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UNIFORM ESTIMATE FOR A SEGMENT FUNCTION IN TERMS OF A POLYNOMIAL STRIP

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Abstract. A problem about a uniform estimate for a continuous segment function in terms of a polynomial strip is considered. The problem reduces to a convex programming problem the target function of which is equal to the sum of the target functions for the inner and outer estimate of the same segment function via a polynomial strip. Convex analysis tools are used to obtain necessary and sufficient conditions for being a solution, and also uniqueness conditions in a form resembling the Chebyshov alternance.

Introduction

Approximation and estimation of many-valued maps can be viewed as a field of research within the nonsmooth analysis (see [1, 2, 3, 4, 5]). As a simplest example of a many-valued map, we mention the segment function (s.f.), i.e., a map assigning a segment of the real axis to each value of its argument. As an object for approximation or an object to be used in a problem setting, the segment function of one variable have been employed for a long time. For example, the problem of approximation of the graph of an s.f. by the graph of a polynomial of a given degree (in the Hausdorff metric of the 2-dimensional space) was considered by Sendov in [6] and also by other authors (see the references in [6]). Also, the "water snakes" problem (see [7, 8]) has long been known, which consists in finding polynomials of a given degree n the graphs of which lie in the graph of an s.f and touch the lower and upper boundary of the latter graph $n+1$ times alternately (as the argument grows). In the paper [9], the problem was treated about the construction of a polynomial strip of minimal width (with respect to the ordinate) that contains the graph of a given s.f.

In this paper, we deal with the problem of finding a uniform estimate on a segment for a continuous s.f. by a simpler s.f. whose graph is a strip of constant width (with respect to the ordinate) with axis given by a polynomial of a given degree.

Our goal is to obtain necessary and sufficient conditions for a function to solve this problem, and also uniqueness conditions in a form resembling the Chebyshev alternance.

The plan of the paper is as follows. In §1 we formalize the problem mathematically and compare it to some known problems. Also in §1, we formulate problems about the inner and outer estimates of an s.f. by a polynomial strip; the results of [9] concerning this problem will be used in the present paper. In §2, our problem is reduced to a convex programming problem the target function of which is the sum of two convex target functions corresponding to the problems on inner and outer estimates of the same s.f. by a polynomial strip. This allows us to state a criterion for solution of the problem under study in terms of convex analysis and to prove the existence of a solution. We employ

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the specific features of the subdifferential of our target function to obtain necessary (§3) and sufficient (§4) conditions for a function to yield a solution, while in §5 we present uniqueness conditions in a form resembling the Chebyshev alternance. In §6 we give some illustrative examples.

§1. Setting of the problem

Let an s.f. $F(t) = [f_1(t), f_2(t)]$ be given on a segment $[c, d]$ by two continuous functions $f_1(t)$ and $f_2(t)$ such that $f_1(t) \le f_2(t)$ for all $t \in [c, d]$. In what follows, we denote by $P_n(A, t) = a_0 + a_1 t + \cdots + a_n t^n$ the polynomial of degree n with the coefficient vector $A = (a_0, a_1, \dots, a_n)$. The main object for us will be the problem

$$\varphi(A, r) \equiv \max_{t\in[c,d]} \max\{|f_1(t) - P_n(A, t) + r|, |f_2(t) - P_n(A, t) - r|\} \to \min_{\substack{A\in\mathbb{R}^{n+1}\\ r\ge 0}}. \tag{1.1}$$

Here, the quantity $\max\{|f_1(t)-P_n(A,t)+r|, |f_2(t)-P_n(A,t)-r|\}$ expresses the Hausdorff distance between the segment $F(t)$ and the segment $\Phi(A, r, t) = [P_n(A, t)-r, P_n(A, t)+r]$, where $r \ge 0$. Thus, problem (1.1) deals with the best approximation, uniform on $[c, d]$, of the s.f. $F(t)$ by a simpler s.f. $\Phi(A, r, t)$ the graph of which is a polynomial strip of width $2r$ with the axis $P_n(A, t)$.

If $f_1(t) \equiv f_2(t)$, $t \in [c, d]$, then, obviously, problem (1.1) reduces to the Chebyshev problem on uniform approximation of a continuous function by a polynomial of a given degree. Problem (1.1) can be associated with the problem on approximation of the graph of an s.f. by the graph of a polynomial in the Hausdorff metric of the 2-dimensional space, treated by Sendov in the book [6] and also by other authors (see the references in [6]). In the present paper, for problem (1.1) it is important that the target function $\varphi(A, r)$ is convex with respect to the variable (A, r) on $\mathbb{R}^{n+1} \times \mathbb{R}_+$, which allows us to use the tools of convex analysis. The example in [6, Chapter IV, §4.1] shows that the problem considered in [6] is not a convex programming problem, under the specific choice of the Hausdorff metric [6, Chapter 2, §2.2].

We shall need some results pertaining to the following problems:

$$\rho(A) \equiv \max_{t\in[c,d]} \max\{P_n(A, t) - f_1(t), f_2(t) - P_n(A, t)\} \to \min_{A\in\mathbb{R}^{n+1}}, \tag{1.2}$$

$$\pi(A) \equiv \max_{t\in[c,d]} \max\{f_1(t) - P_n(A, t), P_n(A, t) - f_2(t)\} \to \min_{A\in\mathbb{R}^{n+1}}. \tag{1.3}$$

The geometric meaning of problem (1.2) is in finding a polynomial strip of the smallest width (with respect to the ordinate) that contains the graph of an s.f. $F(\cdot)$ on the segment $[c, d]$. If a coefficient vector A^* solves problem (1.2), then the graph of the polynomial $P_n(A^*, \cdot)$ gives the axis of the required strip, and its width is $2\rho(A^*)$. We call (1.2) the *outer estimate problem*, it was considered in [9], where criteria of its solvability were obtained, together with some uniqueness conditions for its solution.

Problem (1.3) has geometric interpretation of its own. Namely, if an s.f. $F(\cdot)$ admits at least one polynomial selector (this can be assumed without loss of generality, see [9]), then in problem (1.3) it is required to construct a polynomial strip of maximal width contained in the graph of $F(\cdot)$. We call (1.3) the *inner estimate problem*.

Compared to problems (1.2) and (1.3), problem (1.1) can naturally be called the *uniform estimate problem*.

It should be noted that outer and inner estimates for many-valued maps (m.-v.m.) were studied in many publications (see, e.g., the ellipsoidal estimates in [10, 11]). There are a few papers devoted to uniform estimates (uniform approximation) of an m.-v.m. on a given set. In [12], the uniform approximation problem was considered for a constant convex-valued map given on a segment. In [13], approximation was realized by constant

ball-valued maps. If the role of an m.-v.m. to be approximated is played by an s.f., then the problems treated in [12] and [13] turn into problem (1.1) with $n = 0$.

§2. Reduction to a convex programming problem

In this section, we reduce problem (1.1) to a convex programming problem with simpler formulation and without the parameter r. Also, we state a criterion for being a solution in the language of convex analysis and prove the existence of a solution.

Lemma 2.1. *For any $A \in \mathbb{R}^{n+1}$ and $r \geq 0$, we have*

$$\varphi(A, r) = \max\{\rho(A) - r, \pi(A) + r\}. \tag{2.1}$$

Proof. Indeed, we have

$$\begin{aligned}
\varphi(A, r) = &\max_{t\in[c,d]} \max\{P_n(A,t) - f_1(t) - r, f_2(t) - P_n(A,t) - r, \\
&\qquad f_1(t) - P_n(A,t) + r, P_n(A,t) - f_2(t) + r\} \\
= &\max\{\max_{t\in[c,d]} \max\{P_n(A,t) - f_1(t) - r, f_2(t) - P_n(A,t) - r\}, \\
&\max_{t\in[c,d]} \max\{f_1(t) - P_n(A,t) + r, P_n(A,t) - f_2(t) + r\}\} \\
= &\max\{\max_{t\in[c,d]} \max\{P_n(A,t) - f_1(t), f_2(t) - P_n(A,t)\} - r, \\
&\max_{t\in[c,d]} \max\{f_1(t) - P_n(A,t), P_n(A,t) - f_2(t)\} + r\} \\
= &\max\{\rho(A) - r, \pi(A) + r\}.
\end{aligned}$$

□

Theorem 2.1. *Problem* (1.1) *is equivalent to the problem*

$$\psi(A) \equiv \rho(A) + \pi(A) \to \min_{A\in\mathbb{R}^{n+1}} \tag{2.2}$$

in the following sense. If a pair (A^*, r^*) delivers the minimal value to the function $\varphi(A, r)$ in problem (1.1)*, then A^* is a point of minimum of the function $\psi(A)$ in problem* (2.2)*. Conversely, if A^* is a point of minimum of $\psi(A)$ in* (2.2)*, then the pair (A^*, r^*), where $r^* = (\rho(A^*) - \pi(A^*))/2$, gives the minimal value for $\varphi(A, r)$ in* (1.1)*, and*

$$\varphi(A^*, r^*) = (\rho(A^*) + \pi(A^*))/2.$$

Proof. By (2.1), we can write problem (1.1) in the form

$$\varphi(A, r) = \max\{\rho(A) - r, \pi(A) + r\} \to \min_{\substack{A\in\mathbb{R}^{n+1} \\ r\geq 0}}. \tag{2.3}$$

For any fixed $A \in \mathbb{R}^{n+1}$, the minimal value of $\varphi(A, r)$ with respect to $r \in \mathbb{R}$ is attained at $r = r(A)$, where $r(A) = (\rho(A) - \pi(A))/2$. The inequality $f_1(t) \leq f_2(t)$, which is fulfilled by assumption for all $t \in [c, d]$, implies that $\rho(A) \geq \pi(A)$ for any $A \in \mathbb{R}^{n+1}$. Therefore, we always have $r(A) \geq 0$. Substituting this quantity for r, we arrive at the simple relationship

$$\min_{r\geq 0} \varphi(A, r) = \varphi(A, r(A)) = (\rho(A) + \pi(A))/2,$$

which shows that problem (2.3) and, hence, problem (1.1) are equivalent to problem (2.2). □

It is easily seen that the functions $\rho(A)$ and $\pi(A)$ are convex and finite on $\mathbb{R}^{n+1}$. This allows us to use Theorem 2.1 to state the following criterion for being a solution of problem (1.1) in terms of the subdifferentials $\partial\rho(A)$ and $\partial\pi(A)$ of the functions $\rho(A)$ and $\pi(A)$. We denote $\mathbb{O}_{n+1} = (0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.

Theorem 2.2. *A pair* (A^*, r^*) *delivers the minimal value to the function* $\varphi(A, r)$ *in problem* (1.1) *if and only if* $r^* = (\rho(A^*) - \pi(A^*))/2$ *and*

$$\mathbb{O}_{n+1} \in \partial\rho(A^*) + \partial\pi(A^*). \tag{2.4}$$

Proof. The relation $r^* = (\rho(A^*) - \pi(A^*))/2$ for the optimal pair (A^*, r^*) is a consequence of Theorem 2.1.

Since $\rho(A)$ and $\pi(A)$ are finite and convex (and hence, continuous, see [2, Chapter I, §4]), the Morean–Rockafeller theorem [1, Chapter 2, §3] yields

$$\partial\psi(A) = \partial(\rho(A) + \pi(A)) = \partial\rho(A) + \partial\pi(A)).$$

Since problems (1.1) and (2.2) are equivalent, it remains to observe that, in accordance with a well-known fact of convex analysis (see [1, Chapter IV, §2]), we have

$$\psi(A^*) = \min_{A \in \mathbb{R}^{n+1}} \psi(A) \quad \Longleftrightarrow \quad \mathbb{O}_{n+1} \in \partial\psi(A^*).$$

The theorem is proved. □

Theorem 2.3. *Problem* (1.1) *is solvable.*

Proof. Denote

$$\begin{aligned} G_\rho(\alpha) &= \{A \in \mathbb{R}^{n+1} \,:\, \rho(A) \le \alpha\}, \quad G_\pi(\beta) = \{A \in \mathbb{R}^{n+1} \,:\, \pi(A) \le \beta\}, \\ G^r_\varphi(\gamma) &= \{A \in \mathbb{R}^{n+1} \,:\, \varphi(A, r) \le \gamma\}, \\ G(\gamma) &= \{(A, r) \in \mathbb{R}^{n+1} \times \mathbb{R}_+ \,:\, A \in G_\pi(\gamma),\, r = (\rho(A) - \pi(A))/2\}. \end{aligned}$$

Let $(\widehat{A}, \widehat{r})$ be an arbitrary pair in $\mathbb{R}^{n+1} \times \mathbb{R}_+$, and let $\gamma = \varphi(\widehat{A}, \widehat{r})$. By (2.1), we have

$$G^r_\varphi(\gamma) = G_\rho(\gamma + r) \cap G_\pi(\gamma - r) \subset G_\pi(\gamma). \tag{2.5}$$

Using Theorem 2.1 and formula (2.5), we get

$$\begin{aligned} \operatorname{Arg} \min_{\substack{A \in \mathbb{R}^{n+1} \\ r \ge 0}} \varphi(A, r) &\subset \{(A, r) \in \mathbb{R}^{n+2} \,:\, A \in G^r_\varphi(\gamma),\ r = (\rho(A) - \pi(A))/2\} \\ &\subset \{(A, r) \in \mathbb{R}^{n+2} \,:\, A \in G_\pi(\gamma),\, r = (\rho(A) - \pi(A))/2\} = G(\gamma). \end{aligned} \tag{2.6}$$

In [9] it was shown that the lower Lebesgue sets $G_\rho(\alpha)$ of the function $\rho(A)$ are compact and convex. By analogy, it is not hard to check that the same is true also for the sets $G_\pi(\gamma)$. This implies that the set $G(\gamma)$ is compact, because the convex and finite functions $\rho(A)$ and $\pi(A)$ are continuous. Since the function $\varphi(A, r)$ is continuous and

$$\min_{\substack{A \in \mathbb{R}^{n+1} \\ r \ge 0}} \varphi(A, r) = \min_{(A, r) \in G(\gamma)} \varphi(A, r),$$

which is a consequence of (2.6), it follows that problem (1.1) admits a solution. □

Remark 2.1. As was mentioned in §1, the quantity $\rho(A)$ is half the minimal width of a polynomial strip with the axis $P_n(A, t)$ that contains the graph of the s.f. $F(t)$. If $P_n(A, t) \in F(t)$ for $t \in [c, d]$, then the quantity $\pi(A)$ is half the maximal width (with the minus sign) of a polynomial strip contained in the graph of $F(t)$. There is no loss of generality in assuming that the condition stated above is satisfied. This follows from the fact that the approximation problem for the new s.f. $\widehat{F}(t) = [f_1(t) - \alpha, f_2(t) + \alpha]$, where $\alpha > 0$, remains equivalent to problem (1.1).

Therefore, problem (2.2) and, hence, problem (1.1) can be viewed as constructing a polynomial layer of minimal width that contains the boundary of the s.f. $F(t)$, i.e., the graphs of $f_1(t)$ and $f_2(t)$. In other words, we need to embed the graph of $f_1(t)$ in a polynomial strip of minimal width of the form $[P_n(A, t) - \rho(A), P_n(A, t) + \pi(A)]$,

and simultaneously, the graph of $f_2(t)$ should be embedded in the polynomial strip $[P_n(A,t)-\pi(A), P_n(A,t)+\rho(A)]$, for $t\in[c,d]$.

Observe yet another interpretation of problem (1.1). Its target function can also be written in the form

$$\varphi(A,r)=\max_{t\in[c,d]}\max\{P_n(A,t)-g_1(t,r), g_2(t,r)-P_n(A,t)\},$$

where $g_1(t,r)=\min\{f_1(t)+r, f_2(t)-r\}$, $g_2(t,r)=\max\{f_1(t)+r, f_2(t)-r\}$. Therefore, comparison to (1.2) allows us to view problem (1.1) as the construction of a polynomial strip of minimal width that contains the graph of the s.f. $G(t,r)=[g_1(t,r), g_2(t,r)]$, by using the additional possibility of the vertical shift of the graphs of $f_1(t)$ and $f_2(t)$ by r.

§3. Necessary conditions for being a solution

The inclusion (2.4) is a criterion for being a solution of (1.1) in the language of convex analysis. In this section we use the specific nature of the subdifferentials of $\rho(A)$ and $\pi(A)$ to obtain necessary conditions for being a solution of problem (1.1) in a form comparable to the Chebyshev alternance.

3.1. We present some auxiliary information. Introduce the sets

$$\begin{aligned}
R_1^\rho(A)&=\{t\in[c,d]\,:\,\rho(A)=P_n(A,t)-f_1(t)>f_2(t)-P_n(A,t)\},\\
R_2^\rho(A)&=\{t\in[c,d]\,:\,\rho(A)=f_2(t)-P_n(A,t)>P_n(A,t)-f_1(t)\},\\
R_3^\rho(A)&=\{t\in[c,d]\,:\,\rho(A)=P_n(A,t)-f_1(t)=f_2(t)-P_n(A,t)\},\\
R^\rho(A)&=R_1^\rho(A)\cup R_2^\rho(A)\cup R_3^\rho(A).
\end{aligned}$$

In the paper [9], these sets and also the subdifferential calculus (see [1, 2, 3, 5]) were employed for obtaining a formula for the subdifferential of the function $\rho(A)$:

$$\partial\rho(A)=\operatorname{co}\{\xi^\rho(A,t)(1,t,\dots,t^n)\,:\,t\in R^\rho(A)\}, \tag{3.1}$$

where $\operatorname{co} D$ means the convex hull of a set D, and $\xi^\rho(A,t)$ is a function many-valued for some t. This function is defined on $R^\rho(A)$ by the formula

$$\xi^\rho(A,t)=\begin{cases}1 & \text{if } t\in R_1^\rho(A),\\ -1 & \text{if } t\in R_2^\rho(A),\\ [-1,1] & \text{if } t\in R_3^\rho(A).\end{cases} \tag{3.2}$$

Similarly, denoting

$$\begin{aligned}
R_1^\pi(A)&=\{t\in[c,d]\,:\,\pi(A)=P_n(A,t)-f_2(t)>f_1(t)-P_n(A,t)\},\\
R_2^\pi(A)&=\{t\in[c,d]\,:\,\pi(A)=f_1(t)-P_n(A,t)>P_n(A,t)-f_2(t)\},\\
R_3^\pi(A)&=\{t\in[c,d]\,:\,\pi(A)=P_n(A,t)-f_2(t)=f_1(t)-P_n(A,t)\},\\
R^\pi(A)&=R_1^\pi(A)\cup R_2^\pi(A)\cup R_3^\pi(A),
\end{aligned}$$

we can easily get a formula for the subdifferential of the function $\pi(A)$:

$$\partial\pi(A)=\operatorname{co}\{\xi^\pi(A,t)(1,t,\dots,t^n)\,:\,t\in R^\pi(A)\}, \tag{3.3}$$

where $\xi^\pi(A,t)$ is the following many-valued function defined on $R^\pi(A)$:

$$\xi^\pi(A,t)=\begin{cases}1 & \text{if } t\in R_1^\pi(A),\\ -1 & \text{if } t\in R_2^\pi(A),\\ [-1,1] & \text{if } t\in R_3^\pi(A).\end{cases} \tag{3.4}$$

Note that if the graph of the s.f. $F(t)$ itself is not a polynomial strip, then $R_3^\rho(A)\cap R_3^\pi(A)=\varnothing$, and $R_i^\rho(A)\cap R_j^\pi(A)=\varnothing$ for $i\neq j$.

Let $T \subset \mathbb{R}$ be a set on which a many-valued function $\xi(\cdot) : T \to 2^{\mathbb{R}}$ is defined. The next lemma is a generalization of a statement in [14, Chapter VI, §8].

Lemma 3.1. *The relation*

$$\mathbb{O}_{n+1} \in \operatorname{co}\{\xi(t)(1, t, \dots, t^n) : t \in T\} \tag{3.5}$$

is fulfilled if and only if one of the following conditions is satisfied:

1) *there exists a point* $t_0 \in T$ *such that* $0 \in \xi(t_0)$;

2) *there exists a selector* $\eta(t) \in \xi(t)$ *and a collection* $t_1 < t_2 < \dots < t_{n+2}$ *of points of* T *such that* $\eta(t_i) \neq 0$ *and* $\operatorname{sgn} \eta(t_i) = -\operatorname{sgn} \eta(t_{i+1})$, $i = 1, \dots, n+1$.

Proof. The "only if" part. Suppose (3.5) is true. By the Carathéodory theorem (see [2, Chapter I, §1]), there is a collection of $n+2$ points $T_0 = \{t_i\}_{i=1,\dots,n+1} \subset T$, a corresponding collection of numbers $\{\alpha_i\}_{i=1,\dots,n+1}$ with $\alpha_i \geq 0$, $\sum_{i=1}^{n+2} \alpha_i = 1$, and a selector $\eta(t) \in \xi(t)$, $t \in T$, such that

$$\sum_{i=1}^{n+2} \alpha_i \eta(t_i)(1, t_i, \dots, t_i^n) = \mathbb{O}_{n+1}. \tag{3.6}$$

We assume that the collection T_0 is ordered: $t_1 < t_2 < \dots < t_{n+2}$.

Suppose that condition 1) is not fulfilled. Then $\eta(t_i) \neq 0$, $i = 1, \dots, n+2$. We prove that T_0 and $\eta(t)$ satisfy condition 2).

Indeed, otherwise the collection T_0 can be split into $m \leq n+1$ disjoint and consecutively located collections $T_0 = \bigcup_{k=1,\dots,m} T_k$ such that the function $\eta(t)$ keeps its sign, but changes it when passing to the next collection T_k. Now we pick arbitrary points $x_1 < x_2 < \dots < x_{m-1}$ that separate the collections $\{T_k\}_{k=1,\dots,m}$:

$$\max_{t \in T_k} t < x_k < \min_{t \in T_{k+1}} t, \quad k = 1, \dots, m-1,$$

and consider the polynomial $P_{m-1}(B, t) = \prod_{i=1}^{m-1}(t - x_i)$. The points $\{x_i\}_{i=1,\dots,m}$ are its zeros, it keeps its sign on each of the sets T_k, and it changes the sign when passing from T_k to T_{k+1}. Thus, assuming for definiteness that m is even, we have

$$(-1)^k \langle B, (1, t, \dots, t^{m-1}) \rangle < 0, \quad t \in T_k, \quad k = 1, \dots, m-1. \tag{3.7}$$

Therefore, since the behavior of the selector $\eta(t)$ on the collection of sets T_0 is synchronized with that of the polynomial $P_{m-1}(B, t)$, from (3.7) we deduce that

$$\langle B, \alpha_i \eta(t_i)(1, t_i, \dots, t_i^{m-1}) \rangle < 0, \quad i = 1, 2, \dots, n+2, \tag{3.8}$$

provided $\alpha_i > 0$ (we assume that $\eta(t_1) > 0$ for definiteness). But among the nonnegative numbers $\{\alpha_i\}_{i=1,\dots,n+2}$ these is at least one positive α_i; consequently, (3.8) implies

$$\left\langle B, \sum_{i=1}^{n+2} \alpha_i \eta(t_i)\big(1, t_i, \dots, t_i^{m-1}\big) \right\rangle < 0.$$

Since $m - 1 \leq n$, this contradicts (3.6), proving the "only if" part.

The "if" part. Suppose condition 2) is fulfilled. We prove that

$$\mathbb{O}_{n+1} \in \operatorname{co}\{\eta(t_i)(1, t_i, \dots, t_i^n) : i = 1, \dots, n+2\} \equiv D. \tag{3.9}$$

Suppose this fails. Since the set D is closed, the separation theorem (see [1, Chapter I, §2]) shows that there is a vector $A \neq \mathbb{O}_{n+1}$ such that

$$\langle A, v \rangle < 0, \quad v \in D. \tag{3.10}$$

Then, since the elements $v_i = \eta(t_i)(1, t_i, \dots, t_i^n)$ belong to D, we have

$$\langle A, v_i \rangle < 0, \quad i = 1, \dots, n+2. \tag{3.11}$$

Assuming that $\eta(t_1) > 0$ for definiteness and recalling that the signs of $\eta(t)$ at the points $\{t_i\}_{i=1,\dots,n+2}$ are alternating, from (3.11) we deduce that $(-1)^i P_n(A,t_i) < 0$, $i = 1,\dots,n+2$. Thus, the polynomial $P_n(A,t)$ of degree $m-1 \le n$ changes its sign $n+1$ times at some points located consecutively. Therefore, it has at least $n+1$ zeros, which is impossible because $A \ne \mathbb{O}_{n+1}$. This proves (3.9) and, with it, (3.5).

The fact that condition 1) is sufficient is obvious. The lemma is proved. □

3.2. Now we state necessary conditions for being a solution. For this, we denote

$$R_i(A) = R_i^\rho(A) \cup R_i^\pi(A), \;\; i = 1,2,3; \quad R(A) = \bigcup_{i=1,2,3} R_i(A)$$

and give the following definition.

Definition 3.1. We say that the *$\rho\pi$-alternance* occurs for a coefficient vector $A \in \mathbb{R}^{n+1}$ if in $R_1(A) \cup R_2(A)$ there exists an ordered collection $t_1 < t_2 < \dots < t_{n+2}$ of points such that if $t_i \in R_1(A)\big(R_2(A)\big)$, then $t_{i+1} \in R_2(A)\ \big(R_1(A)\big)$, $i = 1,\dots,n+1$.

Theorem 3.1. *If the function $\varphi(A,r)$ in (1.1) attains its minimum at a pair (A^*,r^*), then $r^* = (\rho(A^*) - \pi(A^*))/2$ and at least one of the conditions below is fulfilled:*
1) *$R_3(A^*) \ne \varnothing$,*
2) *the $\rho\pi$-alternance occurs for the vector A^*.*

Proof. Let (A^*,r^*) be a solution of problem (1.1). Then Theorem 2.2 yields the idensity $r^* = (\rho(A^*) - \pi(A^*))/2$, and relation (2.4) is true, which implies that

$$\mathbb{O}_{n+1} \in \mathrm{co}\{\partial\rho(A^*), \partial\pi(A^*)\}. \tag{3.12}$$

Formulas (3.1)–(3.4) show that

$$\mathrm{co}\{\partial\rho(A^*), \partial\pi(A^*)\} = \mathrm{co}\{\xi(A^*,t)(1,t,\dots,t^n) \,:\, t \in R(A^*)\}, \tag{3.13}$$

where

$$\xi(A^*,t) = \begin{cases} 1 & \text{if } \; t \in R_1(A^*), \\ -1 & \text{if } \; t \in R_2(A^*), \\ [-1,1] & \text{if } \; t \in R_3(A^*). \end{cases} \tag{3.14}$$

Using Lemma 3.1 and (3.13)–(3.14), we conclude that relation (3.12) is fulfilled if and only if either $R_3(A^*) \ne \varnothing$ or the $\rho\pi$-alternance occurs for A^*. The theorem is proved. □

Remark 3.1. In general, the above necessary conditions are not sufficient, because they realize (interpret) relation (3.12), which is weaker than (2.4) (see Examples 6.1 and 6.2).

3.3. Now we want to estimate the range of the possible values of the optimal strip's width for problem (1.1), assuming that we know the solutions of the simpler problems (1.2) and (1.3).

We put

$$\begin{aligned} \rho^* &= \min_{A\in\mathbb{R}^{n+1}} \rho(A), \quad \Omega_\rho = \{A \in \mathbb{R}^{n+1} \,:\, \rho(A) = \rho^*\}, \\ \pi^* &= \min_{A\in\mathbb{R}^{n+1}} \pi(A), \quad \Omega_\pi = \{A \in \mathbb{R}^{n+1} \,:\, \pi(A) = \pi^*\}, \\ \rho^- &= \min_{A\in\Omega_\pi} \rho(A), \quad \pi^- = \min_{A\in\Omega_\rho} \pi(A). \end{aligned} \tag{3.15}$$

Theorem 3.2. *Let d^* be the width of one of the optimal polynomial strips for problem (1.1). Then*

$$\rho^* - \pi^- \le d^* \le \rho^- - \pi^*. \tag{3.16}$$

Proof. Consider the function

$$f(r) = \min_{A\in\mathbb{R}^{n+1}} \varphi(A, r).$$

Since the functions $\rho(A)$ and $\pi(A)$ are convex and finite on $\mathbb{R}^{n+1}$, it follows (see (2.1)) that the function $\varphi(A, r)$ is convex and finite on $\mathbb{R}^{n+2}$. Consequently (see [2, Chapter I, §4]), the function $f(r)$ is convex and finite on $\mathbb{R}_+$. Recalling the definition of $f(t)$ and formula (2.1), we see that

$$f(r) \geq \max\{\rho^* - r, \pi^* + r\}. \tag{3.17}$$

On the other hand,

$$\begin{aligned} f(r) &\equiv \min_{A\in\mathbb{R}^{n+1}} \max\{\rho(A) - r, \pi(A) + r\} \\ &\leq \min_{A\in\Omega_\rho} \max\{\rho(A) - r, \pi(A) + r\} = \max\{\rho^* - r, \pi^- + r\}, \end{aligned} \tag{3.18}$$

and

$$f(r) \leq \min_{A\in\Omega_\pi} \max\{\rho(A) - r, \pi(A) + r\} = \max\{\rho^- - r, \pi^* + r\}. \tag{3.19}$$

Inequalities (3.17)–(3.18) imply that

$$f(r) = \rho^* - r \quad \text{if} \quad 0 \leq r \leq (\rho^* - \pi^-)/2,$$

and from (3.17) and (3.19) we get

$$f(r) = \pi^* + r \quad \text{if} \quad r \geq (\rho^- - \pi^*)/2.$$

Thus, the convex function $f(r)$ decreases linearly on the segment $[0, (\rho^* - \pi^-)/2]$, and increases linearly for $r \geq (\rho^- - \pi^*)/2$. Consequently, its minimal value is attained at some r belonging to the segment $[(\rho^* - \pi^-)/2, (\rho^- - \pi^*)/2]$, and the claim follows. □

§4. Sufficient conditions for being a solution

4.1. We introduce the auxiliary notation

$$\widetilde{R}_i^\rho(A) = R_i^\rho(A) \cup R_3^\rho(A), \quad \widetilde{R}_i^\pi(A) = R_i^\pi(A) \cup R_3^\pi(A), \ \ i = 1, 2.$$

Lemma 4.1. *If*

$$\mathbb{O}_{n+1} \notin \partial\rho(A) + \partial\pi(A), \tag{4.1}$$

then there exists a vector $A^* \neq \mathbb{O}_{n+1}$ *such that the following relations are true*:

$$P_n(A^*, x_\rho) < P_n(A^*, y_\pi) \qquad \textit{for all} \quad x_\rho \in \widetilde{R}_1^\rho(A),\ y_\pi \in \widetilde{R}_2^\pi(A), \tag{4.2}$$

$$P_n(A^*, x_\pi) < P_n(A^*, y_\rho) \qquad \textit{for all} \quad x_\pi \in \widetilde{R}_1^\pi(A),\ y_\rho \in \widetilde{R}_2^\rho(A), \tag{4.3}$$

$$P_n(A^*, x_\rho) + P_n(A^*, x_\pi) < 0 \qquad \textit{for all} \quad x_\rho \in \widetilde{R}_1^\rho(A),\ x_\pi \in \widetilde{R}_1^\pi(A), \tag{4.4}$$

$$P_n(A^*, y_\rho) + P_n(A^*, y_\pi) > 0 \qquad \textit{for all} \quad y_\rho \in \widetilde{R}_2^\rho(A),\ y_\pi \in \widetilde{R}_2^\pi(A). \tag{4.5}$$

Proof. The sum of the compact convex sets $\partial\rho(A)$ and $\partial\pi(A)$ is convex itself. Therefore, by the separation theorem (see [1, Chapter I, § 2]), (4.1) implies the existence of a vector $A^* \neq \mathbb{O}_{n+1}$ such that

$$\langle A^*, v + w\rangle < 0 \quad \text{for all} \quad v \in \partial\rho(A),\ \ w \in \partial\pi(A). \tag{4.6}$$

Obviously, formulas (3.1)–(3.4) for the subdifferentials of $\rho(A)$ and $\pi(A)$ can be written in the form

$$\partial\rho(A) = \operatorname{co}\begin{cases}(1,t,\dots,t^n) & \text{if } t\in\widetilde{R}_1^\rho(A),\\ -(1,t,\dots,t^n) & \text{if } t\in\widetilde{R}_2^\rho(A),\end{cases} \tag{4.7}$$

$$\partial\pi(A) = \operatorname{co}\begin{cases}(1,t,\dots,t^n) & \text{if } t\in\widetilde{R}_1^\pi(A),\\ -(1,t,\dots,t^n) & \text{if } t\in\widetilde{R}_2^\pi(A).\end{cases} \tag{4.8}$$

Then, since $\langle A^*,(1,t,\dots,t^n)\rangle = P_n(A^*,t)$, substitution of the corresponding elements of (4.7), (4.8) in inequality (4.6) yields (4.3)–(4.5). The lemma is proved. □

The following simple fact should also be mentioned.

Lemma 4.2. *If a polynomial $P_n(A,t)$ is nonconstant, then there are at most $n-1$ points $t_1<t_2<\dots<t_m$ such that in the intervals $(-\infty,t_1),(t_1,t_2),\dots,(t_{m-1},t_m)$, (t_m,∞) this polynomial is alternately monotone increasing and monotone decreasing.*

4.2. First, we present some simple but tough sufficient conditions for being a solution of problem (1.1), obtained by realization of the relation

$$\mathbb{O}_{n+1}\in\partial\rho(A^*)\cap\partial\pi(A^*), \tag{4.9}$$

which ensures (2.4).

By a well-known fact of convex analysis (see, e.g., [1, Chapter IV, §2]), relation (4.9) means that $A^*\in\Omega_\rho\cap\Omega_\pi$, i.e., A^* is a solution of (1.2) and (1.3) simultaneously.

Definition 4.1. We say that the *ρ-alternance* occurs for a coefficient vector A if there exists a collection $t_1<t_2<\dots<t_{n+2}$ of points in $R_1^\rho(A)\cup R_2^\rho(A)$ such that $t_i\in R_1^\rho(A)$ $\big(R_2^\rho(A)\big)$ implies $t_{i+1}\in R_2^\rho(A)$ $\big(R_1^\rho(A)\big)$, $i=1,\dots,n+1$.

Definition 4.2. We say that the *π-alternance* occurs for a coefficient vector A if there exists a collection $t_1<t_2<\dots<t_{n+2}$ of points in $R_1^\pi(A)\cup R_2^\pi(A)$ such that $t_i\in R_1^\pi(A)$ $\big(R_2^\pi(A)\big)$ implies $t_{i+1}\in R_2^\pi(A)$ $\big(R_1^\pi(A)\big)$, $i=1,\dots,n+1$.

Theorem 4.1. *Suppose that a coefficient vector A^* satisfies the following conditions*:

1) *$R_3^\rho(A^*)\neq\varnothing$ or the ρ-alternance occurs for A^**;

2) *$R_3^\pi(A^*)\neq\varnothing$ or the π-alternance occurs for A^*.*

Then the pair (A^*,r^*), where $r^*=(\rho(A^*)-\pi(A^*))/2$, gives a minimal value to the function $\varphi(A,r)$ in problem (1.1).

Proof. Lemma 3.1 and formulas (3.1)–(3.4) show that if conditions 1) and 2) are satisfied, then $\mathbb{O}_{n+1}\in\partial\rho(A^*)$ and $\mathbb{O}_{n+1}\in\partial\pi(A^*)$, which ensures (2.4).

Consequently, by Theorem 2.2, the pair (A^*,r^*) with $r^*=(\rho(A^*)-\pi(A^*))/2$ solves problem (1.1). □

Condition 1) is equivalent to $\mathbb{O}_{n+1}\in\partial\rho(A^*)$ and means that $A^*\in\Omega_\rho$. Accordingly, condition 2) means that $A^*\in\Omega_\pi$. Therefore, Theorems 4.1 and 2.1 imply the following statement.

Corollary 4.1. *If $\Omega_\rho\cap\Omega_\pi\neq\varnothing$, then the set of solutions of problem* (1.1) *is the set of pairs $(A^*,(\rho^*-\pi^*)/2)$, where $A^*\in\Omega_\rho\cap\Omega_\pi$.*

Theorem 4.2. *Suppose that for a coefficient vector A^* in the set $R(A^*)$ we can find a collection of $n+1$ pairs of points $\{t_i^{(1)}<t_i^{(2)}\}$, $i=1,\dots,n+1$, such that the following conditions are satisfied*:

1) *if $t_i^{(1)}\in\widetilde{R}_1^\rho(A^*)$ $\big(\widetilde{R}_2^\rho(A^*),\ \widetilde{R}_1^\pi(A^*),\ \widetilde{R}_2^\pi(A^*)\big)$, then, respectively,*

$$t_i^{(2)}\in\widetilde{R}_2^\pi(A^*)\ \big(\widetilde{R}_1^\pi(A^*),\ \widetilde{R}_2^\rho(A^*),\ \widetilde{R}_1^\rho(A^*)\big),\quad i=1,\dots,n+1;$$

2) *if* $t_i^{(2)} \in \widetilde{R}_1^\rho(A^*) \cup \widetilde{R}_1^\pi(A^*)$ $\big(\widetilde{R}_2^\rho(A^*) \cup \widetilde{R}_2^\pi(A^*)\big)$, *then*

$$t_{i+1}^{(1)} \in \widetilde{R}_1^\rho(A^*) \cup \widetilde{R}_1^\pi(A^*) \ \big(\widetilde{R}_2^\rho(A^*) \cup \widetilde{R}_2^\pi(A^*)\big), \quad i = 1, \dots, n;$$

3) $t_i^{(1)} < t_{i+1}^{(1)}$, $t_i^{(2)} < t_{i+1}^{(2)}$, $t_i^{(2)} < t_{i+2}^{(1)}$, $i = 1, \dots, n$.

Then the pair (A^*, r^*), *where* $r^* = (\rho(A^*) - \pi(A^*))/2$, *gives a minimal value to the function* $\varphi(A, r)$ *in problem* (1.1).

Proof. Theorem 2.2 shows that it suffices to prove relation (2.4). Suppose the contrary, i.e.,

$$\mathbb{O}_{n+1} \notin \partial\rho(A^*) + \partial\pi(A^*). \tag{4.10}$$

Then, by Lemma 4.1, there exists a vector $\widehat{A} \neq \mathbb{O}_{n+1}$ such that

$$P_n(\widehat{A}, x_\rho) < P_n(\widehat{A}, y_\pi) \quad \text{for all} \quad x_\rho \in \widetilde{R}_1^\rho(A^*),\ y_\pi \in \widetilde{R}_2^\pi(A^*), \tag{4.11}$$

$$P_n(\widehat{A}, x_\pi) < P_n(\widehat{A}, y_\rho) \quad \text{for all} \quad x_\pi \in \widetilde{R}_1^\pi(A^*),\ y_\rho \in \widetilde{R}_2^\rho(A^*). \tag{4.12}$$

Consider the behavior of the polynomial $P_n(\widehat{A}, t)$ on a collection of pairs $\{t_i^{(1)} < t_i^{(2)}\}$, $i = 1, \dots, n$, satisfying 1)–3). Let $t_1^{(1)} \in \widetilde{R}_1^\rho(A^*)$, for definiteness.

By condition 1), then $t_1^{(2)} \in \widetilde{R}_2^\pi(A^*)$. Consequently, by (4.11),

$$P_n(\widehat{A}, t_1^{(2)}) < P_n(\widehat{A}, t_1^{(2)}). \tag{4.13}$$

Since $t_1^{(2)} \in \widetilde{R}_2^\pi(A^*)$, condition 2) shows that the first point $t_2^{(1)}$ of the next pair lies in $\widetilde{R}_2^\rho(A^*) \cup \widetilde{R}_2^\pi(A^*)$. Also, if $t_2^{(1)} \in \widetilde{R}_2^\rho(A^*)$, then $t_2^{(2)} \in \widetilde{R}_1^\pi(A^*)$ by condition 1), and if $t_2^{(1)} \in \widetilde{R}_2^\pi(A^*)$, then $t_2^{(2)} \in \widetilde{R}_1^\rho(A^*)$. In any case, we can apply (4.11) or (4.12) to get

$$P_n(\widehat{A}, t_2^{(1)}) > P_n(\widehat{A}, t_2^{(2)}). \tag{4.14}$$

Continuing in the same way, we arrive at the inequalities

$$\begin{gathered} P_n(\widehat{A}, t_3^{(1)}) < P_n(\widehat{A}, t_3^{(2)}), \\ P_n(\widehat{A}, t_4^{(1)}) > P_n(\widehat{A}, t_4^{(2)}), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (-1)^n P_n(\widehat{A}, t_{n+1}^{(1)}) < (-1)^n P_n(\widehat{A}, t_{n+1}^{(2)}). \end{gathered} \tag{4.15}$$

Condition 3) means that the segments $\{[t_i^{(1)}, t_i^{(2)}]\}$, $i = 1, \dots, n+1$, are located consecutively, the neighboring segments may intersect, and for each triple of neighboring segments the first segment is disjoint with the third. Therefore, under condition 3) the system of inequalities (4.13)–(4.15) shows that for the polynomial $P_n(\widehat{A}, t)$ we have at least $n+1$ consecutive intervals on which this polynomial is monotone increasing or decreasing alternately. This contradicts Lemma 4.2. The theorem is proved. □

Now we use Theorem 4.2 to obtain some tougher sufficient conditions, but in the form comparable to the necessary conditions (see Theorem 3.1) and also to Chebyshev's alternance.

Definition 4.3. We say that the strict $\rho\pi$-alternance occurs for a coefficient vector $A \in \mathbb{R}^{n+1}$ if in the set $R(A)$ there exists a collection of points $t_1 < t_2 < \dots < t_{n+2}$ such that the following conditions are satisfied:

1) $\{t_i\}_{i=1,\dots,n+2} \subset \widetilde{R}_1^\rho(A) \cup \widetilde{R}_2^\pi(A)$, and if $t_i \in \widetilde{R}_1^\rho(A)$ $\big(\widetilde{R}_2^\pi(A)\big)$, then

$$t_{i+1} \in \widetilde{R}_2^\pi(A) \ \big(\widetilde{R}_1^\rho(A)\big), \quad i = 1, \dots, n+1;$$

2) $\{t_i\}_{i=1,\dots,n+2} \subset \widetilde{R}_2^\rho(A) \cup \widetilde{R}_1^\pi(A)$, and if $t_i \in \widetilde{R}_1^\pi(A)$ $\big(\widetilde{R}_2^\rho(A)\big)$, then

$$t_{i+1} \in \widetilde{R}_2^\rho(A) \ \big(\widetilde{R}_1^\pi(A)\big), \quad i = 1, \dots, n.$$

Theorem 4.3. *If the $\rho\pi$-alternance occurs for a coefficient vector A^*, then the pair (A^*, r^*), where $r^* = (\rho(A^*) - \pi(A^*))/2$, gives a minimal value to the function $\varphi(A, r)$ in problem* (1.1).

Proof. Suppose that the strict $\rho\pi$-alternance occurs for A^* along a sequence $\{t_i\}_{i=1,\dots,n+2}$. We denote

$$t_1^{(1)} = t_1,\ t_1^{(2)} = t_2^{(1)} = t_2,\ t_2^{(2)} = t_3^{(1)} = t_3,\ \dots, t_n^{(2)} = t_{n+1}^{(1)} = t_{n+1},\ t_{n+1}^{(2)} = t_{n+2}.$$

Then it is easy to show that the collection of pairs $\{t_i^{(1)} < t_i^{(2)}\}_{i=1,\dots,n+1}$ satisfies conditions 1)–3) in Theorem 4.2. □

Remark 4.1. If the strict $\rho\pi$-alternance occurs for a vector A, then either $R_3(A) \neq \varnothing$, or the $\rho\pi$-alternance happens, i.e., A satisfies the necessary condition from Theorem 3.1. The converse may fail.

Remark 4.2. Condition 1) in Definition 4.3 is fulfilled if and only if the Chebyshev alternance occurs for the function $f_1(t)$ relative to the polynomial $P_n(A, t) - r$, where $r = (\rho(A) - \pi(A))/2$, so that $P_n(A, t) - r$ is the best Chebyshev approximation for $f_1(t)$. Accordingly, condition 2) talks about the Chebyshev alternance for $f_2(t)$ relative to $P_n(A, t) + r$.

4.3. We present (somewhat degenerate) situations when our sufficient conditions can simultaneously be necessary.

Lemma 4.3. *Let X and Y be closed nonempty subsets of the segment $[c, d]$, and let $X \cap Y = \varnothing$. The relation*

$$\mathbb{O}_{n+1} \in \operatorname{co}\{(1, x, \dots, x^n) : x \in X\} - \operatorname{co}\{(1, y, \dots, y^n) : x \in Y\} \tag{4.16}$$

is fulfilled if and only if in $X \cup Y$ there exists an ordered system of points $t_1 < t_2 < \dots < t_{n+2}$ such that $t_i \in X$ $\big(Y\big)$ implies $t_{i+1} \in Y$ $\big(X\big)$, $i = 1, \dots, n+1$.

Proof. The case of $n = 0$ is obvious. In what follows we assume that $n \geq 1$.

The "*only if*" part. Suppose that (4.16) is true, but there is no point system as in the theorem. Then the segment $[c, d]$ can be split by points

$$c = x_0 < x_1 < \dots < x_{m+1} = d,$$

where $x_i \notin X \cup Y$, $i = 0, \dots, m+1$, and $1 \leq m \leq n$, in such a way that the consecutive intervals (x_i, x_{i+1}), $i = 0, \dots, m$, contain alternately either only points of X, or only points of Y. Let $B \in \mathbb{R}^{m+1}$ be the coefficient vector of the polynomial

$$P_m(B, t) = b \prod_{i=1,\dots,m} (t - x_i).$$

Choosing the sign of b appropriately, we can arrange that this polynomial be negative on X and positive on Y. Consequently, we have

$$P_m(B, x) < P_m(B, y) \quad \text{for all} \quad x \in X,\ y \in Y. \tag{4.17}$$

We write this in the form

$$\langle B, (1, x, \dots, x^m)\rangle < \langle B, (1, y, \dots, y^m)\rangle, \quad x \in X,\ y \in Y. \tag{4.18}$$

We know that

$$\max_{x\in X}\langle B,(1,x,\dots,x^m)\rangle = \max_{v\in\mathrm{co}\{(1,x,\dots,x^m)\,:\,x\in X\}}\langle B,v\rangle,$$
$$\max_{y\in Y}\langle B,(1,y,\dots,y^m)\rangle = \min_{w\in\mathrm{co}\{(1,y,\dots,y^m)\,:\,y\in Y\}}\langle B,w\rangle,$$

see [2, Chapter I, §2]). Therefore, (4.18) implies that

$$\langle B,v-w\rangle<0 \quad \text{for} \quad v\in\mathrm{co}\{(1,x,\dots,x^m)\,:\,x\in X\},$$
$$w\in\mathrm{co}\{(1,y,\dots,y^m)\,:\,y\in Y\}.$$

This contradicts (4.16), because $m\le n$.

The "if" part. Suppose that we have an ordered system $\{t_i\}_{i=1,\dots,n+2}$ as in the lemma, but

$$\mathbb{O}_{n+1}\notin\mathrm{co}\{(1,x,\dots,x^n)\,:\,x\in X\}-\mathrm{co}\{(1,y,\dots,y^n)\,:\,y\in Y\}. \tag{4.19}$$

The difference of two compact convex set is a compact convex set itself. Therefore, by the separation theorem (see [1, Chapter 1, §2]), relation (4.19) implies the existence of a vector $A^*\neq\mathbb{O}_{n+1}$ such that

$$\langle A^*,v-w\rangle<0 \quad \text{for} \quad v\in\mathrm{co}\{(1,x,\dots,x^m)\,:\,x\in X\},$$
$$w\in\mathrm{co}\{(1,y,\dots,y^m)\,:\,y\in Y\}.$$

This yields the inequality

$$\langle A^*,(1,x,\dots,x^m)\rangle<\langle A^*,(1,y,\dots,y^m)\rangle, \quad x\in X,\ y\in Y,$$

which can be written in the form

$$P_n(A^*,x)<P_n(A^*,y), \quad x\in X,\ y\in Y. \tag{4.20}$$

Put

$$\alpha=\max_{x\in X}P_n(A^*,x), \quad \beta=\min_{y\in Y}P_n(A^*,y), \quad \gamma=(\alpha+\beta)/2.$$

Since the sets X and Y are closed, (4.20) shows that $\alpha<\beta$. It follows that the signs of the polynomial $P_n(A^*,t)-\gamma$ at the points $\{t_i\}_{i=1,\dots,n+2}$ alternate, in accordance with the property of these points indicated in the lemma. Therefore, this polynomial has at least $n+1$ zeros, so that $P_n(A^*,t)\equiv\gamma$, contradicting (4.20). The lemma is proved. □

Theorem 4.4. *Suppose that a vector A^* is such that at least one of the sets $\widetilde{R}_i^\rho(A^*)$ and $\widetilde{R}_j^\pi(A^*)$, $i=1,2$, $j=1,2$, is empty. Then the pair (A^*,r^*), where $r^*=(\rho(A^*)-\pi(A^*))/2$, is a solution of problem* (1.1) *if and only if the strict alternance occurs for A^*.*

Proof. For definiteness, let

$$\widetilde{R}_2^\rho(A^*)=R_2^\rho(A^*)\cup R_3^\rho(A^*)=\varnothing.$$

Then formulas (3.1)–(3.4) take the form

$$\partial\rho(A^*)=\mathrm{co}\{(1,t,\dots,t^n)\,:\,t\in R_1^\rho(A^*)\}, \tag{4.21}$$

$$\partial\pi(A^*)=\mathrm{co}\begin{cases}(1,t,\dots,t^n)\,:\,t\in\widetilde{R}_1^\pi(A^*),\\ -(1,t,\dots,t^n)\,:\,t\in\widetilde{R}_2^\pi(A^*).\end{cases} \tag{4.22}$$

There formulas show that for any $v=(v_1,v_2,\dots,v_{n+1})\in\partial\rho(A^*)$ the first component v_1 is equal to 1. Also, if $w=(w_1,w_2,\dots,w_{n+1})\in\partial\pi(A^*)$ and $w_1=-1$, then $w\in\mathrm{co}\{-(1,t,\dots,t^n)\,:\,t\in\widetilde{R}_2^\pi(A^*)\}$. Therefore, in the case under study the relation

$$\mathbb{O}_{n+1}\in\partial\rho(A^*)+\partial\pi(A^*) \tag{4.23}$$

is equivalent to the fact that

$$\mathbb{O}_{n+1} \in \operatorname{co}\{(1,t,\dots,t^n) \,:\, t \in R_1^\rho(A^*)\} - \operatorname{co}\{(1,t,\dots,t^n) \,:\, t \in \widetilde{R}_2^\pi(A^*)\}. \tag{4.24}$$

Observe that $R_1^\rho(A^*) \cap \widetilde{R}_2^\pi(A^*) = \varnothing$, which follows from the definition of these sets. Since, by Theorem (4.23), relation 2.2 expresses a criterion for being a solution of problem (1.1), and we have (4.24), we see that it only remains to apply Lemma 4.3. It should be noted that the set $\widetilde{R}_2^\pi(A^*)$ is closed, as required, by its definition, and the set $R_1^\rho(A^*)$ is closed because the set $R_3^\rho(A^*)$ is assumed to be empty. The theorem is proved. □

§5. Uniqueness conditions

In [9] it was shown that the uniqueness (or nonuniqueness) of a solution of problems (1.2)–(1.3) about outer and inner estimates of an s.f. by a polynomial strip depends on the properties of the s.f. to be estimated, including differential properties. Simple examples tell us that problem (1.1) may fail to have a unique solution.

5.1. Let $D \subset \mathbb{R}^m$ be a set, and let

$$P(D,1) = \{d_1 \in \mathbb{R} \,:\, \exists\, d = (d_1,d_2,\dots,d_m) \in D\},$$
$$P(D,2:m) = \{(d_2,d_3,\dots,d_m) \in \mathbb{R}^{m-1} \,:\, \exists\, d = (d_1,d_2,\dots,d_m) \in D\}$$

be the projections of D to the corresponding coordinate subspaces.

The next statement is verified easily.

Lemma 5.1. *If D is a convex set such that*

$$0 \in \operatorname{int} P(D,1), \quad \mathbb{O}_{m-1} \in \operatorname{int} P(D,2:m),$$

then

$$\mathbb{O}_m \in \operatorname{int} D.$$

Theorem 5.1. *Given a coefficient vector A^*, suppose that there exists a collection of pairs of points $\{t_i^{(1)} < t_i^{(2)}\}$, $i = 1,\dots,n+1$, satisfying conditions* 1)–3) *of Theorem* 4.2 *and, moreover, the condition*

4) $\widetilde{R}_i^\rho(A^*) \neq \varnothing$, $\widetilde{R}_i^\pi(A^*) \neq \varnothing$, $i = 1,2$.

Then the pair (A^*, r^*) with $r^* = (\rho(A^*) - \pi(A^*))/2$ is a unique solution of problem (1.1).

Proof. Denote

$$\begin{aligned}
D_1 &= \operatorname{co}\{d = (1,x,\dots,x^n) - (1,y,\dots,y^n) \,:\, x \in \widetilde{R}_1^\rho(A^*),\ y \in \widetilde{R}_2^\pi(A^*)\},\\
D_2 &= \operatorname{co}\{d = (1,x,\dots,x^n) - (1,y,\dots,y^n) \,:\, x \in \widetilde{R}_1^\pi(A^*),\ y \in \widetilde{R}_2^\rho(A^*)\},\\
D_3 &= \operatorname{co}\{d = (1,x,\dots,x^n) + (1,y,\dots,y^n) \,:\, x \in \widetilde{R}_1^\rho(A^*),\ y \in \widetilde{R}_1^\pi(A^*)\},\\
D_4 &= \operatorname{co}\{d = -(1,x,\dots,x^n) - (1,y,\dots,y^n) \,:\, x \in \widetilde{R}_2^\rho(A^*),\ y \in \widetilde{R}_2^\pi(A^*)\}.
\end{aligned} \tag{5.1}$$

Formulas (4.7), (4.8) for the subdifferentials of $\rho(A)$ and $\pi(A)$ (implied by (3.1)–(3.4)) show that

$$D \equiv \partial\rho(A^*) + \partial\pi(A^*) = \operatorname{co}\{D_i \,:\, i = 1,\dots,4\}. \tag{5.2}$$

a) Denoting $\widehat{D} = P(\operatorname{co}\{D_1,D_2\}, 2:(n+1))$, i.e., in accordance with (5.1),

$$\begin{aligned}
\widehat{D} = \operatorname{co}\{(x-y,\dots,x^n-y^n) \,:\,&\\
(x \in \widetilde{R}_1^\rho(A^*), y \in \widetilde{R}_2^\pi(A^*)) \vee (x \in \widetilde{R}_1^\pi(A^*), y \in \widetilde{R}_2^\rho(A^*))\},&
\end{aligned} \tag{5.3}$$

we prove that

$$\mathbb{O}_n \in \operatorname{int} \widehat{D}. \tag{5.4}$$

Suppose the contrary. Then either $\mathbb{O}_n \notin \widehat{D}$, or $\mathbb{O}_n$ is a boundary point for $\widehat{D}$. Since the set $\widehat{D}$ is convex, we can apply the separation theorem in the first case and the support plane theorem in the second case to show (see [2, Chapter I, §1]) that there exists a vector $B \neq \mathbb{O}_n$ such that

$$\langle B, \widehat{d} \rangle \leq 0 \quad \text{for all} \quad \widehat{d} \in \widehat{D}. \tag{5.5}$$

Plugging the corresponding elements of $\widehat{D}$ in place of $\widehat{d}$ in (5.5) and using (5.3), we get

$$\langle B, (x, x^2, \dots, x^n) \rangle \leq \langle B, (y, y^2, \dots, y^n) \rangle \quad \text{for} \;\; x \in \widetilde{R}_1^{\rho}(A^*), \; y \in \widetilde{R}_2^{\pi}(A^*), \tag{5.6}$$

$$\langle B, (x, x^2, \dots, x^n) \rangle \leq \langle B, (y, y^2, \dots, y^n) \rangle \quad \text{for} \;\; x \in \widetilde{R}_1^{\pi}(A^*), \; y \in \widetilde{R}_2^{\rho}(A^*). \tag{5.7}$$

If we take $A = (0, b_1, \dots, b_n) \in \mathbb{R}^{n+1}$, where $B = (b_1, b_2, \dots, b_n)$, for the role of the coefficient vector, then (5.6) and (5.7) take the form

$$P_n(A, x) \leq P_n(A, y) \quad \text{for} \quad x \in \widetilde{R}_1^{\rho}(A^*), \; y \in \widetilde{R}_2^{\pi}(A^*), \tag{5.8}$$

$$P_n(A, x) \leq P_n(A, y) \quad \text{for} \quad x \in \widetilde{R}_1^{\pi}(A^*), \; y \in \widetilde{R}_2^{\rho}(A^*). \tag{5.9}$$

Now we consider the behavior of the polynomial $P_n(A, t)$ along the sequence of pairs of points $\{t_1^{(1)} < t_i^{(2)}\}_{i=1,\dots,n+1}$. For definiteness, suppose $t_1^{(1)} \in \widetilde{R}_1^{\rho}(A^*)$. Then, using inequalities (5.8), (5.9) and conditions 1)–2), and arguing as in the proof of Theorem 4.2, we obtain

$$\begin{aligned} P_n(A, t_1^{(1)}) \leq P_n(A, t_1^{(2)}), \quad P_n(A, t_2^{(1)}) \geq P_n(A, t_2^{(2)}), \;\; \dots, \\ (-1)^n P_n(A, t_{n+1}^{(1)}) \leq (-1)^n P_n(A, t_{n+1}^{(2)}). \end{aligned} \tag{5.10}$$

Recalling condition 3), we conclude that the above inequalities can be fulfilled only if $P_n(A, t) \equiv \text{const}$. This contradicts the fact that $A = (0, b_1, \dots, b_n) \neq \mathbb{O}_{n+1}$, because $B \neq \mathbb{O}_n$. This proves (5.4).

b) The definitions of the sets D and $\widehat{D}$ show that $\widehat{D} \subset P(D, 2 : (n+1)$. Therefore, (5.4) implies

$$\mathbb{O}_n \in \operatorname{int} P(D, 2 : (n+1)). \tag{5.11}$$

On the other hand, the sets D_3 and D_4, included in D, are nonempty by condition 4), and the first component of each element in D_3 is positive, while the first component of each element in D_4 is negative. Therefore,

$$0 \in \operatorname{int} P(D, 1). \tag{5.12}$$

Now, using relations (5.11), (5.12) and Lemma 5.1, we see that

$$\mathbb{O}_{n+1} \in \operatorname{int}(\partial\rho(A^*) + \partial\pi(A^*)). \tag{5.13}$$

This means (see [2, Chapter II, §5]) that A^* is a unique solution of the convex programming problem (2.2), whence by Theorem 2.1 it follows that the pair (A^*, r^*) with $r^* = (\rho(A^*) - \pi(A^*))/2$ is a unique solution of problem (1.1). The theorem is proved. □

Now we use Theorem 5.1 to obtain a uniqueness condition for a solution in a form comparable to the Chebyshev alternance.

Theorem 5.2. *Given a coefficient vector A^*, suppose that*

1) *the strict $\rho\pi$-alternance occurs for A^*,*
2) $\widetilde{R}_i^{\rho}(A^*) \neq \varnothing$, $\widetilde{R}_i^{\pi}(A^*) \neq \varnothing$, $i = 1, 2$.

Then the pair (A^*, r^*) with $r^* = (\rho(A^*) - \pi(A^*))/2$ is a unique solution of problem (1.1).

Proof. Suppose that the strict $\rho\pi$-alternance is realized along a sequence $t_1 < t_2 < \cdots < t_{n+2}$. For definiteness, assume that $\{t_i\}_{i=1,\dots,n+2} \in \widetilde{R}_1^\rho(A^*) \cup \widetilde{R}_2^\pi(A^*)$, and that $t_{i+1} \in \widetilde{R}_2^\pi(A^*)$ $\big(\widetilde{R}_1^\rho(A^*)\big)$ whenever $t_i \in \widetilde{R}_1^\rho(A^*)$ $\big(\widetilde{R}_2^\pi(A^*)\big)$. We build a sequence of $n+1$ pairs $\{t_i^{(1)} < t_i^{(2)}\}_{i=1,\dots,n+1}$ in the same way as in the proof of Theorem 4.3. Then, using condition 2), it is not hard to check that this sequence of pairs satisfies the assumptions of Theorem 5.1. □

5.2. Conditions 1)–3) of Theorem 4.2, repeated in Theorem 5.1, ensure that relation (2.4) is fulfilled. However, the stronger relation (5.13), which implies the uniqueness of a solution, may fail under these conditions.

The theorem below says that if condition 4) in Theorem 5.1 is violated, than problem (1.1) is solved nonuniquely.

With a coefficient vector $A^* = (a_0^*, a_1^*, \dots, a_n^*)$ we associate the quantities

$$\begin{aligned}
\Delta_\rho &= [\max_{t\in[c,d]}(f_2(t) - P_n(A^*,t)) - \max_{t\in[c,d]}(P_n(A^*,t) - f_1(t))]/2,\\
\Delta_\pi &= [\max_{t\in[c,d]}(f_1(t) - P_n(A^*,t)) - \max_{t\in[c,d]}(P_n(A^*,t) - f_2(t))]/2,\\
\Delta_1 &= \min\{\Delta_\rho, \Delta_\pi\}, \quad \Delta_2 = \max\{\Delta_\rho, \Delta_\pi\},
\end{aligned}$$

and put

$$A^\delta = (a_0^* + \delta, a_1^*, \dots, a_n^*), \quad r_\delta = (\rho(A^\delta) - \pi(A^\delta))/2.$$

Theorem 5.3. *If A^* is an optimal coefficient vector for problem* (1.1) *and at least one of the sets $\widetilde{R}_i^\rho(A^*)$, $i = 1, 2$, and $\widetilde{R}_j^\pi(A^*)$, $j = 1, 2$, is empty, then for any $\delta \in [\Delta_1, \Delta_2]$ the pair (A^δ, r_δ) is a solution of problem* (1.1).

Proof. So, suppose that the pair $(A^*, (\rho(A^*) - \pi(A^*))/2)$ is a solution of (1.1).

1. For definiteness, let $\widetilde{R}_1^\rho(A^*) = \varnothing$. This means that

$$\rho(A^*) = \max_{t\in[c,d]}(f_2(t) - P_n(A^*,t)) > \max_{t\in[c,d]}(P_n(A^*,t) - f_1(t)),$$

whence $\Delta_\rho > 0$ in this case. By Theorem 4.4, the strong $\rho\pi$-alternance occurs; therefore, since $\widetilde{R}_1^\rho(A^*)$ is empty, we have $R_2^\rho(A^*) \neq \varnothing$ and $\widetilde{R}_1^\pi(A^*) \neq \varnothing$. The graph of the polynomial $P_n(A^\delta, t)$ is obtained by a vertical shift of the graph of $P_n(A^*, t)$, and, like for A^*, we have $\widetilde{R}_1^\rho(A^\delta) = \varnothing$, $R_2^\rho(A^\delta) \neq \varnothing$, and $\widetilde{R}_1^\pi(A^\delta) \neq \varnothing$ for $\delta \in [0, \Delta_\rho)$. Therefore, the shift of the graph is accompanied by the corresponding change of the values of functions:

$$\rho(A^\delta) = \rho(A^*) - \delta, \quad \pi(A^\delta) = \pi(A^*) + \delta, \quad \delta \in [0, \Delta_\rho].$$

Thus, $\rho(A^\delta) + \pi(A^\delta) = \rho(A^*) + \pi(A^*)$, and by Theorem 2.1 we see that the pair (A^δ, r_δ) is also a solution of problem (1.1).

2. Now it is easy to make the corresponding conclusions in the other cases by analogy. Observe that:

if $\widetilde{R}_2^\rho(A^*) = \varnothing$, then $\Delta_\rho < 0$;

if $\widetilde{R}_1^\pi(A^*) = \varnothing$, then $\Delta_\pi > 0$;

if $\widetilde{R}_2^\pi(A^*) = \varnothing$, then $\Delta_\pi < 0$.

However, we should keep in mind that the strong $\rho\pi$-alternance prohibits the occurrence of some situations simultaneously. Namely, if $\widetilde{R}_i^\rho(A^*) = \varnothing$, then $\widetilde{R}_i^\pi(A^*) \neq \varnothing$ and *vice versa*, if $\widetilde{R}_i^\pi(A^*) = \varnothing$, then $\widetilde{R}_i^\rho(A^*) \neq \varnothing$. Therefore, for the segment $[\Delta_1, \Delta_2]$ the point 0 is either an inner point, or one of the endpoints. The theorem is proved. □

5.3. We mention some properties of a solution of problem (1.1) in the case of nonuniqueness.

We denote $[r^-, r^+] = \operatorname{Arg}\min_{r\geq 0} f(r)$, $\varphi^* = \min_{\substack{A\in\mathbb{R}^{n+1}\\ r\geq 0}} \varphi(A, r)$.

We are interested in the properties of the map

$$\Omega_\varphi(r) = \{\widehat{A} \in \mathbb{R}^{n+1} \,:\, \varphi(\widehat{A}, r) = \varphi^*\}$$

on the segment $[r^-, r^+]$. Note that, in accordance with this notation,

$$(A^*, r^*) \in \operatorname{Arg}\min_{\substack{A\in\mathbb{R}^{n+1}\\ r\geq 0}} \varphi(A, r) \iff r^* \in [r^-, r^+],\ A^* \in \Omega_\varphi(r^*). \tag{5.14}$$

The behavior of the functions $\rho(A)$ and $\pi(A)$ on the set of solutions of (1.1) can be described as follows.

Theorem 5.4. *If $r \in [r^-, r^+]$ and $A \in \Omega_\varphi(r)$, then*

$$\rho(A) = \varphi^* + r, \quad \pi(A) = \varphi^* - r. \tag{5.15}$$

Proof. Indeed, by Theorem 2.1, for $r \in [r^-, r^+]$ and $A \in \Omega_\varphi(r)$ we have

$$r = (\rho(A) - \pi(A))/2, \quad \varphi^* = (\rho(A) + \pi(A))/2,$$

which yields (5.15). □

Theorem 2.1 shows that, in the pair $(A^*, r(A^*)) \in \operatorname{Arg}\min_{\substack{A\in\mathbb{R}^{n+1}\\ r\geq 0}} \varphi(A, r)$, any coefficient vector A^* gives rise to a unique $r(A^*) = (\rho(A^*) - \pi(A^*))/2$. A question arises as to whether the inverse map $\Omega_\varphi(r)$ is one-valued on $[r^-, r^+]$.

Here, the case where $\Omega_\rho \cap \Omega_\pi \neq \varnothing$ stands apart, because then, by Corollary 4.1, the segment $[r^-, r^+]$ degenerates to the point $r^* = (\rho^* - \pi^*)/2$. In this case the set $\Omega_\varphi(r) = \Omega_\rho \cap \Omega_\varphi$ may fail to be a singleton (see Example 6.4).

Lemma 5.2. *If on an ordered collection of points $t_1 < t_2 < \cdots < t_{n+2}$ the function $\eta(t)$ changes its sign consecutively, i.e., $\operatorname{sgn}\eta(t_i) = -\operatorname{sgn}\eta(t_{i+1})$, $i = 1, \dots, n+1$, then*

$$\mathbb{O}_{n+1} \in \operatorname{int}\operatorname{co}\{\eta(t_i)(1, t_i, \dots, t_i^n) \,:\, i = 1, \dots, n+2\}. \tag{5.16}$$

Proof. By Lemma 3.1,

$$\mathbb{O}_{n+1} \in \operatorname{co}\{\eta(t_i)(1, t_i, \dots, t_i^n) \,:\, i = 1, \dots, n+2\},$$

i.e., there exist $\alpha_i \geq 0$, $i = 1, \dots, n+2$, such that

$$\begin{cases} \displaystyle\sum_{i=1}^{n+2} \alpha_i \eta(t_i)(1, t_i, \dots, t_i^n) = 0, \\ \displaystyle\sum_{i=1}^{n+2} \alpha_i = 1. \end{cases} \tag{5.17}$$

Arguing as in [14, Chapter VI, §8, item 2], we can show that $\alpha_i > 0$ for all $i = 1, \dots, n+2$. After that, it remains to observe that system (5.17), linear in α_i, has a unique solution continuous with respect to the right-hand side. Consequently, there exists $\delta > 0$ such that, for any $\varepsilon \in \mathbb{R}^{n+1}$ with $\|\varepsilon\| < \delta$, the system

$$\begin{cases} \displaystyle\sum_{i=1}^{n+2} \alpha_i(\varepsilon) \eta(t_i)(1, t_i, \dots, t_i^n) = \varepsilon, \\ \displaystyle\sum_{i=1}^{n+2} \alpha_i(\varepsilon) = 1 \end{cases}$$

has a positive solution $\alpha_i(\varepsilon) > 0$, $i = 1, \dots, n+2$. This means that (5.16) is true. □

Theorem 5.5. *If $r^- < r^+$, then the map $\Omega_\varphi(r)$ is one-valued and continuous on the segment $[r^-, r^+]$.*

Proof. Suppose (r^-, r^+). Theorem 5.4 implies

$$\Omega_\varphi(r) \cap \{\Omega_\rho \cup \Omega_\pi\} = \varnothing.$$

Hence, if $\widehat{A} \in \Omega_\varphi(r)$, then

$$\mathbb{O}_{n+1} \notin \partial\rho(\widehat{A}), \quad \mathbb{O}_{n+1} \notin \partial\pi(\widehat{A}).$$

Therefore, formulas (3.1)–(3.4) yield

$$R_3(\widehat{A}) = \varnothing. \tag{5.18}$$

Since the pair $(\widehat{A}, r)$ is a solution and (5.18) is true, Theorem 3.1 ensures that the $\rho\pi$-alternance occurs for $\widehat{A}$: in the set $T = R_1(\widehat{A}) \cup R_2(\widehat{A})$ there exist points $t_1 < t_2 < \dots < t_{n+2}$ such that if $t_i \in R_1(\widehat{A})$ $\big(R_2(\widehat{A})\big)$, then $t_{i+1} \in R_2(\widehat{A})$ $\big(R_1(\widehat{A})\big)$, $i = 1, \dots, n+1$. Note that $R_1(\widehat{A}) \cap R_2(\widehat{A}) = \varnothing$ and define the following function on T:

$$\eta(t) = \begin{cases} 1 & \text{if } \ t \in R_1(\widehat{A}), \\ -1 & \text{if } \ t \in R_2(\widehat{A}). \end{cases} \tag{5.19}$$

Then on the collection of points $\{t_i\}$, $i = 1, \dots, n+2$, this function changes its sign consecutively; hence, we have (5.16) by Lemma 5.2.

On the other hand, since $\rho(\widehat{A}) - r = \pi(\widehat{A}) + r$, we can use (2.1) and a well-known fact of the subdifferential calculus (see [2, Chapter I, §5, item 2]) to express the subdifferential of $\varphi(A, r)$ with respect to A at the point $\widehat{A}$ in the form

$$\partial_A \varphi(\widehat{A}, r) = \mathrm{co}\{\partial\rho(\widehat{A}), \partial\pi(\widehat{A})\}.$$

Therefore, by formulas (3.1)–(3.4) and (5.18), we have

$$\partial_A \varphi(\widehat{A}, r) = \mathrm{co}\{\eta(t)(1, t, \dots, t^n) \ : \ t \in T\}. \tag{5.20}$$

Thus, from (5.16) and (5.20) we obtain the relation

$$\mathbb{O}_{n+1} \in \mathrm{int}\, \partial_A \varphi(\widehat{A}, r).$$

Since the function $\varphi(A, r)$ is convex with respect to $A \in \mathbb{R}^{n+1}$, the last relation implies (see [2, Chapter II, §5, item 2]) that $\widehat{A}$ is a unique point of minimum for the function $\varphi(A, r)$ with respect to A, for a given fixed r. So, we have proved that the map $\Omega_\varphi(r)$ is one-valued for $r \in (r^-, r^+)$. The facts that it is one-valued for $r = r^-$ and $r = r^+$, and also continuity, follow easily from the convexity of the set of all solutions and from what has already been checked. The theorem is proved. □

The next statement is a consequence of Theorems 5.3 and 5.5 and the fact that the set of all solutions is convex.

Corollary 5.1. *If A^* is an optimal coefficient vector for problem* (1.1) *and at least one of the sets $\widetilde{R}_i^\rho(A^*)$, $\widetilde{R}_j^\pi(A^*)$, i, $j = 1, 2$, is empty, then the set of pairs $\{(A^\delta, r_\delta)$, $\delta \in [\Delta_1, \Delta_2]\}$ describes the entire set of solutions of* (1.1).

§6. Examples

As an illustration, we present a series of examples; in the first three of them, instead of defining specific segment functions, we restrict ourselves to indicating the sets $R_i^\rho(A)$ and $R_j^\pi(A)$, $i,j=1,2,3$. In order to have a visual picture on the coordinate plane, it suffices to do the following.

Choose a segment $[c,d]$ arbitrarily, take the graph of an arbitrary polynomial $P_n(A,t)$ for the role of the axis of a polynomial strip, and fix $r>\Delta r>0$ arbitrarily. Then we should place the graph of $f_1(t)$ in the strip $[P_n(A,t)-r-\Delta r, P_n(A,t)-r+\Delta r]$, and the graph of $f_2(t)$ in the strip $[P_n(A,t)+r-\Delta r, P_n(A,t)+r+\Delta r]$ for $t\in[c,d]$. The graphs should touch the boundaries of there strips at points of the sets $R_i^\rho(A)$ and $R_j^\pi(A)$ given beforehand (and only at such points). The following conditions should be satisfied:

if $t\in R_1^\rho(A)$, then $f_1(t)=P_n(A,t)-r-\Delta r$, $f_2(t)<P_n(A,t)+r+\Delta r$;
if $t\in R_2^\rho(A)$, then $f_2(t)=P_n(A,t)+r+\Delta r$, $f_1(t)>P_n(A,t)-r-\Delta r$;
if $t\in R_1^\pi(A)$, then $f_2(t)=P_n(A,t)+r-\Delta r$, $f_1(t)<P_n(A,t)-r+\Delta r$;
if $t\in R_2^\pi(A)$, then $f_1(t)=P_n(A,t)-r+\Delta r$, $f_2(t)>P_n(A,t)+r-\Delta r$;
if $t\in R_3^\rho(A)$, then $f_1(t)=P_n(A,t)-r-\Delta r$, $f_2(t)=P_n(A,t)+r+\Delta r$;
if $t\in R_3^\pi(A)$, then $f_1(t)=P_n(A,t)-r+\Delta r$, $f_2(t)=P_n(A,t)+r-\Delta r$.

With such a construction, obviously,

$$\rho(A)=r+\Delta r,\quad \pi(A)=-r+\Delta r,$$
$$\varphi(A,r)=(\rho(A)+\pi(A))/2=\Delta r,\quad r=(\rho(A)-\pi(A))/2.$$

6.1. We present examples showing that, in general, the necessary conditions stated in Theorem 3.1 are not sufficient.

Example 6.1. Suppose $n=1$, $t_0<t_1$, $R_3^\rho(A)=\{t_0\}$, $R_1^\pi(A)=\{t_1\}$, $R_1^\rho(A)=R_2^\rho(A)=R_2^\pi(A)=R_3^\pi(A)=\varnothing$.

Then $R_3(A)=R_3^\rho(A)\neq\varnothing$, and condition 1) of Theorem 3.1 is satisfied. By formulas (3.1)–(3.4), we have

$$\partial\rho(A)=\mathrm{co}\{(-1,-t_0),(1,t_0)\},\quad \partial\pi(A)=\{(1,t_1)\},$$
$$\partial\rho(A)+\partial\pi(A)=\mathrm{co}\{(0,t_1-t_0),(2,t_0+t_1)\}.$$

Thus, $\mathbb{O}_2\notin\partial\rho(A)+\partial\pi(A)$, and Theorem 2.1 shows that A is not a coefficient vector for any optimal polynomial in problem (1.1).

Example 6.2. We take $n=1$, $t_1<t_2<t_3$, $R_1^\rho(A)=\{t_1\}$, $R_2^\rho(A)=\{t_2\}$, $R_1^\pi(A)=\{t_3\}$, $R_3^\rho(A)=R_3^\pi(A)=\varnothing$, $R_2^\pi(A)=\varnothing$.

Then the $\rho\pi$-alternance occurs for the vector A, i.e., condition 2) of Theorem 3.1 is satisfied. We have

$$\partial\rho(A)=\mathrm{co}\{(1,t_1),(-1,-t_2)\},\quad \partial\pi(A)=\{(1,t_3)\},$$
$$\partial\rho(A)+\partial\pi(A)=\mathrm{co}\{(0,t_3-t_2),(2,t_1+t_3)\}.$$

Thus, $\mathbb{O}_2\notin\partial\rho(A)+\partial\pi(A)$, and hence, the vector A is not optimal also in this case.

6.2. Now we show that, in general, the sufficient conditions given in Theorem 4.2 are not necessary.

Example 6.3. Suppose $n=1$, $t_1<t_2<t_3<t_4$, $R_1^\rho(A)=\{t_1\}$, $R_2^\rho(A)=\{t_2\}$, $R_1^\pi(A)=\{t_3\}$, $R_2^\pi(A)=\{t_4\}$, and $R_3^\rho(A)=R_3^\pi(A)=\varnothing$.

In this case, there is a unique way to form two pairs $\{t_1^{(1)}<t_2^{(1)}\}$ and $\{t_2^{(1)}<t_2^{(1)}\}$ satisfying conditions 1) and 2) of Theorem 4.2, namely, $t_1^{(1)}=t_1$, $t_1^{(2)}=t_4$, $t_2^{(1)}=t_2$,

$t_2^{(2)} = t_3$. However, these pairs do not satisfy condition 3). We have

$$\partial\rho(A) = \mathrm{co}\{(1,t_1),(-1,-t_2)\}, \quad \partial\pi(A) = \mathrm{co}\{(1,t_3),(-1,-t_4)\},$$
$$\partial\rho(A) + \partial\pi(A) = \mathrm{co}\{(2,t_1+t_3),(-2,-t_2-t_4),(0,t_1-t_4),(0,t_3-t_2)\}.$$

Thus,

$$\mathbb{O}_2 \in \mathrm{int}(\partial\rho(A) + \partial\pi(A)),$$

which implies not only the optimality of the vector A, but also its uniqueness. Simultaneously, we have shown that the entire set of conditions 1)–4) of Theorem 5.1 is not necessary for the uniqueness of a solution.

6.3. Now we present an example where the solution is nonunique, but the width of the optimal polynomial strips for different coefficient vectors is one and the same.

Example 6.4. Let $n = 2$, and let a segment function $F(t) = [f_1(t), f_2(t)]$ be given on the segment $[-1,1]$ by the functions

$$f_1(t) = \begin{cases} 0 & \text{if } t \in [-1,0], \\ t/4 & \text{if } t \in [0,1], \end{cases} \qquad f_2(t) = \begin{cases} 1+t & \text{if } t \in [-1,0], \\ 1-t/2 & \text{if } t \in [0,1]. \end{cases}$$

It is easily seen geometrically (and can be checked by verifying that $\mathbb{O}_3 \in \partial\rho(A)$) that the set of solutions of problem (1.2) looks like this:

$$\Omega_\rho = \{A = (a_0,a_1,a_2) \in \mathbb{R}^3 \,:\, a_0 = 0.5,\ -a_1 + a_2 \in [-1,0], a_1 + a_2 \in [-0.5, 0.25]\},$$

with $\rho^* = 0.5$, while the set of solutions of problem (1.3) (the relation $\mathbb{O}_3 \in \partial\pi(A)$ should be checked) looks like this:

$$\Omega_\pi = \{A = (a_0,a_1,a_2) \in \mathbb{R}^3 \,:\, a_0 - a_1 + a_2 = 0, a_0 + a_1 + a_2 \in [0.25, 0.5],\ a_1 - 2a_2 \in [0,1]\},$$

with $\pi^* = 0$. In this case (see Corollary 4.1), the set of optimal coefficient vectors is

$$\Omega_\varphi(r^*) = \Omega_\rho \cap \Omega_\pi = \{A = (a_0,a_1,a_2) \in \mathbb{R}^3 \,:\, a_0 = 0.5,$$
$$a_0 - a_1 + a_2 = 0,\ a_0 + a_1 + a_2 \in [0.25, 0.5]\}$$

with one and the same strip's width $d^* = \rho^* - \pi^* = 0.5$, or $r^* = (\rho^* - \pi^*)/2 = 0.25$.

References

[1] B. N. Pshenichnyĭ, *Convex analysis and extremal problems*, Nauka, Moscow, 1980. (Russian) MR0581125 (81k:90080)

[2] V. F. Dem′yanov and L. V. Vasil′ev, *Nondifferentiable optimization*, Nauka, Moscow, 1981; English transl., Optimization Software, Inc., Publ. Division, New York, 1985. MR0673171 (84d:49002); MR0816531 (86m:49002)

[3] V. F. Dem′yanov and A. M. Rubinov, *Foundations of nonsmooth analysis, and quasidifferential calculus*, Nauka, Moscow, 1990. (Russian) MR1120545 (92j:40001)

[4] E. S. Polovinkin, *Theory of multivalued mappings*, Moskov. Fiz.-Tekhn. Inst., Moscow, 1983. (Russian)

[5] G. G. Magaril-Il′yaev and V. M. Tikhomirov, *Convex analysis and its applications*, Èditorial URSS, Moscow, 2000; English transl., *Convex analysis: theory and applications*, Transl. Math. Monogr., vol. 222, Amer. Math. Soc., Providence, RI, 2003. MR2013877 (2004k:49002)

[6] Bl. Sendov, *Hausdorff approximations*, Bolgar. Akad. Nauk, Sofia, 1979. (Russian) MR0534426 (80j:41004)

[7] V. K. Dzyadyk, *Introduction to the theory of uniform approximation of functions by polynomials*, Nauka, Moscow, 1977; English transl., *Theory of uniform approximation of functions by polynomials*, Walter de Gruyter, Berlin, 2008. MR0612836 (58:29579); MR2447076 (2009f:30001)

[8] S. Karlin and W. Studden, *Tchebycheff systems: with applications in analysis and statistics*, Pure Appl. Math., vol. 15, Intersci. Publ., New York etc., 1966. MR0204922 (34:4757)

[9] I. Yu. Vygodchikova, S. I. Dudov, and E. V. Sorina, *Outer estimation of a segment function by a polynomial strip*, Zh. Vychisl. Mat. Mat. Fiz. **49** (2009), no. 7, 1175–1183; English transl., Comput. Math. Math. Phys. **49** (2009), no. 7, 1119–1127. MR2599389 (2011b:41042)

[10] F. L. Chernous′ko, *Estimation of the phase state of dynamical systems: the method of ellipsoids*, Nauka, Moscow, 1988. (Russian) MR0946472 (89k:49028)

[11] A. B. Kurzhanski and I. Vályi, *Ellipsoidal calculus for estimation and control*, Systems and Control: Foundations and Applications, Birkhäuser Boston, Inc., Boston, MA, 1997. MR1419317 (98b:93003)

[12] M. S. Nikol′skiĭ, *Approximation of a continuous multivalued mapping by constant multivalued mappings*, Vestnik Moskov. Univ. Ser. XV Vychisl. Mat. Kibernet. **1990**, no. 1, 76–79; English transl., Moscow Univ. Comput. Math. Cybernet. **1990**, no. 1, 73–76. MR1051650 (91g:65160)

[13] S. I. Dudov and A. B. Konoplev, *On the approximation of a continuous multivalued mapping by constant multivalued mappings with ball images*, Mat. Zametki **82** (2007), no. 4, 525–529; English transl., Math. Notes **82** (2007), no. 3–4, 469–473. MR2375788 (2009c:49038)

[14] V. F. Dem′yanov and V. N. Malozemov, *Introduction to minimax*, Nauka, Moscow, 1972; English transl., Halsted Press, New York–Toronto, 1974. MR0475822 (57:15407a); MR0475823 (57:15407b)

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