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ESTIMATES FOR FUNCTIONALS WITH A KNOWN, FINITE SET OF MOMENTS, IN TERMS OF MODULI OF CONTINUITY, AND BEHAVIOR OF CONSTANTS, IN THE JACKSON-TYPE INEQUALITIES

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Abstract. A new technique is developed for estimating functionals by moduli of continuity. The generalized Jackson inequality

$$A_{\sigma-0}(f) \le \left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\gamma\pi)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}} \frac{\nu_m^m}{2^{2m}} \right\} \omega_{2m}\Big(f, \frac{\gamma\pi}{\sigma}\Big)$$

is an example of such an estimate. Here $r, m \in \mathbb{N}$, $\sigma, \gamma > 0$, a function f is uniformly continuous and bounded on $\mathbb{R}$, $A_{\sigma-0}$ is the best uniform approximation by entire functions of type less than σ, ω_{2m} is a uniform modulus of continuity of order $2m$, $\mathcal{K}_s$ are the Favard constants, and

$$\nu_m = \frac{8}{\binom{2m}{m}} \sum_{l=0}^{\lfloor (m-1)/2 \rfloor} \frac{\binom{2m}{m-2l-1}}{(2l+1)^2},$$

where $\lfloor x \rfloor$ is the entire part of x. Similar inequalities are obtained for best approximations of periodic functions by splines. In some cases, the constants in inequalities are close to optimal.

§1. Introduction

1.1. An overview of the results. In what follows, C is the space of 2π-periodic continuous functions with the uniform norm, $E_n(f)$ is the best approximation of a function f by trigonometric polynomials of order less than n,

$$\delta_t^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\Big(x + \frac{rt}{2} - kt\Big)$$

is the central difference of order r of f with step t, and

$$\omega_r(f, h) = \sup_{|t| \le h} \|\delta_t^r f\|$$

is the modulus of continuity of f with step h.

The approximation theory deals with a large number of inequalities of the type

$$\Phi(f) \le \sum_{k=1}^{N} D_k \Psi_k(f), \tag{1.1}$$

where Φ and Ψ_k are functionals defined on the space C or on its subset. The functionals Ψ_k in these inequalities are often associated with moduli of continuity. The question

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about the constants D_k in such inequalities is a subject of special investigation. As a rule, this question is very difficult. For this reason, it is often not discussed at all, or only very rough estimates for the constants are established.

The generalized Jackson theorem

$$E_n(f) \le D_r(\gamma)\,\omega_r\Big(f, \frac{\gamma\pi}{n}\Big) \tag{1.2}$$

serves as the classical example of a result of type (1.1). Now, using inequality (1.2) as an example, we shall discuss a method that allows us to establish estimates like (1.1) for a wide class of functionals Φ.

Let $S_h f$ be the 1st order Steklov function of a function f with step h, i.e.,

$$S_h f(x) = \frac{1}{h}\int_{-h/2}^{h/2} f(x+t)\,dt.$$

The operator S_h^r is called the rth order Steklov operator with step h; more often it is denoted by $S_{h,r}$ in the literature. The Steklov functions can be written as

$$S_h^r f = \frac{1}{h^r}\delta_h^r f^{(-r)},$$

where $f^{(-r)}$ is the rth primitive of f, or

$$S_h^r f(x) = \int_{\mathbb{R}} f(x+th)\psi_r(t)\,dt,$$

where ψ_r is the Steklov kernel with the following properties: $\psi_r \ge 0$, ψ_r is even, $\int_{\mathbb{R}}\psi_r = 1$, supp $\psi_r = \big[-\frac{r}{2}, \frac{r}{2}\big]$. Using the Steklov functions, we construct the approximating aggregate

$$\begin{aligned} S_{h,r,m}f = \frac{2}{\binom{2m}{m}}\sum_{j=1}^{m}(-1)^{j-1}\binom{2m}{m-j}S_{jh}^r f &= f + \frac{2(-1)^{m-1}}{\binom{2m}{m}}\int_{\mathbb{R}_+}\delta_{th}^{2m} f\psi_r(t)\,dt \\ &= \frac{2}{\binom{2m}{m}}\sum_{j=1}^{m}(-1)^{j-1}\frac{\binom{2m}{m-j}}{(jh)^r}\delta_{jh}^r f^{(-r)}. \end{aligned} \tag{1.3}$$

Clearly,

$$\big\|f - S_{h,r,m}f\big\| \le \frac{2}{\binom{2m}{m}}\int_{\mathbb{R}_+}\big\|\delta_{th}^{2m} f\big\|\psi_r(t)\,dt \le \frac{1}{\binom{2m}{m}}\omega_{2m}\Big(f, \frac{rh}{2}\Big). \tag{1.4}$$

A well-known method to prove the generalized Jackson inequality consists in using the operator $S_{h,2m,m}$ for intermediate approximation, see [1, p. 57, Theorem 3]. The last expression in (1.3) implies the estimate

$$\Big\|\big(S_{h,2m,m}f\big)^{(2m)}\Big\| \le \Big(\frac{2}{\binom{2m}{m}}\sum_{j=1}^{m}\frac{\binom{2m}{m-j}}{(jh)^{2m}}\Big)\omega_{2m}(f, mh).$$

We employ the following well-known inequalities, see [2, Theorem 4.1.4]:

$$E_n(f) \le \frac{\mathcal{K}_s}{n^s}E_n(f^{(s)}), \tag{1.5}$$

where

$$\mathcal{K}_s = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(s+1)}}{(2k+1)^{s+1}}$$

are the Favard constants. We get

$$\begin{aligned} E_n(f) &\le E_n(f - S_{h,2m,m}f) + E_n(S_{h,2m,m}f) \\ &\le \big\|f - S_{h,2m,m}f\big\| + \frac{\mathcal{K}_{2m}}{n^{2m}}\big\|(S_{h,2m,m}f)^{(2m)}\big\| \\ &\le \frac{1}{\binom{2m}{m}}\Bigg(1 + \frac{\mathcal{K}_{2m}}{(nh)^{2m}}\, 2\sum_{j=1}^{m}\frac{\binom{2m}{m-j}}{j^{2m}}\Bigg)\omega_{2m}(f, mh). \end{aligned}$$

Setting $h = \frac{\gamma\pi}{mn}$, we arrive at inequality (1.2) with constants expressed explicitly:

$$E_n(f) \le \frac{1}{\binom{2m}{m}}\Bigg(1 + \frac{\mathcal{K}_{2m}m^{2m}}{(\gamma\pi)^{2m}}\, 2\sum_{j=1}^{m}\frac{\binom{2m}{m-j}}{j^{2m}}\Bigg)\omega_{2m}\Big(f, \frac{\gamma\pi}{n}\Big). \tag{1.6}$$

The values of constants in (1.6) are often highly excessive.

In the present paper, instead of $S_{h,2m,m}$, we use the operator $S^m_{h,2,m}$. For brevity, we denote $U = S_{h,2,m}$. We write

$$f = \sum_{k=0}^{m-1} U^k(f - Uf) + U^m f.$$

By the triangle inequality,

$$E_n(f) \le \sum_{k=0}^{m-1} E_n\big(U^k(f - Uf)\big) + E_n(U^m f).$$

We estimate the each term, using inequalities (1.5) and the last expression in (1.3):

$$E_n\big(U^k(f - Uf)\big) \le \frac{\mathcal{K}_{2k}}{n^{2k}} E_n\Big(\big(U^k(f - Uf)\big)^{(2k)}\Big) \le \frac{\mathcal{K}_{2k}}{(nh)^{2k}}\, \nu_m^k E_n\big(f - S_{h,2,m}f\big), \tag{1.7}$$

where

$$\nu_m = \frac{8}{\binom{2m}{m}} \sum_{l=0}^{\lfloor (m-1)/2\rfloor} \frac{\binom{2m}{m-2l-1}}{(2l+1)^2},$$

and $\lfloor x\rfloor$ is the integral part of x. To estimate $E_n(U^m f)$, we observe that $\big(S^m_{h,2,m}f\big)^{(2m)}$ can be represented in the form $\frac{1}{h^{2m}} W^m_{h,m}\delta_h^{2m} f$, where $W_{h,m}$ is a summatory operator with coefficients independent of h. Consequently,

$$E_n(U^m f) \le \frac{\mathcal{K}_{2m}}{(nh)^{2m}}\, \tau_m E_n\big(\delta_h^{2m} f\big), \tag{1.8}$$

where τ_m is a quantity dominating the norm of the operator $W_{h,m}$. Calculations show that it is possible to take $\tau_m = \frac{\nu_m^m}{2^{2m}}$.

Summing inequalities (1.7) and (1.8), we get

$$E_n(f) \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(nh)^{2k}}\, \nu_m^k E_n\big(f - S_{h,2,m}f\big) + \frac{\mathcal{K}_{2m}}{(nh)^{2m}}\, \frac{\nu_m^m}{2^{2m}} E_n\big(\delta_h^{2m} f\big).$$

Taking estimate (1.4) into account and setting $h = \frac{\gamma\pi}{n}$, we obtain (1.2):

$$E_n(f) \le \Bigg(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\gamma\pi)^{2k}}\, \nu_m^k + \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}}\, \frac{\nu_m^m}{2^{2m}}\Bigg)\omega_{2m}\Big(f, \frac{\gamma\pi}{n}\Big).$$

For some values of parameters, the constants in the last inequality are close to optimal in a sense.

By a different method, the operators $S_{h,2,m}$ were applied to prove Jackson-type inequalities in [3], and later in [4, 5].

The argument carried out above can be viewed as a concretization of a general method for estimating functionals defined on abstract spaces with a seminorm. This method makes it possible to give a unified treatment to some problems studied earlier separately. In particular, such are the approximation problems for functions defined on the axis, including periodic functions, on the semiaxis, and on the segment. However, we do not want to make our exposition too abstract. So, in this paper we only deal with spaces of functions defined on the axis and use iterations of the operators $S_{h,2,m}$ for intermediate approximation.

Now, we give a brief overview of the results.

In §2 we discuss the properties of linear combinations of the Steklov functions $S_{h,2,m}$. With their help, we establish the general Theorem 2.1, which contains estimates of semi-additive functionals, and deduce its particular cases.

In §3, the results of §2 are applied to the functional of best approximation by entire functions of exponential type. As particular cases, we get some estimates for best approximation of a function in terms of high order continuity moduli of the function itself, its derivatives, and derivatives of the conjugate function. Then we construct linear approximation methods that realize estimates established for best approximations earlier.

In §4 the results of the paper are compared with those known before. It is shown that the new constants are close to optimal is some cases.

In §5, analogs of some results of §3 are established for best approximation by splines, with some loss of generality. Also, we construct linear approximation methods that realize the inequalities obtained.

In §6, we discuss some modifications of the proof technique and deduce several inequalities of the type (1.1).

1.2. Notation. In what follows, in addition to the notation introduced earlier, the symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ denote the sets of reals, nonnegative reals, integers, nonnegative integers, and positive integers, respectively; $[a:b]=[a,b]\cap\mathbb{Z}$.

We use the following notation for operators: I is the identity operator, T^h is the translation operator with step h, i.e., $T^h f = f(\cdot + h)$; the operators of central difference and Steklov were defined in Subsection 1.1; D is the operator of differentiation; if a function f is locally integrable on an interval and $\alpha\in\mathbb{N}$, then $D^{-\alpha}f$ denotes an arbitrary primitive of order α of f. No clarification is needed in expressions like $\delta_h^\alpha D^{-\alpha}f$, because the result does not depend on the choice of a primitive. Also, let J be the operator of trigonometric conjugation, or the Hilbert transform; see, e.g., [6, §79] or [7, §6.1]; instead of Jf the notation $\widetilde{f}$ is often used.

Function spaces are denoted as follows: $UCB(\mathbb{R})$ is the space of bounded and uniformly continuous on functions $\mathbb{R}$, and C is the space of 2π-periodic continuous functions, with the uniform norms $\|\cdot\|=\|\cdot\|_\infty$; if $1\le p<\infty$ then $L_p(\mathbb{R})$ is the space of measurable functions integrable on $\mathbb{R}$ with power p, and L_p denotes the space of measurable and 2π-periodic functions f integrable on the period with power p, with the norms $\|f\|_p=\big(\int_E|f|^p\big)^{1/p}$, where $E=\mathbb{R}$ or $[-\pi,\pi]$, respectively; $L_\infty(\mathbb{R})$ is the space of measurable functions f essentially bounded on $\mathbb{R}$ with the norm

$$\|f\|_\infty=\operatorname*{vrai\,sup}_{x\in\mathbb{R}}|f(x)|,$$

L_∞ is the subspace of 2π-periodic functions belonging to $L_\infty(\mathbb{R})$; $L_{p,\mathrm{loc}}(\mathbb{R})$ is the set of functions that belong to $L_p(E)$ for each segment E. If $r\in\mathbb{N}$, then $W_p^{(r)}(\mathbb{R})$ and $W_{p,\mathrm{loc}}^{(r)}(\mathbb{R})$ are the sets of functions that belong to $L_p(\mathbb{R})$ and $L_{p,\mathrm{loc}}(\mathbb{R})$ and are r-fold integrals of functions that belong to $L_p(\mathbb{R})$ and $L_{p,\mathrm{loc}}(\mathbb{R})$, respectively; $W_p^{(0)}(\mathbb{R})=L_p(\mathbb{R})$,

$W^{(0)}_{p,\mathrm{loc}}(\mathbb{R}) = L_{p,\mathrm{loc}}(\mathbb{R})$. The classes $W^{(r)}_p$ of periodic functions are defined similarly. Unless implied otherwise by the context, function spaces may be real or complex.

Assume that $\mathfrak{M}$ is a closed subspace of $L_p(\mathbb{R})$ $(1 \le p < \infty)$ or $UCB(\mathbb{R})$ $(p = \infty)$, and P is a seminorm defined on $\mathfrak{M}$. We say that the space $(\mathfrak{M}, P)$ is of class $\mathcal{B}$ if the following conditions are fulfilled:

1) the space is shift-invariant, i.e., for every $f \in \mathfrak{M}$ and $h \in \mathbb{R}$ we have $T^h f \in \mathfrak{M}$ and $P(T^h f) = P(f)$;

2) there exists a constant B such that $P(f) \le B\|f\|_p$ for every $f \in \mathfrak{M}$,

As examples of spaces of class $\mathcal{B}$ we mention $\big(UCB(\mathbb{R}), \|\cdot\|_\infty\big)$, $(L_p(\mathbb{R}), \|\cdot\|_p)$ $(1 \le p < \infty)$, the spaces of periodic functions $\big(C, \|\cdot\|_p\big)$ $(1 \le p \le \infty)$, and also more general spaces of uniformly continuous almost-periodic functions [8], with exponents belonging to a fixed set, with various norms (the uniform norm, the norms of Stepanov, Weyl, Besicovitch). The rth order modulus of continuity of a function f with respect to a seminorm P is defined by

$$\omega_r(f,h)_P = \sup_{0\le t\le h} P\big(\delta^r_t(f)\big).$$

The index p attached to the best approximation, to the modulus of continuity, or to similar objects means that $P(f) = \|f\|_p$.

For $s \in \mathbb{Z}_+$ we put

$$\mathfrak{M}^{(s)} = \big\{f \in W^{(s)}_{1,\mathrm{loc}}(\mathbb{R}) \cap \mathfrak{M} \,:\, D^s f \in \mathfrak{M}\big\},$$
$$\widetilde{\mathfrak{M}}^{(s)} = \big\{f \in \mathfrak{M} \,:\, Jf \in W^{(s)}_{1,\mathrm{loc}}(\mathbb{R}),\ D^s Jf \in \mathfrak{M}\big\};$$

and for $s \in \mathbb{N}$ we put $\mathfrak{M}^{(-s)} = \mathfrak{M}$,

$$\widetilde{\mathfrak{M}}^{(-s)} = \big\{f \in \mathfrak{M} \,:\, \forall h > 0\ JS^s_h f \in \mathfrak{M}\big\}.$$

Note that if $\mathfrak{M}$ is a closed shift-invariant subspace of C or L_p with $p \in [1, +\infty)$, then $\widetilde{\mathfrak{M}}^{(-s)} = \mathfrak{M}$ for all $s \in \mathbb{N}$, because $JS^s_h f$ is the convolution of f with an integrable kernel [9]. For each $s \in \mathbb{Z}$ we denote $\mathfrak{M}^{(s,0)} = \mathfrak{M}^{(s)}$, $\mathfrak{M}^{(s,1)} = \widetilde{\mathfrak{M}}^{(s)}$.

If $\alpha \in \mathbb{N}$, then $\delta^\alpha_h D^{-\alpha} Jf$ means the same as $J\delta^\alpha_h D^{-\alpha} f$. In the same sense, we may consider a modulus of continuity of $D^{-\alpha}Jf$. The notation $\delta^\alpha_h D^{-\alpha} Jf$ can be also understood literally if we interpret the primitive in the sense of distributions. This convention allows us to write derivatives and primitives of conjugate functions without respect to the order of operations.

We assume that a composition $G \circ F$ is defined for all x in the domain of F for which $F(x)$ belongs to the domain of G. The functions are assumed to be extended to the points of removable break by continuity; in other cases the symbol $\frac{0}{0}$ is understood as 0.

If $(\mathfrak{M}, P)$ is a space with seminorm, $\mathfrak{N}$ is a subspace of $\mathfrak{M}$, and $U\colon \mathfrak{N} \to \mathfrak{M}$ is a linear operator, then the seminorm of U is defined by

$$N_P(U) = \sup_{f\in\mathfrak{N}} \frac{P(Uf)}{P(f)}.$$

From this definition it is clear that if $N_P(U) < +\infty$, then for every $f \in \mathfrak{N}$ we have

$$P(Uf) \le N_P(U)P(f).$$

If the seminorms of the operators U and V are finite and their product UV is well defined, then

$$N_P(UV) \le N_P(U)N_P(V).$$

Let $\mathcal{F}(\mathfrak{M})$ be the set of semiadditive functionals $\Phi\colon \mathfrak{M} \to \mathbb{R}_+$ (semiadditivity means that $\Phi(f+g) \le \Phi(f) + \Phi(g)$ for every $f, g \in \mathfrak{M}$). If, moreover, $\mathfrak{N} \subset \mathfrak{M}$ and $U\colon \mathfrak{N} \to \mathfrak{M}$

is a linear operator, then we put

$$N_P(\Phi, U) = \sup_{f\in\mathfrak{N}} \frac{\Phi(f)}{P(Uf)}.$$

By this definition, if $N_P(\Phi, U) < +\infty$, then for every $f \in \mathfrak{N}$ we have

$$\Phi(f) \le N_P(\Phi, U)P(Uf). \tag{1.9}$$

§2. The estimates of functionals in the spaces of class $\mathcal{B}$

2.1. Linear combinations of Steklov averages and differentiation. Consider the following linear combinations of Steklov averages:

$$U_{h,m} = \frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m-j} S_{jh}^2.$$

Inequality (1.4) shows that such a choice of the coefficients allows us to represent the deviation $I - U_{h,m}$ as an integrated difference of order $2m$, which is convenient for estimating this deviation in terms of the modulus of continuity. We also note that $U_{h,1} = S_h^2$.

By the definition of the Steklov functions, the operators $U_{h,m}$ can be written as

$$U_{h,m} = \frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \frac{\binom{2m}{m-j}}{(jh)^2} \delta_{jh}^2 D^{-2}.$$

Consequently, the operator $V_{h,m} = h^2 D^2 U_{h,m}$ is summatory:

$$\begin{aligned} V_{h,m} &= \frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \frac{\binom{2m}{m-j}}{j^2} \delta_{jh}^2 \\ &= \frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \frac{\binom{2m}{m-j}}{j^2} \left(T^{jh} + T^{-jh}\right) - \left(\frac{4}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \frac{\binom{2m}{m-j}}{j^2}\right) I. \end{aligned} \tag{2.1}$$

The seminorm of $V_{h,m}$ admits the estimate [3]

$$N_P(V_{h,m}) \le \nu_m,$$

where

$$\nu_m = \frac{8}{\binom{2m}{m}} \sum_{l=0}^{\lfloor (m-1)/2 \rfloor} \frac{\binom{2m}{m-2l-1}}{(2l+1)^2}, \tag{2.2}$$

and $\lfloor x \rfloor$ is the integral part of x. In the same paper, the authors obtained the two-sided estimate

$$\frac{8}{\pi^2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{2m}}{2m+1} \le 1 - \frac{\nu_m}{\pi^2} \le \frac{8}{\pi^2}\sqrt{\frac{\pi}{2}}\frac{\sqrt{2m+1}}{2m}.$$

We note that ν_m does not depend on h, which is reflected in the notation.

On the other hand, since

$$\delta_{jh}^2 = \sum_{s=1-j}^{j-1} \left(j - |s|\right) T^{sh} \delta_h^2,$$

the operators $U_{h,m}$ and $V_{h,m}$ admit separation of the difference factor δ_h^2, and can be written in the form

$$U_{h,m} = \frac{2}{\binom{2m}{m}} \sum_{j=1}^{m} (-1)^{j-1} \frac{\binom{2m}{m-j}}{(jh)^2} \sum_{s=1-j}^{j-1} \big(j - |s|\big) T^{sh} \delta_h^2 D^{-2},$$

$$V_{h,m} = \delta_h^2 W_{h,m}, \tag{2.3}$$

$$W_{h,m} = \sum_{s=1-m}^{m-1} A_{m,s} T^{sh}, \tag{2.4}$$

$$A_{m,s} = \frac{2}{\binom{2m}{m}} \sum_{j=|s|+1}^{m} (-1)^{j-1} \frac{\binom{2m}{m-j}}{j^2} \big(j - |s|\big). \tag{2.5}$$

Remark 2.1. In [4] it was proved that the sequence $\{A_{m,s}\}_s$ is alternating, and so

$$\sum_{s=1-m}^{m-1} |A_{m,s}| = \frac{\nu_m}{4}.$$

By (1.1), the seminorms of the iterations of $V_{h,m}$ and $W_{h,m}$ satisfy

$$N_P\big(V_{h,m}^k\big) \le \nu_m^k, \quad N_P\big(W_{h,m}^k\big) \le \frac{\nu_m^k}{4^k}. \tag{2.6}$$

Moreover, the coefficients of the shifts in the expansions of the initial operators alternate. Hence, equalities occur in (2.6) in the spaces $UCB(\mathbb{R})$ and $L_1(\mathbb{R})$.

Remark 2.2. We mention the following obvious generalization of formula (2.1):

$$D^s U_{h,m}^k = \frac{1}{h^{2k}} W_{h,m}^k \delta_h^{2k} D^{s-2k}. \tag{2.7}$$

2.2. General theorems about estimates of functionals. The quantities

$$N_P(\Phi, D^s) = \sup_{f \in \mathfrak{M}^{(s)}} \frac{\Phi(f)}{P(D^s f)}, \quad N_P(\Phi, D^s J) = \sup_{f \in \widetilde{\mathfrak{M}}^{(s)}} \frac{\Phi(f)}{P(D^s J f)}$$

are called the moments and the conjugate moments of a functional Φ of order $s \in \mathbb{Z}_+$ with respect to the seminorm P. The moments of many important functionals are either known exactly or estimated. This allows us to use these quantities in more complicated estimates.

In what follows, we assume that all moments present in the statements are finite. Instead of assuming the moments to be finite, the reader may accept the convention $0 \cdot (+\infty) = +\infty$. Under this convention, if one of the moments is infinite, then the right-hand sides of the inequalities in question turn to $+\infty$ and the inequalities become trivial.

Theorem 2.1. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m, q \in \mathbb{N}$, $h > 0$, $\{s_k\}_{k=0}^q \subset \mathbb{Z}_+$, $\{\varepsilon_k\}_{k=0}^q \subset \{0,1\}$, $\mathfrak{N} = \bigcap_{k=0}^q \mathfrak{M}^{(s_k - 2k, \varepsilon_k)}$, *and* $\Phi \in \mathcal{F}(\mathfrak{M})$. *Then for every* $f \in \mathfrak{N}$ *we have*

$$\Phi\big(f - U_{h,m}^q f\big) \le \sum_{k=0}^{q-1} \frac{N_P(\Phi, D^{s_k} J^{\varepsilon_k})}{h^{2k}} P\big(W_{h,m}^k \delta_h^{2k} D^{s_k - 2k} (I - U_{h,m}) J^{\varepsilon_k} f\big), \tag{2.8}$$

$$\begin{aligned} \Phi(f) \le \sum_{k=0}^{q-1} \frac{N_P(\Phi, D^{s_k} J^{\varepsilon_k})}{h^{2k}} P\big(W_{h,m}^k \delta_h^{2k} D^{s_k - 2k} (I - U_{h,m}) J^{\varepsilon_k} f\big) \\ + \frac{N_P(\Phi, D^{s_q} J^{\varepsilon_q})}{h^{2q}} P\big(W_{h,m}^q \delta_h^{2q} D^{s_q - 2q} J^{\varepsilon_q} f\big). \end{aligned} \tag{2.9}$$

If, moreover, $\{s_k\}_{k=0}^{\infty} \subset \mathbb{Z}_+$, $\{\varepsilon_k\}_{k=0}^{\infty} \subset \{0,1\}$, $\mathfrak{N} = \bigcap_{k=0}^{\infty} \mathfrak{M}^{(s_k-2k,\varepsilon_k)}$, *and for every* $g \in \mathfrak{N}$ *we have*

$$\frac{N_P(\Phi, D^{s_q} J^{\varepsilon_q})}{h^{2q}} P\big(W_{h,m}^q \delta_h^{2q} D^{s_q-2q} J^{\varepsilon_q} g\big) \underset{q\to\infty}{\longrightarrow} 0, \tag{2.10}$$

then for every $f \in \mathfrak{N}$ *we have*

$$\Phi(f) \le \sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{s_k} J^{\varepsilon_k})}{h^{2k}} P\big(W_{h,m}^k \delta_h^{2k} D^{s_k-2k}(I - U_{h,m}) J^{\varepsilon_k} f\big). \tag{2.11}$$

Proof. We put $U = U_{h,m}$ for brevity. We shall use the identity

$$I - U^q = \sum_{k=0}^{q-1} U^k (I - U).$$

1. We prove (2.8). By semiadditivity, we have

$$\Phi\big(f - U^q f\big) = \Phi\bigg(\sum_{k=0}^{q-1} U^k (I-U) f\bigg) \le \sum_{k=0}^{q-1} \Phi\big(U^k (I-U) f\big).$$

It remains to apply estimate (1.9) to each term:

$$\Phi\big(U^k (I-U) f\big) \le N_P(\Phi, D^{s_k} J^{\varepsilon_k}) P\big(D^{s_k} J^{\varepsilon_k} U^k (I-U) f\big),$$

and then to employ (2.7).

2. In order to prove (2.9), we write

$$\Phi(f) \le \Phi\big(f - U^q f\big) + \Phi\big(U^q f\big).$$

The first term has already been estimated, and the second is estimated in the same way, with the help of (1.9) and (2.7). Inequality (2.11) follows from (2.9), upon letting q tend to ∞. □

Remark 2.3. We turn the reader's attention to the fact that estimation of a functional Φ consists of two independent parts: estimation of the deviation of the power of the approximating operator $\Phi\big(f - U_{h,m}^q f\big)$ and of the power itself $\Phi\big(U_{h,m}^q f\big)$. Estimates of the first type, such as inequality (2.8), are of independent interest but will not be separated in the sequel.

We formulate some special cases of Theorem 2.1.

Theorem 2.2. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\Phi \in \mathcal{F}(\mathfrak{M})$, $m, q \in \mathbb{N}$, $h > 0$, $\beta \in \mathbb{Z}_+$, $\alpha \in [0 : 2q + \beta]$, *and* $\varepsilon, \tau \in \{0,1\}$. *Then for every* $f \in \mathfrak{M}^{(\beta,\tau)} \cap \mathfrak{M}^{(\beta-\alpha,\varepsilon)}$ *we have*

$$\begin{aligned}\Phi(f) \le \sum_{k=0}^{q-1} \frac{N_P(\Phi, D^{2k+\beta} J^{\tau})}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) D^{\beta} J^{\tau} f\big) \\ + \frac{N_P(\Phi, D^{2q+\beta-\alpha} J^{\varepsilon})}{h^{2q}} \frac{\nu_m^q}{2^{2q}} P\big(\delta_h^{2q} D^{\beta-\alpha} J^{\varepsilon} f\big),\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Phi(f) \le \frac{1}{\binom{2m}{m}} \bigg(\sum_{k=0}^{q-1} \frac{N_P(\Phi, D^{2k+\beta} J^{\tau})}{h^{2k}} \nu_m^k\bigg) \omega_{2m}(D^{\beta} J^{\tau} f, h)_P \\ + \frac{N_P(\Phi, D^{2q+\beta-\alpha} J^{\varepsilon})}{h^{2q}} \frac{\nu_m^q}{2^{2q}} \omega_{2q}\big(D^{\beta-\alpha} J^{\varepsilon} f, h\big)_P.\end{aligned} \tag{2.13}$$

If, moreover,

$$\sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta} J^{\tau})}{h^{2k}} \nu_m^k < +\infty, \tag{2.14}$$

then

$$\Phi(f) \le \sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta}J^\tau)}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m})D^\beta J^\tau f\big) \tag{2.15}$$

for every $f \in \mathfrak{M}^{(\beta,\tau)}$.

Proof. To prove (2.12), in Theorem 2.1 we put $s_k = 2k + \beta$, $\varepsilon_k = \tau$ for $k \in [0 : q-1]$, $s_q = 2q + \beta - \alpha$, $\varepsilon_q = \varepsilon$ and use the estimate of the seminorm of the operator $W_{h,m}$. Inequality (2.13) follows from (2.12) with the help of (1.4). The convergence of the series in (2.14) implies that its terms tend to zero. Taking into account the fact that $P(\delta_h^{2q}\varphi) \le 2^{2q}P(\varphi)$ for every $\varphi \in \mathfrak{M}$, writing inequality (2.12) for $\alpha = 0$ and $\varepsilon = \tau$, and letting q tend to ∞, we arrive at (2.15), because the remainder is infinitely small. □

Remark 2.4. Condition (2.14) implies the convergence of the series on the right-hand side of (2.15). In its turn, inequality (2.15) yields the following estimate of Φ in terms of the deviation of the operator $U_{h,m}$:

$$\Phi(f) \le \left(\sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta}J^\tau)}{h^{2k}} \nu_m^k\right) P\big((I - U_{h,m})D^\beta J^\tau f\big), \tag{2.16}$$

where the series on the right also converges. Estimating the right-hand side of (2.16) by the modulus of continuity, we get

$$\Phi(f) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta}J^\tau)}{h^{2k}} \nu_m^k\right) \omega_{2m}\big(D^\beta J^\tau f, h\big)_P. \tag{2.17}$$

However, the constant near the modulus of continuity obtained at the last step is greater than that in inequality (2.13) with $q = m$, $\alpha = 0$, and $\varepsilon = \tau$:

$$\Phi(f) \le \left\{\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k+\beta}J^\tau)}{h^{2k}} \nu_m^k + \frac{N_P(\Phi, D^{2m+\beta}J^\tau)}{h^{2m}} \frac{\nu_m^m}{2^{2m}}\right\} \omega_{2m}(D^\beta J^\tau f, h)_P. \tag{2.18}$$

The constant in (2.17) is greater than that in (2.18) (with the obvious exception when the moments are equal to zero). This follows from comparison of the mth term of the series and the additional term in (2.18). This comparison reduces to the evident inequality $2^{2m} > \binom{2m}{m}$.

For this reason, we formulate estimates by moduli of continuity in the form (2.13). The inequalities of the type (2.16) have their own importance, but we prefer to write their stronger version (2.15).

Remark 2.5. The series in (2.14) is a power series with respect to $\frac{1}{h}$. By the Cauchy–Hadamard formula, the inequality

$$h > \sqrt{\nu_m} \limsup_{k\to\infty} N_P^{1/2k}(\Phi, D^{2k+\beta}J^\tau)$$

ensures the convergence of this series. If the explicit estimates of the moments are known, then we can simplify the convergence conditions by calculating the limits, which will be done for specific applications.

In what follows, we put $\alpha = 0$, $q = m$. The applications that require primitives and their conjugate functions will be studied in a separate paper.

Theorem 2.3. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\Phi \in \mathcal{F}(\mathfrak{M})$, $m \in \mathbb{N}$, $h > 0$, *and* $\beta \in \mathbb{Z}_+$. *Then for every* $f \in \mathfrak{M}^{(\beta)}$ *we have*

$$\Phi(f) \le \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k+\beta})}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) D^\beta f\big) + \frac{N_P(\Phi, D^{2m+\beta})}{h^{2m}} \frac{\nu_m^m}{2^{2m}} P\big(\delta_h^{2m} D^\beta f\big),$$

$$\Phi(f) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k+\beta})}{h^{2k}} \nu_m^k + \frac{N_P(\Phi, D^{2m+\beta})}{h^{2m}} \frac{\nu_m^m}{2^{2m}} \right) \omega_{2m}(D^\beta f, h)_P.$$

If, moreover,

$$\sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta})}{h^{2k}} \nu_m^k < +\infty,$$

then

$$\Phi(f) \le \sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta})}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) D^\beta f\big)$$

for every $f \in \mathfrak{M}^{(\beta)}$.

We state the important case where $\beta = 0$ separately.

Theorem 2.4. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\Phi \in \mathcal{F}(\mathfrak{M})$, $m \in \mathbb{N}$, *and* $h > 0$. *Then for every* $f \in \mathfrak{M}$ *we have*

$$\begin{aligned} \Phi(f) &\le \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k})}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) f\big) + \frac{N_P(\Phi, D^{2m})}{h^{2m}} \frac{\nu_m^m}{2^{2m}} P\big(\delta_h^{2m} f\big), \\ \Phi(f) &\le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k})}{h^{2k}} \nu_m^k + \frac{N_P(\Phi, D^{2m})}{h^{2m}} \frac{\nu_m^m}{2^{2m}} \right) \omega_{2m}(f, h)_P. \end{aligned} \tag{2.19}$$

If, moreover,

$$\sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k})}{h^{2k}} \nu_m^k < +\infty,$$

then

$$\Phi(f) \le \sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k})}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) f\big)$$

for every $f \in \mathfrak{M}$.

For spaces of periodic functions, inequality (2.19) and its concretization for best approximations by trigonometric polynomials and splines (see Theorems 3.1.4 and 5.1.4 below) were announced in [10].

We separate the case where the estimate is given in terms of derivatives of the conjugate function.

Theorem 2.5. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\Phi \in \mathcal{F}(\mathfrak{M})$, $m \in \mathbb{N}$, $h > 0$, *and* $\beta \in \mathbb{Z}_+$. *Then for every* $f \in \widetilde{\mathfrak{M}}^{(\beta)}$ *we have*

$$\begin{aligned} \Phi(f) \le \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k+\beta} J)}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) D^\beta J f\big) \\ + \frac{N_P(\Phi, D^{2m+\beta} J)}{h^{2m}} \frac{\nu_m^m}{2^{2m}} P\big(\delta_h^{2m} D^\beta J f\big), \end{aligned}$$

$$\Phi(f) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k+\beta} J)}{h^{2k}} \nu_m^k + \frac{N_P(\Phi, D^{2m+\beta} J)}{h^{2m}} \frac{\nu_m^m}{2^{2m}} \right) \omega_{2m}(D^\beta J f, h)_P.$$

If, moreover,

$$\sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta})}{h^{2k}} \nu_m^k < +\infty,$$

then

$$\Phi(f) \le \sum_{k=0}^{\infty} \frac{N_P(\Phi, D^{2k+\beta} J)}{h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m}) D^\beta J f\big)$$

for every $f \in \mathfrak{M}^{(\beta)}$.

§3. The Jackson inequalities for best approximations by entire functions of exponential type, and their realization by linear methods

3.1. Estimates for best approximations. For $\sigma > 0$, let $A_{\sigma-0}(f)_P$ ($A_\sigma(f)_P$) denote the best approximation of a function $f \in \mathfrak{M}$ by the set $\mathbf{E}_{\sigma-0}$ ($\mathbf{E}_\sigma$) of entire functions of type smaller (respectively, not greater) than σ in the space $(\mathfrak{M}, P)$. We shall apply the statements proved above to the functionals $A_{\sigma-0}(\cdot)_P$ and $A_\sigma(\cdot)_P$. Recall that the Favard constants $\mathcal{K}_r$ and the conjugate Favard constants $\widetilde{\mathcal{K}}_r$ are defined by

$$\mathcal{K}_r = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}}, \quad \widetilde{\mathcal{K}}_r = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{kr}}{(2k+1)^{r+1}};$$

we put $\mathcal{K}_{r,0} = \mathcal{K}_r$, $\mathcal{K}_{r,1} = \widetilde{\mathcal{K}}_r$. The coefficients ν_m are expressed by (2.2).

The following inequalities of Akhiezer–Krein–Favard type are well known; see [6, § 101] and [9]: if $(\mathfrak{M}, P) \in \mathcal{B}$, $r \in \mathbb{Z}_+$, and $f \in \mathfrak{M}^{(r)}$, then

$$A_{\sigma-0}(f)_P \le \frac{\mathcal{K}_r}{\sigma^r} P(D^r f), \tag{3.1}$$

and if $r \in \mathbb{N}$, $f \in \widetilde{\mathfrak{M}}^{(r)}$, then

$$A_{\sigma-0}(f)_P \le \frac{\widetilde{\mathcal{K}}_r}{\sigma^r} P(D^r J f). \tag{3.2}$$

The quantity $P(\cdot)$ on the right-hand sides of these inequalities can be replaced by $A_{\sigma-0}(\cdot)_P$. A similar refinement is true if we replace $A_{\sigma-0}$ by A_σ. Moreover, if a function f is orthogonal to $\mathbf{E}_\sigma$, then the left-hand sides of (3.1) and (3.2) can be replaced by $P(f)$.

Our theorems are numerated as follows: if a general theorem in Subsection 2.2 has number 2.x, then its concretization for best approximations has number 3.1.x.

Theorem 3.1.1. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m, q \in \mathbb{N}$, $\sigma, h > 0$, $\{s_k\}_{k=0}^q \subset \mathbb{Z}_+$, $\{\varepsilon_k\}_{k=0}^q \subset \{0,1\}$, $\varepsilon_k = 0$ *whenever* $s_k = 0$, *and* $\mathfrak{N} = \bigcap_{k=0}^q \mathfrak{M}^{(s_k-2k,\varepsilon_k)}$. *Then for every* $f \in \mathfrak{N}$ *we have*

$$\begin{aligned} A_{\sigma-0}(f)_P \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{s_k,\varepsilon_k}}{\sigma^{s_k} h^{2k}} A_{\sigma-0}\big(W_{h,m}^k \delta_h^{2k} D^{s_k-2k}(I-U_{h,m}) J^{\varepsilon_k} f\big)_P \\ + \frac{\mathcal{K}_{s_q,\varepsilon_q}}{\sigma^{s_q} h^{2q}} A_{\sigma-0}\big(W_{h,m}^q \delta_h^{2q} D^{s_q-2q} J^{\varepsilon_q} f\big)_P, \end{aligned} \tag{3.3}$$

$$\begin{aligned} A_{\sigma-0}(f)_P \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{s_k,\varepsilon_k}}{\sigma^{s_k} h^{2k}} P\big(W_{h,m}^k \delta_h^{2k} D^{s_k-2k}(I-U_{h,m}) J^{\varepsilon_k} f\big) \\ + \frac{\mathcal{K}_{s_q,\varepsilon_q}}{\sigma^{s_q} h^{2q}} P\big(W_{h,m}^q \delta_h^{2q} D^{s_q-2q} J^{\varepsilon_q} f\big). \end{aligned} \tag{3.4}$$

If, moreover, $\{s_k\}_{k=0}^\infty \subset \mathbb{Z}_+$, $\{\varepsilon_k\}_{k=0}^\infty \subset \{0,1\}$, $\varepsilon_k = 0$ *whenever* $s_k = 0$,

$$\mathfrak{N} = \bigcap_{k=0}^{\infty} \mathfrak{M}^{(s_k-2k,\varepsilon_k)},$$

and

$$\frac{1}{\sigma^{s_q} h^{2q}} A_{\sigma-0}\big(W_{h,m}^q \delta_h^{2q} D^{s_q-2q} J^{\varepsilon_q} g\big)_P \underset{q\to\infty}{\longrightarrow} 0 \tag{3.5}$$

for every $g \in \mathfrak{N}$, then for every $f \in \mathfrak{N}$ we have

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{s_k,\varepsilon_k}}{\sigma^{s_k} h^{2k}} A_{\sigma-0}\big(W^k_{h,m}\delta_h^{2k} D^{s_k-2k}(I-U_{h,m})J^{\varepsilon_k} f\big)_P, \tag{3.6}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{s_k,\varepsilon_k}}{\sigma^{s_k} h^{2k}} P\big(W^k_{h,m}\delta_h^{2k} D^{s_k-2k}(I-U_{h,m})J^{\varepsilon_k} f\big). \tag{3.7}$$

In inequalities (3.3), (3.4), (3.6), *and* (3.7), $A_{\sigma-0}$ *can be replaced by* A_σ. *For the functions f orthogonal to $\mathbf{E}_\sigma$, the left-hand sides of* (3.4) *and* (3.7) *can be replaced by* $P(f)$.

Proof. Since $1 \le \mathcal{K}_{s,\varepsilon} \le 2$, relation (2.10) turns into (3.5). Estimates (3.4) and (3.7) are obtained by applying Theorem 2.1 to the functional $\Phi(f) = A_{\sigma-0}(f)_P$ and by using the fact that, in virtue of (3.1) and (3.2), $N_P(\Phi, D^s J^\varepsilon) \le \mathcal{K}_{s,\varepsilon}\sigma^{-s}$, with the exception of the case where $s = 0$, $\varepsilon = 1$. To obtain inequalities (3.3) and (3.6) (which are formally stronger than (3.4) and (3.7)), we can apply (3.4) and (3.7) to the space $\big(\mathfrak{M}, A_{\sigma-0}(\cdot)_P\big)$, which also belongs to the class $\mathcal{B}$, and use the identity

$$A_{\sigma-0}(f)_{A_{\sigma-0}(\cdot)_P} = A_{\sigma-0}(f)_P.$$

The claim about A_σ is proved similarly.

The inequalities for functions orthogonal to $\mathbf{E}_\sigma$ are obtained by applying the inequalities already proved to the space $\big(\mathfrak{M}_\sigma^\perp, P\big)$ of functions that belong to $\mathfrak{M}$ and are orthogonal to $\mathbf{E}_\sigma$ (which is also of class $\mathcal{B}$), and to the functional $\Phi = P$. While doing this, one must take into account that the moments are estimated as before. □

Now we state several special cases of Theorem 3.1.1.

Theorem 3.1.2. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m, q \in \mathbb{N}$, $\sigma, h > 0$, $\beta \in \mathbb{Z}_+$, $\alpha \in [0 : 2q+\beta]$, $\varepsilon, \tau \in \{0,1\}$, $\tau = 0$ *whenever* $\beta = 0$, *and* $\varepsilon = 0$ *whenever* $\alpha = 2q+\beta$. *Then for every* $f \in \mathfrak{M}^{(\beta,\tau)} \cap \mathfrak{M}^{(\beta-\alpha,\varepsilon)}$ *we have*

$$\begin{aligned} A_{\sigma-0}(f)_P \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta}h^{2k}}\,\frac{\nu_m^k}{2^{2k}}\,A_{\sigma-0}\big(\delta_h^{2k}(I-U_{h,m})D^\beta J^\tau f\big)_P \\ + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{\sigma^{2q+\beta-\alpha}h^{2q}}\,\frac{\nu_m^q}{2^{2q}}\,A_{\sigma-0}\big(\delta_h^{2q}D^{\beta-\alpha}J^\varepsilon f\big)_P, \end{aligned} \tag{3.8}$$

$$\begin{aligned} A_{\sigma-0}(f)_P \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta}h^{2k}}\,\frac{\nu_m^k}{2^{2k}}\,P\big(\delta_h^{2k}(I-U_{h,m})D^\beta J^\tau f\big) \\ + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{\sigma^{2q+\beta-\alpha}h^{2q}}\,\frac{\nu_m^q}{2^{2q}}\,P\big(\delta_h^{2q}D^{\beta-\alpha}J^\varepsilon f\big), \end{aligned} \tag{3.9}$$

$$\begin{aligned} A_{\sigma-0}(f)_P \le \left(\frac{1}{\binom{2m}{m}}\sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta}h^{2k}}\,\nu_m^k\right)\omega_{2m}\big(D^\beta J^\tau f, h\big)_P \\ + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{\sigma^{2q+\beta-\alpha}h^{2q}}\,\frac{\nu_m^q}{2^{2q}}\,\omega_{2q}\big(D^{\beta-\alpha}J^\varepsilon f, h\big)_P. \end{aligned} \tag{3.10}$$

If, moreover, $h > \frac{\sqrt{\nu_m}}{\sigma}$, *then for every* $f \in \mathfrak{M}^{(\beta,\tau)}$ *we have*

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta}h^{2k}}\,\frac{\nu_m^k}{2^{2k}}\,A_{\sigma-0}\big(\delta_h^{2k}(I-U_{h,m})D^\beta J^\tau f\big)_P, \tag{3.11}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta}h^{2k}}\,\frac{\nu_m^k}{2^{2k}}\,P\big(\delta_h^{2k}(I-U_{h,m})D^\beta J^\tau f\big). \tag{3.12}$$

In inequalities (3.8)–(3.12), $A_{\sigma-0}$ *can be replaced by* A_σ. *For the functions f orthogonal to $\mathbf{E}_\sigma$, the left-hand sides of* (3.9), (3.10), *and* (3.12) *can be replaced by* $P(f)$.

To prove Theorem 3.1.2, we must take into account that the inequality $h > \frac{\sqrt{\nu_m}}{\sigma}$ is equivalent to the convergence of the series (2.14), which takes the form

$$\sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta} h^{2k}}\, \nu_m^k.$$

Theorem 3.1.3. *Suppose that $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, $\sigma, h > 0$, and $\beta \in \mathbb{Z}_+$. Then for every $f \in \mathfrak{M}^{(\beta)}$ we have*

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{\sigma^{2k+\beta} h^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, A_{\sigma-0}\big(\delta_h^{2k}(I - U_{h,m}) D^\beta f\big)_P + \frac{\mathcal{K}_{2m+\beta}}{\sigma^{2m+\beta} h^{2m}}\, \frac{\nu_m^m}{2^{2m}}\, A_{\sigma-0}\big(\delta_h^{2m} D^\beta f\big)_P, \tag{3.13}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{\sigma^{2k+\beta} h^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, P\big(\delta_h^{2k}(I - U_{h,m}) D^\beta f\big) + \frac{\mathcal{K}_{2m+\beta}}{\sigma^{2m+\beta} h^{2m}}\, \frac{\nu_m^m}{2^{2m}}\, P\big(\delta_h^{2m} D^\beta f\big), \tag{3.14}$$

$$A_{\sigma-0}(f)_P \le \frac{1}{\sigma^\beta} \left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{(\sigma h)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m+\beta}}{(\sigma h)^{2m}}\, \frac{\nu_m^m}{2^{2m}} \right\} \omega_{2m}\big(D^\beta f, h\big)_P. \tag{3.15}$$

If, moreover, $h > \frac{\sqrt{\nu_m}}{\sigma}$, then for every $f \in \mathfrak{M}^{(\beta)}$ we have

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k+\beta}}{\sigma^{2k+\beta} h^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, A_{\sigma-0}\big(\delta_h^{2k}(I - U_{h,m}) D^\beta f\big)_P, \tag{3.16}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k+\beta}}{\sigma^{2k+\beta} h^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, P\big(\delta_h^{2k}(I - U_{h,m}) D^\beta f\big). \tag{3.17}$$

In inequalities (3.13)–(3.17), $A_{\sigma-0}$ can be replaced by A_σ. For the functions f orthogonal to $\mathbf{E}_\sigma$, the left-hand sides of (3.14), (3.15), and (3.17) can be replaced by $P(f)$.

Theorem 3.1.4. *Suppose that $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, and σ, $h > 0$. Then for every $f \in \mathfrak{M}$ we have*

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, A_{\sigma-0}\big(\delta_h^{2k}(I - U_{h,m}) f\big)_P + \frac{\mathcal{K}_{2m}}{(\sigma h)^{2m}}\, \frac{\nu_m^m}{2^{2m}}\, A_{\sigma-0}\big(\delta_h^{2m} f\big)_P, \tag{3.18}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, P\big(\delta_h^{2k}(I - U_{h,m}) f\big) + \frac{\mathcal{K}_{2m}}{(\sigma h)^{2m}}\, \frac{\nu_m^m}{2^{2m}}\, P\big(\delta_h^{2m} D^\beta f\big), \tag{3.19}$$

$$A_{\sigma-0}(f)_P \le \left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m}}{(\sigma h)^{2m}}\, \frac{\nu_m^m}{2^{2m}} \right\} \omega_{2m}\big(f, h\big)_P. \tag{3.20}$$

If, moreover, $h > \frac{\sqrt{\nu_m}}{\sigma}$, then for every $f \in \mathfrak{M}$ we have

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, A_{\sigma-0}\big(\delta_h^{2k}(I - U_{h,m}) f\big)_P, \tag{3.21}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}}\, \frac{\nu_m^k}{2^{2k}}\, P\big(\delta_h^{2k}(I - U_{h,m}) f\big). \tag{3.22}$$

In inequalities (3.18)–(3.22), $A_{\sigma-0}$ *can be replaced by* A_σ. *For the functions* f *orthogonal to* $\mathbf{E}_\sigma$, *the left-hand sides of* (3.19), (3.20), *and* (3.22) *can be replaced by* $P(f)$.

For comparison, we mention that in [4] the following generalization of inequalities (3.21) and (3.22) was established. Under the same conditions, if $p \in \mathbb{Z}_+$, then for every $f \in \mathfrak{M}$ we have

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \binom{k+p}{p} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}} \frac{\nu_m^k}{2^{2k}} A_{\sigma-0}\big(\delta_h^{2k}(I-U_{h,m})^{p+1}(f)\big)_P, \tag{3.23}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \binom{k+p}{p} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I-U_{h,m})^{p+1}(f)\big). \tag{3.24}$$

In inequalities (3.23) and (3.24), $A_{\sigma-0}$ can be replaced by A_σ. For the functions f orthogonal to $\mathbf{E}_\sigma$, the left-hand side of (3.24) can be replaced by $P(f)$.

Inequalities (3.21) and (3.22) were established in a different way in [5], where, moreover, their sharpness was proved in some cases.

In the present paper, we do not leave the parameter p free, because otherwise additional efforts are needed to estimate the right-hand side by higher order moduli of continuity.

Theorem 3.1.5. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, $\sigma, h > 0$, *and* $\beta \in \mathbb{N}$. *Then for every* $f \in \widetilde{\mathfrak{M}}^{(\beta)}$ *we have*

$$\begin{aligned} A_{\sigma-0}(f)_P \le \sum_{k=0}^{m-1} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{\sigma^{2k+\beta}h^{2k}} \frac{\nu_m^k}{2^{2k}} A_{\sigma-0}\big(\delta_h^{2k}(I-U_{h,m})D^\beta Jf\big)_P \\ + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{\sigma^{2m+\beta}h^{2m}} \frac{\nu_m^m}{2^{2m}} A_{\sigma-0}\big(\delta_h^{2m}D^\beta Jf\big)_P, \end{aligned} \tag{3.25}$$

$$\begin{aligned} A_{\sigma-0}(f)_P \le \sum_{k=0}^{m-1} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{\sigma^{2k+\beta}h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I-U_{h,m})D^\beta Jf\big) \\ + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{\sigma^{2m+\beta}h^{2m}} \frac{\nu_m^m}{2^{2m}} P\big(\delta_h^{2m}D^\beta Jf\big), \end{aligned} \tag{3.26}$$

$$A_{\sigma-0}(f)_P \le \frac{1}{\sigma^\beta}\left\{\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{(\sigma h)^{2k}}\nu_m^k + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{(\sigma h)^{2m}} \frac{\nu_m^m}{2^{2m}}\right\}\omega_{2m}\big(D^\beta Jf, h\big)_P. \tag{3.27}$$

If, moreover, $h > \frac{\sqrt{\nu_m}}{\sigma}$, *then for every* $f \in \mathfrak{M}^{(\beta)}$ *we have*

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{\sigma^{2k+\beta}h^{2k}} \frac{\nu_m^k}{2^{2k}} A_{\sigma-0}\big(\delta_h^{2k}(I-U_{h,m})D^\beta Jf\big)_P, \tag{3.28}$$

$$A_{\sigma-0}(f)_P \le \sum_{k=0}^{\infty} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{\sigma^{2k+\beta}h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I-U_{h,m})D^\beta Jf\big). \tag{3.29}$$

In inequalities (3.25)–(3.29), $A_{\sigma-0}$ *can be replaced by* A_σ. *For the functions* f *orthogonal to* $\mathbf{E}_\sigma$, *the left-hand sides of* (3.26), (3.27), *and* (3.29) *can be replaced by* $P(f)$.

Observe that the case where $\beta = 0$ was excluded in Theorem 3.1.5.

3.2. Realization of estimates for best approximations by linear methods. Suppose that $r \in \mathbb{N}$, $\sigma > 0$. The Akhiezer–Krein–Favard operators $X_{\sigma,r}$ and the Akhiezer–Krein–Favard conjugate operators $\widetilde{X}_{\sigma,r}$ are convolution operators with the multiplier

functions [6, Section 87]

$$\xi_{\sigma,r}(y) = \begin{cases} \Big(\dfrac{y}{2\sigma}\Big)^r \displaystyle\sum_{l=-\infty}^{\infty} \frac{(-1)^{l(r+1)}}{(l+\frac{y}{2\sigma})^r} & \text{if } |y| \le \sigma, \\ 0 & \text{if } |y| > \sigma, \end{cases}$$

$$\widetilde{\xi}_{\sigma,r}(y) = \begin{cases} \Big(\dfrac{y}{2\sigma}\Big)^r \displaystyle\sum_{l=-\infty}^{\infty} (-1)^{lr} \frac{-\operatorname{sign}(y(l+\frac{y}{2\sigma}))}{(l+\frac{y}{2\sigma})^r} & \text{if } |y| \le \sigma, \\ 0 & \text{if } |y| > \sigma, \end{cases}$$

respectively. We assume $X_{\sigma,0}$ to be the zero operator. It is known that if $(\mathfrak{M}, P) \in \mathcal{B}$, $r \in \mathbb{Z}_+$, $f \in \mathfrak{M}^{(r)}$, then

$$P\big(f - X_{\sigma,r} f\big) \le \frac{\mathcal{K}_r}{\sigma^r} P(D^r f), \tag{3.30}$$

and if $r \in \mathbb{N}$, $f \in \widetilde{\mathfrak{M}}^{(r)}$, then

$$P\big(f - \widetilde{X}_{\sigma,r} f\big) \le \frac{\widetilde{\mathcal{K}}_r}{\sigma^r} P(D^r J f). \tag{3.31}$$

Put $X_{\sigma,r}^{(0)} = X_{\sigma,r}$, $X_{\sigma,r}^{(1)} = \widetilde{X}_{\sigma,r}$. The reader should remember that the tilde over ξ does not mean trigonometric conjugation, and $\widetilde{X}_{\sigma,r}$ is an indivisible notation of the operator, which is not $JX_{\sigma,r}$.

On the basis of $X_{\sigma,r}$, it is possible to construct linear convolution operators that realize the estimates in the theorems of Subsection 3.1. We shall denote them by $Y_{\sigma,m,q}$ or (if $m = q$) $Y_{\sigma,m}$, omitting the other parameters, so that, in what follows, $Y_{\sigma,m,q}$ and $Y_{\sigma,m}$ denote different operators in different statements.

The theorems are numerated as follows: if a theorem in Subsection 3.1 for best approximations has number 3.1.x, then the analogous theorem for a linear method has number 3.2.x.

Theorem 3.2.1. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m, q \in \mathbb{N}$, $\sigma, h > 0$, $\{s_k\}_{k=0}^q \subset \mathbb{Z}_+$, $\{\varepsilon_k\}_{k=0}^q \subset \{0,1\}$, $\varepsilon_k = 0$ *whenever* $s_k = 0$, $\mathfrak{N} = \bigcap_{k=0}^q \mathfrak{M}^{(s_k - 2k, \varepsilon_k)}$, *and*

$$Y_{\sigma,m,q} = \sum_{k=0}^{q-1} X_{\sigma,s_k}^{(\varepsilon_k)} U_{h,m}^k (I - U_{h,m}) + X_{\sigma,s_q}^{(\varepsilon_q)} U_{h,m}^q.$$

Then for every $f \in \mathfrak{N}$ *we have*

$$P\big(f - Y_{\sigma,m,q} f\big) \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{s_k,\varepsilon_k}}{\sigma^{s_k} h^{2k}} P\big(W_{h,m}^k \delta_h^{2k} D^{s_k - 2k} (I - U_{h,m}) J^{\varepsilon_k} f\big) + \frac{\mathcal{K}_{s_q,\varepsilon_q}}{\sigma^{s_q} h^{2q}} P\big(W_{h,m}^q \delta_h^{2q} D^{s_q - 2q} J^{\varepsilon_q} f\big).$$

Proof. Using the formula

$$I = \sum_{k=0}^{q-1} U^k (I - U) + U^q,$$

where $U = U_{h,m}$, and the definition of $Y_{\sigma,m,q}$, we write the deviation as

$$I - Y_{\sigma,m,q} = \sum_{k=0}^{q-1} \big(I - X_{\sigma,s_k}^{(\varepsilon_k)}\big) U_{h,m}^k (I - U_{h,m}) + \big(I - X_{\sigma,s_q}^{(\varepsilon_q)}\big) U_{h,m}^q.$$

It remains to use the triangle inequality, estimates (3.30) and (3.31), and formula (2.7). $\square$

Now we formulate several special cases of Theorem 3.2.1.

Theorem 3.2.2. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m, q \in \mathbb{N}$, $\sigma, h > 0$, $\beta \in \mathbb{Z}_+$, $\alpha \in [0 : 2q+\beta]$, $\varepsilon, \tau \in \{0, 1\}$, $\tau = 0$ *whenever* $\beta = 0$, $\varepsilon = 0$ *whenever* $\alpha = 2q + \beta$, *and*

$$Y_{\sigma,m,q} = \sum_{k=0}^{q-1} X^{(\tau)}_{\sigma,2k+\beta} U^k_{h,m}(I - U_{h,m}) + X^{(\varepsilon)}_{\sigma,2q+\beta-\alpha} U^q_{h,m}.$$

Then for every $f \in \mathfrak{M}^{(\beta,\tau)} \cap \mathfrak{M}^{(\beta-\alpha,\varepsilon)}$ *we have*

$$P\big(f - Y_{\sigma,m,q} f\big) \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta} h^{2k}} \frac{\nu^k_m}{2^{2k}} P\big(\delta^{2k}_h (I - U_{h,m}) D^\beta J^\tau f\big) + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{\sigma^{2q+\beta-\alpha} h^{2q}} \frac{\nu^q_m}{2^{2q}} P\big(\delta^{2q}_h D^{\beta-\alpha} J^\varepsilon f\big),$$

$$P\big(f - Y_{\sigma,m,q} f\big) \le \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta,\tau}}{\sigma^{2k+\beta} h^{2k}} \nu^k_m\right) \omega_{2m}\big(D^\beta J^\tau f, h\big)_P + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{\sigma^{2q+\beta-\alpha} h^{2q}} \frac{\nu^q_m}{2^{2q}} \omega_{2q}\big(D^{\beta-\alpha} J^\varepsilon f, h\big)_P.$$

Theorem 3.2.3. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, $\sigma, h > 0$, $\beta \in \mathbb{Z}_+$, *and*

$$Y_{\sigma,m} = \sum_{k=0}^{m-1} X_{\sigma,2k+\beta} U^k_{h,m}(I - U_{h,m}) + X_{\sigma,2m+\beta} U^m_{h,m}.$$

Then for every $f \in \mathfrak{M}^{(\beta)}$ *we have*

$$P\big(f - Y_{\sigma,m} f\big) \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{\sigma^{2k+\beta} h^{2k}} \frac{\nu^k_m}{2^{2k}} P\big(\delta^{2k}_h (I - U_{h,m}) D^\beta f\big) + \frac{\mathcal{K}_{2m+\beta}}{\sigma^{2m+\beta} h^{2m}} \frac{\nu^m_m}{2^{2m}} P\big(\delta^{2m}_h D^\beta f\big),$$

$$P\big(f - Y_{\sigma,m} f\big) \le \frac{1}{\sigma^\beta} \left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{(\sigma h)^{2k}} \nu^k_m + \frac{\mathcal{K}_{2m+\beta}}{(\sigma h)^{2m}} \frac{\nu^m_m}{2^{2m}} \right\} \omega_{2m}\big(D^\beta f, h\big)_P.$$

Theorem 3.2.4. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, $\sigma, h > 0$, *and*

$$Y_{\sigma,m} = \sum_{k=1}^{m-1} X_{\sigma,2k} U^k_{h,m}(I - U_{h,m}) + X_{\sigma,2m} U^m_{h,m}. \tag{3.32}$$

Then for every $f \in \mathfrak{M}$ *we have*

$$P\big(f - Y_{\sigma,m} f\big) \le \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}} \frac{\nu^k_m}{2^{2k}} P\big(\delta^{2k}_h (I - U_{h,m}) f\big) + \frac{\mathcal{K}_{2m}}{(\sigma h)^{2m}} \frac{\nu^m_m}{2^{2m}} P\big(\delta^{2m}_h D^\beta f\big),$$

$$P\big(f - Y_{\sigma,m} f\big) \le \left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}} \nu^k_m + \frac{\mathcal{K}_{2m}}{(\sigma h)^{2m}} \frac{\nu^m_m}{2^{2m}} \right\} \omega_{2m}\big(f, h\big)_P.$$

In (3.32) we begin summation from one because $X_{\sigma,0}$ is the zero operator.

Theorem 3.2.5. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, $\sigma, h > 0$, $\beta \in \mathbb{N}$, *and*

$$Y_{\sigma,m} = \sum_{k=0}^{m-1} \widetilde{X}_{\sigma,2k+\beta} U^k_{h,m}(I - U_{h,m}) + \widetilde{X}_{\sigma,2m+\beta} U^m_{h,m}.$$

Then for every $f \in \widetilde{\mathfrak{M}}^{(\beta)}$ *we have*

$$P\big(f - Y_{\sigma,m}f\big) \le \sum_{k=0}^{m-1} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{\sigma^{2k+\beta}h^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m})D^\beta f\big) + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{\sigma^{2m+\beta}h^{2m}} \frac{\nu_m^m}{2^{2m}} P\big(\delta_h^{2m} D^\beta f\big),$$

$$P\big(f - Y_{\sigma,m}f\big) \le \frac{1}{\sigma^\beta}\left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{(\sigma h)^{2k}} \nu_m^k + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{(\sigma h)^{2m}} \frac{\nu_m^m}{2^{2m}} \right\} \omega_{2m}\big(D^\beta Jf, h\big)_P.$$

Remark 3.1. All the operators $Y_{\sigma,m,q}$ constructed above vanish on the functions f orthogonal to $\mathbf{E}_\sigma$. Therefore, for such functions the left-hand sides of the inequalities in Theorems 3.2.1–3.2.5 can be replaced by $P(f)$. This was already established in another way in Subsection 3.1.

Remark 3.2. In [5], the authors constructed a convolution operator with values in $\mathbf{E}_\sigma$ that realizes inequality (3.22). The same technique can be applied to construct convolution operators that realize inequalities (3.7), (3.12), (3.17), and (3.29) in the theorems of Subsection 3.1. There is another approach to the inequalities mentioned, in which they are reduced to estimates of best approximations on classes of convolutions [6, §100]. This approach can lead to sharper estimates, which can also be obtained by linear approximation methods.

Remark 3.3. In a standard way (e.g., approximating f by its Fejér integral), statements of this section can be extended from the sets of continuous functions to the sets $L_\infty(\mathbb{R})$ and L_p $(1 \le p \le \infty)$ with the seminorms $\|\cdot\|_p$, $\omega_s(\cdot, h)_p$, $A_\sigma(\cdot)_p$, $A_{\sigma-0}(\cdot)_p$.

§4. Discussion of the results

With the exception of the space $L_2(\mathbb{R})$, before the authors' papers [4, 5], the Jackson inequalities with explicitly written constants were established only in the periodic case. Therefore, we shall compare results only for spaces of periodic functions. For definiteness, we restrict ourselves to the space C. All the upper estimates are valid in an arbitrary space of periodic functions of class $\mathcal{B}$. The exceptions will be mentioned separately.

It is convenient to write the step h in Jackson-type inequalities in the form $\frac{\gamma\pi}{\sigma}$, where $\gamma > 0$.

4.1. Jackson inequalities for a function itself. Setting $h = \frac{\gamma\pi}{\sigma}$, we write inequality (3.20) as

$$A_{\sigma-0}(f) \le \left\{ \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\gamma\pi)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}} \frac{\nu_m^m}{2^{2m}} \right\} \omega_{2m}\Big(f, \frac{\gamma\pi}{\sigma}\Big). \tag{4.1}$$

We denote by $\varkappa_{2m}(\gamma)$ the best constant in the generalized Jackson inequality

$$E_n(f) \le K \cdot \omega_{2m}\Big(f, \frac{\gamma\pi}{n}\Big),$$

i.e., we put

$$\varkappa_{2m}(\gamma) = \sup_{n\in\mathbb{N}} \sup_{f\in C} \frac{E_n(f)}{\omega_{2m}(f, \frac{\gamma\pi}{n})}.$$

Inequality (4.1) can be written in the form $\varkappa_{2m}(\gamma) \le A_{2m}(\gamma)$, where

$$A_{2m}(\gamma) = \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\gamma\pi)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}} \frac{\nu_m^m}{2^{2m}}.$$

In [1, p. 57, Theorem 3], the estimate $\varkappa_{2m}(\gamma) \le B_{2m}(\gamma)$ was established, where

$$B_{2m}(\gamma) = \frac{1}{\binom{2m}{m}}\left(1 + \frac{\mathcal{K}_{2m} m^{2m}}{(\gamma\pi)^{2m}}\, 2\sum_{j=1}^{m} \frac{\binom{2m}{m-j}}{j^{2m}}\right).$$

The way to obtain this estimate was explained in the Introduction.

Next, to formulate a series of known results, we introduce some additional notation. For $m \in \mathbb{Z}_+$ we put

$$T_m(t) = t\prod_{l=1}^{m}(t^2 - l^2), \quad Q_m(t) = \prod_{l=0}^{m-1}(t^2 - l^2).$$

We shall need the identities (see [11])

$$T_m'(0) = (-1)^m (m!)^2, \qquad T_m'''(0) = (-1)^{m-1} 6 (m!)^2 \sum_{j=1}^{m} \frac{1}{j^2},$$

$$T_m^{(2m+1)}(0) = (2m+1)!, \qquad T_m^{(2m-1)}(0) = -\frac{(2m-1)!}{4}\binom{2m+2}{3},$$

$$Q_m^{(2m)}(0) = (2m)!, \qquad Q_{m+1}^{(2m)}(0) = -\frac{(2m)!}{4}\binom{2m+2}{3}.$$

In [11, Theorem 5], the quantity $\varkappa_{2m}(\gamma)$ was estimated in two ways. Namely, the following inequalities were established: $\varkappa_{2m}(\gamma) \le C_{2m}(\gamma)$ and $\varkappa_{2m}(\gamma) \le D_{2m}(\gamma)$, where

$$C_{2m}(\gamma) = \frac{1}{(2m+1)!}\sum_{l=0}^{m}\binom{2m+1}{2l+1}|T_m^{(2l+1)}(0)|\frac{\mathcal{K}_{2l}}{(\gamma\pi)^{2l}},$$

and $D_{2m,\gamma}$ for $m \ne 2$ is defined by

$$\begin{aligned} D_{2m}(\gamma) &= R_m + r_m(\gamma), \\ r_m(\gamma) &= \frac{1}{(\gamma\pi)^{2m}}\sum_{l=m}^{2m-1}\frac{|Q_l^{(2m)}(0)|}{(2l)!}\min\{\mathcal{K}_{2m}2^{2l-2m}, \mathcal{K}_{4m-2l}(\gamma\pi)^{2l-2m}\}, \\ R_m &= \frac{1}{(4m-2)!}\sum_{j=0}^{m-1}\frac{2^{2m-2j-2}}{2j+1}|T_{2m-1}^{(2j+1)}(0)|. \end{aligned} \tag{4.2}$$

For $m = 2$, a sharper estimate was obtained in [11, Theorem 3]:

$$D_4(\gamma) = \begin{cases} \dfrac{121}{360} + \dfrac{15 + 2\min\{5, 12\gamma^2\}}{1152\gamma^4}, & 0 < \gamma \le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{5} + \dfrac{2352\gamma^2 - 1568\sqrt{2}\,\gamma + 713}{5760\gamma^4}, & \gamma \ge \dfrac{1}{\sqrt{2}}. \end{cases}$$

For $0 < \gamma \le \frac{1}{\sqrt{2}}$ this quantity coincides with that given by formula (4.2), and for $\gamma > \frac{1}{\sqrt{2}}$ it turns out to be smaller.

While comparing the results, one should bear in mind that in [11] the step of the modulus of continuity was written as $\frac{\gamma\pi}{2n}$, so that γ should be replaced by 2γ in the statements from [11]. We also mention that in [11] the author estimated best approximations by odd order moduli of continuity of the first derivative, and by integrated moduli of continuity. Later, the estimates containing $C_{2m}(\gamma)$ and those in terms of the integrated moduli of continuity were included in the book [12, p. 200–203].

Yet another method of proof of the Jackson inequality was applied in [13, p. 158, Corollary 4]. There, the constants were expressed explicitly for $\gamma = 1$, but they are greater than $A_{2m}(1)$, and we shall not give their values here.

TABLE 1

m	$A_{2m}(1)$	$B_{2m}(1)$	$C_{2m}(1)$	$D_{2m}(1)$
2	$0,30093$	$0,44879$	$0,49219$	$0,34714$
3	$0,11614$	$1,5068$	$0,39768$	$0,16432$
4	$0,039893$	$14,112$	$0,32475$	$0,070703$
5	$0,012788$	$221,42$	$0,26770$	$0,029931$
6	$0,0039170$	$5141,4$	$0,22258$	$0,012718$
7	$0,0011623$	16567	$0,18650$	$0,0054672$
8	$0,00033685$	$7,0768 \cdot 10^6$	$0,15731$	$0,0023831$

TABLE 2

m	$A_{2m}(2)$	$B_{2m}(2)$	$C_{2m}(2)$	$D_{2m}(2)$
2	$0,19590$	$0,18423$	$0,24561$	$0,26170$
3	$0,06101$	$0,072762$	$0,11952$	$0,16058$
4	$0,017765$	$0,069355$	$0,059444$	$0,070031$
5	$0,0049962$	$0,22203$	$0,029841$	$0,029810$
6	$0,0013752$	$1,2563$	$0,015065$	$0,012696$
7	$0,00037292$	$10,112$	$0,0076397$	$0,0054632$
8	$0,00010004$	$107,99$	$0,0038902$	$0,0023823$

TABLE 3

m	$A_{2m}(1/2)$	$B_{2m}(1/2)$	$C_{2m}(1/2)$	$D_{2m}(1/2)$
2	$0,98149$	$4,6806$	$1,625$	$0,62778$
3	$0,87403$	$93,282$	$2,6139$	$0,32441$
4	$0,75892$	$3609,1$	$4,2588$	$0,16085$
5	$0,66064$	$2,2673 \cdot 10^5$	$7,0473$	$0,079735$
6	$0,58085$	$2,1059 \cdot 10^7$	$11,809$	$0,039998$
7	$0,51601$	$2,7144 \cdot 10^9$	$19,988$	$0,020337$
8	$0,46249$	$4,6379 \cdot 10^{11}$	$34,105$	$0,010462$

We also notice that all the estimates mentioned are realized by linear approximation methods.

We present the tables of constants for $\gamma = 1$, $\gamma = 2$ and $\gamma = \frac{1}{2}$. For $m = 1$ all four quantities coincide, as well as the methods used to obtain them, and are equal to $\frac{1}{2} + \frac{1}{8\gamma^2}$. For this reason, the tables include the values m beginning with 2. Here are the constants for $m = 2$ expressed as rational fractions (excluding $D_4(1)$ and $D_4(2)$, whose expression involves $\sqrt{2}$):

$$A_4(1) = \frac{65}{216}, \qquad B_4(1) = \frac{517}{1152}, \qquad C_4(1) = \frac{63}{128},$$
$$A_4(2) = \frac{677}{3456}, \qquad B_4(2) = \frac{3397}{18432}, \qquad C_4(2) = \frac{503}{2048},$$
$$A_4\Big(\frac{1}{2}\Big) = \frac{53}{54}, \qquad B_4\Big(\frac{1}{2}\Big) = \frac{337}{72}, \qquad C_4\Big(\frac{1}{2}\Big) = \frac{13}{8}, \qquad D_4\Big(\frac{1}{2}\Big) = \frac{113}{180}.$$

The tables contain approximate values with five significant digits, the last digit is rounded up.

Let us compare the new results with those already known.

First, for m fixed and γ sufficiently large we have

$$B_{2m}(\gamma) < A_{2m}(\gamma) < C_{2m}(\gamma) < D_{2m}(\gamma). \tag{4.3}$$

This is implied by the following expansions as $\gamma \to +\infty$:

$$\begin{aligned}
A_{2m}(\gamma) &= \frac{1}{\binom{2m}{m}}\Big(1 + \frac{\nu_m}{8\gamma^2}\Big) + O(\gamma^{-4}),\\
B_{2m}(\gamma) &= \frac{1}{\binom{2m}{m}} + O(\gamma^{-2m}),\\
C_{2m}(\gamma) &= \frac{1}{\binom{2m}{m}}\left(1 + \frac{2m(2m-1)}{8\gamma^2}\sum_{j=1}^{m}\frac{1}{j^2}\right) + O(\gamma^{-4}),\\
D_{2m}(\gamma) &= R_m + O(\gamma^{-2m}), \quad m \neq 2, \quad D_4(\gamma) > \frac{1}{5},
\end{aligned}$$

and by comparison of coefficients. Holding the first term in the sum R_m, we get the inequality

$$R_m > \frac{2^{2m-2}}{(4m-2)!}\,|T'_{2m-1}(0)| = \frac{2^{2m-2}}{(4m-2)!}\big((2m-1)!\big)^2 = \frac{2^{2m-2}}{\binom{4m-2}{2m-1}} > \frac{1}{\binom{2m}{m}}.$$

The last inequality is proved by induction. Comparison of the coefficients of γ^{-2} in $A_{2m}(\gamma)$ and $C_{2m}(\gamma)$ is obvious.

Tracks of inequality (4.3) can be seen in Table 2 for $m = 2$.

Second, for m fixed and γ sufficiently small we have

$$D_{2m}(\gamma) < C_{2m}(\gamma) < A_{2m}(\gamma) < B_{2m}(\gamma).$$

This is implied by the following expansions as $\gamma \to 0+$:

$$\begin{aligned}
A_{2m}(\gamma) &= \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}}\,\frac{\nu_m^m}{2^{2m}} + O(\gamma^{2-2m}),\\
B_{2m}(\gamma) &= \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}}\,\frac{m^{2m}2}{\binom{2m}{m}}\sum_{j=1}^{m}\frac{\binom{2m}{m-j}}{j^{2m}} + O(\gamma^{2-2m}),\\
C_{2m}(\gamma) &= \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}} + \frac{\mathcal{K}_{2m-2}}{(\gamma\pi)^{2m-2}}\,\frac{\binom{2m+2}{3}}{8} + O(\gamma^{4-2m}),\\
D_{2m}(\gamma) &= \frac{\mathcal{K}_{2m}}{(\gamma\pi)^{2m}} + \frac{\mathcal{K}_{2m-2}}{(\gamma\pi)^{2m-2}}\,\frac{\binom{2m+2}{3}}{4(2m+2)(2m+1)} + O(\gamma^{4-2m})
\end{aligned}$$

and by simple comparison of coefficients. The value $\gamma = \frac{1}{2}$ in Table 3 is not sufficiently small to provide the middle inequality.

Third, in [3], Foucart, Kryakin, and Shadrin showed that $\varkappa_{2m}(\gamma) \geq \frac{1}{\binom{2m}{m}}$ for all $\gamma > 0$. For $\gamma = \frac{2m}{\pi}$ this estimate had been obtained before by G. I. Natanson, see [1, p. 111]. The quantities $\frac{1}{\binom{2m}{m}}$ are the Whitney constants (see [14, Chapter 2]) for the uniform norm on $\mathbb{R}$. By the Stirling formula,

$$\frac{1}{\binom{2m}{m}} \sim \frac{\sqrt{\pi m}}{2^{2m}}. \tag{4.4}$$

In [3], for every $\gamma > 1$, a constant c_γ was found such that $\varkappa_{2m}(\gamma) \leq \frac{c_\gamma}{\binom{2m}{m}}$. In particular, $c_2 < 5$. Thus, the exact asymptotic order of $\varkappa_m(\gamma)$ for $\gamma > 1$, $m \to \infty$, was found. In

the same paper, the authors showed that

$$\varkappa_{2m}(1) = O\left(\frac{\sqrt{m}\,\ln(m+1)}{\binom{2m}{m}}\right). \tag{4.5}$$

In [4, 5] the constant c_γ was reduced and the inequality

$$\varkappa_{2m}(\gamma) \le \frac{1}{\binom{2m}{m}}\left(\cos\frac{\sqrt{\nu_m}}{2\gamma}\right)^{-1} = \frac{1}{\binom{2m}{m}}\sum_{k=0}^{\infty}\frac{\mathcal{K}_{2k}}{(\gamma\pi)^{2k}}\nu_m^k. \tag{4.6}$$

was proved. This made it possible to eliminate the logarithmic factor in (4.5):

$$\varkappa_{2m}(1) = O\left(\frac{\sqrt{m}}{\binom{2m}{m}}\right). \tag{4.7}$$

Inequality (4.6) also follows from estimate (3.22) of the present paper. If $\gamma < 1$, then the series in (4.6) fails to converge for some m (and it may diverge for all m). Therefore, the methods of [3] and [4, 5] based on series expansions are not valid for estimates with $\gamma < 1$.

From Remark 2.4 it follows that, for $\gamma \ge 1$, Theorem 3.1.4 also gives a better result, i.e.,

$$A_{2m}(\gamma) < \frac{1}{\binom{2m}{m}}\left(\cos\frac{\sqrt{\nu_m}}{2\gamma}\right)^{-1}. \tag{4.8}$$

All the said shows that the constants obtained in [3] are greater than both the right-hand side of (4.6) and $A_{2m}(\gamma)$. Therefore, they are not included in the tables above. Thus, *a fortiori*, the sequence $A_{2m}(\gamma)$ has the same exact order of asymptotics for $\gamma > 1$, $m \to \infty$. Unfortunately, replacing the remainder of the series by the only term does not improve the factor of $\frac{1}{\binom{2m}{m}}$ in the main term of the asymptotics. The main advantage is achieved for small m.

In [3], the authors compared their estimates with $B_{2m}(\gamma)$ and found that for small m the quantities $B_{2m}(\gamma)$ are smaller. The values $B_{2m}(\gamma)$ increase rapidly as m increases, which is easily seen from the tables. We shall not study the asymptotics of estimates as $m \to \infty$. Some order relations for the constants in Jackson inequalities in the spaces L_p were obtained in [15].

4.2. Jackson inequities for derivatives. Setting $h = \frac{\gamma\pi}{\sigma}$, we write inequality (3.15) as

$$A_{\sigma-0}(f) \le \frac{1}{\sigma^\beta}\left\{\frac{1}{\binom{2m}{m}}\sum_{k=0}^{m-1}\frac{\mathcal{K}_{2k+\beta}}{(\gamma\pi)^{2k}}\nu_m^k + \frac{\mathcal{K}_{2m+\beta}}{(\gamma\pi)^{2m}}\,\frac{\nu_m^m}{2^{2m}}\right\}\omega_{2m}\Big(D^\beta f, \frac{\gamma\pi}{\sigma}\Big). \tag{4.9}$$

For $r, \beta \in \mathbb{Z}_+$ and $\gamma > 0$, we denote by $\varkappa_{r,\beta}(\gamma)$ the best constant in the generalized Jackson inequality

$$E_n(f) \le \frac{K}{n^\beta}\cdot\omega_r\Big(D^\beta f, \frac{\gamma\pi}{n}\Big),$$

i.e., we put

$$\varkappa_{r,\beta}(\gamma) = \sup_{n\in\mathbb{N}}\sup_{f\in C^{(\beta)}}\frac{n^\beta E_n(f)}{\omega_r(D^\beta f, \frac{\gamma\pi}{n})}.$$

Obviously, $\varkappa_{r,\beta}(\gamma)$ decreases monotonically with respect to γ. We make two other simple observations about the behavior of $\varkappa_{r,\beta}(\gamma)$, which are based on the properties of the moduli of continuity. The inequality $\omega_r \le 2\omega_{r-1}$ yields

$$\varkappa_{r-1,\beta}(\gamma) \le 2\varkappa_{r,\beta}(\gamma).$$

Applying this inequality r times and recalling that $\varkappa_{0,\beta}(\gamma) = \mathcal{K}_\beta$, we get

$$\varkappa_{r,\beta}(\gamma) \geq \frac{\mathcal{K}_\beta}{2^r}.$$

In particular,

$$\liminf_{\beta\to\infty} \varkappa_{r,\beta}(\gamma) \geq \frac{4}{\pi 2^r}. \tag{4.10}$$

The inequality

$$\omega_r(f,h) \leq h\omega_{r-1}(f',h)$$

implies

$$\varkappa_{r-1,\beta+1}(\gamma) \leq \gamma\pi\varkappa_{r,\beta}(\gamma).$$

Applying this inequality r times, we get

$$\varkappa_{r,\beta}(\gamma) \geq \frac{\mathcal{K}_{\beta+r}}{(\gamma\pi)^r}.$$

In particular,

$$\liminf_{\beta\to\infty} \varkappa_{r,\beta}(\gamma) \geq \frac{4}{\pi(\gamma\pi)^r}. \tag{4.11}$$

Inequality (4.9) can be written as $\varkappa_{2m,\beta}(\gamma) \leq A_{2m,\beta}(\gamma)$, where

$$A_{2m,\beta}(\gamma) = \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{(\gamma\pi)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m+\beta}}{(\gamma\pi)^{2m}} \frac{\nu_m^m}{2^{2m}}.$$

Now we exclude the case of $\beta = 0$ from consideration, see the discussion in Subsection 4.1, assuming that $\beta \in \mathbb{N}$. We list the results we know about the values $\varkappa_{r,\beta}(\gamma)$. In [13, p. 167, Theorem 2 and p. 265, Theorem 4], the estimates $\varkappa_{r,\beta}(\gamma) \leq G_{r,\beta}(\gamma)$ and $\varkappa_{r,\beta}(\gamma) \leq H_{r,\beta}(\gamma)$ were established, where $G_{r,\beta}(\gamma)$ and $H_{r,\beta}(\gamma)$ are defined as follows.

If $\gamma \in (0,1]$, then

$$G_{r,\beta}(\gamma) = \begin{cases} \dfrac{4(r+\beta)(r+\beta+1)}{\pi 2^r \beta(\beta+1)\sin^r \frac{\gamma\pi}{2}} & \text{if } r+\beta \text{ is even,} \\ \dfrac{4(r+\beta+1)(r+\beta+2)}{\pi 2^r \beta(\beta+1)\sin^r \frac{\gamma\pi}{2}} & \text{if } r+\beta \text{ is odd.} \end{cases}$$

If $\gamma > 1$, then $G_{r,\beta}(\gamma) = G_{r,\beta}(1)$.

The quantities $H_{r,\beta}(\gamma)$ are expressed in terms of the norms of the Riesz operators $R_{\sigma,s}$. Those are convolution operators whose multiplier functions are

$$\rho_{\sigma,s}(y) = \begin{cases} 1 - \frac{|y|^s}{\sigma^s} & \text{if} \quad |y| \leq \sigma, \\ 0 & \text{if} \quad |y| \geq \sigma. \end{cases}$$

The estimate

$$\frac{4}{\pi^2}\ln(s+1) + 0,1 - \frac{1}{s+1} \leq \sup_{n\in\mathbb{N}} \|R_{n,s}\|_{C\to C} \leq \frac{4}{\pi^2}\ln\Big(s+\frac{1}{2}\Big) + 1,59$$

was established in [16] for the norms of the Riesz operators in the space C of periodic functions. Clearly, for every $\sigma > 0$,

$$\sup_{n\in\mathbb{N}} \|R_{n,s}\|_{C\to C} = \frac{2}{\pi}\int_0^{+\infty} \left|\int_0^1 (1-y^s)\cos yt\,dy\right| dt = N_\infty(R_{\sigma,s}) = N_1(R_{\sigma,s}) \geq N_P(R_{\sigma,s}).$$

The first identity was proved in [17, p. 168, Theorem 2], while the other relations in the chain are standard estimates of a convolution, see [9]. Since the norms $N_\infty(R_{\sigma,s})$ do not depend on σ, we denote them by $\|R_s\|$.

If $r+\beta$ is even, $\gamma \in (0,1]$, then

$$H_{r,\beta}(\gamma) = \frac{2(r+\beta)\|R_{r+\beta}\|}{2^r \beta \sin^r \frac{\gamma\pi}{2}}.$$

If $\gamma > 1$, then $H_{r,\beta}(\gamma) = H_{r,\beta}(1)$. For the spaces $(\mathfrak{M}, P)$ this definition involves the upper bound $\sup_{n\in\mathbb{N}} N_P(R_{n,r+\beta})$.

We also mention a result for an odd order modulus of continuity of the first derivative; this result was obtained in [11, Theorem 5] and states that $\varkappa_{2m-1,1}(\gamma) \le D_{2m-1,1}(\gamma)$, where

$$D_{2m-1,1}(\gamma) = \frac{\gamma\pi}{(2m-1)!}\sum_{l=0}^{m}\binom{2m}{2k}|\Lambda_m^{(2l)}(0)|\frac{\mathcal{K}_{2l}}{(2k+1-2l)(\gamma\pi)^{2l}},$$

$$\Lambda_m(t) = \prod_{j=1}^{m}\Big(t^2 - \Big(j-\frac{1}{2}\Big)^2\Big).$$

We proceed to comparison of results. While considering the majorant $H_{r,\beta}(\gamma)$, we should bear in mind that it is defined only if $r+\beta$ is even.

First, we notice that, by (4.10) and (4.11), for fixed r and γ the estimate $G_{r,\beta}(\gamma)$ gives the best possible asymptotics as $\beta \to \infty$.

Second, for fixed m and γ and for β sufficiently large,

$$G_{2m,\beta}(\gamma) < A_{2m,\beta}(\gamma) < H_{2m,\beta}(\gamma).$$

The estimate $H_{2m,\beta}(\gamma)$ is rough for large β, because $H_{2m,\beta}(\gamma) \to +\infty$ as $\beta \to \infty$. The left inequality follows easily from comparison of the limits of $A_{2m,\beta}(\gamma)$ and $G_{2m,\beta}(\gamma)$ as $\beta \to \infty$.

Third, it is clear that, if β and γ are fixed and $r \to \infty$, then the inequality $H_{r,\beta}(\gamma) < G_{r,\beta}(\gamma)$ is true. On the other hand, for fixed β and $\gamma \ge 1$ and for m sufficiently large, we have $A_{2m,\beta}(\gamma) < H_{2m,\beta}(\gamma)$. To prove this, it should be noted that $A_{2m,\beta}(1) < \frac{K\sqrt{m}}{\binom{2m}{m}}$ (see the definition of $A_{2m,\beta}(\gamma)$ and relations (4.4) and (4.7)), that $A_{2m,\beta}(\gamma) < \frac{K}{\binom{2m}{m}}$ for $\gamma > 1$, and $H_{2m,\beta}(\gamma)$ has an additional factor of logarithmic growth.

The main terms of the estimates of $A_{2m,\beta}(\gamma)$, $G_{2m,\beta}(\gamma)$, and $H_{2m,\beta}(\gamma)$ are ordered differently as $\gamma \to 0+$ and $\gamma \to +\infty$, depending on m and β. We shall not study this issue.

4.3. Jackson inequalities for derivatives of a conjugate function. Setting $h = \frac{\gamma\pi}{\sigma}$, we write inequality (3.27) as

$$A_{\sigma-0}(f) \le \frac{1}{\sigma^\beta}\Big\{\frac{1}{\binom{2m}{m}}\sum_{k=0}^{m-1}\frac{\widetilde{\mathcal{K}}_{2k+\beta}}{(\gamma\pi)^{2k}}\nu_m^k + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{(\gamma\pi)^{2m}}\frac{\nu_m^m}{2^{2m}}\Big\}\omega_{2m}\Big(D^\beta Jf, \frac{\gamma\pi}{\sigma}\Big). \tag{4.12}$$

For $r \in \mathbb{Z}_+$, $\beta \in \mathbb{N}$, and $\gamma > 0$, we denote by $\widetilde{\varkappa}_{r,\beta}(\gamma)$ the best constant in the generalized Jackson inequality

$$E_n(f) \le \frac{K}{n^\beta}\cdot\omega_r\Big(D^\beta Jf, \frac{\gamma\pi}{n}\Big),$$

i.e., we put

$$\widetilde{\varkappa}_{r,\beta}(\gamma) = \sup_{n\in\mathbb{N}}\sup_{f\in\widetilde{C}^{(\beta)}}\frac{n^\beta E_n(f)}{\omega_r(D^\beta Jf, \frac{\gamma\pi}{n})}.$$

Inequality (4.12) can be written as $\widetilde{\varkappa}_{2m,\beta}(\gamma) \le \widetilde{A}_{2m,\beta}(\gamma)$, where

$$\widetilde{A}_{2m,\beta}(\gamma) = \frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\widetilde{\mathcal{K}}_{2k+\beta}}{(\gamma\pi)^{2k}} \nu_m^k + \frac{\widetilde{\mathcal{K}}_{2m+\beta}}{(\gamma\pi)^{2m}} \frac{\nu_m^m}{2^{2m}}.$$

All the results we know about the quantities $\widetilde{\varkappa}_{r,\beta}(\gamma)$ are contained in [13, p. 168, Theorem 3 and p. 265, Theorem 5]. There, the estimates $\widetilde{\varkappa}_{r,\beta}(\gamma) \le \widetilde{G}_{r,\beta}(\gamma)$ and $\widetilde{\varkappa}_{r,\beta}(\gamma) \le \widetilde{H}_{r,\beta}(\gamma)$ were established, where $\widetilde{G}_{r,\beta}(\gamma)$ and $\widetilde{H}_{r,\beta}(\gamma)$ are defined as follows.

If $\gamma \in (0,1]$, then

$$\widetilde{G}_{r,\beta}(\gamma) = \begin{cases} \dfrac{4(r+\beta)(r+\beta+1)}{\pi 2^r \beta(\beta+1)\sin^r \frac{\gamma\pi}{2}} & \text{if} \quad r+\beta \text{ is odd,} \\ \dfrac{4(r+\beta+1)(r+\beta+2)}{\pi 2^r \beta(\beta+1)\sin^r \frac{\gamma\pi}{2}} & \text{if} \quad r+\beta \text{ is even.} \end{cases}$$

If $r+\beta$ is odd, $\gamma \in (0,1]$, then

$$\widetilde{H}_{r,\beta}(\gamma) = \frac{2(r+\beta)\|R_{r+\beta}\|}{2^r \beta \sin^r \frac{\gamma\pi}{2}},$$

where, as in Subsection 4.2, the $\|R_{r+\beta}\|$ are the norms of the Riesz operators. If $\gamma > 1$, then $\widetilde{G}_{r,\beta}(\gamma) = \widetilde{G}_{r,\beta}(1)$, $\widetilde{H}_{r,\beta}(\gamma) = \widetilde{H}_{r,\beta}(1)$.

The conclusions similar to those in Subsection 4.2 are true for the quantity $\widetilde{\varkappa}_{r,\beta}(\gamma)$ and for its estimates. Considering the majorant $\widetilde{H}_{r,\beta}(\gamma)$, one should bear in mind that it is defined only if $r+\beta$ is odd.

§5. Jackson inequalities for best approximation by splines, and their realization by linear methods

5.1. Estimates for best approximations. In this section, for $n \in \mathbb{N}$ we denote by $\mathbf{S}_{2n,\gamma}$ the $2n$-dimensional space of 2π-periodic splines of order $\gamma \in \mathbb{Z}_+$ and defect 1 with respect to the uniform partition $\frac{k\pi}{n}$ $(k \in \mathbb{Z})$. In other words, for $\gamma \in \mathbb{N}$, this is the set of functions of class $C^{(\gamma-1)}$ whose restriction to each interval $\left(\frac{k\pi}{n}, \frac{(k+1)\pi}{n}\right)$ is an algebraic polynomial of order at most γ. For $\gamma = 0$, this is the set of functions that are constant on the intervals mentioned above. It is known [2, Corollary 2.3.6] that $\mathbf{S}_{2n,\gamma}$ coincides with the set of functions of the form

$$s(t) = \beta + \sum_{j=0}^{2n-1} \beta_j d_{\gamma+1}\Big(t - \frac{j\pi}{n}\Big), \quad \sum_{j=0}^{2n-1} \beta_j = 0,$$

where $d_{\gamma+1}$ is the Bernoulli kernel of order $\gamma+1$. The set of splines in $\mathbf{S}_{2n,\gamma}$ for which $\sum_{j=0}^{2n-1}(-1)^j\beta_j = 0$ is denoted by $\mathbf{S}^{\times}_{2n,\gamma}$. The dimension of the space $\mathbf{S}^{\times}_{2n,\gamma}$ equals $2n-1$. For $\gamma \in \mathbb{Z}_+$, we denote by $E_{n,\gamma}$ and $E^{\times}_{n,\gamma}$ the best approximations by the spaces $\mathbf{S}_{2n,\gamma}$ and $\mathbf{S}^{\times}_{2n,\gamma}$, respectively, and put $E_{n,-1}(f)_p = E^{\times}_{n,-1}(f)_p = \|f\|_p$.

The inequalities of Akhiezer–Krein–Favard type for best spline approximation are amply known only for the classes $W_p^{(r)}$ of periodic functions. Therefore, while using our general theorems, we restrict ourselves to those cases. We apply the statements proved to the functionals $E_{n,\gamma}(\cdot)_p$ and $E^{\times}_{n,\gamma}(\cdot)_p$, $p \in [1,+\infty]$. Recall that $\mathcal{K}_r$ denotes the Favard constants and the coefficients ν_m are defined by (2.2).

It is known [2, 18] that

$$E_{n,\gamma}(f)_p \le \frac{\mathcal{K}_r}{n^r}\|D^r f\|_p, \quad \gamma \ge r-1, \tag{5.1}$$

$$E^{\times}_{n,\gamma}(f)_p \le \frac{\mathcal{K}_r}{n^r}\|D^r f\|_p, \quad \gamma \ge r. \tag{5.2}$$

Inequality (5.1) for $p=1$ and $p=+\infty$ is contained in [2, Subsection 5.2.4 and Proposition 5.4.9]; see the historical comments there; inequality (5.2) for $p=1$ and $p=+\infty$ can be found in [19]; the general case was considered in [18, Lemma 7].

Theorem 5.1.1. *Suppose that $p \in [1,+\infty]$, $n,m,q,\gamma \in \mathbb{N}$, $h>0$, $\{s_k\}_{k=0}^q \subset \mathbb{Z}_+$, $\gamma \ge \max_{0\le k\le q} s_k - 1$, and $\mathfrak{N} = \bigcap_{k=0}^q W_p^{(s_k-2k)}$. Then for every $f \in \mathfrak{N}$ we have*

$$E_{n,\gamma}(f)_p \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{s_k}}{n^{s_k}h^{2k}}\big\|W^k_{h,m}\delta_h^{2k}D^{s_k-2k}(I-U_{h,m})f\big\|_p + \frac{\mathcal{K}_{s_q}}{n^{s_q}h^{2q}}\big\|W^q_{h,m}\delta_h^{2q}D^{s_q-2q}f\big\|_p. \tag{5.3}$$

If $\gamma \ge \max_{0\le k\le q} s_k$, then E can be replaced by $E^{\times}$.

Proof. To prove (5.3), Theorem 2.1 should be applied to the functionals $\Phi(f) = E_{n,\gamma}(f)_p$ and $\Phi(f) = E^{\times}_{n,\gamma}(f)_p$; we should take into account that, by (5.1) and (5.2), $N_p(\Phi, D^s) \le \mathcal{K}_s n^{-s}$, for $\gamma \ge s-1$ and $\gamma \ge s$, respectively. □

Remark 5.1. Theorem 5.1.1 can be refined. Namely, under the same conditions we have

$$E_{n,\gamma}(f)_p \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{s_k}}{n^{s_k}h^{2k}} E_{n,\gamma-s_k}\big(W^k_{h,r,m}\delta_h^{2k}D^{s_k-2k}(I-S_{h,r,m})f\big)_p + \frac{\mathcal{K}_{s_q}}{n^{s_q}h^{2q}} E_{n,\gamma-s_q}\big(W^q_{h,r,m}\delta_h^{2q}D^{s_q-2q}f\big)_p.$$

If $\gamma \ge \max_{0\le k\le q} s_k$, then E can be replaced by $E^{\times}$. To prove this, we can use the inequalities (see the comments to (5.1) and (5.2))

$$E_{n,\gamma}(f)_p \le \frac{\mathcal{K}_r}{n^r} E_{n,\gamma-r}(D^r f)_p, \quad \gamma \ge r-1,$$
$$E^{\times}_{n,\gamma}(f)_p \le \frac{\mathcal{K}_r}{n^r} E^{\times}_{n,\gamma-r}(D^r f)_p, \quad \gamma \ge r.$$

In what follows, while formulating special cases of Theorem 5.1.1, we do not mention this improvement.

We present several particular cases of Theorem 5.1.1.

Theorem 5.1.2. *Suppose that $p \in [1,+\infty]$, $n,m,q,\gamma \in \mathbb{N}$, $h>0$, $\beta \in \mathbb{Z}_+$, $\alpha \in [0 : 2q+\beta]$, and $\gamma \ge 2q+\beta-1-\min\{2,\alpha\}$. Then for every $f \in W_p^{(\beta)}$ we have*

$$E_{n,\gamma}(f)_p \le \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta}}{n^{2k+\beta}h^{2k}}\,\frac{\nu_m^k}{2^{2k}}\,\big\|\delta_h^{2k}(I-U_{h,m})D^\beta f\big\|_p + \frac{\mathcal{K}_{2q+\beta-\alpha}}{n^{2q+\beta-\alpha}h^{2q}}\,\frac{\nu_m^q}{2^{2q}}\,\big\|\delta_h^{2q}D^{\beta-\alpha}f\big\|_p,$$

$$E_{n,\gamma}(f)_p \le \Bigg(\frac{1}{\binom{2m}{m}}\sum_{k=0}^{q-1}\frac{\mathcal{K}_{2k+\beta,\tau}}{n^{2k+\beta}h^{2k}}\,\nu_m^k\Bigg)\omega_{2m}\big(D^\beta f,h\big)_p + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{n^{2q+\beta-\alpha}h^{2q}}\,\frac{\nu_m^q}{2^{2q}}\,\omega_{2q}\big(D^{\beta-\alpha}f,h\big)_p.$$

If $\gamma \ge 2q+\beta-\min\{2,\alpha\}$, then E can be replaced by $E^{\times}$.

Theorem 5.1.3. *Suppose that $p \in [1,+\infty]$, $n,m,\gamma \in \mathbb{N}$, $h>0$, $\beta \in \mathbb{Z}_+$, and $\gamma \geq 2m+\beta-1$. Then for every $f \in W_p^{(\beta)}$ we have*

$$E_{n,\gamma}(f)_p \leq \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{n^{2k+\beta}h^{2k}} \frac{\nu_m^k}{2^{2k}} \left\|\delta_h^{2k}(I-U_{h,m})D^\beta f\right\|_p + \frac{\mathcal{K}_{2m+\beta}}{n^{2m+\beta}h^{2m}} \frac{\nu_m^m}{2^{2m}} \left\|\delta_h^{2m}D^\beta f\right\|_p,$$

$$E_{n,\gamma}(f)_p \leq \frac{1}{n^\beta}\left\{\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{(nh)^{2k}}\nu_m^k + \frac{\mathcal{K}_{2m+\beta}}{(nh)^{2m}} \frac{\nu_m^m}{2^{2m}}\right\}\omega_{2m}\big(D^\beta f,h\big)_p.$$

If $\gamma \geq 2m+\beta$, then E can be replaced by $E^\times$.

Theorem 5.1.4. *Suppose that $p \in [1,+\infty]$, $n,m,\gamma \in \mathbb{N}$, $h>0$, and $\gamma \geq 2m-1$. Then for every $f \in L_p$ we have*

$$E_{n,\gamma}(f)_p \leq \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(nh)^{2k}} \frac{\nu_m^k}{2^{2k}} \left\|\delta_h^{2k}(I-U_{h,m})f\right\|_p + \frac{\mathcal{K}_{2m}}{(nh)^{2m}} \frac{\nu_m^m}{2^{2m}} \left\|\delta_h^{2m}f\right\|_p,$$

$$E_{n,\gamma}(f)_p \leq \left\{\frac{1}{C_{2m}^m} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(nh)^{2k}}\nu_m^k + \frac{\mathcal{K}_{2m}}{(nh)^{2m}} \frac{\nu_m^m}{2^{2m}}\right\}\omega_{2m}\big(f,h\big)_p.$$

If $\gamma \geq 2m$, then E can be replaced by $E^\times$.

5.2. Realization of estimates for best approximations by linear methods. Like for approximation by entire functions, the estimates in Subsection 5.1 can be obtained by linear approximation methods.

For $n,r,\gamma \in \mathbb{N}$, $\gamma \geq r$, we denote by $X_{n,r,\gamma}$ the linear operators that act from L to $\mathbf{S}_{2n,\gamma}^\times$ and realize the estimate

$$\left\|f - X_{n,r,\gamma}f\right\|_p \leq \frac{\mathcal{K}_r}{n^r}\|f^{(r)}\|_p \tag{5.4}$$

for all $p \in [1,+\infty]$, $f \in W_p^{(r)}$. The linear operators with values in $\mathbf{S}_{2n,\gamma}$ that realize estimate (5.4) are denoted by $X_{n,r,r-1}$. From [2, Subsection 5.2.4] we know that the role of $X_{n,r,r-1}$ can be played by interpolational operators. The operators $X_{n,r,\gamma}$ for $\gamma \geq r$ were constructed in [19], estimates (5.4) for them were obtained in [19] ($p=1,\infty$) and in [18, Lemma 7] ($1<p<+\infty$). We assume that $X_{n,0,\gamma}$ is the zero operator.

With the help of $X_{n,r,\gamma}$, it is possible to construct linear operators realizing the estimates that occur in the theorems of Subsection 5.1. We shall denote them by $Y_{n,m,q,\gamma}$ or (if $q=m$) by $Y_{n,m,\gamma}$, omitting the other parameters, so that, in the sequel, $Y_{n,m,q,\gamma}$ and $Y_{n,m,\gamma}$ denote different operators in different statements.

Our theorems are numerated as follows: if a theorem for best approximations in Subsection 5.1 has the number 5.1.x, then a similar theorem for a linear method has the number 5.2.x.

Theorem 5.2.1. *Suppose that $p \in [1,+\infty]$, $n,m,q,\gamma \in \mathbb{N}$, $h>0$, $\{s_k\}_{k=0}^q \subset \mathbb{Z}_+$, $\gamma \geq \max_{0\leq k\leq q} s_k - 1$, $\mathfrak{N} = \bigcap_{k=0}^q W_p^{(s_k-2k)}$, and*

$$Y_{n,m,q,\gamma} = \sum_{k=0}^{q-1} X_{n,s_k,\gamma}U_{h,m}^k(I-U_{h,m}) + X_{n,s_q,\gamma}U_{h,m}^q.$$

Then for every $f \in \mathfrak{N}$ we have

$$\left\|f - Y_{n,m,q,\gamma}f\right\|_p \leq \sum_{k=0}^{q-1} \frac{\mathcal{K}_{s_k}}{n^{s_k}h^{2k}}\left\|W_{h,m}^k\delta_h^{2k}D^{s_k-2k}(I-U_{h,m})f\right\|_p + \frac{\mathcal{K}_{s_q}}{n^{s_q}h^{2q}}\left\|W_{h,m}^q\delta_h^{2q}D^{s_q-2q}f\right\|_p.$$

Proof. The proof is similar to that of Theorem 3.2.1; inequalities (5.4) should be used. □

We formulate some special cases of Theorem 5.2.1.

Theorem 5.2.2. *Suppose that* $p \in [1, +\infty]$, $n, m, q, \gamma \in \mathbb{N}$, $h > 0$, $\beta \in \mathbb{Z}_+$, $\alpha \in [0 : 2q + \beta]$, $\gamma \geq 2q + \beta - 1 - \min\{2, \alpha\}$, *and*

$$Y_{n,m,q,\gamma} = \sum_{k=0}^{q-1} X_{n,2k+\beta,\gamma} U_{h,m}^k (I - U_{h,m}) + X_{n,2q+\beta-\alpha,\gamma} U_{h,m}^q.$$

Then for every $f \in W_p^{(\beta)}$ *we have*

$$\begin{aligned}\big\|f - Y_{n,m,q,\gamma} f\big\|_p \leq \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta}}{n^{2k+\beta} h^{2k}} \frac{\nu_m^k}{2^{2k}} \big\|\delta_h^{2k} (I - U_{h,m}) D^\beta f\big\|_p \\ + \frac{\mathcal{K}_{2q+\beta-\alpha}}{n^{2q+\beta-\alpha} h^{2q}} \frac{\nu_m^q}{2^{2q}} \big\|\delta_h^{2q} D^{\beta-\alpha} f\big\|_p,\end{aligned}$$

$$\begin{aligned}\big\|f - Y_{n,m,q,\gamma} f\big\|_p \leq \left(\frac{1}{\binom{2m}{m}} \sum_{k=0}^{q-1} \frac{\mathcal{K}_{2k+\beta,\tau}}{n^{2k+\beta} h^{2k}} \nu_m^k\right) \omega_{2m}\big(D^\beta f, h\big)_p \\ + \frac{\mathcal{K}_{2q+\beta-\alpha,\varepsilon}}{n^{2q+\beta-\alpha} h^{2q}} \frac{\nu_m^q}{2^{2q}} \omega_{2q}\big(D^{\beta-\alpha} f, h\big)_p.\end{aligned}$$

Theorem 5.2.3. *Suppose that* $p \in [1, +\infty]$, $n, m, \gamma \in \mathbb{N}$, $h > 0$, $\beta \in \mathbb{Z}_+$, $\gamma \geq 2m + \beta - 1$, *and*

$$Y_{n,m,\gamma} = \sum_{k=0}^{m-1} X_{n,2k+\beta,\gamma} U_{h,m}^k (I - U_{h,m}) + X_{n,2m+\beta,\gamma} U_{h,m}^m.$$

Then for every $f \in W_p^{(\beta)}$ *we have*

$$\big\|f - Y_{n,m,\gamma} f\big\|_p \leq \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{n^{2k+\beta} h^{2k}} \frac{\nu_m^k}{2^{2k}} \big\|\delta_h^{2k} (I - U_{h,m}) D^\beta f\big\|_p + \frac{\mathcal{K}_{2m+\beta}}{n^{2m+\beta} h^{2m}} \frac{\nu_m^m}{2^{2m}} \big\|\delta_h^{2m} D^\beta f\big\|_p,$$

$$\big\|f - Y_{n,m,\gamma} f\big\|_p \leq \frac{1}{n^\beta} \left\{\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k+\beta}}{(nh)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m+\beta}}{(nh)^{2m}} \frac{\nu_m^m}{2^{2m}}\right\} \omega_{2m}\big(D^\beta f, h\big)_p.$$

Theorem 5.2.4. *Suppose that* $p \in [1, +\infty]$, $n, m, \gamma \in \mathbb{N}$, $h > 0$, $\gamma \geq 2m - 1$, *and*

$$Y_{n,m,\gamma} = \sum_{k=1}^{m-1} X_{n,2k,\gamma} U_{h,m}^k (I - U_{h,m}) + X_{n,2m,\gamma} U_{h,m}^m. \tag{5.5}$$

Then for every $f \in L_p$ *we have*

$$\big\|f - Y_{n,m,\gamma} f\big\|_p \leq \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(nh)^{2k}} \frac{\nu_m^k}{2^{2k}} \big\|\delta_h^{2k} (I - U_{h,m}) f\big\|_p + \frac{\mathcal{K}_{2m}}{(nh)^{2m}} \frac{\nu_m^m}{2^{2m}} \big\|\delta_h^{2m} f\big\|_p,$$

$$\big\|f - Y_{n,m,\gamma} f\big\|_p \leq \left\{\frac{1}{\binom{2m}{m}} \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(nh)^{2k}} \nu_m^k + \frac{\mathcal{K}_{2m}}{(nh)^{2m}} \frac{\nu_m^m}{2^{2m}}\right\} \omega_{2m}\big(f, h\big)_p.$$

In (5.5), summation starts from one because $X_{n,0,\gamma}$ is the zero operator.

Remark 5.2. In [20, Remark 4], a $(2n-1)$-dimensional subspace $\mathbf{Q}_{2n-1}^\times$ of the space L_∞ was constructed with the following property: if $f \perp \mathbf{Q}_{2n-1}^\times$, then $X_{n,r,\gamma} f = 0$ for every $r \in [1 : \gamma]$. Therefore, under the conditions on the inequalities for $E^\times$, for such functions f the left-hand sides of the inequalities of this subsection can be replaced by $\|f\|_p$.

Remark 5.3. It is known that $X_{n,r,\gamma} \longrightarrow_{\gamma\to\infty} X_{n,r}$, e.g., in the operator norm from L to C, see [19, Remark 11]. Therefore, the estimates of §3 that concern approximation by trigonometric polynomials in the spaces L_p can be obtained by the limit passage in the estimates for approximations by splines.

In conclusion of this section, we emphasize that, elsewhere, we have not met Jackson inequalities for best approximations by splines and high order moduli of continuity with constants written explicitly.

§6. Additional estimates

In this section we introduce yet two other versions of estimates of functionals, based on the same general technique. In contrast to the previous statements, we shall not formulate every special case of this modifications systematically, but restrict ourselves to several examples.

6.1. Using the moments of smaller order. While deducing Theorem 2.2 from Theorem 2.1, we used the moment of order $2k+\beta$ to estimate the kth term. In the case of best approximations this yielded the factor $\sigma^{-2k-\beta}h^{-2k}$.

We can achieve a similar effect if we use the moment of order $2k-2m+\beta$, and get the missing factor h^{-2m} by estimating the deviation $I-U_{h,m}$. The estimate of a functional by modulus of continuity of order $2m$ is now provided by the difference operator of order $2k$. This operation can be applied to the terms with numbers $k \ge m$, so that we must put $q > m$. For definiteness, we formulate a claim of the type of Theorem 2.4, which involves no conjugate functions or derivatives.

Theorem 6.1. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\Phi \in \mathcal{F}(\mathfrak{M})$, $m, q \in \mathbb{N}$, $q > m$, *and* $h > 0$. *Then for every* $f \in \mathfrak{M}$ *we have*

$$\begin{aligned}\Phi(f) \le \sum_{k=0}^{m-1} \frac{N_P(\Phi, D^{2k})}{h^{2k}}\,\frac{\nu_m^k}{2^{2k}}\,P\big(\delta_h^{2k}(I-U_{h,m})f\big)\\ + \sum_{k=m}^{q-1} \frac{N_P(\Phi, D^{2k-2m})}{h^{2k-2m}}\,\frac{\nu_m^k}{2^{2k}}\,\frac{1}{(2m+1)(m+1)\binom{2m}{m}}\,P\big(\delta_h^{2k}f\big)\\ + \frac{N_P(\Phi, D^{2q})}{h^{2q}}\,\frac{\nu_m^q}{2^{2q}}\,P\big(\delta_h^{2q}f\big).\end{aligned}$$

Proof. In Theorem 2.1, we put $\alpha=\beta=0$, $\varepsilon_k=0$ for all k, $s_k=2k$ for $k\in[0:m-1]$ and for $k=q$, and $s_k=2k-2m$ for $k\in[m:q-1]$. We continue the estimation in the inequality (2.8). For $k\in[0:m-1]$ and $k=q$, we estimate the terms with the help of the right-hand inequality in (2.6), as earlier. For $k\in[m:q-1]$ we have

$$P\big(W_{h,m}^k\delta_h^{2k}D^{-2m}(I-U_{h,m})f\big) \le \frac{\nu_m^k}{2^{2k}}\,P\big(\varphi - U_{h,m}\varphi\big),$$

where $\varphi=\delta_h^{2k}D^{-2m}f$. Now it remains to use the inequality

$$P\big(\varphi-U_{h,m}\varphi\big) \le \frac{2}{\binom{2m}{m}}\int_0^h t^{2m}\frac{1}{h}\Big(1-\frac{t}{h}\Big)\,dt\,P(D^{2m}\varphi) = \frac{h^{2m}}{(2m+1)(m+1)\binom{2m}{m}}P(D^{2m}\varphi).$$

□

We concretize Theorem 6.1 for best approximations, restricting ourselves to one inequality.

Theorem 6.2. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\Phi \in \mathcal{F}(\mathfrak{M})$, $m, q \in \mathbb{N}$, $q > m$, σ, *and* $h > 0$. *Then for every* $f \in \mathfrak{M}$ *we have*

$$\begin{aligned} A_{\sigma-0}(f)_P \le{}& \sum_{k=0}^{m-1} \frac{\mathcal{K}_{2k}}{(\sigma h)^{2k}} \frac{\nu_m^k}{2^{2k}} P\big(\delta_h^{2k}(I - U_{h,m})f\big) \\ &+ \sum_{k=m}^{q-1} \frac{\mathcal{K}_{2k-2m}}{(\sigma h)^{2k-2m}} \frac{\nu_m^k}{2^{2k}} \frac{1}{(2m+1)(m+1)\binom{2m}{m}} P\big(\delta_h^{2k} f\big) + \frac{\mathcal{K}_{2k}}{(\sigma h)^{2q}} \frac{\nu_m^q}{2^{2q}} P\big(\delta_h^{2q} f\big). \end{aligned} \tag{6.1}$$

If we estimate the right-hand side by $\omega_{2m}(f,h)_P$, Theorem 6.2 leads to a loss in comparison with Theorem 3.1.4, but the right-hand sides of (3.19) and (6.1) themselves are noncomparable.

6.2. Expansions with respect to differences of increasing order. The sequence $\{2^{-l}P(\delta_h^l f)\}_{l=0}^{\infty}$ decreases and is nonnegative. So, it has a finite limit

$$\eta(f,h)_P = \lim_{l\to\infty} 2^{-l} P\big(\delta_h^l f\big).$$

The following refinements of (3.1) and (3.2) are known:

$$A_{\sigma-0}(f)_P \le \sum_{l=0}^{\infty} a_{\sigma,r,l} P(\delta_{\frac{\pi}{\sigma}}^l D^r f) + \left(\frac{\mathcal{K}_r}{\sigma^r} - \sum_{l=0}^{\infty} 2^l a_{\sigma,r,l}\right)\eta\Big(D^r f, \frac{\pi}{\sigma}\Big)_P, \tag{6.2}$$

$$A_{\sigma-0}(f)_P \le \sum_{l=0}^{\infty} \widetilde{a}_{\sigma,r,l} P(\delta_{\frac{\pi}{\sigma}}^l D^r J f) + \left(\frac{\widetilde{\mathcal{K}}_r}{\sigma^r} - \sum_{l=0}^{\infty} 2^l \widetilde{a}_{\sigma,r,l}\right)\eta\Big(D^r J f, \frac{\pi}{\sigma}\Big)_P. \tag{6.3}$$

Here the $a_{\sigma,r,l}$ and $\widetilde{a}_{\sigma,r,l}$ are some positive coefficients. Moreover, the factors of η are nonnegative. We shall not give explicit expressions for these coefficients; the numbers $a_{\sigma,r,l}$ are linear combinations of the Favard constants. The left-hand sides of (6.2) and (6.3) can be replaced by $P(f - X_{\sigma,r} f)$ and $P\big(f - \widetilde{X}_{\sigma,r} f\big)$, respectively. Inequality (6.2) was obtained in the periodic case by Zhuk [13, Corollary 2], and in the nonperiodic case by Merlina [21]; inequality (6.3) was obtained by Vinogradov [9, Corollary 6]. Applying relations (6.2) and (6.3) instead of (3.1) and (3.2), we get subtler although bulkier estimates. We give an example.

Theorem 6.3. *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $m \in \mathbb{N}$, *and* $\sigma, h > 0$. *Then for every* $f \in \mathfrak{M}$ *we have*

$$\begin{aligned} A_{\sigma-0}(f)_P \le{}& \sum_{k=0}^{m-1} \frac{\nu_m^k}{(2h)^{2k}} \bigg\{ \sum_{l=0}^{\infty} a_{\sigma,2k,l} P\Big(\delta_{\frac{\pi}{\sigma}}^l \delta_h^{2k}(I - U_{h,m})f\Big) \\ &+ \left(\frac{\mathcal{K}_{2k}}{\sigma^{2k}} - \sum_{l=0}^{\infty} 2^l a_{\sigma,2k,l}\right)\eta\Big(\delta_h^{2k}(I - U_{h,m})f, \frac{\pi}{\sigma}\Big)_P \bigg\} \\ &+ \frac{\nu_m^m}{(2h)^{2m}} \bigg\{ \sum_{l=0}^{\infty} a_{\sigma,2m,l} P\Big(\delta_{\frac{\pi}{\sigma}}^l \delta_h^{2m} f\Big) + \left(\frac{\mathcal{K}_{2m}}{\sigma^{2m}} - \sum_{l=0}^{\infty} 2^l a_{\sigma,2m,l}\right)\eta\Big(\delta_h^{2m} f, \frac{\pi}{\sigma}\Big)_P \bigg\}. \end{aligned} \tag{6.4}$$

As usual, applying the inequalities proved above to the seminorm $A_{\sigma-0}(\cdot)_P$, we get estimates that are formally stronger and whose right-hand sides involve best approximations. The left-hand side of (6.4) can be replaced by $P(f - Y_{\sigma,m} f)$, where the operator $Y_{\sigma,m}$ is defined by (3.32). In [22], spline analogs of inequality (6.2) were obtained. Their application makes it possible to refine the theorems of §5.

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