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SYSTEMS OF SUBSPACES IN HILBERT SPACE THAT OBEY CERTAIN CONDITIONS, ON THEIR PAIRWISE ANGLES

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Abstract. The paper is devoted to systems of subspaces $H_1, \dots, H_n$ of a complex Hilbert space H that satisfy the following conditions: for every index $i > 1$, the angle $\theta_{1,i} \in (0, \pi/2)$ between H_1 and H_i is fixed; the projections onto H_{2k} and H_{2k+1} commute for $1 \le k \le m$ (m is a fixed nonnegative number satisfying $m \le (n-1)/2$); all other pairs H_i, H_j are orthogonal. The main tool in the study is a construction of a system of subspaces in a Hilbert space on the basis of its Gram operator (the G-construction).

§1. Introduction

1.1. Systems of subspaces. In many algebraic problems, it is important to study systems $L = (V; V_1, \dots, V_n)$ of subspaces $V_1, \dots, V_n$ in a finite-dimensional linear space V, $n \in \mathbb{N}$. In particular, we mention a description of indecomposable quads of subspaces in V (see [1]) or description of irreducible representations in V of finite partially ordered sets (see, e.g., the papers in the collection [9]).

Let H be a complex Hilbert space, and let H_k, $1 \le k \le n$, be a family of its subspaces. In functional analysis and mathematical physics, it is also important to study the systems

$$S = (H; H_1, \dots, H_n)$$

(or equivalently, the families of the corresponding orthogonal projections $P_1, \dots, P_n$). This matter has been treated in many publications (see, e.g., [10] and the references therein).

All indecomposable systems S of subspaces are described, up to unitary equivalence, for $n \le 2$. If $n = 1$, every indecomposable system S is unitarily equivalent either to $S_0 = (\mathbb{C}; 0)$ or to $S_1 = (\mathbb{C}; \mathbb{C})$. If $n = 2$ (see, e.g., [3]), then, up to unitary equivalence, there are four indecomposable couples of subspaces, $S_{00} = (\mathbb{C}; 0, 0)$, $S_{01} = (\mathbb{C}; 0, \mathbb{C})$, $S_{10} = (\mathbb{C}; \mathbb{C}, 0)$, $S_{11} = (\mathbb{C}; \mathbb{C}, \mathbb{C})$, and a family $S_\varphi = (H; H_1, H_2)$, $\varphi \in (0, \pi/2)$, of indecomposable subspaces in a two-dimensional space such that, in some orthonormal basis $\{e_1, e_2\}$ in H, the subspace H_1 is spanned by e_1 and H_2 is spanned by $x = \cos\varphi e_1 + \sin\varphi e_2$.

For $n \ge 3$, the problem of describing indecomposable systems of subspaces up to unitary equivalence is $*$-wild, see [4, 5, 8]; the problem of describing the triples $S = (H; H_1, H_2, H_3)$ such that $H_2 \perp H_3$ is also $*$-wild (see [4, 5] about $*$-wild problems).

Thus, it is natural to consider various restricted classes of subspace systems, trying, so far as possible, to describe (up to unitary equivalence) all indecomposable systems in the class chosen.

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1.2. Some classes of subspace systems. In [7], the physicists Tempelrey and Lieb introduced the algebras

$$\begin{aligned}\mathbb{C}\langle p_1, p_2, \dots, p_n \,|\, p_j^2 = p_j,\ j = 1, 2, \dots, n;\\ p_i p_j p_i = \nu p_i,\ |i-j| = 1;\\ p_i p_j = p_j p_i,\ |i-j| \geq 2\rangle,\end{aligned}$$

$\nu \in \mathbb{C}$, in connection with models of statistical physics. If $\nu = \tau_0^2 \in (0,1)$, these algebras can be treated as $*$-algebras if they are endowed with the involution determined by $p_j^* = p_j$, $1 \leq j \leq n$. Let π be a $*$-representation of such a $*$-algebra in a Hilbert space H, and let H_i be the images of the projections $P_i = \pi(p_i)$. Thus, we have obtained a system $S = (H; H_1, \dots, H_n)$ that satisfies the following conditions:

(1) the "neighboring" couples of subspaces *form an angle of* θ_0, $\tau_0 = \cos\theta_0$, i.e., we have

$$P_i P_{i+1} P_i = \tau_0^2 P_i, \quad P_{i+1} P_i P_{i+1} = \tau_0^2 P_{i+1}, \quad i = 1, \dots, n-1;$$

(2) the other couples of subspaces "commute", i.e., $P_i P_j = P_j P_i$.

Let A_n be the graph whose vertices are $\{1, 2, \dots, n\}$ and whose edges are $\{i, i+1\}$, $1 \leq i \leq n-1$. Then the conditions on the angle between subspaces correspond to pairs i, j of vertices joined by an edge in A_n, and the commutation conditions correspond to pairs i, j not joined by an edge.

Consider the class of subspace systems determined by a graph Γ and a function τ on its edges. Let Γ be a graph without loops and multiple links and with vertices $\{1, 2, \dots, n\}$. Let E denote the set of edges of Γ, and let $\bar{E}$ be the set of pairs $\{i, j\}$ of vertices not joined by an edge in Γ. Suppose two functions $\theta : E \to (0, \pi/2) : \{i, j\} \mapsto \theta_{\{i,j\}}$ and $\tau = \cos\theta : E \to (0, 1) : \{i, j\} \mapsto \tau_{\{i,j\}}$ are defined on the edges of Γ. We introduce the set $\mathrm{Sys}(\Gamma, \tau)$ of subspace systems S such that

(1) if $\{i, j\} \in E$, then H_i and H_j form an angle of $\theta_{\{i,j\}}$, i.e., $P_i P_j P_i = \tau_{\{i,j\}}^2 P_i$ and $P_j P_i P_j = \tau_{\{i,j\}}^2 P_j$;

(2) if $\{i, j\} \in \bar{E}$, then H_i and H_j "commute", i.e., $P_i P_j = P_j P_i$.

The systems $S \in \mathrm{Sys}(\Gamma, \tau)$ can be viewed as $*$-representations of the corresponding $*$-algebra

$$\begin{aligned}\mathcal{TL}_{\Gamma,\tau} = \mathbb{C}\langle p_1, p_2, \dots, p_n \,|\, p_j^2 = p_j^* = p_j,\ j = 1, 2, \dots, n;\\ p_i p_j p_i = \tau_{\{i,j\}}^2 p_i,\ \{i, j\} \in E;\\ p_i p_j = p_j p_i,\ \{i, j\} \in \bar{E}\rangle.\end{aligned}$$

Note that if we "forget" the involution, i.e., replace the conditions $p_j^2 = p_j^* = p_j$ by $p_j^2 = p_j$ in the definition of $\mathcal{TL}_{\Gamma,\tau}$, then we obtain a *projective algebra* (see [2, §6]).

A smaller class of subspace systems arises if for every pair i, j of vertices not joined by an edge in Γ commutation is strengthened to *orthogonality*: $P_i P_j = P_j P_i = 0$. The set of such subspace systems ("simple" subspace systems) is denoted by $\mathrm{Sys}(\Gamma, \tau, \perp)$. These systems can be viewed as $*$-representations of the corresponding $*$-algebra $\mathcal{TL}_{\Gamma,\tau,\perp}$ (a factor-algebra of $\mathcal{TL}_{\Gamma,\tau}$). The algebras $\mathcal{TL}_{\Gamma,\tau,\perp}$ and the classes $\mathrm{Sys}(\Gamma, \tau, \perp)$ have been studied in a series of papers (see the survey [10]).

A class "intermediate" between $\mathrm{Sys}(\Gamma, \tau)$ and $\mathrm{Sys}(\Gamma, \tau, \perp)$ arises naturally. Let E^c be a subset of $\bar{E}$. Denote by $\mathrm{Sys}(\Gamma, E^c, \tau)$ the set of subspace systems $S = (H; H_1, \dots, H_n)$ such that

(1) if $\{i, j\} \in E$, then H_i, H_j form an angle of $\theta_{\{i,j\}}$;

(2) if $\{i, j\} \in E^c$, then H_i, H_j "commute";

(3) if $\{i,j\} \in \bar{E} \setminus E^c$, then H_i, H_j are orthogonal.

In the present paper, we study the classes $\mathrm{Sys}(K_{1,N}, E^c_m, \tau)$, where $N \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $2m \le N$, $K_{1,N}$ is a "star" with N rays, τ is an arbitrary function on the edges of $K_{1,N}$, and the set E^c_m consists of the pairs $\{2k, 2k+1\}$ of vertices, $1 \le k \le m$. Before stating the problem and the main results more accurately, we recall some necessary definitions.

1.3. Principal definitions. A system $S = (H; H_1, \dots, H_n)$ is said to be *decomposable* if there exists an orthogonal decomposition $H = H' \oplus H''$ into a direct sum of two nonzero subspaces H', H'', and subspace systems $S' = (H'; H'_1, \dots, H'_n)$, $S'' = (H''; H''_1, \dots, H''_n)$ such that $H_k = H'_k \oplus H''_k$ for $1 \le k \le n$. A subspace system is *indecomposable* if it is not decomposable. It is well known that a system S is indecomposable if and only if it is *irreducible*, i.e., if and only if the following is true: whenever a bounded linear operator $A : H \to H$ satisfies $AP_k = P_k A$, $1 \le k \le n$, we have $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

Subspace systems $S = (H; H_1, \dots, H_n)$ and $S' = (H'; H'_1, \dots, H'_n)$ are said to be *unitarily equivalent* if there exists a unitary operator $U : H \to H'$ such that $H'_k = U(H_k)$ for $1 \le k \le n$. This condition is equivalent to the identity $P'_k = UP_kU^*$, i.e., $UP_k = P'_kU$.

Given a subspace system $S = (H; H_1, \dots, H_n)$, the vector

$$(\dim H; \dim H_1, \dots, \dim H_n),$$

(whose components are cardinal numbers) is called the *generalized dimension* of the system. To lighten the notation, in the case where all $\dim H_k$ are equal for a system S, we attribute the term generalized dimension to the ordered pair $(\dim H; \dim H_1)$ of cardinal numbers.

A *zero* subspace system is a system S such that $H_k = 0$ for $1 \le k \le n$. All other systems are said to be *nonzero*.

1.4. Statement of the problem and the main results. Let m and r be nonnegative integers; we put $N = 2m + r$. Consider the "star" with N rays, i.e., the graph $K_{1,N}$ whose set $V = \{1, 2, \dots, N+1\}$ of vertices is enumerated in such a way that the vertex 1 is linked with all other vertices. Thus, the set E of edges is equal to

$$\big\{\{1,k\} \,|\, k \in V \setminus \{1\}\big\}.$$

Denote by E^c_m the set $\{\{2k, 2k+1\} \,|\, 1 \le k \le m\}$ of pairs of vertices (in the figure, the vertices of these pairs are joined by dotted lines).

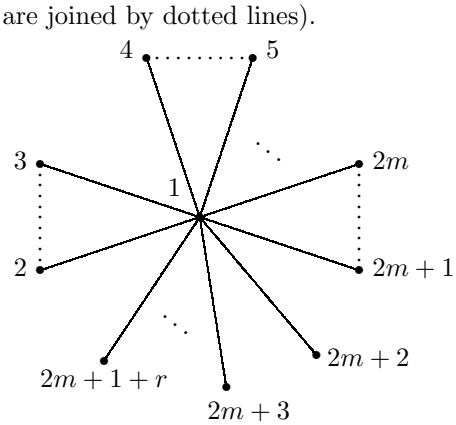

FIGURE 1. The graph $K_{1,N}$ with "dotted edges" E^c_m.

To every edge $\{1,k\}$, we attribute an angle $\theta_{\{1,k\}} \in (0, \pi/2)$, i.e., we define a function $\theta : E \to (0, \pi/2)$. Next, we introduce the function $\tau = \cos\theta : E \to (0,1)$, i.e., $\tau_{\{1,k\}} = \cos\theta_{\{1,k\}} \in (0,1)$.

The above graph $K_{1,N}$, the set E_m^c, and the function τ make it possible to "visualize" the conditions imposed on the subspace systems $S = (H; H_1, \dots, H_n)$, $n = N+1$, treated in this paper.

(Ang): Conditions on the angles (corresponding to the pairs of vertices joined by an edge). For every $k = 2, 3, \dots, N+1$ the subspaces H_1 and H_k form an angle of $\theta_{\{1,k\}}$, i.e., $P_1P_kP_1 = \tau^2_{\{1,k\}}P_1$ and $P_kP_1P_k = \tau^2_{\{1,k\}}P_k$.

(Com): Commutation conditions (corresponding to the pairs of vertices joined by dotted lines). For every $k = 1, 2, \dots, m$, the orthogonal projections P_{2k} and P_{2k+1} commute, i.e., $P_{2k}P_{2k+1} = P_{2k+1}P_{2k}$.

(Ort): Orthogonality conditions (corresponding to the pairs of vertices not joined by an edge or a dotted line). If a pair of different vertices i, j is not joined by an edge and does not belong to E_m^c, then H_i and H_j are orthogonal, i.e., $P_iP_j = 0$.

It should be noted that, for $m = 0$ the subspace system in question is the "simple" system related to the graph $K_{1,N}$ and the function τ on its edges.

The subspace systems described above can be viewed as $*$-representations in a Hilbert space of the $*$-algebra $\mathcal{TL}_{K_{1,N},E_m^c,\tau}$ determined by generators and relations:

$$\begin{aligned}\mathcal{TL}_{K_{1,N},E_m^c,\tau} = \mathbb{C}\langle p_1, p_2, \dots, p_{N+1} \,|\, & p_j^2 = p_j^* = p_j,\ j \in V,\\ & p_ip_jp_i = \tau^2_{\{i,j\}}p_i,\ \{i,j\} \in E,\\ & p_ip_j = p_jp_i,\ \{i,j\} \in E_m^c,\\ & p_ip_j = 0,\ \{i,j\} \notin E \cup E_m^c\rangle.\end{aligned}$$

One-to-one correspondence between the subspace systems S satisfying (Ang), (Com), and (Ort) and the $*$-representations π of the $*$-algebra $\mathcal{TL}_{K_{1,N},E_m^c,\tau}$ in a Hilbert space H is determined by the formula $H_k = \operatorname{Im}\pi(p_k)$, $k \in V$.

The main results of the paper are the following:

(1) a description (up to unitary equivalence) of the subspace systems S satisfying (Ang), (Com), and (Ort) (see Subsection 4.1);
(2) a description (up to unitary equivalence) of all *irreducible* subspace systems satisfying (Ang), (Com), and (Ort) (see Theorem 1 and Subsection 4.2).

Note that for $m \geq 3$ three situations are possible, depending on the parameters $\tau_{\{1,k\}}$: either there are finitely many unitarily nonequivalent irreducible systems of subspaces (a *finite* problem), or there are infinitely many unitarily nonequivalent irreducible systems of subspaces but they admit a description (a *tame* problem), or the problem of description of all irreducible systems up to unitary equivalence is "hopeless" in a sense (a *wild* problem).

The main tool used in the paper to describe subspace systems is a construction of a system of subspaces in a Hilbert space on the basis of its Gram operator (the G-construction, see §2). We note that the G-construction and the results of §2 (except those in Subsection 2.5) are stated for arbitrary subspace systems. The statements in Subsection 2.5 can be strengthened, but this would have led to bulky claims and more complicated proofs.

For certain values of the parameters $\tau_{\{1,k\}}$, the question about the wildness of a problem under study reduces to the question about the wildness of the description problem (up to unitary equivalence) for irreducible triples of projections P_1, P_2, P_3 satisfying $P_1 + P_2 + P_3 \leq (1+\varepsilon)I$. In §3, it will be shown (see Proposition 6) that this problem is $*$-wild.

§2. The Gram operator of a subspace system, and the G-construction

2.1. The G-construction for a system of subspaces. Let $H_{0,k}$, $1 \leq k \leq n$, be a collection of nonzero Hilbert spaces. We introduce the Hilbert space $\widetilde{H} = H_{0,1} \oplus \dots \oplus H_{0,n}$

and denote by $\langle\cdot,\cdot\rangle_0$ the scalar product in it. For $k = 1, \dots, n$, we define the subspace

$$\widetilde{H}_k = \big\{(0, \dots, 0, x_k, 0, \dots, 0) \mid x_k \in H_{0,k}\big\},$$

where x_k occurs at the kth place. Next, we define a unitary operator $\Gamma_k : H_{0,k} \to \widetilde{H}_k$ by

$$\Gamma_k x_k = (0, \dots, 0, x_k, 0, \dots, 0), \quad x_k \in H_{0,k}.$$

Let $B : \widetilde{H} \to \widetilde{H}$ be a bounded nonnegative selfadjoint operator such that for its block decomposition $B = (B_{i,j} : H_{0,j} \to H_{0,i},\ 1 \le i, j \le n)$ we have

$$B_{k,k} = I_{H_{0,k}}, \quad 1 \le k \le n. \tag{1}$$

Put $\widetilde{H}_0 = \operatorname{Ker} B$. By using the operator B, we introduce a scalar product in the linear space $\widetilde{H}/\widetilde{H}_0$:

$$\langle x + \widetilde{H}_0, y + \widetilde{H}_0\rangle = \langle Bx, y\rangle_0, \quad x, y \in \widetilde{H}.$$

Clearly, this definition is consistent because it does not depend on the choice of representatives of the equivalence classes. Let H be the completion of $\widetilde{H}/\widetilde{H}_0$ with respect to this scalar product.

We define a bounded linear operator $\rho : \widetilde{H} \to H$ by

$$\rho(x) = x + \widetilde{H}_0.$$

Put

$$H_k = \rho(\widetilde{H}_k) = \{z + \widetilde{H}_0 \mid z \in \widetilde{H}_k\}, \quad 1 \le k \le n.$$

Since for an arbitrary $z \in \widetilde{H}_k$ we have

$$\|z + \widetilde{H}_0\| = \sqrt{\langle Bz, z\rangle_0} = \|z\|_0,$$

we see that H_k is a subspace of H. Moreover,

$$\rho_k = \rho \restriction_{\widetilde{H}_k} : \widetilde{H}_k \to H_k, \quad 1 \le k \le n,$$

is a unitary operator. Next, since $H_1 + \dots + H_n = \rho(\widetilde{H}) = \widetilde{H}/\widetilde{H}_0$, the sum $H_1 + \dots + H_n$ is dense in H.

The resulting subspace system $(H; H_1, \dots, H_n)$ will be denoted by $\mathcal{G}(H_{0,1}, \dots, H_{0,n}; B)$, and the construction of it itself will be called a G-construction.

2.2. An arbitrary system of subspaces as the result of a G-construction.

Definition 1. Let $S = (K; K_1, \dots, K_n)$ be a system of subspaces of a Hilbert space K, and let Q_i denote the orthogonal projection onto K_i, $1 \le i \le n$. The operator $G = G(S) : \bigoplus_{i=1}^n K_i \to \bigoplus_{i=1}^n K_i$ determined by its block decomposition $G_{i,j} = Q_i \restriction_{K_j} : K_j \to K_i$, $1 \le i, j \le n$, is called the Gram operator of the system S.

Proposition 1. *Let $S = (K; K_1, \dots, K_n)$ be a system of nonzero subspaces of a Hilbert space K such that $K_1 + \dots + K_n$ is dense in K. Then the system $\mathcal{G}(K_1, \dots, K_n; G(S))$ is unitarily equivalent to S.*

Proof. The definition of $G = G(S)$ shows that $G_{i,i} = I_{K_i}$ and $G_{i,j}^* = G_{j,i}$, $1 \le i, j \le n$. We claim that the operator G is nonnegative. Denote by $\langle\cdot,\cdot\rangle$ the scalar product in K. For arbitrary $x_j \in K_j$ and $y_i \in K_i$, we have

$$\langle G_{i,j} x_j, y_i\rangle = \langle Q_i x_j, y_i\rangle = \langle x_j, y_i\rangle.$$

Then for arbitrary $x = (x_1, \dots, x_n) \in \widetilde{H}$ and $y = (y_1, \dots, y_n) \in \widetilde{H}$ we obtain

$$\langle Gx, y\rangle_0 = \Big\langle \sum_{k=1}^n x_k, \sum_{k=1}^n y_k \Big\rangle.$$

This implies that the operator G is nonnegative and its kernel $\operatorname{Ker} G$ consists of the vectors $x \in \widetilde{H}$ such that $x_1 + \cdots + x_n = 0$.

We introduce an operator $U : \widetilde{H}/\widetilde{H}_0 \to \sum_{i=1}^n K_i$ by the formula

$$U(x + \widetilde{H}_0) = \sum_{k=1}^n x_k, \quad x = (x_1, \dots, x_n) \in \widetilde{H}, \quad x_i \in K_i, \quad 1 \le i \le n.$$

This definition is consistent because $\widetilde{H}_0 = \operatorname{Ker} G$. Since

$$\langle x + \widetilde{H}_0, y + \widetilde{H}_0 \rangle = \langle Gx, y \rangle_0 = \left\langle \sum_{k=1}^n x_k, \sum_{k=1}^n y_k \right\rangle$$

and $K_1 + \cdots + K_n$ is dense in K, we see that U is a linear operator preserving the scalar product and having range dense in K. Consequently, U has a unique extension by continuity up to a unitary operator $\bar{U} : H \to K$. Since $\bar{U}(H_i) = K_i$ for all $1 \le i \le n$, the proposition follows. □

2.3. An equivalence criterion for subspace systems resulting from a G-construction.

Proposition 2. *Two subspace systems*

$$\mathcal{G}(H_{0,1}, \dots, H_{0,n}; B) \quad \text{and} \quad \mathcal{G}(H'_{0,1}, \dots, H'_{0,n}; B')$$

are unitarily equivalent if and only if there exists a collection of unitary operators $U_{0,k} : H'_{0,k} \to H_{0,k}$, $1 \le k \le n$, such that for arbitrary i, j we have

$$B'_{i,j} = U^*_{0,i} B_{i,j} U_{0,j}. \tag{2}$$

Introducing the unitary operator

$$\widetilde{U} = \operatorname{diag}(U_{0,1}, \dots, U_{0,n}) : \widetilde{H}' \to \widetilde{H},$$

we can rewrite (2) *in the form $B' = \widetilde{U}^* B \widetilde{U}$.*

Proof. 1. Suppose that the subspace systems

$$(H, H_1, \dots, H_n) = \mathcal{G}(H_{0,1}, \dots, H_{0,n}; B)$$

and

$$(H', H'_1, \dots, H'_n) = \mathcal{G}(H'_{0,1}, \dots, H'_{0,n}; B')$$

are unitarily equivalent. Then there exists a unitary operator $U : H' \to H$ such that $U(H'_k) = H_k$, $1 \le k \le n$. Consider the unitary operator $U_k = U \restriction_{H'_k} : H'_k \to H_k$. Define a unitary operator $U_{0,k} : H'_{0,k} \to H_{0,k}$ by

$$U_{0,k} = \Gamma_k^{-1} \rho_k^{-1} U_k \rho'_k \Gamma'_k.$$

For every $1 \le k \le n$, the diagram

$$\begin{array}{ccc} H'_{0,k} & \xrightarrow{U_{0,k}} & H_{0,k} \\ {\scriptstyle \Gamma'_k}\downarrow & & {\scriptstyle \Gamma_k}\downarrow \\ \widetilde{H}'_k & \longrightarrow & \widetilde{H}_k \\ {\scriptstyle \rho'_k}\downarrow & & {\scriptstyle \rho_k}\downarrow \\ H'_k & \xrightarrow{U_k} & H_k \end{array} \tag{3}$$

is commutative.

We prove the formula $B'_{i,j} = U^*_{0,i} B_{i,j} U_{0,j}$, $1 \le i, j \le n$. Since the diagram (3) is commutative, for every $x \in H'_{0,k}$ we have

$$\Gamma_k U_{0,k} x + \widetilde{H}_0 = U_k(\Gamma'_k x + \widetilde{H}'_0) = U(\Gamma'_k x + \widetilde{H}'_0).$$

Let $x \in H'_{0,j}$, $y \in H'_{0,i}$, then

$$\langle B'_{i,j} x, y\rangle = \langle B'\Gamma'_j x, \Gamma'_i y\rangle_0 = \langle \Gamma'_j x + \widetilde{H}'_0, \Gamma'_i y + \widetilde{H}'_0\rangle,$$

$$\langle B_{i,j} U_{0,j} x, U_{0,i} y\rangle = \langle B\Gamma_j U_{0,j} x, \Gamma_i U_{0,i} y\rangle_0 = \langle \Gamma_j U_{0,j} x + \widetilde{H}_0, \Gamma_i U_{0,i} y + \widetilde{H}_0\rangle.$$

Now, since U is unitary, we see that $\langle B'_{i,j} x, y\rangle = \langle B_{i,j} U_{0,j} x, U_{0,i} y\rangle$. Since $x \in H'_{0,j}$ and $y \in H'_{0,i}$ are arbitrary, we obtain $B'_{i,j} = U^*_{0,i} B_{i,j} U_{0,j}$.

2. Conversely, suppose we are given a collection of unitary operators

$$U_{0,k} : H'_{0,k} \to H_{0,k}, \quad 1 \le k \le n,$$

with $B' = \widetilde{U}^* B \widetilde{U}$. Define an operator $U : \widetilde{H}'/\widetilde{H}'_0 \to \widetilde{H}/\widetilde{H}_0$ by

$$U(x + \widetilde{H}'_0) = \widetilde{U} x + \widetilde{H}_0, \quad x \in \widetilde{H}'.$$

This definition is consistent because $\widetilde{U}(\operatorname{Ker} B') = \operatorname{Ker} B$.

For arbitrary x, $y \in \widetilde{H}'$, we have

$$\langle x + \widetilde{H}'_0, y + \widetilde{H}'_0\rangle = \langle B'x, y\rangle_0 = \langle \widetilde{U}^* B \widetilde{U} x, y\rangle_0 = \langle B\widetilde{U}x, \widetilde{U}y\rangle_0 = \langle \widetilde{U}x + \widetilde{H}_0, \widetilde{U}y + \widetilde{H}_0\rangle.$$

Thus, U is a linear operator preserving the scalar product, and its image $U(\widetilde{H}'/\widetilde{H}'_0) = \widetilde{H}/\widetilde{H}_0$ is dense in H. Consequently, it extends uniquely by continuity up to a unitary operator $\bar{U} : H' \to H$. Clearly, $\bar{U}(H'_k) = H_k$ for all $1 \le k \le n$, and the proposition follows. □

2.4. Relationship between the properties of B and the properties of the subspace system $\mathcal{G}(H_{0,1}, \dots, H_{0,n}; B)$. Suppose we are given a subspace system

$$S = (H; H_1, \dots, H_n) = \mathcal{G}(H_{0,1}, \dots, H_{0,n}; B).$$

Denote by P_i the orthogonal projection onto H_i, $1 \le i \le n$. Let $G = G(S)$ be the Gram operator of the system S, $G_{i,j} = P_i \restriction_{H_j} : H_j \to H_i$, $1 \le i, j \le n$.

Proposition 3. *There exists a collection of unitary operators $U_{0,k} : H_k \to H_{0,k}$, $1 \le k \le n$, such that $G_{i,j} = U^*_{0,i} B_{i,j} U_{0,j}$, $1 \le i,\ j \le n$.*

Proof. Proposition 1 shows that S is unitarily equivalent to the system $\mathcal{G}(H_1, \dots, H_n; G)$. Now, the claim is a consequence of Proposition 2. □

Proposition 3 makes it possible to relate the properties of the system S to the properties of B. In all examples that follow, we denote by α, β a pair of different indices among $\{1, 2, \dots, n\}$.

Example 1. *Orthogonality condition.* The subspaces H_α and H_β are orthogonal if and only if $P_\alpha P_\beta = 0$, which is equivalent to the condition $G_{\alpha,\beta} = 0$ or to the condition $B_{\alpha,\beta} = 0$.

Example 2. *The angle between subspaces.* The subspaces H_α and H_β are at an angle of $\theta_0 \in (0, \pi/2)$ if and only if

$$P_\alpha P_\beta P_\alpha = \tau_0^2 P_\alpha, \quad P_\beta P_\alpha P_\beta = \tau_0^2 P_\beta,$$

i.e.,

$$G_{\alpha,\beta} G_{\beta,\alpha} = \tau_0^2 I_{H_\alpha}, \quad G_{\beta,\alpha} G_{\alpha,\beta} = \tau_0^2 I_{H_\beta},$$

where $\tau_0 = \cos\theta_0$.

Since $G_{i,j} = U_{0,i}^* B_{i,j} U_{0,j}$, $1 \le i,j \le n$, for some collection of unitary operators $U_{0,k} : H_k \to H_{0,k}$, $1 \le k \le n$, it follows that the last condition is equivalent to

$$B_{\alpha,\beta} B_{\beta,\alpha} = \tau_0^2 I_{H_{0,\alpha}}, \quad B_{\beta,\alpha} B_{\alpha,\beta} = \tau_0^2 I_{H_{0,\beta}}.$$

These two identities are equivalent to the property for $B_{\alpha,\beta}/\tau_0$ to be unitary.

Example 3. *Commutation condition.* The condition $P_\alpha P_\beta = P_\beta P_\alpha$ is equivalent to the condition

$$P_\alpha P_\beta = P_\alpha P_\beta P_\alpha.$$

Since $H_1 + \cdots + H_n$ is dense in H, this is also equivalent to the condition

$$P_\alpha P_\beta P_i \restriction_{H_i} = P_\alpha P_\beta P_\alpha P_i \restriction_{H_i}, \quad 1 \le i \le n.$$

These identities can be rewritten as

$$G_{\alpha,\beta} G_{\beta,i} = G_{\alpha,\beta} G_{\beta,\alpha} G_{\alpha,i}, \quad 1 \le i \le n,$$

which is equivalent to

$$B_{\alpha,\beta} B_{\beta,i} = B_{\alpha,\beta} B_{\beta,\alpha} B_{\alpha,i}, \quad 1 \le i \le n.$$

The last identity is fulfilled automatically for $i = \alpha$, and if it is fulfilled for $i = \beta$, then $B_{\alpha,\beta}$ is a partial isometry.

2.5. Irreducibility of the subspace system $\mathcal{G}(H_{0,1}, \dots, H_{0,n}; B)$. Suppose

$$S = (H; H_1, \dots, H_n) = \mathcal{G}(H_{0,1}, \dots, H_{0,n}; B).$$

Denote by P_k the orthogonal projection onto H_k, $1 \le k \le n$.

2.5.1. The descent of an operator. Let $C : H \to H$ be a bounded linear operator commuting with all P_k, $1 \le k \le n$, i.e., the spaces H_k and $H_k^\perp$ are invariant under C for $1 \le k \le n$. We introduce the operators $C_k = C \restriction_{H_k} : H_k \to H_k$, $1 \le k \le n$. Next, define operators $C_{0,k} : H_{0,k} \to H_{0,k}$ by the formula

$$C_{0,k} = \Gamma_k^{-1} \rho_k^{-1} C_k \rho_k \Gamma_k.$$

The collection of the operators $C_{0,k}$, $1 \le k \le n$, will be called the *descent* of C. This definition shows that the diagram

$$\begin{array}{ccc} H_{0,k} & \xrightarrow{C_{0,k}} & H_{0,k} \\ {\scriptstyle \Gamma_k}\downarrow & & {\scriptstyle \Gamma_k}\downarrow \\ \widetilde{H}_k & \longrightarrow & \widetilde{H}_k \\ {\scriptstyle \rho_k}\downarrow & & {\scriptstyle \rho_k}\downarrow \\ H_k & \xrightarrow{C_k} & H_k \end{array}$$

is commutative for $1 \le k \le n$.

Proposition 4. *For all $1 \le i,j \le n$, we have*

$$C_{0,i} B_{i,j} = B_{i,j} C_{0,j}. \tag{4}$$

Proof. Let $i,j \in \{1, \dots, n\}$. Consider arbitrary $x \in H_{0,j}$, $y \in H_{0,i}$. Then

$$\rho_j \Gamma_j C_{0,j} x = C_j \rho_j \Gamma_j x = C \rho_j \Gamma_j x.$$

Since $C_{0,i}^* = \Gamma_i^{-1} \rho_i^{-1} C_i^* \rho_i \Gamma_i$, we have

$$\rho_i \Gamma_i C_{0,i}^* y = C_i^* \rho_i \Gamma_i y = C^* \rho_i \Gamma_i y.$$

Now,

$$\begin{aligned}\langle B_{i,j}C_{0,j}x, y\rangle &= \langle B\Gamma_j C_{0,j}x, \Gamma_i y\rangle_0 = \langle \rho_j\Gamma_j C_{0,j}x, \rho_i\Gamma_i y\rangle \\ &= \langle C\rho_j\Gamma_j x, \rho_i\Gamma_i y\rangle = \langle \rho_j\Gamma_j x, C^*\rho_i\Gamma_i y\rangle = \langle \rho_j\Gamma_j x, \rho_i\Gamma_i C_{0,i}^* y\rangle \\ &= \langle B\Gamma_j x, \Gamma_i C_{0,i}^* y\rangle_0 = \langle B_{i,j}x, C_{0,i}^* y\rangle = \langle C_{0,i}B_{i,j}x, y\rangle.\end{aligned}$$

Thus, $\langle B_{i,j}C_{0,j}x, y\rangle = \langle C_{0,i}B_{i,j}x, y\rangle$ for arbitrary $x \in H_{0,j}$, $y \in H_{0,i}$. This implies (4). □

For a sequence $l = (i_1, i_2, \dots, i_k)$ of indices, we introduce the operator

$$B_l = B_{i_1,i_2} \dots B_{i_{k-1},i_k} : H_{0,i_k} \to H_{0,i_1}.$$

Corollary 1. *For every sequence $l = (i, \dots, j)$ of indices, we have $C_{0,i}B_l = B_l C_{0,j}$.*

2.5.2. Lifting of a collection of operators. Let $C_{0,k} : H_{0,k} \to H_{0,k}$, $1 \le k \le n$, be a collection of unitary operators such that (4) is fulfilled for arbitrary i, j. Define the unitary operator $\widetilde{C} = \operatorname{diag}(C_{0,1}, \dots, C_{0,n}) : \widetilde{H} \to \widetilde{H}$. By (4), we have $\widetilde{C}B = B\widetilde{C}$, i.e., $B = \widetilde{C}^* B\widetilde{C}$.

We introduce an operator $C : \widetilde{H}/\widetilde{H}_0 \to \widetilde{H}/\widetilde{H}_0$ by the formula

$$C(x + \widetilde{H}_0) = \widetilde{C}x + \widetilde{H}_0, \quad x \in \widetilde{H}.$$

This definition is consistent because $\widetilde{C}(\operatorname{Ker} B) = \operatorname{Ker} B$.

For arbitrary x, $y \in \widetilde{H}$, we have

$$\langle x + \widetilde{H}_0, y + \widetilde{H}_0\rangle = \langle Bx, y\rangle_0 = \langle \widetilde{C}^* B\widetilde{C}x, y\rangle_0 = \langle B\widetilde{C}x, \widetilde{C}y\rangle_0 = \langle \widetilde{C}x + \widetilde{H}_0, \widetilde{C}y + \widetilde{H}_0\rangle.$$

Since $\widetilde{H}/\widetilde{H}_0$ is dense in H, we see that C is a linear operator preserving the scalar product and having dense range in H, $C(\widetilde{H}/\widetilde{H}_0) = \widetilde{H}/\widetilde{H}_0$. Thus, C has a unique extension by continuity up to a unitary operator (still denoted by C), $C : H \to H$. This C is called the lifting of the collection $C_{0,k}$, $1 \le k \le n$. The definition of C shows that $C(H_k) = H_k$, $1 \le k \le n$. Since C is unitary, we see that $C(H_k^\perp) = H_k^\perp$, $1 \le k \le n$.

Clearly, the descent of C coincides with the collection $C_{0,k}$, $1 \le k \le n$.

2.5.3. Irreducibility of a subspace system S. For a sequence $l = (i_1, \dots, i_{k-1}, i_k)$ of indices, we define the conjugate system $l^* = (i_k, i_{k-1}, \dots, i_1)$. Clearly, $B_l^* = B_{l^*}$. For a pair l, l' of sequences of indices such that the last index in l coincides with the first index in l', i.e., $l = (i_1, \dots, i_{k-1}, i_k)$, $l' = (i_k, i_{k+1}, \dots, i_m)$, we define their product by the formula $ll' = (i_1, \dots, i_{k-1}, i_k, i_{k+1}, \dots, i_m)$. Clearly, $B_{ll'} = B_l B_{l'}$.

Let $\alpha \in \{1, \dots, n\}$. Denote by $\mathcal{L}_\alpha$ the set of all sequences l of indices whose first and last elements are equal to α. We note that the operators B_l, $l \in \mathcal{L}_\alpha$, constitute a $*$-set, i.e., A^* belongs to this set whenever A does.

Proposition 5. *Let α have the property that for every k, $1 \le k \le n$, there exists a sequence $l = (\alpha, \dots, k)$ of indices for which the operator B_l is invertible. The following statements are equivalent*:

(1) *the subspace system $S = (H; H_1, \dots, H_n)$ is irreducible,*
(2) *the set of the operators B_l, $l \in \mathcal{L}_\alpha$, is irreducible.*

Proof. (1) ⇒ (2). Suppose the contrary. Then there exists an operator $C_{0,\alpha} : H_{0,\alpha} \to H_{0,\alpha}$ different from $\lambda I_{H_{0,\alpha}}$, $\lambda \in \mathbb{C}$, such that $C_{0,\alpha}B_l = B_l C_{0,\alpha}$ for arbitrary $l \in \mathcal{L}_\alpha$. Since the operators B_l, $l \in \mathcal{L}_\alpha$, form a $*$-set, the operator $C_{0,\alpha}$ can be chosen unitary. By assumption, for every k there exists a sequence $l(k) = (k, \dots, \alpha)$ of indices such that $B_{l(k)}$ is invertible. We introduce the operator

$$C_{0,k} = B_{l(k)} C_{0,\alpha} B_{l(k)}^{-1} : H_{0,k} \to H_{0,k}, \quad 1 \le k \le n.$$

Note that the definition is consistent for $k = \alpha$ because $C_{0,\alpha}$ and $B_{l(\alpha)}$ commute.

We show that $C_{0,k}$ is unitary. Clearly, $C_{0,k}^* = (B_{l(k)}^*)^{-1} C_{0,\alpha}^* B_{l(k)}^*$. Thus, $C_{0,k}$ is invertible and

$$\begin{aligned} C_{0,k}^* C_{0,k} &= (B_{l(k)}^*)^{-1} C_{0,\alpha}^* B_{l(k)}^* B_{l(k)} C_{0,\alpha} B_{l(k)}^{-1} \\ &= (B_{l(k)}^*)^{-1} C_{0,\alpha}^* C_{0,\alpha} B_{l(k)}^* B_{l(k)} B_{l(k)}^{-1} = I_{H_{0,k}}, \end{aligned}$$

implying that $C_{0,k}$ is unitary.

We show that $C_{0,i} B_{i,j} = B_{i,j} C_{0,j}$ for every indices i, j. This identity is equivalent to the identity $B_{l(i)} C_{0,\alpha} B_{l(i)}^{-1} B_{i,j} = B_{i,j} B_{l(j)} C_{0,\alpha} B_{l(j)}^{-1}$, which is equivalent to

$$B_{l(i)}^* B_{l(i)} C_{0,\alpha} B_{l(i)}^{-1} B_{i,j} = B_{l(i)}^* B_{i,j} B_{l(j)} C_{0,\alpha} B_{l(j)}^{-1}. \tag{5}$$

Since the sequences $l(i)^* l(i)$ and $l(i)^*(i,j) l(j)$ of indices start with and terminate at α, the operators $B_{l(i)}^* B_{l(i)}$ and $B_{l(i)}^* B_{i,j} B_{l(j)}$ commute with $C_{0,\alpha}$. Therefore, the two sides of (5) are equal to $C_{0,\alpha} B_{l(i)}^* B_{i,j}$, and (5) is true.

We lift the family $C_{0,k}$, $1 \le k \le n$, of unitary operators, to a unitary operator $C : H \to H$. Then $CP_k = P_k C$, $1 \le k \le n$. Since the subspace system S is irreducible, for some $\lambda \in \mathbb{C}$ we have $C = \lambda I_H$. Therefore, $C_{0,\alpha} = \lambda I_{H_{0,\alpha}}$, a contradiction.

(2) $\Rightarrow$ (1). Suppose that a linear operator $C : H \to H$ commutes with all P_k, $1 \le k \le n$. Consider the collection $C_{0,k}$, $1 \le k \le n$, of operators obtained from C by descent. Since $C_{0,\alpha} B_l = B_l C_{0,\alpha}$ for every $l \in \mathcal{L}_\alpha$, there exists $\lambda \in \mathbb{C}$ with $C_{0,\alpha} = \lambda I_{H_{0,\alpha}}$. Consider an arbitrary $1 \le k \le n$ and choose a sequence $l = (\alpha, \dots, k)$ of indices for which the operator B_l is invertible. Since $C_{0,\alpha} B_l = B_l C_{0,k}$, we have $B_l (C_{0,k} - \lambda I_{H_{0,k}}) = 0$, whence $C_{0,k} = \lambda I_{H_{0,k}}$. It follows that $Cx = \lambda x$ for $x \in \rho(\widetilde{H})$. Since $\rho(\widetilde{H})$ is dense in H, we see that $C = \lambda I_H$. Thus, the system S is irreducible. $\square$

§3. The wildness of the description problem for triples of orthogonal projections

It is well known that the problem of describing irreducible triples of orthogonal projections up to unitary equivalence is at least as complex as that of describing irreducible pairs of bounded selfadjoint operators (again up to unitary equivalence), i.e., it is $*$-wild (see, e.g., [5]). In Subsection 4.2 we shall need a refinement of the statement about wildness.

Proposition 6. *For every $\varepsilon > 0$, the problem of describing irreducible triples of orthogonal projections P_1, P_2, P_3 with*

$$P_1 + P_2 + P_3 \le (1 + \varepsilon) I \tag{6}$$

up to unitary equivalence is $*$-wild.

The proof of this statement presented below is based on certain ideas suggested in [6].

Consider the following construction of subspace systems in a Hilbert space. Let $n \in \mathbb{N}$, $n \ge 2$, let L be a Hilbert space, and let

$$\widehat{B} = (B_{i,j} : L \to L \mid 2 \le j \le n, 1 \le i \le j-1)$$

be a collection of continuous linear operators. We introduce the Hilbert space

$$K = \bigoplus_{i=1}^{n} L$$

and the collection of its subspaces

$$K_1 = \{(v, 0, \dots 0) \mid v \in L\},$$
$$K_j = \{(B_{1,j}v, \dots, B_{j-1,j}v, v, 0, \dots, 0) \mid v \in L\}, \quad 2 \le j \le n.$$

The subspace system $(K; K_1, \dots, K_n)$ will be denoted by $\mathcal{S}_n(L, \widehat{B})$.

We present some properties of this construction to be used in the proof of Proposition 6.

For $j = 1, \dots, n$, we introduce the subspaces

$$L_j = \{(0, \dots, 0, v, 0, \dots, 0) \mid v \in L\},$$

where v occurs at the jth place.

The following statement was proved in [6, Theorem 2]. We supply it with a shorter proof.

Lemma 1. *Let*

$$(K; K_1, \dots, K_n) = \mathcal{S}_n(L, \widehat{B}), \quad (K'; K'_1, \dots, K'_n) = \mathcal{S}_n(L', \widehat{B}').$$

The following statements are true.

(1) *If a unitary operator $U : K \to K'$ satisfies $U(K_j) = K'_j$, $1 \le j \le n$, then $U = \operatorname{diag}(U_1, \dots, U_n)$ for some unitary operators $U_j : L \to L'$, $1 \le j \le n$; moreover, $U_i B_{i,j} = B'_{i,j} U_j$ for all $i < j$.*

(2) *Suppose a collection of unitary operators $U_j : L \to L'$, $1 \le j \le n$, satisfies $U_i B_{i,j} = B'_{i,j} U_j$ for all $i < j$. Then the operator $U = \operatorname{diag}(U_1, \dots, U_n) : K \to K'$ is unitary and $U(K_j) = K'_j$, $1 \le j \le n$.*

Proof. We prove the first statement of the lemma. Since $U(K_j) = K'_j$, $1 \le j \le n$, we have

$$U(K_1 + \dots + K_j) = K'_1 + \dots + K'_j, \quad 1 \le j \le n.$$

It can easily be observed that

$$K_1 + \dots + K_j = L_1 \oplus \dots \oplus L_j, \quad K'_1 + \dots + K'_j = L'_1 \oplus \dots \oplus L'_j$$

for every j, $1 \le j \le n$. Therefore,

$$U(L_1 \oplus \dots \oplus L_j) = L'_1 \oplus \dots \oplus L'_j, \quad 1 \le j \le n.$$

Consequently, $U(L_j) = L'_j$, $1 \le j \le n$. Thus,

$$U = \operatorname{diag}(U_1, \dots, U_n)$$

for some unitary operators $U_j : L \to L'$, $1 \le j \le n$.

Next,

$$U(K_j) = \{(U_1 B_{1,j} v, \dots, U_{j-1} B_{j-1,j} v, U_j v, 0, \dots, 0) \mid v \in L\}.$$

Since $U(K_j) = K'_j$, we have $U_i B_{i,j} v = B'_{i,j} U_j v$, $v \in L$, i.e., $U_i B_{i,j} = B'_{i,j} U_j$.

We prove the second statement of the lemma. Clearly, U is unitary. For an arbitrary j, $1 \le j \le n$, we have

$$\begin{aligned} U(K_j) &= \{(U_1 B_{1,j} v, \dots, U_{j-1} B_{j-1,j} v, U_j v, 0, \dots, 0) \mid v \in L\} \\ &= \{(B'_{1,j} U_j v, \dots, B'_{j-1,j} U_j v, U_j v, 0 \dots, 0) \mid v \in L\} \\ &= \{(B'_{1,j} v', \dots, B'_{j-1,j} v', v', 0 \dots, 0) \mid v' \in L'\}. \end{aligned}$$

Thus, $U(K_j) = K'_j$. □

Denote by P_j and Q_j the orthogonal projections onto K_j and L_j, respectively.

Lemma 2. *For an arbitrary $\varepsilon > 0$, there exists $\delta = \delta(n, \varepsilon) > 0$ such that if $\|B_{i,j}\| \le \delta$ for all $i < j$, then*

$$\|P_j - Q_j\| \le \varepsilon, \quad j = 1, \dots, n.$$

This lemma is a direct consequence of Lemma 3, which gives explicit formulas for the block decomposition of P_j, $j = 1, \dots, n$. Note that $Q_j = \operatorname{diag}(0, \dots, 0, I_L, 0, \dots, 0)$, where I_L occurs at the jth place.

Lemma 3. *Suppose that $n \in \mathbb{N}$ and L is a Hilbert space. Let $A_i : L \to L$, $1 \le i \le n$, be a collection of continuous linear operators such that the operator*

$$A = \sum_{k=1}^{n} A_k^* A_k$$

is invertible. Then the set

$$W = \{(A_1 v, \dots, A_n v) \mid v \in L\}$$

is a subspace of the Hilbert space $K = \bigoplus_{i=1}^{n} L$, and for the block decomposition of the orthogonal projection P_W we have

$$P_W = \left(A_i A^{-1} A_j^* : L \to L,\ 1 \le i, j \le n\right). \tag{7}$$

Proof. Clearly, W is a linear subspace. We show that it is closed. Since A is nonnegative and invertible, we have $A \ge \lambda I_L$ for some $\lambda > 0$. Therefore, for every $v \in L$ we have

$$\sum_{k=1}^{n} \|A_k v\|^2 = \sum_{k=1}^{n} \langle A_k^* A_k v, v \rangle = \langle Av, v \rangle \ge \lambda \|v\|^2.$$

This readily implies that W is closed. Thus, W is a subspace of the Hilbert space K.

We prove (7). For an arbitrary vector $x = (x_1, \dots, x_n) \in K$, there exists $v \in L$ such that $P_W x = (A_1 v, \dots, A_n v)$. The vector $x - P_W x$ belongs to $W^\perp$, i.e., the vectors $(x_1 - A_1 v, \dots, x_n - A_n v)$ and $(A_1 u, \dots, A_n u)$ are orthogonal for arbitrary $u \in L$. Consequently,

$$0 = \sum_{k=1}^{n} \langle x_k - A_k v, A_k u \rangle = \sum_{k=1}^{n} \langle A_k^* x_k - A_k^* A_k v, u \rangle = \left\langle \sum_{k=1}^{n} A_k^* x_k - Av, u \right\rangle.$$

Thus,

$$Av = \sum_{k=1}^{n} A_k^* x_k, \quad v = A^{-1} \sum_{k=1}^{n} A_k^* x_k,$$

and the ith component of $P_W x$ is equal to

$$A_i v = A_i A^{-1} \sum_{j=1}^{n} A_j^* x_j = \sum_{j=1}^{n} A_i A^{-1} A_j^* x_j.$$

This proves (7). □

Proof of Proposition 6. We fix $\varepsilon > 0$ and choose $\delta = \delta(3, \varepsilon/3) > 0$ in accordance with Lemma 2. Given a pair of selfadjoint operators A and B acting in a Hilbert space L and satisfying $\|A\| < 1$, $\|B\| < 1$, we put $\widehat{B} = (B_{1,2}, B_{1,3}, B_{2,3})$, where

$$B_{1,2} = \delta I_L, \quad B_{1,3} = \frac{\delta}{2}(A + iB), \quad B_{2,3} = \delta I_L.$$

Consider the subspace triple

$$(K; K_1, K_2, K_3) = \mathcal{S}_3(L, \widehat{B}).$$

Then, by Lemma 2, we have

$$\|P_1 - Q_1\| \le \frac{\varepsilon}{3}, \quad \|P_2 - Q_2\| \le \frac{\varepsilon}{3}, \quad \|P_3 - Q_3\| \le \frac{\varepsilon}{3}.$$

Consequently, $\|P_1 + P_2 + P_3 - I_K\| \le \varepsilon$, whence $P_1 + P_2 + P_3 \le (1+\varepsilon)I_K$.

Thus, given a pair of selfadjoint operators A, B with norms smaller than 1, we have constructed a triple of orthogonal projections satisfying (6).

We show that this construction possesses the following properties.

(P1) Two triples (P_1, P_2, P_3) and (P_1', P_2', P_3') are unitarily equivalent if and only if the corresponding pairs (A, B) and (A', B') are unitarily equivalent.

(P2) The triple (P_1, P_2, P_3) is irreducible if and only if the corresponding pair (A, B) is irreducible.

In order to verify (P1) and (P2), we shall show that the construction described above has the following property.

(P) If a unitary operator $U : K \to K'$ intertwines two triples (P_1, P_2, P_3) and (P_1', P_2', P_3'), i.e., satisfies $UP_j = P_j'U$, $j = 1, 2, 3$, then $U = \operatorname{diag}(V, V, V)$ for some unitary operator $V : L \to L'$ that intertwines the pairs (A, B) and (A', B'), i.e., satisfies $VA = A'V$ and $VB = B'V$. Conversely, if a unitary operator $V : L \to L'$ intertwines the pairs (A, B) and (A', B'), then the unitary operator $U = \operatorname{diag}(V, V, V) : K \to K'$ intertwines the triples (P_1, P_2, P_3) and (P_1', P_2', P_3').

Indeed, let U intertwine the triples (P_1, P_2, P_3) and (P_1', P_2', P_3'). Then $U(K_j) = K_j'$, $j = 1, 2, 3$. The first statement in Lemma 1 shows that $U = \operatorname{diag}(U_1, U_2, U_3)$ for some unitary operators $U_j : L \to L'$, $j = 1, 2, 3$, satisfying $U_j B_{j,k} = B_{j,k}' U_k$ for all $j < k$. Putting $j = 1$, $k = 2$, we obtain $U_1 = U_2$. Putting $j = 2$, $k = 3$, we obtain $U_2 = U_3$. The formula $U_1 B_{1,3} = B_{1,3}' U_3$ is equivalent to the formula $U_1(A + iB)U_1^* = A' + iB'$. It follows that $U_1 A U_1^* = A'$, $U_1 B U_1^* = B'$, i.e., $U_1 A = A' U_1$, $U_1 B = B' U_1$. Thus, $U = \operatorname{diag}(U_1, U_1, U_1)$ and U_1 intertwines the pairs (A, B) and (A', B').

Conversely, suppose that V intertwines the pairs (A, B) and (A', B'). Then $V(A+iB) = (A' + iB')V$. Let $U = \operatorname{diag}(V, V, V)$. The second statement in Lemma 1 shows that $U(K_j) = K_j'$, $j = 1, 2, 3$. Therefore, $UP_jU^* = P_j'$, i.e., $UP_j = P_j'U$, $j = 1, 2, 3$.

To verify (P1), we observe the following.

(1) Two triples (P_1, P_2, P_3) and (P_1', P_2', P_3') are unitarily equivalent if and only if there exists a unitary operator $U : K \to K'$ that intertwines these triples.
(2) Two pairs (A, B) and (A', B') are unitarily equivalent if and only if there exists a unitary operator $V : L \to L'$ that intertwines these pairs.

Now it is easily seen that (P1) is a consequence of (P).

To verify (P2), we observe the following.

(1) A triple (P_1, P_2, P_3) is irreducible if and only if for every unitary operator $U : K \to K$ that intertwines this triple with itself there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $U = \lambda I_K$.
(2) A pair (A, B) is irreducible if and only if for every unitary operator $V : L \to L$ that intertwines this pair with itself there exists $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $V = \lambda I_L$.

Now it is easily seen that (P2) is a consequence of (P).

So, the initial problem involves the problem of describing unitarily nonequivalent irreducible pairs of selfadjoint operators A, B with $\|A\| < 1$, $\|B\| < 1$. The latter is equivalent to the "hopeless" problem of describing unitarily nonequivalent irreducible pairs of selfadjoint operators A_0, B_0. The relevant bijection is given by the formulas

$$A = \frac{2}{\pi} \arctan A_0, \quad B = \frac{2}{\pi} \arctan B_0,$$

i.e.,

$$A_0 = \tan\left(\frac{\pi}{2}A\right), \quad B_0 = \tan\left(\frac{\pi}{2}B\right).$$

Clearly, the pairs (A, B) and (A', B') are unitarily equivalent if and only if the corresponding pairs (A_0, B_0) and (A_0', B_0') are unitarily equivalent; the pair (A, B) is irreducible if and only if the pair (A_0, B_0) is irreducible.

Thus, the initial problem involves a "hopeless" subproblem, and is "hopeless" itself. □

§4. Description of the systems satisfying (Ang), (Com), and (Ort)

In this section we

(1) describe all subspace systems satisfying (Ang), (Com), and (Ort);
(2) describe all unitarily nonequivalent *irreducible* systems satisfying (Ang), (Com), and (Ort);
(3) as an example, present a description of all unitarily nonequivalent irreducible subspaces satisfying (Ang), (Com), and (Ort) in the case of $m = 3$ and $r = 1$.

First, we show that there is no loss of generality in assuming that $\tau_{\{1,2k\}} = \tau_{\{1,2k+1\}}$ for all $1 \le k \le m$.

Lemma 4. *Let M, M_1, M_2 be subspaces of H. Suppose that the following conditions are satisfied*:

(1) *M and M_i form an angle of $\varphi_i \in [0, \pi/2)$, $i = 1, 2$;*
(2) *the orthogonal projections onto M_1 and M_2 commute.*

If $\varphi_1 \ne \varphi_2$, then M_1 and M_2 are orthogonal.

Proof. Let Q, Q_1, Q_2 denote the orthogonal projections onto M, M_1, M_2, respectively. Put $\mu_k = \cos\varphi_k$, $k = 1, 2$. Then

$$Q_1Q_2 = Q_1Q_2Q_1 = \frac{1}{\mu_2^2}Q_1Q_2QQ_2Q_1 = \frac{1}{\mu_2^2}Q_2Q_1QQ_1Q_2 = \frac{\mu_1^2}{\mu_2^2}Q_2Q_1Q_2 = \frac{\mu_1^2}{\mu_2^2}Q_1Q_2,$$

whence $Q_1Q_2 = 0$. □

So, reducing m if necessary, we may assume that $\tau_{\{1,2k\}} = \tau_{\{1,2k+1\}}$ for all $1 \le k \le m$. We define

$$\tau_k = \begin{cases} \tau_{\{1,2k\}} = \tau_{\{1,2k+1\}}, & 1 \le k \le m; \\ \tau_{\{1,k+m+1\}}, & m+1 \le k \le m+r. \end{cases}$$

4.1. Description of all subspace systems $S = (H; H_1, \dots, H_{N+1})$ that satisfy (Ang), (Com), and (Ort). We start with several obvious remarks.

1. The zero system $S = (H; 0, \dots, 0)$ satisfies all the required conditions. In the sequel, we consider nonzero subspace systems. We observe that if a system $S = (H; H_1, \dots, H_{N+1})$ satisfies (Ang) and $H_k = 0$ for some k, then $H_1 = \dots = H_{N+1} = 0$. Thus, if a system is nonzero, then $H_k \ne 0$ for $1 \le k \le N+1$.

2. Let $S = (H; H_1, \dots, H_{N+1})$ be a nonzero subspace system satisfying (Ang), (Com), and (Ort). Suppose that $H_1 + \dots + H_{N+1}$ is not dense in H. We introduce the systems

$$S' = (H'; H_1, \dots, H_{N+1}) \quad \text{and} \quad S'' = (H \ominus H'; 0, \dots, 0),$$

where $H' = \overline{H_1 + \dots + H_{N+1}}$. Then $S = S' \oplus S''$, S' satisfies (Ang), (Com), and (Ort), and S'' is a zero system. Therefore, we may restrict ourselves to the case where $H_1 + \dots + H_{N+1}$ is dense in H.

3. Now, let $S = (H; H_1, \dots, H_{N+1})$ be a subspace system satisfying (Ang), (Ort), and (Com) and such that $H_k \ne 0$, $1 \le k \le N+1$, and $H_1 + \dots + H_{N+1}$ is dense in H. Let

$G = (G_{i,j},\ 1 \le i,j \le N+1)$ be the Gram operator of S. Then S is unitarily equivalent to the system $\mathcal{G}(H_1,\dots,H_{N+1};G)$. Since H_1 and H_k form an angle of $\theta_{\{1,k\}}$, the operator $G_{1,k}/\tau_{\{1,k\}}$ is unitary. We define unitary operators $U_{0,k} : H_1 \to H_k$, $1 \le k \le N+1$, by the formulas

$$U_{0,1} = I_{H_1}, \quad U_{0,k} = G^*_{1,k}/\tau_{\{1,k\}}, \quad 2 \le k \le N+1.$$

Put $B_{i,j} = U^*_{0,i} G_{i,j} U_{0,j}$, $1 \le i,j \le N+1$, then $B_{1,k} = \tau_{\{1,k\}} I_{H_1}$, $2 \le k \le N+1$. Let the operator $B : \bigoplus_{k=1}^{N+1} H_1 \to \bigoplus_{k=1}^{N+1} H_1$ be determined by the block decomposition $B = (B_{i,j})$. Proposition 2 implies that $\mathcal{G}(H_1,\dots,H_{N+1};G)$ is unitarily equivalent to $\mathcal{G}(H_1,\dots,H_1;B)$, whence S is unitarily equivalent to $\mathcal{G}(H_1,\dots,H_1;B)$.

Now, let $S = (H;H_1,\dots,H_{N+1}) = \mathcal{G}(H_0,\dots,H_0;B)$ for some Hilbert space H_0 and an operator $B : \bigoplus_{k=1}^{N+1} H_0 \to \bigoplus_{k=1}^{N+1} H_0$ such that $B_{1,k} = \tau_{\{1,k\}} I_{H_0}$, $2 \le k \le N+1$. We are going to describe all B such that the subspace system S satisfies (Ang), (Com), and (Ort). For this, we invoke the results of Subsection 2.4.

Condition (Ang) is equivalent to the property of $B_{1,k}/\tau_{\{1,k\}}$ to be unitary for $2 \le k \le N+1$. Since $B_{1,k} = \tau_{\{1,k\}} I_{H_0}$, this condition is fulfilled.

Condition (Ort) is equivalent to the fact that $B_{i,j} = 0$ for all couples of distinct i and j with $\{i,j\} \notin E \cup E^c_m$.

Consider condition (Com). The requirement that $P_{2k}P_{2k+1} = P_{2k+1}P_{2k}$, $1 \le k \le m$, is equivalent to the formula

$$B_{2k,2k+1}B_{2k+1,i} = B_{2k,2k+1}B_{2k+1,2k}B_{2k,i} \quad 1 \le i \le N+1. \tag{8}$$

For $i = 1$ we obtain $\tau_k B_{2k,2k+1} = \tau_k B_{2k,2k+1} B_{2k+1,2k}$, implying that $B_{2k,2k+1}$ is an orthogonal projection.

For $i = 2k$ formula (8) is true automatically.

For $i = 2k+1$, we obtain $B_{2k,2k+1} = B_{2k,2k+1}B_{2k+1,2k}B_{2k,2k+1}$. This condition is fulfilled because $B_{2k,2k+1}$ is an orthogonal projection.

For $i \ne 1$, $2k$, $2k+1$ the two sides of (8) are equal to zero.

Thus, B has the form

$$B = \begin{pmatrix} I & \tau_1 I & \tau_1 I & \dots & \tau_m I & \tau_m I & \tau_{m+1} I & \dots & \tau_{m+r} I \\ \tau_1 I & I & Q_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \tau_1 I & Q_1 & I & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & \dots & 0 \\ \tau_m I & 0 & 0 & 0 & I & Q_m & 0 & \dots & 0 \\ \tau_m I & 0 & 0 & 0 & Q_m & I & 0 & \dots & 0 \\ \tau_{m+1} I & 0 & 0 & 0 & 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \tau_{m+r} I & 0 & 0 & 0 & 0 & 0 & 0 & \dots & I \end{pmatrix},$$

for some collection of orthogonal projections Q_k, $1 \le k \le m$; we shall use the notation $B(Q_1,\dots,Q_m)$.

In order that the G-construction be applicable to a collection Q_k of orthogonal projections, it is necessary that the operator $B(Q_1,\dots,Q_m)$ be nonnegative. To find out when this happens, we need a lemma.

Lemma 5. *Let K be a Hilbert space, and let $A_1,\dots,A_n$ be nonnegative invertible operators in K. Suppose that $y \in K$ and $\mu_k > 0$, $1 \le k \le n$. If $u_k \in K$, $1 \le k \le n$, and $\sum_{k=1}^n \mu_k u_k = y$, then*

$$\sum_{k=1}^n \langle A_k u_k, u_k \rangle \ge \left\langle \left(\sum_{j=1}^n \mu_j^2 A_j^{-1} \right)^{-1} y, y \right\rangle;$$

moreover, equality occurs if and only if

$$u_k = \mu_k A_k^{-1}\Big(\sum_{j=1}^n \mu_j^2 A_j^{-1}\Big)^{-1} y, \quad 1 \le k \le n.$$

Proof. We denote $F(u_1,\dots,u_n) = \sum_{k=1}^n \langle A_k u_k, u_k\rangle$. Consider

$$F(u_1+h_1,\dots,u_n+h_n) - F(u_1,\dots,u_n) = \sum_{k=1}^n (\langle A_k u_k, h_k\rangle + \langle A_k h_k, u_k\rangle + \langle A_k h_k, h_k\rangle).$$

If $u_1,\dots,u_n \in K$ satisfy

$$\sum_{k=1}^n \mu_k u_k = y, \quad \mu_1^{-1} A_1 u_1 = \dots = \mu_n^{-1} A_n u_n, \tag{9}$$

then for arbitrary $h_1,\dots,h_n \in K$ with $\sum_{k=1}^n \mu_k h_k = 0$ we have

$$\sum_{k=1}^n \langle A_k u_k, h_k\rangle = \sum_{k=1}^n \langle \mu_k^{-1} A_k u_k, \mu_k h_k\rangle = 0,$$
$$\sum_{k=1}^n \langle A_k h_k, u_k\rangle = \sum_{k=1}^n \langle \mu_k h_k, \mu_k^{-1} A_k u_k\rangle = 0;$$

consequently,

$$F(u_1+h_1,\dots,u_n+h_n) - F(u_1,\dots,u_n) = \sum_{k=1}^n \langle A_k h_k, h_k\rangle.$$

It follows that $F(u_1+h_1,\dots,u_n+h_n) \ge F(u_1,\dots,u_n)$ with equality if and only if $h_1 = \dots = h_n = 0$, i.e., $F(v_1,\dots,v_n) \ge F(u_1,\dots,u_n)$ for arbitrary $v_1,\dots,v_n \in K$ such that $\sum_{k=1}^n \mu_k v_k = y$, with equality only if $v_k = u_k$, $1 \le k \le n$.

We find $u_1,\dots,u_n$ satisfying (9). Let $\mu_k^{-1} A_k u_k = x$, $1 \le k \le n$. Then $u_k = \mu_k A_k^{-1} x$. We have $\sum_{k=1}^n \mu_k u_k = \sum_{k=1}^n \mu_k^2 A_k^{-1} x = y$, whence $x = \big(\sum_{k=1}^n \mu_k^2 A_k^{-1}\big)^{-1} y$. Thus, $u_k = \mu_k A_k^{-1}\big(\sum_{j=1}^n \mu_j^2 A_j^{-1}\big)^{-1} y$, $1 \le k \le n$, and

$$\begin{aligned} F(u_1,\dots,u_n) &= \sum_{k=1}^n \Big\langle \mu_k \Big(\sum_{j=1}^n \mu_j^2 A_j^{-1}\Big)^{-1} y, \mu_k A_k^{-1}\Big(\sum_{j=1}^n \mu_j^2 A_j^{-1}\Big)^{-1} y \Big\rangle \\ &= \Big\langle \Big(\sum_{j=1}^n \mu_j^2 A_j^{-1}\Big)^{-1} y, y \Big\rangle. \end{aligned}$$

□

The preceding lemma allows us to prove the following statement.

Proposition 7. *Let*

$$\xi(\tau) = 1 - \sum_{k=1}^{m+r} \tau_k^2.$$

The operator $B = B(Q_1,\dots,Q_m)$ is nonnegative if and only if

$$\sum_{k=1}^m \tau_k^2 R_k \le \xi(\tau) I, \tag{10}$$

where $R_k = I - Q_k$, $1 \le k \le m$.

Proof. We write the nonnegativity condition:

$$\langle Bx, x\rangle \geq 0, \quad x = (z, x_1, y_1, \dots, x_m, y_m, v_1, \dots, v_r) \in \bigoplus_{k=1}^{N+1} H_0.$$

Denoting

$$z_0 = z_0(x_1, y_1, \dots, x_m, y_m, v_1, \dots, v_r) = \sum_{k=1}^{m} \tau_k(x_k + y_k) + \sum_{k=1}^{r} \tau_{k+m} v_k,$$

$$B_0 = B_0(x_1, y_1, \dots, x_m, y_m) = \sum_{k=1}^{m} \left(\|x_k\|^2 + \|y_k\|^2 + 2\,\mathrm{Re}\langle\, Q_k x_k, y_k \,\rangle\right),$$

we obtain the condition

$$\langle\, Bx, x \,\rangle = \|z\|^2 + 2\,\mathrm{Re}\langle\, z, z_0 \,\rangle + B_0 + \sum_{k=1}^{r} \|v_k\|^2 \geq 0.$$

This is equivalent to the inequality

$$\|z + z_0\|^2 - \|z_0\|^2 + B_0 + \sum_{k=1}^{r} \|v_k\|^2 \geq 0.$$

The minimum of the left-hand side over z is attained at $z = -z_0$. Therefore, the operator B is nonnegative if and only if

$$-\|z_0\|^2 + B_0 + \sum_{k=1}^{r} \|v_k\|^2 \geq 0. \tag{11}$$

Let

$$z_k = \frac{x_k + y_k}{2}, \quad \delta_k = \frac{x_k - y_k}{2}, \tag{12}$$

then $x_k = z_k + \delta_k$ and $y_k = z_k - \delta_k$. Clearly,

$$z_0 = 2\sum_{k=1}^{m} \tau_k z_k + \sum_{k=1}^{r} \tau_{k+m} v_k. \tag{13}$$

Furthermore,

$$\begin{aligned} \|x_k\|^2 + \|y_k\|^2 + 2\,\mathrm{Re}\langle\, Q_k x_k, y_k \,\rangle &= \|R_k x_k\|^2 + \|R_k y_k\|^2 + \|Q_k(x_k + y_k)\|^2 \\ &= 2\|R_k z_k\|^2 + 2\|R_k \delta_k\|^2 + 4\|Q_k z_k\|^2 \\ &= \langle\, (2I + 2Q_k) z_k, z_k \,\rangle + 2\|R_k \delta_k\|^2. \end{aligned} \tag{14}$$

By using (13) and (14), we rewrite (11) in the form

$$\sum_{k=1}^{m} (\langle 2(I + Q_k) z_k, z_k \,\rangle + 2\|R_k \delta_k\|^2) + \sum_{k=1}^{r} \|v_k\|^2 - \left\| 2\sum_{k=1}^{m} \tau_k z_k + \sum_{k=1}^{r} \tau_{k+m} v_k \right\|^2 \geq 0.$$

This holds true for all vectors

$$z_1, \dots, z_m, \quad \delta_1, \dots, \delta_m, \quad v_1, \dots, v_r$$

if and only if the inequality

$$\sum_{k=1}^{m} \langle\, 2(I + Q_k) z_k, z_k \,\rangle + \sum_{k=1}^{r} \|v_k\|^2 \geq \left\| 2\sum_{k=1}^{m} \tau_k z_k + \sum_{k=1}^{r} \tau_{k+m} v_k \right\|^2 \tag{15}$$

is fulfilled for all vectors $z_1, \dots, z_m$, $v_1, \dots, v_r$.

We fix $2\sum_{k=1}^{m}\tau_k z_k+\sum_{k=1}^{r}\tau_{k+m}v_k=y$ and denote

$$A_k=\begin{cases}2(I+Q_k), & 1\le k\le m;\\ I, & m+1\le k\le m+r;\end{cases}\qquad \mu_k=\begin{cases}2\tau_k, & 1\le k\le m;\\ \tau_k, & m+1\le k\le m+r.\end{cases}$$

Lemma 5 shows that, for y fixed, the minimal value of the left-hand side of (15) is equal to $\langle\,(\sum_{k=1}^{m+r}\mu_k^2A_k^{-1})^{-1}y,y\,\rangle$. Thus, B is nonnegative if and only if

$$\left(\sum_{k=1}^{m+r}\mu_k^2A_k^{-1}\right)^{-1}\ge I,\quad \text{i.e.,}\quad \sum_{k=1}^{m+r}\mu_k^2A_k^{-1}\le I. \tag{16}$$

Since $(2(I+Q_k))^{-1}=\dfrac{1}{4}(I+R_k)$, we see that

$$\sum_{k=1}^{m+r}\mu_k^2A_k^{-1}=\sum_{k=1}^{m}4\tau_k^2\cdot\frac{1}{4}(I+R_k)+\sum_{k=m+1}^{m+r}\tau_k^2I=\sum_{k=1}^{m+r}\tau_k^2I+\sum_{k=1}^{m}\tau_k^2R_k.$$

Consequently, (16) can be rewritten in the form

$$\sum_{k=1}^{m+r}\tau_k^2I+\sum_{k=1}^{m}\tau_k^2R_k\le I,\quad \text{i.e.,}\quad \sum_{k=1}^{m}\tau_k^2R_k\le\left(1-\sum_{k=1}^{m+r}\tau_k^2\right)I. \qquad\square$$

The next statement describes $\operatorname{Ker}B$; it is a consequence of the proof of Proposition 7 and Lemma 5. We recall that z_k, δ_k, $1\le k\le m$, were defined by (12).

Proposition 8. *Let* (10) *be fulfilled. The element*

$$x=(z,x_1,y_1,\dots,x_m,y_m,v_1,\dots,v_r)$$

belongs to $\operatorname{Ker}B$ *if and only if*

(1) $z=-(2\sum_{k=1}^{m}\tau_kz_k+\sum_{k=1}^{r}\tau_{k+m}v_k)$;
(2) $\delta_k\in\operatorname{Im}Q_k=\operatorname{Ker}R_k$ *for all* $1\le k\le m$;
(3) $z_k=\frac{1}{2}\tau_k(I+R_k)y$, $1\le k\le m$, *and* $v_k=\tau_{k+m}y$, $1\le k\le r$, *where* $y\in\operatorname{Ker}(\xi(\tau)I-\sum_{k=1}^{m}\tau_k^2R_k)$.

Corollary 2. *Suppose* (10) *is fulfilled and* H_0 *is finite-dimensional. Then*

$$\dim\operatorname{Ker}B=\sum_{k=1}^{m}\dim\operatorname{Im}Q_k+\dim\operatorname{Ker}\left(\xi(\tau)I-\sum_{k=1}^{m}\tau_k^2R_k\right). \tag{17}$$

In the case under consideration, the criterion of unitary equivalence for subspace systems (Proposition 2) can be stated in terms of the orthogonal projections $Q_1,\dots,Q_m$.

Proposition 9. *Two subspace systems*

$$\mathcal{G}(H_0,\dots,H_0;B(Q_1,\dots,Q_m))\quad\text{and}\quad\mathcal{G}(H_0',\dots,H_0';B(Q_1',\dots,Q_m'))$$

are unitarily equivalent if and only if the collections of orthogonal projections Q_k, $1\le k\le m$, *and* Q_k', $1\le k\le m$, *are unitarily equivalent.*

Thus, by using the G-construction for subspace systems, we have established one-to-one correspondence between the nonzero subspace systems S satisfying (Ang), (Com), and (Ort) and such that $H_1+\dots+H_{N+1}$ is dense in H, and the collections of m orthogonal projections $R_1,\dots,R_m$ in a Hilbert space H_0 that satisfy (10) (both subspace systems and collections of orthogonal projections are considered up to unitary equivalence).

It should be noted that the condition $\xi(\tau)\ge 0$ is necessary for (10) to be fulfilled. Thus, if $\xi(\tau)<0$, there is no nonzero subspace systems S satisfying (Ang), (Com), and (Ort). In the sequel, we assume that $\xi(\tau)\ge 0$.

4.2. Description of all unitarily nonequivalent irreducible systems S satisfying (Ang), (Com), and (Ort). Before we proceed to the matter, observe that

- up to unitary equivalence, there is a unique zero irreducible system of subspaces, namely
$$S = (\mathbb{C}^1; 0, \dots, 0);$$
- for every nonzero irreducible system of subspaces $S = (H; H_1, \dots, H_n)$, the sum $H_1 + \dots + H_n$ is dense in H.

Furthermore, the irreducibility criterion for subspace systems (Proposition 5) can be formulated in terms of the orthogonal projections $Q_1, \dots, Q_m$ in the case in question.

Proposition 10. *The subspace system*
$$\mathcal{G}(H_0, \dots, H_0; B(Q_1, \dots, Q_m))$$
is irreducible if and only if the collection of orthogonal projections Q_k, $1 \le k \le m$, is irreducible.

Thus, up to unitary equivalence, all nonzero irreducible systems of subspaces satisfying (Ang), (Com), and (Ort), are of the form $S = \mathcal{G}(H_0, \dots, H_0; B(Q_1, \dots, Q_m))$, where H_0 is a Hilbert space, and $Q_1, \dots, Q_m$ is an irreducible system of orthogonal projections in H_0 such that the orthogonal projections $R_k = I - Q_k$, $1 \le k \le m$, satisfy (10).

Since two systems S and S' are unitarily equivalent if and only if the collections $Q_1, \dots, Q_m$ and $Q'_1, \dots, Q'_m$ of orthogonal projections are unitarily equivalent, we see that the problem under study reduces to that of describing unitarily nonequivalent irreducible collections of orthogonal projections $R_1, \dots, R_m$ satisfying (10).

If $\xi(\tau) = 0$, then $R_1 = \dots = R_m = 0$, i.e., $Q_1 = \dots = Q_m = I$. Since the collection $R_1, \dots, R_m$ is irreducible, we see that $H_0 = \mathbb{C}^1$. Using (17), we obtain $\dim \operatorname{Ker} B = m+1$. The definition of the G-construction directly implies that $\dim H = (N+1) - (m+1) = m + r$ and $\dim H_k = \dim H_0 = 1$ for all $1 \le k \le N+1$. Therefore, the generalized dimension of the system S is equal to $(m + r; 1)$.

Consider the case where $\xi(\tau) > 0$.

We split the index set $M = \{1, 2, \dots, m\}$ into three parts:
$$M_l = \{k \in M \mid \tau_k^2 < \xi(\tau)\},$$
$$M_e = \{k \in M \mid \tau_k^2 = \xi(\tau)\},$$
$$M_g = \{k \in M \mid \tau_k^2 > \xi(\tau)\}.$$

There is no loss of generality in assuming that $k_1 < k_2 < k_3$ for every $k_1 \in M_l$, $k_2 \in M_e$, $k_3 \in M_g$.

If $i \in M_g$, then, clearly, $R_i = 0$.

If $i \in M_e$, then $R_i R_j = 0$ for all $j \ne i$. Since the collection $R_1, \dots, R_m$ is irreducible, we see that either $R_i = 0$ or $R_i = I$. If $R_i = I$, then $R_j = 0$ for all $j \ne i$.

Suppose that $R_i = I$ and $R_j = 0$ for some $i \in M_e$ and all $j \ne i$. Since the collection $R_1, \dots, R_m$ is irreducible, we have $H_0 = \mathbb{C}^1$. By (17) we obtain $\dim \operatorname{Ker} B = m$. Therefore, $\dim H = (N+1) - m = m + r + 1$ and $\dim H_k = 1$ for all $1 \le k \le N+1$. Thus, the generalized dimension of S is equal to $(m + r + 1; 1)$.

So, we have obtained $|M_e|$ unitarily nonequivalent irreducible systems S satisfying (Ang), (Com), and (Ort), which correspond to the elements $i \in M_e$. It remains to consider the case where $R_i = 0$ for all $i \in M_e$. Then (10) can be rewritten in the form
$$\sum_{k \in M_l} \tau_k^2 R_k \le \xi(\tau) I. \tag{18}$$

Consider the following variants for $|M_l|$.

1. Let $|M_l| \geq 3$. Then $1,\ 2,\ 3 \in M_l$. Denote $\tau_{\max} = \max\{\tau_1, \tau_2, \tau_3\}$, and let $\varepsilon = \xi(\tau)/\tau_{\max}^2 - 1 > 0$. Let $R_k = 0$ for $k \in M_l$, $k \geq 4$. For every three projections R_1, R_2, R_3 with

$$R_1 + R_2 + R_3 \leq (1+\varepsilon)I,$$

we obtain

$$\tau_1^2 R_1 + \tau_2^2 R_2 + \tau_3^2 R_3 \leq \tau_{\max}^2 (R_1 + R_2 + R_3) \leq \xi(\tau) I.$$

Thus, we have proved that if $|M_l| \geq 3$, then the problem of describing unitarily nonequivalent irreducible collections of orthogonal projections $R_1, \dots, R_m$ satisfying (18) involves a "hopeless" subproblem (as was shown in Proposition 6); consequently, the problem itself is "hopeless".

2. Let $M_l = \varnothing$. Then $R_1 = \dots = R_m = 0$, i.e., $Q_1 = \dots = Q_m = I$. Since the collection $R_1, \dots, R_m$ is irreducible, we have $H_0 = \mathbb{C}^1$. From (17) we deduce that $\dim \operatorname{Ker} B = m$. Then $\dim H = (N+1) - m = m + r + 1$; $\dim H_k = 1$ for all $1 \leq k \leq N+1$. Thus, the generalized dimension of S is equal to $(m + r + 1; 1)$.

3. Let $|M_l| = 1$. Then (18) takes the form $\tau_1^2 R_1 \leq \xi(\tau) I$. Since $\tau_1^2 < \xi(\tau)$, this inequality is fulfilled for all orthogonal projections R_1. Since the collection $R_1, \dots, R_m$ is irreducible and $R_2 = \dots = R_m = 0$, we see that $H_0 = \mathbb{C}^1$ and either $R_1 = 0$, or $R_1 = I$.

If $R_1 = 0$, then $Q_1 = \dots = Q_m = I$. By (17) we obtain $\dim \operatorname{Ker} B = m$. Then $\dim H = (N+1) - m = m + r + 1$ and $\dim H_k = 1$ for all $1 \leq k \leq N+1$. Thus, the generalized dimension of S is $(m + r + 1; 1)$.

If $R_1 = I$ and $R_2 = \dots = R_m = 0$, we have $Q_1 = 0$, $Q_2 = \dots = Q_m = I$. By (17) we have $\dim \operatorname{Ker} B = m - 1$. Therefore, $\dim H = (N+1) - (m-1) = m + r + 2$ and $\dim H_k = 1$ for all $1 \leq k \leq N+1$. The generalized dimension of the system S is $(m + r + 2; 1)$.

4. Let $|M_l| = 2$. In this case, (18) takes the form

$$\tau_1^2 R_1 + \tau_2^2 R_2 \leq \xi(\tau) I. \tag{19}$$

We refer to a well-known description of all irreducible pairs of orthogonal projections R_1, R_2 (not necessarily obeying (19)) in a Hilbert space H_0 up to unitary equivalence (see, e.g., [3]). They split into two portions:

(1) four irreducible couples of orthogonal projections in $H_0 = \mathbb{C}^1$:

$$\pi_{00} : R_1 = 0, R_2 = 0; \quad \pi_{01} : R_1 = 0, R_2 = I;$$
$$\pi_{10} : R_1 = I, R_2 = 0; \quad \pi_{11} : R_1 = I, R_2 = I;$$

(2) the family of pairs π_φ, $\varphi \in (0, \pi/2)$, in $H_0 = \mathbb{C}^2$:

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos^2\varphi & \cos\varphi \sin\varphi \\ \cos\varphi \sin\varphi & \sin^2\varphi \end{pmatrix}. \tag{20}$$

Out of these pairs, we must choose those satisfying (19). Consider the following variants.

4.1. If $H_0 = \mathbb{C}^1$, $R_1 = R_2 = 0$, then (19) is true and the generalized dimension of the system S is $(m + r + 1; 1)$ because $\dim \operatorname{Ker} B = m$.

4.2. If $H_0 = \mathbb{C}^1$ and $R_1 = I$, $R_2 = 0$ or $R_1 = 0$, $R_2 = I$, then (19) is true and the generalized dimension of S is $(m + r + 2; 1)$ because $\dim \operatorname{Ker} B = m - 1$.

4.3. Let $H_0 = \mathbb{C}^1$, and let $R_1 = R_2 = I$. Then (19) is fulfilled if and only if $\xi(\tau) \geq \tau_1^2 + \tau_2^2$. In this case, the generalized dimension of S is equal to

(1) $(m + r + 2; 1)$ if $\xi(\tau) = \tau_1^2 + \tau_2^2$, because $\dim \operatorname{Ker} B = m - 1$;
(2) $(m + r + 3; 1)$ if $\xi(\tau) > \tau_1^2 + \tau_2^2$, because $\dim \operatorname{Ker} B = m - 2$.

4.4. Consider the situation where $H_0 = \mathbb{C}^2$ and the orthogonal projections R_1 and R_2 are defined by (20), $\varphi \in (0, \pi/2)$. Then (19) is fulfilled if and only if the (2×2)-matrix (operator)

$$M = \xi(\tau)I - \tau_1^2 R_1 - \tau_2^2 R_2 = \begin{pmatrix} \xi(\tau) - \tau_1^2 - \tau_2^2 \cos^2\varphi & -\tau_2^2 \cos\varphi \sin\varphi \\ -\tau_2^2 \cos\varphi \sin\varphi & \xi(\tau) - \tau_2^2 \sin^2\varphi \end{pmatrix}$$

is nonnegative definite. This is equivalent to the requirement that the diagonal entries and the determinant of M be nonnegative. Now, $(M)_{1,1} \geq 0$ if and only if

$$\cos^2\varphi \leq \frac{\xi(\tau) - \tau_1^2}{\tau_2^2}.$$

Next, $(M)_{2,2} > 0$ for every φ. It is easy to obtain the following expression for the determinant:

$$\det M = (\xi(\tau) - \tau_1^2)(\xi(\tau) - \tau_2^2) - \tau_1^2 \tau_2^2 \cos^2\varphi.$$

Thus, the condition $\det M \geq 0$ can be rewritten in the form

$$\cos^2\varphi \leq \frac{\xi(\tau) - \tau_1^2}{\tau_2^2} \cdot \frac{\xi(\tau) - \tau_2^2}{\tau_1^2} = \eta(\tau).$$

Consider the following variants.

4.4.1. Suppose that $\xi(\tau) \geq \tau_1^2 + \tau_2^2$. Then the matrix M is positive definite for every $\varphi \in (0, \pi/2)$. Using (17), we obtain $\dim H = 2(N+1) - (2m-2) = 2m + 2r + 4$, $\dim H_k = \dim H_0 = 2$ for all $1 \leq k \leq N+1$. The generalized dimension of the system S is $(2m + 2r + 4; 2)$.

4.4.2. Suppose that $\xi(\tau) < \tau_1^2 + \tau_2^2$. We define the angle

$$\varphi(\tau) = \arccos\sqrt{\eta(\tau)} \in (0, \pi/2).$$

The matrix M is nonnegative definite if and only if $\varphi \in [\varphi(\tau), \pi/2)$.

If $\varphi \in (\varphi(\tau), \pi/2)$, then M is positive definite and the generalized dimension of S is equal to $(2m + 2r + 4; 2)$.

If $\varphi = \varphi(\tau)$, then $\operatorname{Ker} M$ is one-dimensional, whence $\dim \operatorname{Ker} B = 2m - 1$ and the generalized dimension of S is $(2m + 2r + 3; 2)$.

4.3. Classification theorem. We state the results obtained in Subsection 4.2 in the form of a theorem.

Theorem 1. *If $\xi(\tau) < 0$, there is no nonzero systems of subspaces satisfying* (Ang), (Com), *and* (Ort).

If $\xi(\tau) = 0$, there exists a unique irreducible system (up to unitary equivalence) of subspaces that satisfies (Ang), (Com), *and* (Ort). *Its generalized dimension is $(m+r; 1)$.*

If $\xi(\tau) > 0$, then, up to unitary equivalence, all nonzero irreducible systems of subspaces satisfying (Ang), (Com), *and* (Ort), *are described in the following way:*

(1) $M_l = \varnothing$:
  (a) $|M_e| + 1$ *systems of generalized dimension* $(m + r + 1; 1)$.
(2) $|M_l| = 1$:
  (a) $|M_e| + 1$ *systems of generalized dimension* $(m + r + 1; 1)$;
  (b) *one system of generalized dimension* $(m + r + 2; 1)$.
(3) $|M_l| = 2$, $\sum_{k \in M_l} \tau_k^2 > \xi(\tau)$:
  (a) $|M_e| + 1$ *systems of generalized dimension* $(m + r + 1; 1)$;
  (b) *two systems of generalized dimension* $(m + r + 2; 1)$;
  (c) *an infinite family of systems of generalized dimension* $(2m + 2r + 4; 2)$ *parametrized by an angle* $\varphi \in (\varphi(\tau), \pi/2)$;

(d) *one system of generalized dimension* $(2m+2r+3;2)$ *corresponding to the angle* $\varphi=\varphi(\tau)$.

(4) $|M_l|=2$ *and* $\sum_{k\in M_l}\tau_k^2=\xi(\tau)$:
   (a) $|M_e|+1$ *systems of generalized dimension* $(m+r+1;1)$;
   (b) *three systems of generalized dimension* $(m+r+2;1)$;
   (c) *an infinite family of systems of generalized dimension* $(2m+2r+4;2)$ *parametrized by an angle* $\varphi\in(0,\pi/2)$.

(5) $|M_l|=2$ *and* $\sum_{k\in M_l}\tau_k^2<\xi(\tau)$:
   (a) $|M_e|+1$ *systems of generalized dimension* $(m+r+1;1)$;
   (b) *two systems of generalized dimension* $(m+r+2;1)$;
   (c) *one system of generalized dimension* $(m+r+3;1)$;
   (d) *an infinite family of systems of generalized dimension* $(2m+2r+4;2)$ *parametrized by an angle* $\varphi\in(0,\pi/2)$.

If $|M_l|\geq 3$, *then the description problem for all unitarily nonequivalent irreducible systems* S *satisfying* (Ang), (Com), *and* (Ort), *is* $*$*-wild.*

Corollary 3. *If* $m\geq 3$ *and* $\tau_k<1/\sqrt{m+r+1}$ *for* $k=1,2,\dots,m+r$, *then the problem of description of all unitarily nonequivalent irreducible systems of subspaces* S *satisfying* (Ang), (Com), *and* (Ort) *is* $*$*-wild.*

4.4. Example. As an example, we present a description (up to unitary equivalence) of all nonzero irreducible subspace systems $S=(H;H_1,\dots,H_8)$ satisfying (Ang), (Com), and (Ort) in the case where $m=3$, $r=1$, and the function $\tau=\tau(\tau_0)$, $\tau_0\in(0,1/3)$, is defined as follows: $\tau_1=\tau_0$, $\tau_2=\sqrt{2}\tau_0$, $\tau_3=2\tau_0$, $\tau_4=3\tau_0$. In this case, $\xi(\tau)=1-16\tau_0^2$.

If $\tau_0\in(0,1/\sqrt{20})$, then the description problem in question is $(*)$-wild.

For $\tau_0\in[1/\sqrt{20},1/4]$, all subspace systems under study are described in the following way.

(1) $\tau_0=1/\sqrt{20}$:
   (a) two systems of generalized dimension $(5;1)$;
   (b) two systems of generalized dimension $(6;1)$;
   (c) one system of generalized dimension $(7;1)$;
   (d) an infinite family of systems of generalized dimension $(12;2)$ parametrized by an angle $\varphi\in(0,\pi/2)$.

(2) $\tau_0\in(1/\sqrt{20},1/\sqrt{19})$:
   (a) one system of generalized dimension $(5;1)$;
   (b) two systems of generalized dimension $(6;1)$;
   (c) one system of generalized dimension $(7;1)$;
   (d) an infinite family of systems of generalized dimension $(12;2)$ parametrized by an angle $\varphi\in(0,\pi/2)$.

(3) $\tau_0=1/\sqrt{19}$:
   (a) one system of generalized dimension $(5;1)$;
   (b) three systems of generalized dimension $(6;1)$;
   (c) an infinite family of systems of generalized dimension $(12;2)$ parametrized by an angle $\varphi\in(0,\pi/2)$.

(4) $\tau_0\in(1/\sqrt{19},1/\sqrt{18})$:
   (a) one system of generalized dimension $(5;1)$;
   (b) two systems of generalized dimension $(6;1)$;

(c) an infinite family of systems of generalized dimension $(12;2)$ parametrized by an angle $\varphi \in (\varphi(\tau), \pi/2)$, where

$$\varphi(\tau) = \arccos\left(\frac{(1-17\tau_0^2)(1-18\tau_0^2)}{2\tau_0^4}\right)^{1/2} \in (0, \frac{\pi}{2}),$$

(d) one system of generalized dimension $(11;2)$. This system corresponds to the angle $\varphi = \varphi(\tau)$.

(5) $\tau_0 = 1/\sqrt{18}$:
  (a) two systems of generalized dimension $(5;1)$;
  (b) one system of generalized dimension $(6;1)$.

(6) $\tau_0 \in (1/\sqrt{18}, 1/\sqrt{17})$:
  (a) one system of generalized dimension $(5;1)$;
  (b) one system of generalized dimension $(6;1)$.

(7) $\tau_0 = 1/\sqrt{17}$:
  (a) two systems of generalized dimension $(5;1)$;

(8) $\tau_0 \in (1/\sqrt{17}, 1/4)$:
  (a) one system of generalized dimension $(5;1)$;

(9) $\tau_0 = 1/4$:
  (a) one system of generalized dimension $(4;1)$.

If $\tau_0 \in (1/4, 1/3)$, there are no systems in question.

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