, \alpha_2),$$

where

$$L_x(\alpha_0, \alpha_1, \alpha_2) := a(x)\alpha_0 + \langle b(x), \alpha_1\rangle + \langle j(x), \alpha_2\rangle. \tag{10}$$

Then (9) implies that K_x and H_x are of one of the following two forms:

$$\begin{aligned} &\text{(a)} \quad H_x(\alpha_0, \alpha_1, \alpha_2) = d(x)\alpha_0 + \langle c(x), \alpha_1\rangle + \langle k(x), \alpha_2\rangle, \\ &\qquad\quad K_x(\alpha_0, \alpha_1, \alpha_2) = \frac{1}{2}H_x(\alpha_0, \alpha_1, \alpha_2)^2; \\ &\text{(b)} \quad H_x(\alpha_0, \alpha_1, \alpha_2) = e(x)\big(e^{d(x)\alpha_0 + \langle c(x), \alpha_1\rangle + \langle k(x), \alpha_2\rangle} - 1\big), \\ &\qquad\quad K_x(\alpha_0, \alpha_1, \alpha_2) = e(x)H_x(\alpha_0, \alpha_1, \alpha_2). \end{aligned} \tag{11}$$

We return to function values. Our choice was $\alpha_0 = g(x) = \ln|f(x)|$ for $f(x) \neq 0$. Therefore, $\alpha_1 = g'(x) = \frac{f'(x)}{f(x)}$, $\alpha_2 = g''(x) = \frac{f''(x)}{f(x)} - \frac{f'(x)\otimes f'(x)}{f(x)^2}$. As before, $f'(x) \otimes f'(x)$ denotes the bilinear form $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle u, f'(x)\rangle\langle v, f'(x)\rangle$. Using (11) and the representation (7) of A by H, for *positive* functions $f \in C^2(\mathbb{R}^n, \mathbb{R})$ we find that one of the following holds true:

$$\begin{aligned} &\text{(a)} \quad (Af)(x) = d(x)f(x)\ln|f(x)| + \langle c(x), f'(x)\rangle \\ &\qquad\qquad + \Big\langle k(x), f''(x) - \frac{f'(x)\otimes f'(x)}{f(x)}\Big\rangle, \\ &\text{(b)} \quad (Af)(x) = e(x)f(x)\Bigg(\exp\Big(d(x)\ln|f(x)| + \Big\langle c(x), \frac{f'(x)}{f(x)}\Big\rangle \\ &\qquad\qquad + \Big\langle k(x), \frac{f''(x)}{f(x)} - \frac{f'(x)\otimes f'(x)}{f(x)^2}\Big\rangle\Big) - 1\Bigg). \end{aligned} \tag{12}$$

Since A acts on all $C^2(\mathbb{R}^n, \mathbb{R})$-functions, including those with values $f(x) = 0$, cases (a) , (b) of (12) will only yield well-defined formulas for all f on $C^2(\mathbb{R}^n, \mathbb{R})$ with Af being continuous if they are well defined for $f(x) = 0$. This shows that $k(x) = 0$ is required in case (a), and that also $c(x) = 0$ in case (b). Therefore, in (11), H_x and K_x do not, in fact, depend on α_2, and writing $H_x(\alpha_0, \alpha_1)$, $K_x(\alpha_0, \alpha_1)$ only we have the simplified formulas

$$\begin{aligned} &\text{(a)} \quad H_x(\alpha_0, \alpha_1) = d(x)\alpha_0 + \langle c(x), \alpha_1\rangle, \quad K_x(\alpha_0, \alpha_1) = \frac{1}{2}H_x(\alpha_0, \alpha_1)^2; \\ &\text{(b)} \quad H_x(\alpha_0, \alpha_1) = e(x)(e^{d(x)\alpha_0} - 1), \quad K_x(\alpha_0, \alpha_1) = e(x)H_x(\alpha_0, \alpha_1). \end{aligned} \tag{13}$$

(iv) Formula (7) describes Tf/f in terms of $G_x = L_x + K_x$. Therefore, we can write $Tf(x) = S'f(x) + T'f(x)$, where

$$\begin{aligned} &S'f(x) = a(x)f(x)\ln|f(x)| + \langle b(x), f'(x)\rangle + \Big\langle j(x), f''(x) - \frac{f'(x)\otimes f'(x)}{f(x)}\Big\rangle, \\ &T'f(x) = f(x)K_x((\ln|f|)(x), (\ln|f|)'(x)). \end{aligned} \tag{14}$$

The formula for $S'f$ gives all solutions of the homogeneous equation when $A = 0$, and the third term is one that only shows up when the functions on which T acts are never zero. Formulas (13) yield the following possibilities for $T'f(x)$ with $f(x) > 0$:

(a) $T'f(x) = \frac{1}{2}f(x)(d(x)\ln|f(x)| + \langle c(x), \frac{f'(x)}{f(x)}\rangle)^2$;

(b) $T'f(x) = e(x)^2 f(x)\,(|f(x)|^{d(x)} - 1)$.

Case (b) yields a well-defined formula for $T'f$, with $T'f$ being continuous also in the case of $f(x) = 0$, provided that $d(x) \geq -1$. However, the continuity of $Tf = S'f + T'f \in C(\mathbb{R}^n, \mathbb{R})$ in this case requires, by (13) and (14), that $j(x) = 0$. In case (a), since

$$\langle c(x), f'(x)\rangle^2 = \langle c(x) \otimes c(x), f'(x) \otimes f'(x)\rangle,$$

we can guarantee that Tf is continuous for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$ only if

$$\Big\langle j(x), f''(x) - \frac{f'(x) \otimes f'(x)}{f(x)}\Big\rangle + \frac{1}{2}\Big\langle c(x) \otimes c(x), \frac{f'(x) \otimes f'(x)}{f(x)}\Big\rangle + d(x)\langle c(x), f'(x)\rangle \ln|f(x)|$$

is well defined also when $f(x) = 0$, which requires that $j(x) = \frac{1}{2}\, c(x) \otimes c(x)$ and $d(x) = 0$ or $c(x) = 0$. This way we get two formulas for the operators T and A; taken together with case (b), we have shown that there are three possible cases for T and A, namely,

$$\begin{aligned} Tf(x) &= Sf(x) + T_1 f(x), \\ Sf(x) &= a(x) f(x) \ln|f(x)| + \langle b(x), f'(x)\rangle, \end{aligned}$$

$$\text{(15)} \qquad \begin{aligned} &\text{(a)} \quad T_1 f(x) = \frac{1}{2}\langle c(x) \otimes c(x), f''(x)\rangle,\ Af(x) = \langle c(x), f'(x)\rangle; \\ &\text{(b)} \quad T_1 f(x) = \frac{1}{2} d(x)^2 f(x)(\ln|f(x)|)^2,\ Af(x) = d(x) f(x) \ln|f(x)|; \\ &\text{(c)} \quad T_1 f(x) = e(x) Af(x),\ Af(x) = e(x) f(x)(|f(x)|^{d(x)} - 1). \end{aligned}$$

In (a), we may write also

$$T_1 f(x) = \frac{1}{2} f''(x)(c(x), c(x)) = \frac{1}{2}\langle f''(x) c(x), c(x)\rangle,$$

viewing the action of $f''(x)$ as bilinear or given by the Hessian matrix.

(v) We wrote $|f(x)|$ even though $f(x) \geq 0$ was assumed. As written, the formulas remain true if $f(x) < 0$, except for one case that must be added to (c) in (15): we know that only $\operatorname{sgn} f(x)$ might additionally affect the solutions. For constant functions $f(x) = \alpha, g(x) = \beta$, in terms of F and B equation (2) means that

$$F_x(\alpha\beta, 0, 0) = F_x(\alpha, 0, 0)\beta + F_x(\beta, 0, 0)\alpha + B_x(\alpha, 0, 0) B_x(\beta, 0, 0).$$

The solutions of this equation are given by Lemma 6, with $\alpha = f(x)$. Parts (b) and (c) correspond to one another. Part (c) of Lemma 6 gives a *possible* additional factor $\operatorname{sgn} f(x)$ for $|f(x)|^{d(x)}$ so that (c) should be replaced by the more general formulas

(c′) $T_1 f(x) = e(x)^2 f(x)(\{\operatorname{sgn} f(x)\}\,|f(x)|^{d(x)} - 1)$,
$Af(x) = e(x) f(x)(\{\operatorname{sgn} f(x)\}\,|f(x)|^{d(x)} - 1)$.

Clearly, $d(x) \geq -1$ is needed so that $T_1 f, Af \in C(\mathbb{R}^n, \mathbb{R})$.

(vi) The extension of the formulas for T and A from the nonzero functions in $C^2(\mathbb{R}^n, \mathbb{R})$ to all of $C^2(\mathbb{R}^n, \mathbb{R})$ is unique. For $x_0 \in \mathbb{R}^n$ and $f \in C^2(\mathbb{R}^n, \mathbb{R})$ with $\alpha_0 = f(x_0) = 0$, we assume that f is nonzero in $J \setminus \{x_0\}$ for a suitable open set $J \subset \mathbb{R}^n$ with $x_0 \in J$. Thus for $x \in J \setminus \{x_0\}$,

$$(Tf)(x) = F_x\big(f(x),\, f'(x),\, f''(x)\big),$$

because the proofs of Propositions 3 and 4 involved only local arguments. Since the coefficients appearing in the formulas for Tf, i.e., in F, were shown to be continuous, and since Tf is assumed to be continuous, we conclude that

$$(Tf)(x_0) = F_x\big(f(x_0),\, f'(x_0),\, f''(x_0)\big),$$

i.e., $(Tf)(x_0)$ is expressed by the same formulas for $f(x_0) = 0$ as for $f(x) \neq 0$, and the extension is unique. This is also true if the zero set of f has x_0 as an accumulation point and, in particular, if f is zero in a neighborhood of x_0, in which case both sides of the last equation are zero. The same statement applies to $(Af)(x_0)$. This ends the proof of Theorem 2. □

§5. The Laplace operator

Now we show that the Laplace operator is characterized by the operator equation (1), orthogonal invariance, and annihilation of affine functions.

Proof of Theorem 1.

(i) We assume that $T : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$ and $A : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}^n)$ satisfy the operator equation

$$T(f \cdot g)(x) = (Tf)(x) \cdot g(x) + f(x) \cdot (Tg)(x) + \langle (Af)(x), (Ag)(x) \rangle \tag{1}$$

for $f, g \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$, and that A is nondegenerate. For $i \in \{1, \dots, n\}$, define the component operators $A_i : C^2(\mathbb{R}^n, \mathbb{R})$ by $(Af)(x) = ((A_i f)(x))_{i=1}^n \in \mathbb{R}^n$. Consider the equation for an operator $T_1 : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$,

$$T_1(fg)(x) = (T_1 f)(x) \cdot g(x) + f(x) \cdot (T_1 g)(x) + (A_1 f)(x) \cdot (A_1 g)(x),$$

the general solution of which is given by Theorem 2, and subtract this from (1) to get

$$\widetilde{T}(f \cdot g)(x) = (\widetilde{T} f)(x) \cdot g(x) + f(x) \cdot (\widetilde{T} g)(x) + \langle (\widetilde{A} f)(x), (\widetilde{A} g)(x) \rangle$$

with $\widetilde{T} f := Tf - T_1 f$, $\widetilde{A} f := (0, A_2 f, \dots, A_n f)$.

Continuing inductively, we see that (1) decomposes into n scalar equations

$$T_i(f \cdot g)(x) = (T_i f)(x) \cdot g(x) + f(x) \cdot (T_i g)(x) + (A_i f)(x) \cdot (A_i g)(x) \tag{2}$$

for $i = 1, \dots, n$ with $Tf(x) = \sum_{i=1}^n (T_i f)(x)$, $Af(x) = (A_i f(x))_{i=1}^n$, $T_i : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$, $A_i : C^2(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R})$; the general solution of (1) is given by the sum of general solutions of (2), which we know by Theorem 2.

The homogeneous part of the solution, i.e., the general solution of

$$S(f \cdot g)(x) = (Sf)(x) \cdot g(x) + f(x) \cdot (Sg)(x)$$

is of the form $(Sf)(x) = \langle b(x), f'(x) \rangle + a(x) f(x) \ln |f(x)|$. Combining this with sums of terms from the second or third type of operators in Theorem 2 only yields expressions involving $f(x)$, except for the part $\langle b(x), f'(x) \rangle$ in $Sf(x)$ involving $f'(x)$. All terms involving $f(x)$ only are quickly seen to be zero by the assumption that T is zero on all constant functions, for which $f'(x) = 0$, $f''(x) = 0$. Since T is also assumed to be zero on all linear functions $f(x) = \langle w, x \rangle$, we see that $b(x) = 0$ as well, i.e., the term involving $f'(x)$ is zero. Therefore, under the additional assumption that T annihilates the affine functions, the general solution of (1) is given by

$$Tf(x) = \frac{1}{2} \sum_{i=1}^n \langle f''(x) c_i(x), c_i(x) \rangle, \quad Af(x) = (\langle f'(x), c_i(x) \rangle)_{i=1}^n \tag{16}$$

for suitable functions $c_i \in C(\mathbb{R}^n, \mathbb{R}^n)$. Hence, T is a sum of multiples of second order directional derivatives, and A a sequence of multiples of the corresponding first order directional derivatives.

(ii) Assume that T is $O(n)$-invariant, i.e., $T(f \circ u) = (Tf) \circ u$ for all orthogonal transformations $u \in O(n)$. By the chain rule we have $(f \circ u)''(x) = u^*(f''(u(x))u)$

interpreting the Hessian $f''(u(x)) \in L(\mathbb{R}^n, \mathbb{R}^n)$ as a matrix. Hence, using (16), we obtain

$$2T(f \circ u)(x) = \sum_{i=1}^{n} \langle (f \circ u)''(x) c_i(x),\, c_i(x) \rangle = \sum_{i=1}^{n} \langle f''(u(x)) u(c_i(x)),\, u(c_i(x)) \rangle$$
$$2((Tf) \circ u)(x) = \sum_{i=1}^{n} \langle f''(u(x))\, c_i(u(x)),\, c_i(u(x)) \rangle.$$

Since $T(f \circ u) = (Tf) \circ u$ is required for all $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and all $u \in O(n)$, given u, $f''(u(x))$ can be an arbitrary preassigned matrix B. Therefore, for any $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ and any $u \in O(n)$, $x \in \mathbb{R}^n$, we require that

$$\sum_{i=1}^{n} \langle B\, c_i(u(x)),\, c_i(u(x)) \rangle = \sum_{i=1}^{n} \langle B\, u(c_i(x)),\, u(c_i(x)) \rangle. \tag{17}$$

Let $c_i(x) := (c_{ip}(x))_{p=1}^n$ and $C(x) := (c_{ip}(x))_{i,p=1}^n \in L(\mathbb{R}^n, \mathbb{R}^n)$. Then (17) is equivalent to

$$\operatorname{trace}(B\, C^*(u(x)) C(u(x)) = \operatorname{trace}(B\, u\, C^*(x) C(x) u^*)$$

for all $B \in L(\mathbb{R}^n, \mathbb{R}^n)$, and hence, for all $u \in O(n)$,

$$C^*(u(x)) C(u(x)) = u\, C^*(x) C(x) u^*. \tag{18}$$

For a fixed $x \in \mathbb{R}^n$, consider $O_{n-1} := \{u \in O(n) \mid u(x) = x\} \simeq O(n-1)$. Any $u \in O_{n-1}$ maps $H := x^\perp$ into itself and O_{n-1} acts transitively on H. By (18), any $u \in O(n-1)$ commutes with $C^*(x)C(x)$, and therefore, $C^*(x)C(x)|_H$ is a multiple of the identity on H and also maps x into a multiple of x. Hence, for some $\lambda(x), \mu(x) \in \mathbb{R}$ we have

$$C^*(x)C(x) = \lambda(x)\, \mathrm{Id} + \mu(x) P_x,$$

where $P_x : \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto x; $P_x = \left\langle \cdot, \frac{x}{\|x\|} \right\rangle \frac{x}{\|x\|}$ $(x \neq 0)$. By (16), with $\Delta f(x) = \operatorname{trace}(f''(x))$,

$$\begin{aligned} Tf(x) &= \frac{1}{2} \operatorname{trace}(f''(x) C^*(x) C(x)) = \frac{\lambda(x)}{2} \operatorname{trace}(f''(x)) + \frac{\mu(x)}{2} \operatorname{trace}(f''(x) P_x) \\ &= \frac{\lambda(x)}{2} (\Delta f)(x) + \frac{\mu(x)}{2} \frac{\langle f''(x)x, x \rangle}{\|x\|^2} = \frac{\lambda(x)}{2} (\Delta f)(x), \end{aligned}$$

where we have used the fact that $f''(x)$ is zero on the linear functions. Hence, (1) implies that

$$\langle Af(x), Ag(x) \rangle = T(f \cdot g)(x) - (Tf)(x) \cdot g(x) - f(x) \cdot Tg(x) = \frac{\lambda(x)}{2} \langle f'(x),\, g'(x) \rangle.$$

This means that $\lambda(x) \geq 0$ and $Af(x) = \sqrt{\lambda(x)}\, f'(x)$. Clearly, $\lambda \in C(\mathbb{R}^n, \mathbb{R})$ is continuous because T has image in $C(\mathbb{R}^n, \mathbb{R})$. Since both T and the Laplacian are $O(n)$-invariant, so must be λ. This means that there is a continuous function $d : [0, \infty) \to \mathbb{R}$ such that $\sqrt{\lambda(x)} = d(\|x\|)$. Therefore,

$$Tf(x) = \frac{d(\|x\|)^2}{2} (\Delta f)(x), \quad Af(x) = d(\|x\|) f'(x).$$

This ends the proof of Theorem 1. □

Appendix

Proof of Lemma 5. (i) If $H = 0$, G is additive and we are in case (a) with $\gamma = 0$. Thus, assume that $H \neq 0$ and choose $b \in \mathbb{R}^m$ with $H(b) \neq 0$. For $\alpha \in \mathbb{R}^m$, let

$$g(\alpha) := G(\alpha + b) - G(\alpha) - G(b), \quad h(\alpha) := H(\alpha + b) - H(\alpha).$$

Equation (4) means that

$$g(\alpha + \beta) = g(\alpha) + h(\alpha)H(\beta) \tag{19}$$

for all $\alpha, \beta \in \mathbb{R}^m$.

Hence, for $\alpha = 0$, we have $g(\beta) - g(0) = h(0)H(\beta)$. Inserting this back into (19), we find

$$h(0)(H(\alpha + \beta) - H(\alpha)) = h(\alpha)H(\beta). \tag{20}$$

If $h(0) = 0$, (20) yields $h = 0$ because $H(b) \neq 0$, and $g = g(0)$ is constant. Note that, by (4), $g(\alpha) = H(b)H(\alpha)$. Thus, also H is constant, $H = g(0)/H(b) =: \gamma$. Let $d(\alpha) := G(\alpha) + \gamma^2$. Then, by (4),

$$d(\alpha + \beta) = G(\alpha + \beta) + \gamma^2 = (G(\alpha) + G(\beta) + \gamma^2) + \gamma^2 = d(\alpha) + d(\beta),$$

i.e., d is additive on $\mathbb{R}^m$ and G and H are of the form given in (a).

(ii) Now, assume that $h(0) \neq 0$. Then putting $\alpha = 0$ in (20), we see that $H(0) = 0$. Moreover,

$$H(\alpha + \beta) = H(\alpha) + \frac{h(\alpha)}{h(0)} H(\beta). \tag{21}$$

First, we consider the case where $h = h(0)$ is a constant function. Then $H(\alpha) = c[\alpha]$ is additive, and $F(\alpha) := G(\alpha) - \frac{1}{2}(c[\alpha])^2$ satisfies

$$\begin{aligned} F(\alpha + \beta) &= G(\alpha + \beta) - \frac{1}{2}(c[\alpha] + c[\beta])^2 \\ &= (G(\alpha) + G(\beta) + H(\alpha)H(\beta)) - \frac{1}{2}(c[\alpha])^2 - \frac{1}{2}(c[\beta])^2 - c[\alpha]c[\beta] \\ &= F(\alpha) + F(\beta). \end{aligned}$$

Hence, $F(\alpha) = d[\alpha]$ is additive on $\mathbb{R}^m$, and G and H are of the form as in (b),

$$G(\alpha) = \frac{1}{2}(c[\alpha])^2 + d[\alpha], \quad H(\alpha) = c[\alpha].$$

(iii) Now assume that $h(0) \neq 0$ and that h is not constant. Choose $\alpha_0 \in \mathbb{R}^m$ with $h(\alpha_0) \neq h(0)$. Since the left-hand side of (21) is symmetric in α and β, we see that

$$H(\alpha) + \frac{h(\alpha)}{h(0)} H(\beta) = H(\beta) + \frac{h(\beta)}{h(0)} H(\alpha).$$

Hence, for $\beta = \alpha_0$, we have $H(\alpha) = \frac{H(\alpha_0)}{h(\alpha_0) - h(0)}(h(\alpha) - h(0))$, and by (19),

$$g(\alpha) - g(0) = h(0)H(\alpha) = \gamma(h(\alpha) - h(0)), \tag{22}$$

where $\gamma := h(0)H(\alpha_0)/(h(\alpha_0) - h(0))$. For $\gamma = 0$, i.e., $H(\alpha_0) = 0$, we again are in case (a). So we may assume that $\gamma \neq 0$. By (19) and (22),

$$\begin{aligned} \gamma(h(\alpha + \beta) - h(0)) &= g(\alpha + \beta) - g(0) = g(\alpha) - g(0) + h(\alpha)H(\beta) \\ &= \gamma(h(\alpha) - h(0)) + \frac{h(\alpha)}{h(0)} \gamma(h(\beta) - h(0)) \\ &= \gamma\Big(\frac{h(\alpha)h(\beta)}{h(0)} - h(0)\Big). \end{aligned}$$

Hence, $\widetilde{h}(\alpha) := h(\alpha)/h(0)$ satisfies $\widetilde{h}(\alpha+\beta) = \widetilde{h}(\alpha)\,\widetilde{h}(\beta)$ and $\widetilde{h}(\alpha) = \widetilde{h}\big(\frac{\alpha}{2}\big)^2 \geq 0$. Therefore, $c(\alpha) := \ln \widetilde{h}(\alpha)$ is additive and

$$h(\alpha) = h(0)e^{c[\alpha]}.$$

Then the formula for $H(\alpha)$ before (22) gives

$$H(\alpha) = \gamma\big(e^{c[\alpha]} - 1\big). \tag{23}$$

Put $F(\alpha) := G(\alpha) - \gamma^2(e^{c[\alpha]} - 1)$. Then (4) and (23) imply that $F(\alpha+\beta) = F(\alpha) + F(\beta)$, i.e., F is additive, $F(\alpha) = d[\alpha]$. Therefore,

$$G(\alpha) = \gamma^2(e^{c[\alpha]} - 1) + d[\alpha]$$

and this case yields case (c) of the lemma. □

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