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ABSOLUTELY CONTINUOUS SPECTRUM OF A ONE-PARAMETER FAMILY OF SCHRÖDINGER OPERATORS

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Abstract. Under certain conditions on the potential V, it is shown that the absolutely continuous spectrum of the Schrödinger operator $-\Delta + \alpha V$ is essentially supported on $[0, +\infty)$ for almost every $\alpha \in \mathbb{R}$.

§1. Main results

We study the absolutely continuous spectrum of a Schrödinger operator

$$H = -\Delta + \alpha V, \quad \alpha \in \mathbb{R}, \tag{1}$$

acting in the space $L^2(\mathbb{R}^d)$. Our main result is as follows.

Theorem 1.1. *Let V be a real-valued bounded potential such that*

$$\int_{\mathbb{R}^d} \frac{V^2}{|x|^{d-1}}\, dx < \infty. \tag{2}$$

Let W be the Fourier transform of the square integrable function $V|x|^{-(d-1)/2}$. Assume that

$$\int_{|\xi|<\delta} \frac{|W(\xi)|^2}{|\xi|^2}\, d\xi < \infty, \tag{3}$$

for some small $\delta > 0$. Then the absolutely continuous spectrum of the operator (1) is essentially supported by the interval $[0, \infty)$ for almost every $\alpha \in \mathbb{R}$. That is, the spectral projection $E(\Omega)$ corresponding to any set $\Omega \subset [0, \infty)$ is different from zero, $E(\Omega) \neq 0$, whenever the Lebesgue measure of Ω is positive.

Condition (2) was first mentioned in [21], where it was conjectured that the same statement holds true without assumption (3) (and for all α). This conjecture was proved only in the case where $d = 1$ (see [3]).

The original idea to study the a.c. spectrum of a family of operators $-\Delta + \alpha V$ belongs to S. Denisov [7]. However, the class of potentials considered in [7] does not coincide with that described by conditions (2) and (3). As a matter of fact, the results of [7] neither imply the results of the present paper nor follow from them. While the investigations of the absolutely continuous spectrum are usually very complicated, we hope that our paper will be of interest for mathematicians because of the unexpected simplicity of the arguments used in the proof. Our methods make it possible to obtain the following statement.

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Theorem 1.2. *Let V be a real-valued bounded potential representable in the form*

$$V(x) = |x|^{(d-1)/2} \operatorname{div} Q(x), \quad Q : \mathbb{R}^d \mapsto \mathbb{R}^d, \tag{4}$$

where Q satisfies the condition

$$\int_{\mathbb{R}^d} |Q|^2 \, dx < \infty. \tag{5}$$

Then the absolutely continuous spectrum of $-\Delta + \alpha V$ is essentially supported by the interval $[0, \infty)$ for almost every $\alpha \in \mathbb{R}$.

The reader has probably noticed that, after we already imposed condition (2), the assumption that $\delta > 0$ in (3) is small or finite is not needed because $W \in L^2$. However, we prefer to keep δ in (3) by some psychological reason: this way we see that the problematic region is the set of small ξ. Observe that (2) and (3) imply that $V(x) = |x|^{(d-1)/2} \operatorname{div} Q(x)$, where Q satisfies (5). Indeed, set

$$Q(x) = \nabla \Delta^{-1} \big(V |x|^{(1-d)/2} \big).$$

Then, obviously, $Q \in L^2$ and $\operatorname{div} Q = V|x|^{(1-d)/2}$. Therefore, Theorem 1.2 is more general compared to Theorem 1.1.

As an application of Theorem 1.2, we formulate the following statement.

Theorem 1.3. *Let*

$$V = \sum_{n \in \mathbb{Z}^d} v_n \omega_n \chi(x - n), \quad x \in \mathbb{R}^d, \ d \geq 3,$$

where χ is the characteristic function of the unit cube $[0,1)^d$, and the ω_n are bounded identically distributed independent random variables with zero expectations $\mathbb{E}[\omega_n] = 0$. Suppose that the real coefficients v_n satisfy the condition

$$\sum_{n \neq 0} \frac{v_n^2}{|n|^{d-1}} < \infty.$$

Then the absolutely continuous spectrum of $-\Delta + \alpha V$ is essentially supported by $[0, \infty)$ almost surely for almost every $\alpha \in \mathbb{R}$.

In order to prove this result, it suffices to observe that V satisfies the conditions of Theorem 1.2 by one of the results of [20]. Theorem 1.3 is proved for the first time. Close but slightly different theorems can be also found in [1, 2], and [5].

§2. Auxiliary material

Notation. Throughout the text, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and the imaginary part of a complex number z. The notation $\mathbb{S}$ stands for the unit sphere in $\mathbb{R}^d$. Its area is denoted by $|\mathbb{S}|$. For a selfadjoint operator $B = B^*$ and a vector g in a Hilbert space, the expression $((B - k - i0)^{-1} g, g)$ is always understood as the limit

$$\big((B - k - i0)^{-1} g, g\big) = \lim_{\varepsilon \to 0} \big((B - k - i\varepsilon)^{-1} g, g\big), \quad \varepsilon > 0, \ k \in \mathbb{R}.$$

The following simple and very well-known statement plays quite an important role in our proof.

Lemma 2.1. *Let B be a selfadjoint operator in a separable Hilbert space $\mathfrak{H}$, and let $g \in \mathfrak{H}$. Then the function*

$$\eta(k) := \operatorname{Im}\big((B - k - i0)^{-1} g, g\big) \geq 0$$

is integrable over $\mathbb{R}$. *Moreover,*

$$\int_{-\infty}^{\infty} \eta(k)\, dk \le \pi \|g\|^2.$$

At the beginning of the proof of Theorem 1.2 we shall assume that V is compactly supported. We shall obtain certain estimates on the derivative of the spectral measure of the operator $-\Delta + \alpha V$ for compactly supported potentials and then these estimates will be extended to the case of an arbitrary potential V satisfying the conditions of Theorem 1.2. We approximate V by compactly supported functions. It is important not to destroy the inequalities obtained previously for "nice" V. Therefore, the way we select approximations plays a crucial role in our proof.

Now we describe our choice of compactly supported functions V_n approximating the given potential V. Approximations of V by V_n will correspond to approximations of Q by some vector potentials Q_n.

Lemma 2.2. *Suppose V satisfies the conditions of Theorem* 1.2. *Then, for any $\varepsilon > 0$, there exists a number $R > 1$ and a sequence of smooth compactly supported vector potentials $Q_n : \mathbb{R}^d \to \mathbb{R}^d$ satisfying*

$$\int_{\mathbb{R}^d} |Q_n|^2\, dx \le \varepsilon, \tag{6}$$

and such that the sequence of the functions

$$V_n = |x|^{(d-1)/2} \operatorname{div} Q_n \tag{7}$$

converges in L^2_{loc} to a function $\widetilde{V} \in L^\infty(\mathbb{R}^d)$ such that $\widetilde{V}(x) = V(x)$, for $|x| > R$. Moreover, the Q_n can be chosen so that

$$\operatorname{supp} V_n \subset \{x \in \mathbb{R}^d \,:\, |x| > 1\}.$$

Proof. Let Q be the vector potential as in Theorem 1.2. Clearly, our ability to select Q_n satisfying (6) has something to do with the fact that

$$\lim_{L\to\infty} \int_{|x|>L} |Q|^2\, dx = 0.$$

We choose a function $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\zeta(x) = \begin{cases} 1 & \text{if } |x| < 1; \\ 0 & \text{if } |x| > 2. \end{cases} \tag{8}$$

The temporary choice of the sequence Q_n (denoted by $\widetilde{Q}_n$) will be related to the "tails" of Q:

$$\widetilde{Q}_n(x) = \big(1 - \zeta(x/L)\big)\zeta(x/n)Q(x),$$

where ζ is the same as in (8) and L is sufficiently large. It is easily seen that one can choose L so as to ensure that

$$\int_{\mathbb{R}^d} |\widetilde{Q}_n|^2\, dx < \varepsilon.$$

However, this choice is still not quite satisfactory, because it is not clear why the limit of the sequence $\widetilde{V}_n = |x|^{(d-1)/2} \operatorname{div} \widetilde{Q}_n$ is bounded. Denote this limit by $\check{V}$. Since, clearly,

$$\check{V} - v_0 \in L^\infty(\mathbb{R}^d) \quad \text{for} \quad v_0(x) = -L^{-1}|x|^{(d-1)/2}\nabla\zeta(x/L)\cdot Q(x),$$

the only reason why $\check{V}$ might be unbounded is the unboundedness of the function v_0. Surprisingly, this problem can be resolved by a further small perturbation of $\widetilde{V}$. Indeed, we can find a vector potential $\mathcal{A}$ with

$$\|\mathcal{A}\|_{L^2} < \varepsilon \tag{9}$$

such that the function

$$v_0(x) - |x|^{(d-1)/2} \operatorname{div} \mathcal{A}(x) \tag{10}$$

is bounded and compactly supported. This is possible because any function v obeying the three conditions

$$\int_{\mathbb{R}^d} \frac{v(x)}{|x|^{(d-1)/2}}\, dx = 0, \quad \int_{\mathbb{R}^d} \frac{|x|\,|v(x)|}{|x|^{(d-1)/2}}\, dx < \varepsilon, \quad \int_{\mathbb{R}^d} \frac{|v(x)|^2}{|x|^{d-1}}\, dx < \varepsilon \tag{11}$$

is representable in the form

$$v = |x|^{(d-1)/2} \operatorname{div} \mathcal{A}(x) \ \text{ with } \ \|\mathcal{A}\|_{L^2} < C\varepsilon. \tag{12}$$

Indeed, let w be the Fourier transform of $v/|x|^{(d-1)/2}$. Then the first of the three conditions implies that $w(0) = 0$. The second condition says that $|\nabla w| < \varepsilon$, which indicates that $|w(\xi)| < \varepsilon|\xi|$. Finally, the third of the conditions (11) can be written in the form

$$\int_{\mathbb{R}^d} |w(\xi)|^2\, d\xi < \varepsilon.$$

Therefore, (11) leads to the estimate

$$\int_{\mathbb{R}^d} \frac{|w(\xi)|^2}{|\xi|^2}\, d\xi < C\varepsilon,$$

which is equivalent to (12). On the other hand, since $v_0 \in L^2$ vanishes outside of the spherical shell $\{x : L < |x| < 2L\}$, we can always find v satisfying (11) and such that $v_0 - v$ is a bounded compactly supported function. So, we have found a vector potential $\mathcal{A}$ obeying (9) such that the function (10) is bounded and compactly supported. It remains to set

$$Q_n = \widetilde{Q}_n - \zeta_n \mathcal{A}, \quad \zeta_n(x) := \zeta(x/n). \qquad \square$$

The next statement explains why we need to construct a sequence of vector potentials mentioned in Lemma 2.2.

Proposition 2.1. *Let $\widetilde{V}$ be the limit of the sequence V_n constructed in Lemma 2.2. Then the absolutely continuous parts of the operators $-\Delta + \alpha V$ and $-\Delta + \alpha\widetilde{V}$ are unitarily equivalent.*

Proof. The difference $V - \widetilde{V}$ is a bounded compactly supported function. Therefore, in accordance with Proposition 1.4 of the paper [12],

$$(-\Delta + \alpha V - \lambda)^{-n} - (-\Delta + \alpha\widetilde{V} - \lambda)^{-n} \in \mathfrak{S}_1$$

is a trace class operator for some $n \in \mathbb{N}$ and $\lambda < 0$. Now the statement of this proposition follows by the Kato–Rosenblum theorem. $\square$

Let P_{ac} and $\widetilde{P}_{\mathrm{ac}}$ be the orthogonal projections onto the absolutely continuous subspaces of $-\Delta+\alpha V$ and $-\Delta+\alpha\widetilde{V}$, respectively. Proposition 2.1 says that there exists an isometric operator S such that

$$S^* P_{\mathrm{ac}}(-\Delta + \alpha V - z)^{-1} S = \widetilde{P}_{\mathrm{ac}}(-\Delta + \alpha\widetilde{V} - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{13}$$

Here, $S^*S = \widetilde{P}_{\mathrm{ac}}$ and $SS^* = P_{\mathrm{ac}}$.

§3. Proof of Theorem 1.2

Our proof is based on the relationship between the derivative of the spectral measure and the so-called scattering amplitude. Both objects should be introduced properly. While the spectral measure can be defined for any selfadjoint operator, the scattering coefficient will be introduced only for the Schrödinger operator. Let f be a vector in the Hilbert space $\mathfrak{H}$ and H a selfadjoint operator in $\mathfrak{H}$. It turns out that the quadratic form of the resolvent of H can be written as a Cauchy integral:

$$\big((H-z)^{-1}f, f\big) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t-z}, \quad \operatorname{Im} z \neq 0.$$

The measure μ in this representation is called the spectral measure of H corresponding to the element f.

Now we introduce the scattering amplitude. First, assume that the support of the potential V is compact. Take any compactly supported function f. Then (see [23, p. 40–42])

$$(H-z)^{-1}f = \frac{e^{ik|x|}}{|x|^{(d-1)/2}}\big(A_f(k,\theta)+O(|x|^{-1})\big), \text{ as } |x|\to\infty, \ k^2=z, \ \operatorname{Im} k \geq 0, \ \theta = \frac{x}{|x|}.$$

Clearly, the relation

$$\mu'(\lambda) = \pi^{-1} \lim_{z\to\lambda+i0} \operatorname{Im}((H-z)^{-1}f, f) = \pi^{-1} \lim_{z\to\lambda+i0} \operatorname{Im} z \|(H-z)^{-1}f\|^2$$

implies that (see [23, p. 40–42] again)

$$\pi\mu'(\lambda) = \sqrt{\lambda}\int_{\mathbb{S}} |A_f(k,\theta)|^2\, d\theta, \quad k^2=\lambda>0. \tag{14}$$

Formula (14) is a very important equality that relates the absolutely continuous spectrum to the so-called extended states. The rest of the proof will be devoted to a lower estimate for $|A_f(k,\theta)|$.

First, consider the case where $d=3$. For our purposes, it suffices to assume that f is the characteristic function of the unit ball. Traditionally, H is viewed as an operator obtained by a perturbation of

$$H_0 = -\Delta.$$

In its turn, $(H-z)^{-1}$ can be viewed as an operator obtained by a perturbation of $(H_0-z)^{-1}$. Often, the theory of such perturbations is based on the second resolvent identity

$$(H-z)^{-1} = (H_0-z)^{-1} - (H-z)^{-1}\alpha V(H_0-z)^{-1}, \tag{15}$$

which turns out to be useful for our argument. As a consequence of (15), we see that

$$A_f(k,\theta) = F(k) - A_g(k,\theta), \quad z = k^2+i0, \ k>0, \tag{16}$$

where $g(x) = \alpha V(x)(H_0-z)^{-1}f$ and $F(k)$ is defined by

$$(H_0-z)^{-1}f = e^{ik|x|}\frac{F(k)}{|x|^{(d-1)/2}} \quad \text{for} \quad |x|>1 \quad (\text{recall that } \ d=3). \tag{17}$$

Without loss of generality, we may assume that $V(x)=0$ inside the unit ball. In this case,

$$g = F(k)h_k, \quad \text{where} \quad h_k(x) = \alpha V(x)e^{ik|x|}|x|^{(1-d)/2}. \tag{18}$$

In accordance with (16),

$$2\int_{\mathbb{S}} |A_f(k,\theta)|^2\, d\theta \geq |F(k)|^2|\mathbb{S}| - 2\int_{\mathbb{S}} |A_g(k,\theta)|^2\, d\theta,$$

which can be written in the form

$$2\pi\mu'(\lambda) \geq |F(k)|^2\big(|\mathbb{S}|\sqrt{\lambda} - 2\,\mathrm{Im}\big((H-z)^{-1}h_k,\, h_k\big)\big), \quad z = \lambda + i0, \tag{19}$$

by (14) and (18). Therefore, to establish the presence of the absolutely continuous spectrum, we need to show that the quantity $\mathrm{Im}\big((H-z)^{-1}h_k,\, h_k\big)$ is small. The chain of arguments that have led us to this conclusion was suggested by Boris Vainberg. The method developed by the author in the previous version of the paper was much longer.

Theorem 3.1. *Suppose $0 < \lambda_1 < \lambda_2 < \infty$ and $0 < \alpha_1 < \alpha_2 < \infty$. Assume that V is a bounded compactly supported potential such that*

$$\mathrm{supp}(V) \subset \{x \in \mathbb{R}^d \,:\, |x| > 1\}.$$

Let f be the characteristic function of the unit ball and μ the spectral measure of H constructed for f. Then

$$\mathrm{meas}\Big\{(\lambda,\alpha) \in [\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2] \,:\, \pi\mu'(\lambda) < \frac{|\mathbb{S}|\sqrt{\lambda}}{4}\big|F(\sqrt{\lambda})\big|^2\Big\} \leq C\int_{\mathbb{R}^d} \frac{|W(\xi)|^2}{|\xi|^2}\,d\xi, \tag{20}$$

where W is the Fourier transform of the function $V|x|^{-(d-1)/2}$. The constant C in this inequality depends on the edges λ_j and α_j, but is independent of V.

Proof. By (19), relation (20) follows from the inequality

$$\begin{aligned} \mathrm{meas}\Big\{(\lambda,\alpha) \in [\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2] \,:\, \mathrm{Im}\big((H-z)^{-1}h_k,\, h_k\big) > \frac{|\mathbb{S}|\sqrt{\lambda}}{4}\Big\} \\ \leq C\int_{\mathbb{R}^d} \frac{|W(\xi)|^2}{|\xi|^2}\,d\xi, \end{aligned} \tag{21}$$

In order to establish (21), we shall obtain an integral estimate of the quantity η_0 defined by

$$\alpha^2 k^{-2}\eta_0(k,\alpha) := \frac{1}{k}\,\mathrm{Im}\big((H-z)^{-1}h_k,\, h_k\big) \geq 0.$$

It is clear that $\eta_0(k,\alpha)$ is positive for all real $k \neq 0$. Here we agree that $z = k^2 \pm i0$ if $\pm k > 0$. This is quite convenient. Since $\eta_0 > 0$, we can conclude that η_0 is small on a rather large set if the integral of this function is small. That is why we try to estimate

$$J(V) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta_0(k,\alpha)}{(\alpha^2+k^2)}\,\frac{dk\,d\alpha}{|k|} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta_0(k,tk)}{k^2(t^2+1)}\,dk\,dt. \tag{22}$$

We shall show that

$$J(V) \leq \pi^2\int_{\mathbb{R}^d} \frac{|W(\xi)|^2}{|\xi|^2}\,d\xi. \tag{23}$$

By Chebyshev's inequality, this will imply (21). Now, we employ a couple of tricks, one of which has an artificial nature and will be appreciated not immediately but a bit later. Instead of dealing with the operator H, we deal with $H + \varepsilon I$, where $\varepsilon > 0$ is a small parameter. First, we obtain an integral estimate for the quantity

$$\eta_\varepsilon(k,\alpha) = \frac{k}{\alpha^2}\,\mathrm{Im}\big((H+\varepsilon-z)^{-1}h_k,\, h_k\big).$$

Then this estimate will be carried over to the case of $\varepsilon = 0$. This is possible due to the following arguments. Let X be the operator of multiplication by the characteristic function of the support of V. It is well known (see the Appendix) that

$$X(H-z)^{-1}X, \quad z = k^2,$$

admits a meromorphic extension to the cut plane $\mathbb{C}\setminus\{z=iy,\ y\le 0\}$ as a function of k. The set of real poles of this function is a set of measure zero. Consequently,

$$\eta_0(k,\alpha)=\lim_{\varepsilon\to 0}\eta_\varepsilon(k,\alpha)\qquad \text{a.e. on } \ \mathbb{R}\times\mathbb{R}.$$

Therefore, by the Fatou lemma,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_0(k,\alpha)}{(\alpha^2+k^2)}\,\frac{dk\,d\alpha}{|k|}\le\liminf_{\varepsilon\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\eta_\varepsilon(k,\alpha)}{(\alpha^2+k^2)}\,\frac{dk\,d\alpha}{|k|}.$$

The estimate of the integral on the right-hand side will be uniform in ε, which will allow us to say that a similar inequality holds true for $\varepsilon=0$.

The second trick is to set $\alpha=kt$ and represent η_ε in the form

$$\eta_\varepsilon(k,kt)=\operatorname{Im}\bigl((B+1/k-i0)^{-1}A_\varepsilon^{-1/2}v,\ A_\varepsilon^{-1/2}v\bigr),\tag{24}$$

where $v=V|x|^{(1-d)/2}$, $A_\varepsilon=-\Delta+\varepsilon I$, and B is the bounded selfadjoint operator defined by

$$B=A_\varepsilon^{-1/2}\Bigl(-2i\frac{\partial}{\partial r}-\frac{i(d-1)}{|x|}+tV\Bigr)A_\varepsilon^{-1/2}.\tag{25}$$

The symbol r in (25) denotes the radial variable $r=|x|$. It is easily seen that B is not only selfadjoint, but also bounded. Note that it is the parameter ε that makes B bounded.

In order to justify (24), we introduce the operator U of multiplication by the function $\exp(ik|x|)$. This operator is unitary only if k is real. If k belongs to the upper half-plane, then U^{-1} is an unbounded operator. However, this fact does not bring additional difficulties, because we apply the operator U^{-1} only to functions that decay sufficiently fast at infinity. Using this notation, we can represent η_ε in the form

$$\eta_\varepsilon(k,tk)=k\operatorname{Im}\bigl(U^{-1}(H+\varepsilon-k^2\mp i0)^{-1}Uv,\ v\bigr),\quad \pm k>0.$$

The idea of the proof of (24) is to employ the formula

$$U^{-1}(H+\varepsilon-z)^{-1}U=(U^{-1}HU+\varepsilon-z)^{-1},$$

and then use the fact that H is a differential operator whose commutator with the multiplication operator U can be found easily. Now we give a rigorous proof of (24). Clearly, it suffices to check the statement below.

Proposition 3.1. *Let V be a compactly supported real potential. Let k be a point in the upper half-plane, and let $z=k^2$. Then the closure of the operator $U^{-1}(H+\varepsilon-z)^{-1}U$ defined on $C_0^\infty(\mathbb{R}^d)$ is a bounded operator that satisfies*

$$kU^{-1}(H+\varepsilon-z)^{-1}U=A_\varepsilon^{-1/2}(B+1/k)^{-1}A_\varepsilon^{-1/2},\quad \alpha=kt,\tag{26}$$

where $A_\varepsilon=-\Delta+\varepsilon$ and B is defined by (25).

In its turn, this proposition is implied by the following claim.

Proposition 3.2. *Under the assumptions of Proposition* 3.1*, let $u\in C_0^\infty(\mathbb{R}^d)$ be a compactly supported function. Then*

$$\psi=k(H+\varepsilon-z)^{-1}Uu,\quad \alpha=kt,\ \operatorname{Im}z\ne 0,$$

is representable in the form

$$\psi=Uw$$

with $w\in L^2(\mathbb{R}^d)$. Moreover,

$$w=A_\varepsilon^{-1/2}(B+1/k)^{-1}A_\varepsilon^{-1/2}u,$$

where $A_\varepsilon=-\Delta+\varepsilon$ and B is the bounded operator defined by (25).

Proof. We view ψ as a solution of the differential equation

$$(H+\varepsilon-z)\psi = kUu.$$

Clearly, $\psi = O(e^{-\operatorname{Im}\sqrt{z-\varepsilon}|x|})$ as $|x|\to\infty$. Consider the function $w = e^{-ik|x|}\psi$. It is easy to check that w is a solution of the differential equation

$$-\Delta w + \varepsilon w + ktVw - 2ik\frac{\partial w}{\partial r} - \frac{ik(d-1)w}{|x|} = k\,u.$$

Moreover, w decays at infinity as $O\big(e^{-(\operatorname{Im}\sqrt{k^2-\varepsilon}-\operatorname{Im}k)|x|}\big)$. Consequently, $w\in D(A_\varepsilon)$ and, in accordance with our notation,

$$A_\varepsilon w + kA_\varepsilon^{1/2}BA_\varepsilon^{1/2}w = k\,u.$$

Since $\big(A_\varepsilon + kA_\varepsilon^{1/2}BA_\varepsilon^{1/2}\big)^{-1} = k^{-1}A_\varepsilon^{-1/2}(B+1/k)^{-1}A_\varepsilon^{-1/2}$, the claim follows. □

Look at formula (24). Since B is a selfadjoint operator, $\pi^{-1}\eta_\varepsilon(k,kt)$ coincides with the derivative of the spectral measure of the operator B corresponding to the element $A_\varepsilon^{-1/2}v$. By Lemma 2.1,

$$\int_{-\infty}^{\infty}\eta_\varepsilon(k,kt)\,k^{-2}\,dk \le \pi\|A_\varepsilon^{-1/2}v\|^2.$$

This implies that $J(V)$ (see (22)) satisfies

$$J(V)\le \pi^2\lim_{\varepsilon\to 0}\|A_\varepsilon^{-1/2}v\|^2 = \pi^2\int\frac{|W(\xi)|^2}{|\xi|^2}\,d\xi.$$

Omitting the middle part, we see that this inequality coincides with (23). The proof of Theorem 3.1 is complete. □

§4. Semicontinuity of the entropy. End of the proof of Theorem 1.2

Now, we complete the proof of Theorem 1.2 and mention what ingredients were missing. We need to transfer the estimates obtained in the previous section to the case of V with an infinite support.

Theorem 4.1. *Let V and $E(\cdot)$ be the same as in Theorem 1.2. Suppose $0<\lambda_1<\lambda_2<\infty$ and $0<\alpha_1<\alpha_2<\infty$. Then for any $\tau>0$ and $\alpha\in[\alpha_1,\alpha_2]$, we can choose $f_{\tau,\alpha}\in L^2(\mathbb{R}^d)$ so that*

$$\operatorname{meas}\Big\{(\lambda,\alpha)\in[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2]\,:\,\frac{d}{d\lambda}\big(E(-\infty,\lambda)f_{\tau,\alpha},f_{\tau,\alpha}\big)=0\Big\}<\tau. \tag{27}$$

Proof. Let V_n be the sequence of functions constructed in Lemma 2.2 with $\varepsilon=\tau/C$, where C is the same as in (20). Define measures μ_n by setting

$$\big((-\Delta+\alpha V_n - z)^{-1}f,f\big) = \int_{-\infty}^{\infty}\frac{d\mu_n(t)}{t-z},\quad \operatorname{Im}z\ne 0,$$

where f is the characteristic function of the unit ball $\{x:|x|<1\}$. Each of the elements of the sequence V_n satisfies the conditions of Theorem 3.1 with the quantity

$$\int_{\mathbb{R}^d}\frac{|W_n(\xi)|^2}{|\xi|^2}\,d\xi\le\varepsilon$$

in place of the integral on the right-hand side of (20). (Here W_n is the Fourier transform of $V_n|x|^{-(d-1)/2}$.) Therefore,

$$\pi\mu_n'(\lambda) > \big|F(\sqrt{\lambda})\big|^2\frac{|\mathbb{S}|\sqrt{\lambda}}{4} \tag{28}$$

on a set of pairs (λ, α) of very large Lebesgue measure. By Theorem 3.1, the measure of this set is not smaller than $(\lambda_2 - \lambda_1)(\alpha_2 - \alpha_1) - C\varepsilon$. Denote by χ_n the characteristic function of the intersection of this set with the rectangle $[\lambda_1, \lambda_2] \times [\alpha_1, \alpha_2]$. Thus, inequality (28) is valid on the support of χ_n. By the way, since the function F is analytic, it can have only a discrete set of zeros. Therefore, the right-hand side of (28) is positive for almost every $\lambda > 0$.

We study the behavior of χ_n as $n \to \infty$. The difficulty of the situation is that χ_n may change with the growth of n. However, since the unit ball in any Hilbert space is compact in the weak topology, there is no loss of generality in assuming that the χ_n converge weakly in L^2 to a square integrable function χ. In a sense, we can say that χ_n does not change much if n is sufficiently large. Now the situation is less hopeless, because the limit χ preserves the properties of the sequence χ_n. The necessary information about the limit χ can easily be obtained from the information about χ_n. It is clear that $0 \le \chi \le 1$ and $\chi > 0$ on a set of very large measure $(\lambda_2 - \lambda_1)(\alpha_2 - \alpha_1) - C\varepsilon$. Indeed, let $\widetilde{\chi}$ be the characteristic function of the set where $\chi > 1 + \varepsilon_0$. Since $\iint \chi_n \widetilde{\chi}\, d\lambda\, d\alpha \le \iint \widetilde{\chi}\, d\lambda\, d\alpha$, we obtain

$$(1 + \varepsilon_0) \iint \widetilde{\chi}\, d\lambda\, d\alpha \le \iint \widetilde{\chi}\, d\lambda\, d\alpha,$$

which is possible only if $\widetilde{\chi} = 0$ almost everywhere. Consequently, $\chi \le 1$, so that we can judge about the size of the set where $\chi > 0$ by the value of the integral $\iint \chi\, d\lambda\, d\alpha = \lim_{n\to\infty} \iint \chi_n\, d\lambda\, d\alpha$.

It is also known (cf. [19]) that if V_n converge to $\widetilde{V} \in L^\infty$ in L^2_{loc}, then

$$\mu_n \to \mu \quad \text{as} \quad n \to \infty \tag{29}$$

weakly for any fixed α. Here μ is the spectral measure of $-\Delta + \alpha\widetilde{V}$ constructed for the same element f as before. We see that each of the sequences μ_n and χ_n has a limit, however they converge in a weak sense, which brings additional difficulties. Therefore we need to find a quantity that not only depends on a pair of measures (semi)continuously with respect to the weak topology, but also is infinite whenever the derivative μ' of one of the measures vanishes on a large set. Such a quantity is the entropy, defined by

$$S = \int_{\alpha_1}^{\alpha_2} \int_{\lambda_1}^{\lambda_2} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda, \alpha)}\Big) \chi(\lambda, \alpha)\, d\lambda\, d\alpha.$$

Its properties were studied thoroughly in [11]. It can diverge only to negative infinity, but if it is finite, then $\mu' > 0$ almost everywhere on the set $\{(\lambda, \alpha) : \chi > 0\}$. We can formulate a more general definition.

Definition. Let ρ, ν be finite Borel measures on a compact Hausdorff space X. We define the entropy of ρ relative to ν by

$$S(\rho|\nu) = \begin{cases} -\infty, & \text{if } \rho \text{ is not } \nu\text{-ac}, \\ -\int_X \log\left(\frac{d\rho}{d\nu}\right) d\rho, & \text{if } \rho \text{ is } \nu\text{-ac}. \end{cases} \tag{30}$$

Theorem 4.2 (cf. [11])**.** *The entropy $S(\rho|\nu)$ is jointly upper semicontinuous in ρ and ν with respect to the weak topology. That is, if $\rho_n \to \rho$ and $\nu_n \to \nu$ as $n \to \infty$, then*

$$S(\rho|\nu) \ge \limsup_{n\to\infty} S(\rho_n|\nu_n).$$

Relation (29) means that

$$\int \phi(\lambda, \alpha)\, d\mu_n \to \int \phi(\lambda, \alpha)\, d\mu \quad \text{as} \quad n \to \infty$$

for any fixed α and any continuous compactly supported function ϕ. By the Lebesgue dominated convergence theorem, we obtain

$$\iint \phi(\lambda,\alpha)\, d\mu_n\, d\alpha \to \iint \phi(\lambda,\alpha)\, d\mu\, d\alpha \quad \text{as} \quad n\to\infty,$$

which means that the sequence of measures

$$\widetilde{\nu}_n\,(\Omega) := \iint_{(\lambda,\alpha)\in\Omega} d\mu_n\, d\alpha$$

converges weakly to some limit $\widetilde{\nu}$. Now we choose a continuous compactly supported function $\theta(\lambda,\alpha)$ with the properties

$$\theta(\lambda,\alpha) = \begin{cases} 1 & \text{if} \ \ (\lambda,\alpha)\in[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2], \\ 0 & \text{if} \ \ (\lambda,\alpha)\notin[\lambda_1-\delta,\lambda_2+\delta]\times[\alpha_1-\delta,\alpha_2+\delta], \end{cases}$$

where $0<\delta<\min\{\lambda_1,\alpha_1\}$, and then define two measures ν_n and ν by setting

$$d\,\nu_n = \theta\, d\widetilde{\nu}_n, \quad d\,\nu = \theta d\,\widetilde{\nu}.$$

Now, we treat the measures ν_n and ν as measures on a **compact** set $[\lambda_1-\delta,\lambda_2+\delta]\times[\alpha_1-\delta,\alpha_2+\delta]$. Observe that a function ϕ continuous on this rectangle may fail to vanish on its boundary, still

$$\int \phi d\nu_n \to \int \phi\, d\nu \quad \text{as} \quad n\to\infty, \quad \text{for all} \quad \phi\in C([\lambda_1-\delta,\lambda_2+\delta]\times[\alpha_1-\delta,\alpha_2+\delta]),$$

because $\theta\phi$ is continuous on $\mathbb{R}\times\mathbb{R}$. Therefore, Theorem 4.2 with

$$\rho_n(\Omega) = \iint_\Omega \chi_n(\lambda,\alpha)\, d\lambda\, d\alpha, \quad \text{and} \quad \rho(\Omega) = \iint_\Omega \chi(\lambda,\alpha)\, d\lambda\, d\alpha,$$

implies that

$$\int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \log\Big(\frac{\mu'(\lambda)}{\chi(\lambda,\alpha)}\Big)\chi(\lambda,\alpha)\, d\lambda\, d\alpha \\ \geq \limsup_{n\to\infty}\int_{\alpha_1}^{\alpha_2}\int_{\lambda_1}^{\lambda_2} \log\Big(\frac{\mu_n'(\lambda)}{\chi_n(\lambda,\alpha)}\Big)\chi_n(\lambda,\alpha)\, d\lambda\, d\alpha > -\infty,$$

because logarithmic integrals are semicontinuous with respect to the weak convergence of measures. This proves that $\mu'>0$ on the support of χ, which means that

$$\text{meas}\{(\lambda,\alpha)\in[\lambda_1,\lambda_2]\times[\alpha_1,\alpha_2] \,:\, \mu'(\lambda)=0\} \leq C\varepsilon. \tag{31}$$

Recall that the measure μ in (31) is the spectral measure of the operator with the potential $\widetilde{V}$, which is different from V if ε is very small. However, relation (13) shows that this does not really matter, because the condition

$$\mu'(\lambda) = \pi^{-1}\,\text{Im}\,\big((-\Delta+\alpha\widetilde{V}-\lambda-i0)^{-1}f,f\big) > 0, \ \text{ for a.e. } \lambda\in\Omega\subset\mathbb{R} \text{ and some } f\in L^2,$$

implies that

$$\frac{d}{d\lambda}\big(E(-\infty,\lambda)\widetilde{f},\widetilde{f}\big) = \pi^{-1}\,\text{Im}\,\big((-\Delta+\alpha V-\lambda-i0)^{-1}\widetilde{f},\widetilde{f}\big) > 0, \\ \text{for a.e. } \lambda\in\Omega \text{ and } \widetilde{f} = S\widetilde{P}_{\text{ac}}f.$$

So, (31) leads to (27) with $\tau = C\varepsilon$. This completes the proof of Theorem 4.1. □

End of the proof of Theorem 1.2. We show that the absolutely continuous spectrum fills the interval $[\lambda_1, \lambda_2]$ for a.e. $\alpha \in [\alpha_1, \alpha_2]$. This means we need to show that the operator $E(\delta)$ is positive as soon as $\delta \subset [\lambda_1, \lambda_2]$ has positive Lebesgue measure. If the contrary was true, we would find a family of sets $\Omega(\alpha) \subset [\lambda_1, \lambda_2]$ of positive Lebesgue measure such that $E(\Omega(\alpha)) = 0$ for all $\alpha \in \Theta$. Here, $\Theta \subset [\alpha_1; \alpha_2]$ either has positive Lebesgue measure, or is non-measurable. Without loss of generality, we may assume that $\Omega(\alpha) = \varnothing$ for $\alpha \notin \Theta$. In this case,

$$K := \bigcup_{\alpha \in \Theta} \Omega(\alpha) \times \{\alpha\} = \bigcup_{\alpha \in [\alpha_1, \alpha_2]} \Omega(\alpha) \times \{\alpha\}.$$

Now, K cannot be a set of zero measure, because otherwise $\int_{\alpha_1}^{\alpha_2} |\Omega(\alpha)|\, d\alpha = 0$. On the other hand, it is easy to show that, for a fixed $\alpha \in \Theta$,

$$\Omega(\alpha) \subset \Big\{ \lambda \in [\lambda_1, \lambda_2] \,:\, \frac{d}{d\lambda}(E(-\infty, \lambda) f_{\tau,\alpha}, f_{\tau,\alpha}) = 0 \Big\} \tag{32}$$

up to a set of measure zero. But since sets of zero measure do not matter, we can slightly correct $\Omega(\alpha)$ removing a tiny part of it, so that (32) would be true for all τ running over a countable set. Such an inclusion implies that

$$K \subset \Big\{ (\lambda, \alpha) \in [\lambda_1, \lambda_2] \times [\alpha_1, \alpha_2] \,:\, \frac{d}{d\lambda}(E(-\infty, \lambda) f_{\tau,\alpha}, f_{\tau,\alpha}) = 0 \Big\}.$$

By (27), $|K|$ is smaller than any positive number. So, $|K| = 0$ and we obtain a contradiction. We have established that the a.c. spectrum fills the interval $[\lambda_1, \lambda_2]$ for a.e. $\alpha \in [\alpha_1, \alpha_2]$. In order to complete the proof of Theorem 1.2 for $d = 3$, if suffices to use the freedom of the choice of these intervals.

Now, if $d \neq 3$, then no identity of the form (17) can be true. The ratio on the right-hand side is only the asymptotics of the function on the left-hand side, so that (17) holds true only up to terms of smaller order. If we want to avoid the difficulty of dealing with these terms, we need to replace $H_0 = -\Delta$ by the operator

$$H_0 = -\Delta - \frac{\kappa_d \widetilde{\chi}}{|x|^2} P_0, \quad \kappa_d = \Big(\frac{d-2}{2}\Big)^2 - \frac{1}{4},$$

where P_0 is the projection onto the space of spherically symmetric functions and $\widetilde{\chi}$ is the characteristic function of the complement of the unit ball. Provided H_0 is defined as above, relation (17) is valid without terms of smaller order and all proofs of the statements in this paper can be repeated literally. However, the definition of B should also be changed. Namely, we must set $A_\varepsilon = H_0 + \varepsilon$ in (25). Finally, it is easy to check that

$$(H_0 + \alpha V - i)^{-1} - (-\Delta + \alpha V - i)^{-1} = (H_0 + \alpha V - i)^{-1} \frac{\kappa_d \widetilde{\chi}}{|x|^2} P_0 (-\Delta + \alpha V - i)^{-1} \tag{33}$$

is an operator belonging to the trace class. Therefore, the a.c. spectra of $H_0 + \alpha V$ and $-\Delta + \alpha V$ have the same properties.

Note that the statement that the operator (33) belongs to the trace class was in fact established in [8]. We repeat these simple arguments again. Since V is bounded, so are the operators

$$\big((-\Delta + i)(H_0 + \alpha V + i)^{-1}\big)^* \quad \text{and} \quad (-\Delta - i)(-\Delta + \alpha V - i)^{-1}. \tag{34}$$

Consequently, it suffices to prove that

$$(-\Delta - i)^{-1} \frac{\kappa_d \widetilde{\chi}}{|x|^2} P_0 (-\Delta - i)^{-1} \in \mathfrak{S}_1 \tag{35}$$

is a trace class operator (because (33) is obtained by multiplying the operator (35) by the two bounded factors (34)). Now, since P_0 commutes with $-\Delta$, and the product $P_0(-\Delta)$ can be viewed as a one-dimensional Schrödinger operator, we can apply one-dimensional arguments to show that

$$(1+|x|^2)^{-1/2}P_0(-\Delta-i)^{-1}$$

has a square integrable kernel (see [8]). Thus, (35) is a product of two Hilbert–Schmidt operators. □

The semicontinuity of the logarithmic integrals as in (30) was discovered by R. Killip and B. Simon in [11]. The reason why S is semicontinuous is that S is representable as the infimum of the difference of two integrals with respect to the measures ν and ρ:

$$S(\rho|\nu)=\inf_F\left(\int F(x)\,d\nu-\int(1+\log F(x))\,d\rho\right),\quad \min_x F(x)>0.$$

In conclusion of this section, we would like to draw the reader's attention to the papers [1, 2, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17], which contain important information on the absolutely continuous spectrum of multidimensional Schrödinger operators. Surveys of these results can be found in [9] and [18].

§5. Appendix

Here we discuss the analytic properties of the resolvent of H.

Proposition 5.1. *Assume that V is a bounded compactly supported potential. Let χ be the characteristic function of a compact set containing the support of V. Then the operator-valued function*

$$T(k)=\chi(H-k^2)^{-1}\chi,\quad \alpha=kt,$$

admits a meromorphic continuation to the plane with the cut along the half-line $\{z=iy,\ y\le 0\}$.

Proof. Indeed, if $V=0$, then the proof of the claim can be found in [22]. Note that

$$T_0(k)=\chi(H_0-k^2)^{-1}\chi,\quad k\in\mathbb{C}\setminus\{z=iy,\ y\le 0\},$$

is an integral operator whose kernel depends on k analytically. Moreover, the results of [22] clearly say that $T_0(k)$ is compact. The relation

$$\chi(H-k^2)^{-1}\chi=\chi(H_0-k^2)^{-1}\chi-tk\chi(H_0-k^2)^{-1}V\chi(H-k^2)^{-1}\chi$$

implies that

$$T(k)=\big(I+tkT_0(k)V\big)^{-1}T_0(k).$$

Now, the statement follows from the analytic Fredholm alternative. □

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