Алгебра и анализ
Том 25 (2013), № 1

St. Petersburg Math. J.
Vol. 25 (2014), No. 1, Pages 117–149
S 1061-0022(2013)01282-9
Article electronically published on November 20, 2013

WEYL ASYMPTOTICS FOR THE SPECTRUM OF THE MAXWELL OPERATOR IN LIPSCHITZ DOMAINS OF ARBITRARY DIMENSION

N. FILONOV

To the memory of M. Sh. Birman

Abstract. The eigenvalues of the multidimensional Maxwell operator in a domain in the Euclidean space are shown to obey the Weyl asymptotics.

Introduction

0.1. Maxwell operator. It is well known that many differential operators in bounded domains possess a power-type spectral asymptotics. For instance, this is so for the problem of electro-magnetic oscillations of a resonator $\Omega \subset \mathbb{R}^3$ with ideally conducting boundary. The relevant frequencies correspond to the spectrum of the Maxwell operator in Ω. This operator acts by the formula

$$\mathcal{M}\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} i\varepsilon^{-1}\operatorname{curl} H \\ -i\mu^{-1}\operatorname{curl} E \end{pmatrix}, \tag{0.1}$$

where E and H are the electric and magnetic components of the field that are solenoidal:

$$\operatorname{div}(\varepsilon E) = \operatorname{div}(\mu H) = 0 \tag{0.2}$$

and obey the conditions of ideal conductivity on the boundary:

$$E_\tau\big|_{\partial\Omega} = 0, \quad (\mu H)_\nu\big|_{\partial\Omega} = 0. \tag{0.3}$$

Here $\varepsilon(x)$ and $\mu(x)$ are positive definite matrix-valued functions describing the dielectric and the magnetic permeability of the medium filling Ω; the index τ means the tangential component of a vector on the boundary, and the index ν means the normal component. In 1952, H. Weyl defined a Maxwell operator (with constant coefficients) in a domain in the Euclidean n-space with arbitrary n, by using the language of differential forms. However, he did not pose the question about spectral asymptotics. An equivalent version of Weyl's definition looks like this:

$$\mathcal{M}\begin{pmatrix} \omega \\ \theta \end{pmatrix} = \begin{pmatrix} (-1)^{k+1}\star d\theta \\ \star d\omega \end{pmatrix} \tag{0.4}$$

under the conditions

$$d(\star\omega) = 0, \quad d(\star\theta) = 0, \quad j^*\omega = 0, \quad j^*(\star\theta) = 0. \tag{0.5}$$

Here ω is a k-form, θ is an $(n-k-1)$-form, d is the operator of exterior differentiation, $\star$ is the Hodge operator, and j is the embedding of $\partial\Omega$ into $\bar{\Omega}$, so that $j^*\omega$ is the restriction of ω to the boundary in the sense of forms.

2010 *Mathematics Subject Classification.* Primary 35P20.

Key words and phrases. Maxwell operator, Weyl spectral asymptotics, domains with Lipschitz boundary, differential forms, ratios of quadratic forms.

Supported by the RFBR grant 11-01-00458-a and by the grant NSh-357.2012.1.

If the boundary $\partial\Omega$ and the coefficients ε and μ are sufficiently smooth, then the operator (0.1)–(0.3) is well defined on the fields in the Sobolev space $W_{2,1}(\Omega,\mathbb{C}^3)$ and is selfadjoint. But if, for instance, $\partial\Omega$ contains incoming edges, then, even for $\varepsilon=\mu=\mathbb{1}$ (an empty resonator), the Maxwell operator, if defined only on the fields in $W_{2,1}$, fails to be selfadjoint (it is symmetric with infinite defect indices). A detailed discussion of difficulties that arise in the work with the Maxwell system in the nonsmooth three-dimensional case can be found in [22] and [11]. N. Weck (see [30]) carried Weyl's definition over to the case of a Riemannian manifold with possibly nonsmooth boundary. Thus, the Maxwell operator can be studied in at least three cases: in a three-dimensional domain, in a multidimensional domain, and on a manifold (of arbitrary dimension). In the present paper we treat the second case.

0.2. The domain of the Maxwell operator. A key role in the theory of the Maxwell operator is played by the space of differential forms

$$F^k_{\mathrm{el}}=\{\omega\in L^k_2\,:\,\|\omega\|_{F^k}<\infty,\ j^*\omega=0\} \tag{0.6}$$

(see §2 below). Here k is the valency of the form ω, L^k_2 is the space of square-integrable k-forms Ω (see (1.3) below), $j^*\omega$ is the restriction of ω to the boundary $\partial\Omega$, and

$$\|\omega\|^2_{F^k}=\|d\omega\|^2_{L_2}+\|d(\star\omega)\|^2_{L_2}+\|\omega\|^2_{L_2}. \tag{0.7}$$

In the physical case, Ω is a domain in $\mathbb{R}^3$, $k=1$, and

$$F_{\mathrm{el}}=\{E\in L_2(\Omega,\mathbb{C}^3)\,:\,\operatorname{curl}E\in L_2,\ \operatorname{div}E\in L_2,\ E_\tau\big|_{\partial\Omega}=0\} \tag{0.8}$$

is the space of electric fields. If the boundary $\partial\Omega$ is smooth, then the space F^k_{el} embeds in $W_{2,1}$ and the norm (0.7) is equivalent to the $W_{2,1}$-norm (see [17]); in the physical case, we have

$$F_{\mathrm{el}}=\{E\in W_{2,1}(\Omega,\mathbb{C}^3)\,:\,E_\tau\big|_{\partial\Omega}=0\}.$$

If the boundary is not smooth, then F_{el} does not embed in $W_{2,1}$. In [9], for domains Ω with Lipschitz boundary it was shown that every field E in F_{el} is representable as the sum of a term in $W_{2,1}$ and the gradient of a weak solution of a scalar Dirichlet problem, specifically,

$$E=u+\nabla\varphi,\quad u\in\mathring{W}_{2,1}(\Omega,\mathbb{C}^3),\quad \varphi\in\mathring{W}_{2,1}(\Omega,\mathbb{C}),\quad \Delta\varphi\in L_2(\Omega). \tag{0.9}$$

We shall prove an analog of this decomposition theorem for k-forms in a Lipschitz domain $\Omega\subset\mathbb{R}^n$ for arbitrary k and n (see Theorem 4.8 below).

In the smooth case, the spaces (0.6), (0.8) embed in $L_2(\Omega)$ compactly by the embedding theorems. The compactness of this embedding in nonsmooth situations has been investigated in a variety of papers. We mention Picard's result (see [25]): the embedding $F^k_{\mathrm{el}}\subset L^k_2$ is compact for Lipschitzian manifolds with boundary (see Corollary 4.10 below, and also [26] and the references therein). It follows that the spectrum of the Maxwell operator on a Lipschitz manifold is discrete.

It should also be noted that the solvability of the stationary Maxwell system

$$\mathcal{M}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}-\lambda\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\quad j^*\omega_1=\psi,$$

in various classes of functions is well studied, see, e.g., [32, 29, 20, 26, 24].

0.3. Spectral asymptotics for the Maxwell operator. The spectrum of the Maxwell operator is discrete and symmetric about zero (see Subsection 2.2 below). When talking about the eigenvalue asymptotics, we shall consider only positive eigenvalues m_k. We introduce the counting function $N(\lambda)$ equal to the number (with multiplicities) of eigenvalues for the Maxwell operator in the interval $(0, \lambda)$. In 1912, Weyl [31] proved that, in a domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and for $\varepsilon = \mu = \mathbb{1}$, the eigenvalues of the operator (0.1)–(0.3) obey the law

$$m_k \sim \left(\frac{3\pi^2}{\operatorname{meas}\Omega}\right)^{1/3} k^{1/3}, \quad k \to +\infty,$$

or, equivalently,

$$N(\lambda) \sim \frac{\operatorname{meas}\Omega}{3\pi^2}\,\lambda^3, \quad \lambda \to +\infty. \tag{0.10}$$

It was not until 1976 (see [2]) that this result was extended to the case of nonsmooth ε and μ and smooth boundary $\partial\Omega$. The asymptotic formula becomes more involved than (0.10), see (3.14) and (3.15) below. In 1987 (see [10]) Birman and Solomyak proved formula (0.10) in the opposite situation, where $\partial\Omega$ is only Lipschitz but $\varepsilon = \mu = \mathbb{1}$. It should be noted that, for a nonsmooth boundary, Weyl's variation technique is not applicable directly. Only in 2007, the approaches of [2] and [10] were combined: in [5] the result was obtained for the Maxwell operator in domains with Lipschitz boundary and nonsmooth (specifically, L_∞) coefficients.

We mention also the papers [27] and [28], where for an empty ($\varepsilon = \mu = \mathbb{1}$) resonator Ω the following remainder estimates were obtained:

- if $\partial\Omega \in C^\infty$, then
$$N(\lambda) = \frac{\operatorname{meas}\Omega}{3\pi^2}\,\lambda^3 + O(\lambda^2), \quad \lambda \to +\infty,$$
(under the additional assumption that the billiard flow in Ω is nonperiodic, O can be replaced with o in the remainder estimate);
- if $\partial\Omega$ is Lipschitz, then
$$N(\lambda) = \frac{\operatorname{meas}\Omega}{3\pi^2}\,\lambda^3\left(1 + O(\lambda^{-2/5})\right), \quad \lambda \to +\infty.$$

In [15] and in the present paper, the main objective is to obtain a spectral asymptotics for the Maxwell operator with nonsmooth coefficients that acts on k-forms on an n-dimensional compact Lipschitz manifold. The main result of this paper is Theorem 3.5, where we justify the asymptotics for domains with Lipschitz boundary in $\mathbb{R}^n$. In [15] it was shown how a similar result for manifolds can be deduced from this. Basically, the construction is done as in [5, 3], where $n = 3$, $k = 1$, but there are two distinctions. First, the proof of the decomposition (0.9) becomes more involved: it is necessary to invoke an analog of the Poincaré lemma concerning representation of closed forms with estimates of Sobolev norms. Such a version of the Poincaré lemma was proved by Mitrea in [24]. Second, for $k = 1$ the problem reduces to the question about the spectral asymptotics for a certain ratio of quadratic forms on $\mathring{W}_{2,1}$; this asymptotics was obtained in [1]. For $k > 1$ we must prove the spectral asymptotics of the corresponding ratio anew (see also Remark 7.4).

The paper consists of two chapters. In Chapter I, we recall the calculus of differential forms (§1), give the definition of the Maxwell operator (§2) , and state the result (§3). Next, in §4 we describe the space (0.6) analytically (Theorem 4.8). In §§5, 6, we reduce the spectral asymptotics question for the Maxwell operators to a similar question for certain quadratic forms (Theorem 6.5). In Chapter II, we prove Theorem 6.5.

The author is grateful to M. N. Demchenko, M. Z. Solomyak, and A. B. Pushnitskiĭ for useful discussions.

Chapter I

§1. Differential forms

Throughout, we assume that the dimension n of the Euclidean space we work with is at least 2. The case of $n = 1$ is trivial, see Subsection 3.4 below.

1.1. Calculus of differential forms. We denote by $\mathcal{F}^k$, $k = 0, 1, \dots, n$, the space of complex-valued antisymmetric k-linear forms on $\mathbb{R}^n$, $\dim \mathcal{F}^k = \binom{n}{k}$. The basis elements of $\mathcal{F}^k$ will be denoted by $dx^{i_1} \wedge \dots \wedge dx^{i_k}$, $i_1 < \dots < i_k$. The space $\mathcal{F}^k$ is Hilbert under the norm

$$|\theta|^2_{\mathcal{F}^k} = \sum_{i_1 < \dots < i_k} |\theta_{i_1 \dots i_k}|^2, \tag{1.1}$$

where $\theta = \sum_{i_1 < \dots < i_k} \theta_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

The Hodge operator $\star \equiv \star_k : \mathcal{F}^k \to \mathcal{F}^{n-k}$ is defined on the basis elements by the formula

$$\star(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \varepsilon(i_1, \dots, i_k, j_1, \dots, j_{n-k}) dx^{j_1} \wedge \dots \wedge dx^{j_{n-k}},$$

where $\{j_1, \dots, j_{n-k}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$, $j_1 < \dots < j_{n-k}$, and ε on the right is the sign of the corresponding rearrangement. We mention the formulas

$$\begin{gathered} \theta \wedge \star \bar{\theta} = |\theta|^2 dx^1 \wedge \dots \wedge dx^n, \quad \theta \in \mathcal{F}^k, \\ |\xi \wedge \theta|^2 + |\bar{\xi} \wedge \star \theta|^2 = |\xi|^2 |\theta|^2, \quad \xi \in \mathcal{F}^1, \ \theta \in \mathcal{F}^k. \end{gathered} \tag{1.2}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$ (i.e., for every $x \in \partial\Omega$ there exists a neighborhood U_x such that $\Omega \cap U_x$ is the subgraph of a Lipschitz function). We introduce the space

$$L^k_p \equiv L_p(\Omega, \mathcal{F}^k) = \left\{ \omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \ \omega_{i_1 \dots i_k} \in L_p(\Omega) \right\}, \ 1 \le p \le \infty. \tag{1.3}$$

For $\omega \in L^n_1$, the following integral is defined in a standard way:

$$\int_\Omega \omega = \int_\Omega \omega_{1 \dots n}(x)\, dx^1 \wedge \dots \wedge dx^n = \int_\Omega \omega_{1 \dots n}(x)\, d^n x,$$

where $d^n x$ is Lebesgue measure in $\mathbb{R}^n$.

The external differentiation d is defined on forms with smooth coefficients by the formula

$$(d\omega)_{i_1 \dots i_k}(x) = \sum_{l=1}^{k} (-1)^{l-1} \frac{\partial}{\partial x^{i_l}} \omega_{i_1 \dots (i_l) \dots i_k}(x), \quad i_1 < \dots < i_k,$$

where $\{i_1 \dots (i_l) \dots i_k\}$ stands for the collection $\{i_1, \dots, i_k\}$ in which i_l is dropped. On L^k_1, the operator d is defined in the distributional sense:

$$d\omega = \varphi \quad \Leftrightarrow \quad \int_\Omega \omega \wedge d\theta = (-1)^{k+1} \int_\Omega \varphi \wedge \theta, \quad \theta \in C^\infty_0(\Omega, \mathcal{F}^{n-k-1}).$$

If $\omega \in L^n_1$, we put $d\omega = 0$ by definition. Next, we introduce the space

$$\mathcal{W}^k_p = \{\omega \in L^k_p(\Omega) \, : \, d\omega \in L^{k+1}_p(\Omega)\}.$$

Note that $\mathcal{W}^0_p = W_{p,1}(\Omega, \mathbb{C})$ is the usual Sobolev space of scalar functions, and $\mathcal{W}^n_p = L^n_p$. In [18] it was shown that the spaces $\mathcal{W}^k_p$ are complete with respect to the natural norm $\|d\omega\|_{L_p} + \|\omega\|_{L_p}$.

Let $j : \partial\Omega \to \bar{\Omega}$ be the natural embedding, and let ω be a k-form in Ω. In the smooth case, the relation $j^*\omega = 0$ is equivalent to the integral identity $\int_{\partial\Omega} j^*(\omega \wedge \theta) = 0$ for all $(n-k-1)$-forms θ in Ω. Taking the Stokes formula into account, we introduce the following definition.

Definition 1.1. Let $\omega \in \mathcal{W}_p^k$ with $k < n$ and $1 < p < \infty$. We write $j_\Omega^*\omega = 0$ if

$$\int_\Omega \omega \wedge d\theta = (-1)^{k+1} \int_\Omega d\omega \wedge \theta, \quad \theta \in \mathcal{W}_{p'}^{n-k-1},$$

where $p' = p/(p-1)$. For $\omega \in \mathcal{W}_p^n$ we agree that $j_\Omega^*\omega = 0$ by definition.

Remark 1.2. It is easily seen that the relation $j_\Omega^*\omega = 0$ implies $j_\Omega^*(d\omega) = 0$.

1.2. Coefficients. Let $\mathrm{Lin}_k = \mathrm{Lin}(\mathcal{F}^k, \mathcal{F}^{n-k})$ be the space of linear mappings from $\mathcal{F}^k$ to $\mathcal{F}^{n-k}$. We denote by M_+^k the set of all $\beta \in \mathrm{Lin}_k$ with the following properties:

- $\overline{\beta\,\omega} = \beta\,\bar{\omega}$ for all $\omega \in \mathcal{F}^k$;
- the function $b(\omega)$ determined by the identity
$$\omega \wedge \beta\,\bar{\omega} = b(\omega)\, dx^1 \wedge \cdots \wedge dx^n,$$
is positive for all $\omega \neq 0$.

Clearly, $\star_k \in M_+^k$ and if $\beta \in M_+^k$, then $\omega \wedge \beta\,\theta = \theta \wedge \beta\,\omega$ for all $\omega, \theta \in \mathcal{F}^k$. It is convenient to rewrite the last identity as follows:

$$\omega \wedge \beta\,\theta = (-1)^{k(n-k)} \beta\,\omega \wedge \theta, \quad \omega, \theta \in \mathcal{F}^k. \tag{1.4}$$

Next, if $\beta \in M_+^k$, then the inverse mapping $\alpha : \mathcal{F}^{n-k} \to \mathcal{F}^k$ exists, $\alpha \circ \beta = \mathrm{id}$. *We denote by β^{-1} this inverse supplied with a sign, namely* $\beta^{-1} := (-1)^{k(n-k)}\alpha$. Then $\beta^{-1} \in M_+^{n-k}$.

Lemma 1.3. *Suppose that $\alpha > 0$, ξ is a real 1-form, $h \in \mathcal{F}^l$, and $\beta \in M_+^{l+1}$. Then*

$$(-1)^l \bar{h} \wedge \xi \wedge \beta(\xi \wedge h) + (-1)^{(l-1)(n-l)} \alpha \bar{h} \star (\xi \wedge \star(\xi \wedge \star h)) \geq \min\bigl(\alpha, \|\beta\|_{\mathrm{Lin}_{l+1}}^{-1}\bigr) |\xi|^2 |h|^2. \tag{1.5}$$

Proof. We have

$$(-1)^l \bar{h} \wedge \xi \wedge \beta(\xi \wedge h) + (-1)^{(l-1)(n-l)} \alpha \bar{h} \star (\xi \wedge \star(\xi \wedge \star h)) \geq \|\beta\|_{\mathrm{Lin}_{l+1}}^{-1} |\xi \wedge h|^2 + \alpha |\xi \wedge \star h|^2.$$

Then (1.5) follows from (1.2). □

Now, we put

$$M_+^k(\Omega) = \bigl\{\beta \in L_\infty(\Omega, M_+^k) \,:\, \beta^{-1} \in L_\infty(\Omega, M_+^{n-k})\bigr\}$$

($M_+^k(\Omega)$ is well defined because M_+^k is a subset of the normed space Lin_k). Clearly, $(\star_k)^{-1} = \star_{n-k}$, $\star_k \in M_+^k(\Omega)$, and

$$\beta \in M_+^k(\Omega) \quad \Leftrightarrow \quad \beta^{-1} \in M_+^{n-k}(\Omega).$$

Every function $\beta_k \in M_+^k(\Omega)$ gives rise to a scalar product in $L_2^k = L_2^k(\beta_k)$:

$$(\omega, \theta)_{\beta_k} := \int_\Omega \omega \wedge \beta_k \bar{\theta}. \tag{1.6}$$

It is easily seen that

$$\int_\Omega \omega \wedge \beta_k \bar{\omega} \geq \|\beta_k^{-1}\|_{L_\infty}^{-1} \int_\Omega \omega \wedge \star\bar{\omega}. \tag{1.7}$$

Suppose that $0 \leq k \leq n-1$ and $\beta_k \in M_+^k(\Omega)$, $\beta_{k+1} \in M_+^{k+1}(\Omega)$. We introduce the differentiation operator $d_k : L_2^k(\beta_k) \to L_2^{k+1}(\beta_{k+1})$ by setting

$$d_k\omega = d\omega, \quad \mathrm{Dom}\, d_k = \{\omega \in \mathcal{W}_2^k \,:\, j^*\omega = 0\}. \tag{1.8}$$

Clearly, $C_0^\infty(\Omega,\mathcal{F}^k)\subset \operatorname{Dom} d_k$. Therefore, d_k is densely defined in $L_2^k(\beta_k)$. The following simple fact was proved in [15].

Lemma 1.4. *The operator d_k is closed. Its adjoint acts in accordance with the formula*

$$d_k^*\theta=(-1)^{k(n-k-1)+1}\beta_k^{-1}d(\beta_{k+1}\theta),\ \ \operatorname{Dom} d_k^*=\{\theta\in L_2^{k+1}\,:\,d(\beta_{k+1}\theta)\in L_2^{n-k}\}. \tag{1.9}$$

1.3. Problem setting. Let $0\le k\le n-1$, $\beta_k\in M_+^k(\Omega)$, and let $\beta_{k+1}\in M_+^{k+1}(\Omega)$. In the space $L_2^k(\beta_k)\oplus L_2^{n-k-1}(\beta_{k+1}^{-1})$, consider the operator

$$\mathcal{M}_k(\beta_k,\beta_{k+1})=\begin{pmatrix}0 & d_k^*\\ d_k & 0\end{pmatrix}, \tag{1.10}$$

where d_k and d_k^* are given by (1.8), (1.9). The operator $\mathcal{M}_k$ is selfadjoint, and its spectrum is symmetric about zero because

$$\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix}0 & d_k^*\\ d_k & 0\end{pmatrix}=-\begin{pmatrix}0 & d_k^*\\ d_k & 0\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}.$$

The kernel of $\mathcal{M}_k$ is infinite-dimensional. As we shall see below, the spectrum is discrete outside zero. Our purpose is the study of the asymptotic behavior of the eigenvalues of $\mathcal{M}_k$ at infinity. In the next section, we "split out" the kernel of $\mathcal{M}_k$ and obtain the "genuine" Maxwell operator, more suitable to work with.

§2. Maxwell operator

2.1. Spaces of differential forms. We introduce a series of spaces of k-forms. Put

$$G^k=d\mathcal{W}_2^{k-1}=\{d\varphi\,:\,\varphi\in\mathcal{W}_2^{k-1}\},\quad G_0^k=\{d\varphi\,:\,\varphi\in\mathcal{W}_2^{k-1},j^*\varphi=0\}$$

for $k\ge 1$ and $G^0=G_0^0=\{0\}$. Next,

$$J^k(\beta_k)=\{\omega\in L_2^k\,:\,d(\beta_k\omega)=0\},\quad J_0^k(\beta_k)=\{\omega\in J^k(\beta_k)\,:\,j^*(\beta_k\omega)=0\}.$$

Clearly, $J^0=J_0^0=L_2^0$. Also, it is easy to show that the spaces J are orthogonal to the corresponding spaces G with respect to the scalar product (1.6):

$$L_2^k(\beta_k)=G_0^k\oplus_{\beta_k}J^k=G^k\oplus_{\beta_k}J_0^k.$$

In different mathematical fields, these decompositions are related to the names of Weyl, Helmholtz, and Hodge.

Below we assume that $k\le n-1$. We introduce the Hilbert space

$$F^k(\beta_k)=\{\omega\in\mathcal{W}_2^k\,:\,d(\beta_k\omega)\in L_2^{n-k+1}\}$$

with the norm

$$\|\omega\|_{F^k}^2=\int_\Omega d\omega\wedge\beta_{k+1}(\overline{d\omega})+d(\beta_k\omega)\wedge\star_{n-k+1}d(\overline{\beta_k\omega})+\omega\wedge\beta_k\bar\omega$$

(if $k=0$, the second summand disappears), and its "electric" and "magnetic" subspaces

$$F_{\mathrm{el}}^k(\beta_k)=\{\omega\in F^k(\beta_k)\,:\,j^*\omega=0\},\quad F_{\mathrm{m}}^k(\beta_k)=\{\omega\in F^k(\beta_k)\,:\,j^*(\beta_k\omega)=0\}.$$

For $k=0$, these spaces coincide with the usual Sobolev spaces of scalar functions, $F_{\mathrm{el}}^0=\mathring{W}_{2,1}(\Omega)$, $F_m^0=F^0=W_{2,1}(\Omega)$. Next, we put

$$\Phi_{\mathrm{el}}^k(\beta_k)=F_{\mathrm{el}}^k(\beta_k)\cap J^k(\beta_k),\qquad E_{\mathrm{el}}^k(\beta_k)=F_{\mathrm{el}}^k(\beta_k)\cap G_0^k,$$
$$\Phi_{\mathrm{m}}^k(\beta_k)=F_m^k(\beta_k)\cap J_0^k(\beta_k),\qquad E_{\mathrm{m}}^k(\beta_k)=F_m^k(\beta_k)\cap G^k.$$

Obviously,

$$\begin{aligned}F_{\mathrm{el}}^k&=\Phi_{\mathrm{el}}^k\oplus_{\beta_k}E_{\mathrm{el}}^k,\\ F_m^k&=\Phi_{\mathrm{m}}^k\oplus_{\beta_k}E_m^k,\end{aligned} \tag{2.1}$$

$$\Phi^0_{\mathrm{el}} = \mathring{W}_{2,1}(\Omega), \quad \Phi^0_{\mathrm{m}} = W_{2,1}(\Omega), \quad E^0_{\mathrm{el}} = E^0_{\mathrm{m}} = \{0\}.$$

Finally, we put

$$K^k(\beta_k) = \bigl\{\omega \in \Phi^k_{\mathrm{el}}(\beta_k) \,:\, d\omega = 0\bigr\} = \bigl\{\omega \in L^k_2(\beta_k) \,:\, d\omega = 0, d(\beta_k\omega) = 0, j^*\omega = 0\bigr\}.$$

The space $K^k(\beta_k)$ depends on the topological properties of Ω. Below we shall show (see Corollary 6.2) that it is always finite-dimensional (maybe, trivial) for domains with Lipschitz boundary. For d_k, we have $\ker d_k = G^k_0 \oplus_{\beta_k} K^k(\beta_k)$.

2.2. Maxwell operator. Suppose that $k \le n-1$, $\beta_k \in M^k_+(\Omega)$, and $\beta_{k+1} \in M^{k+1}_+(\Omega)$. We introduce the operator

$$R_k(\beta_k, \beta_{k+1}) \,:\, J^k(\beta_k) \to J^{n-k-1}_0(\beta^{-1}_{k+1}), \;\; R_k\omega = \beta_{k+1}d\omega, \;\; \operatorname{Dom} R_k = \Phi^k_{\mathrm{el}}(\beta_k).$$

The operator R_k is closed, and its kernel $\ker R_k$ is equal to $K^k(\beta_k)$. It is easy to describe the adjoint operator:

$$\begin{aligned} &R_k(\beta_k, \beta_{k+1})^* \,:\, J^{n-k-1}_0(\beta^{-1}_{k+1}) \to J^k(\beta_k), \\ &R^*_k\theta = (-1)^{k(n-k-1)+1}\beta^{-1}_k d\theta, \quad \operatorname{Dom} R^*_k = \Phi^{n-k-1}_m(\beta^{-1}_{k+1}). \end{aligned}$$

Definition 2.1. We define the Maxwell operator $M_k(\beta_k, \beta_{k+1})$, $k = 0, 1, \dots, n-1$, to be the following block operator in the Hilbert space $J^k(\beta_k) \oplus J^{n-k-1}_0(\beta^{-1}_{k+1})$:

$$\begin{aligned} M_k(\beta_k, \beta_{k+1}) &= \begin{pmatrix} 0 & R^*_k \\ R_k & 0 \end{pmatrix}, \\ M_k \begin{pmatrix} \omega \\ \theta \end{pmatrix} &= \begin{pmatrix} (-1)^{k(n-k-1)+1}\beta^{-1}_k d\theta \\ \beta_{k+1}d\omega \end{pmatrix}, \\ \operatorname{Dom} M_k(\beta_k, \beta_{k+1}) &= \Phi^k_{\mathrm{el}}(\beta_k) \oplus \Phi^{n-k-1}_m(\beta^{-1}_{k+1}). \end{aligned}$$

The operator $M_k(\beta_k, \beta_{k+1})$ is selfadjoint, its kernel is finite-dimensional, and its spectrum outside zero coincides with that of $\mathcal{M}_k(\beta_k, \beta_{k+1})$ (see Subsection 1.3). The spectrum is symmetric about zero and discrete. It is convenient to describe the asymptotic behavior of its eigenvalues in terms of the counting function $N(\lambda, M_k(\beta_k, \beta_{k+1}))$, where $N(\lambda, A)$ is the number (with multiplicity) of eigenvalues of a selfadjoint operator A in the interval $(0, \lambda)$.

Remark 2.2. Clearly,

$$M^2_k = \begin{pmatrix} R^*_k R_k & 0 \\ 0 & R_k R^*_k \end{pmatrix},$$

and the operators $R^*_k R_k$ and $R_k R^*_k$ are unitarily equivalent on the orthogonal complements of their kernels; therefore,

$$N(\lambda, M_k(\beta_k, \beta_{k+1})) = N(\lambda^2, R_k(\beta_k, \beta_{k+1})^* R_k(\beta_k, \beta_{k+1})). \tag{2.2}$$

We recall that our definition of the counting function $N(\lambda)$ does not take zero eigenvalues into account.

We formulate the main result.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, and let*

$$0 \le k \le n-1, \quad \beta_k \in M^k_+(\Omega), \quad \beta_{k+1} \in M^{k+1}_+(\Omega).$$

Then the counting function for the positive eigenvalues of the Maxwell operator obeys the power-type asymptotic law

$$N(\lambda, M_k(\beta_k, \beta_{k+1})) \sim \varkappa\lambda^n, \quad \lambda \to +\infty,$$

$\varkappa > 0$.

Here and below, the symbol $\sim$ means that the ratio of the quantities on the right and on the left of it tends to 1.

In terms of the eigenvalues λ_m, the claim means that

$$\lambda_m(M_k(\beta_k,\beta_{k+1})) \sim \varkappa^{-1/n} m^{1/n}, \quad m \to +\infty.$$

There is an explicit (though cumbersome) expression for $\varkappa$ in terms of the coefficients of the operators β_k, β_{k+1}. The formula is presented in the next section (Theorem 3.5).

§3. Algebraic problems

3.1. "Complicated" algebraic problem. This problem will be required for the proof of Theorems 2.3 and 3.5. We note that for the calculation itself of the coefficient in the spectral asymptotics for the Maxwell operator, only a particular case (to be discussed in the next subsection) of this problem is required.

Recall that the space $\mathcal{F}^k$ of k-forms is a Hilbert space with respect to the norm (1.1). For a linear map $\gamma \in \mathrm{Lin}_k$, we denote by γ^* its adjoint, and by γ^+ the operator

$$\gamma^+ = \star_k \gamma^* \star_k \,:\, \mathcal{F}^k \to \mathcal{F}^{n-k}.$$

Then

$$\bar\omega \wedge \gamma^+\theta = \overline{\gamma\omega} \wedge \theta, \quad \omega,\theta \in \mathcal{F}^k.$$

For the positive definite operator β_k, we have $\beta_k^+ = (-1)^{k(n-k)}\beta_k$.

Let $\alpha > 0$, and let ξ be a real 1-form. Suppose that $\beta_k(x) \in M_+^k$, $\beta_{k+1}(x) \in M_+^{k+1}$, and $\gamma(x) \in \mathrm{Lin}_k$. For $k \geq 1$, we consider the problem

$$\begin{cases} \gamma(x)^+\beta_k(x)^{-1}\gamma(x)h + i(-1)^{k(n-k)+1}\gamma(x)^+(\xi\wedge\psi) \\ \qquad = \lambda\big((-1)^k\xi\wedge\beta_{k+1}(x)(\xi\wedge h) + (-1)^{(k-1)(n-k)}\alpha\star(\xi\wedge\star(\xi\wedge\star h))\big), \\ i\xi\wedge\beta_k(x)(\xi\wedge\psi) = \xi\wedge\gamma(x)h, \quad \xi\wedge\star\psi = 0. \end{cases} \tag{3.1}$$

Here $\lambda \in \mathbb{C}$ is an eigenvalue and $h \in \mathcal{F}^k$ is an eigenvector of the problem. Note that the last two equations in (3.1) can be united in a single equation for $\psi \in \mathcal{F}^{k-1}$:

$$(-1)^{k-1}\xi\wedge\beta_k(x)(\xi\wedge\psi) + (-1)^{k(n-k+1)}\star(\xi\wedge\star(\xi\wedge\star\psi)) = (-1)^k i\xi\wedge\gamma(x)h. \tag{3.2}$$

Indeed, clearly, (3.1) implies (3.2). Conversely, let ψ satisfy (3.2). Then we multiply (3.2) by $\xi\wedge\star(\xi\wedge\star\psi)$ to obtain

$$\xi\wedge\star(\xi\wedge\star\psi)\wedge\star(\xi\wedge\star(\xi\wedge\star\psi)) = 0 \quad\Rightarrow\quad \xi\wedge\star(\xi\wedge\star\psi) = 0.$$

Multiplying the last identity by $\star\psi$, we arrive at $\xi\wedge\star\psi = 0$. Thus, (3.1) holds true. We see that ψ is uniquely determined,

$$\psi = Q(\beta_k(x),\xi)^{-1}\left((-1)^k i\xi\wedge\gamma(x)h\right),$$

where $Q(\beta_k,\xi)$ stands for the operator acting from $\mathcal{F}^{k-1}$ to $\mathcal{F}^{n-k+1}$ as follows:

$$Q(\beta_k,\xi)\psi = (-1)^{k-1}\xi\wedge\beta_k(x)(\xi\wedge\psi) + (-1)^{k(n-k+1)}\star(\xi\wedge\star(\xi\wedge\star\psi)).$$

It is invertible by Lemma 1.3.

Lemma 3.1. *System* (3.1) *possesses the following properties*:

1) *the quadratic form of the left-hand side of the first equation is nonnegative*;
2) *the quadratic form of the right-hand side of the first equation is positive*;
3) *the eigenvalues are nonnegative*;
4) *the eigenvalues are homogeneous of order* -2 *in* ξ;
5) *if* $\gamma(x) = 0$, *then all eigenvalues are zero.*

Proof. 1) For the first summand on the left, we have

$$\bar{h} \wedge \gamma^+ \beta_k^{-1} \gamma h = \overline{\gamma h} \wedge \beta_k^{-1} \gamma h \geq 0.$$

For the second summand on the left, we have

$$\begin{aligned} i(-1)^{k(n-k)+1} \bar{h} \wedge \gamma^+(\xi \wedge \psi) &= i(-1)^{(k-1)(n-k)+1} \xi \wedge \overline{\gamma h} \wedge \psi \\ &= (-1)^{(k-1)(n-k)} \xi \wedge \beta_k(\xi \wedge \bar{\psi}) \wedge \psi = (\xi \wedge \psi) \wedge \beta_k(\xi \wedge \bar{\psi}) \geq 0. \end{aligned}$$

2) is a consequence of Lemma 1.3.
3) follows from 1) and 2).
Statements 4) and 5) are obvious. □

Let $\{\lambda_j(x,\xi)\}_{j=1}^{\binom{n}{k}}$ be the eigenvalues of problem (3.1), counted with multiplicities. If $k = 0$, the form ψ is absent in (3.1) and h is a complex number. So, the problem reduces to the equation

$$\gamma(x)^+ \beta_k(x)^{-1} \gamma(x) 1 = \lambda \xi \wedge \beta_1(x) \xi,$$

which determines a unique eigenvalue $\lambda_1(\xi)$. Clearly, statements 3), 4), and 5) of Lemma 3.1 are also true for this eigenvalue λ_1.

We introduce the functions $n(\lambda; x, \xi) = \#\{j : \lambda_j(x,\xi) > \lambda\}$, $\lambda > 0$, and

$$\nu(\lambda; x) = (2\pi)^{-n} \int_{\mathbb{R}^n} n(\lambda; x, \xi)\, d^n\xi, \tag{3.3}$$

where we have identified the real 1-forms and the vectors in $\mathbb{R}^n$ in a standard way.

Lemma 3.2. *Suppose that $\beta_k \in M_+^k(\Omega)$, $\beta_{k+1} \in M_+^{k+1}(\Omega)$, and $\gamma \in L_n(\Omega, \mathrm{Lin}_k)$. Then the function $\nu(\lambda; x)$ defined by (3.3) is homogeneous of degree $(-n/2)$ in λ, i.e., $\nu(\lambda; x) = \lambda^{-n/2} \nu(x)$. The function $\nu(\,.\,) = \nu(1, \,.\,)$ is calculated by the formula*

$$\nu(x) = \frac{1}{n(2\pi)^n} \int_{|\theta|=1} \sum_{j=1}^{\binom{n}{k}} \lambda_j(x,\theta)^{n/2}\, dS(\theta), \tag{3.4}$$

where $dS(\theta)$ is Lebesgue measure on the unit sphere of $\mathbb{R}^n$.

Proof. For $\xi = \rho\theta$ with $\rho > 0$, by the homogeneity of λ_j we obtain

$$\lambda_j(x,\xi) > \lambda \quad \Leftrightarrow \quad \lambda_j(x,\theta) > \rho^2 \lambda.$$

Consequently,

$$\begin{aligned} (2\pi)^n \nu(\lambda; x) &= \sum_j \int_{\{\xi \in \mathbb{R}^n \,:\, \lambda_j(x,\xi) > \lambda\}} d^n\xi = \sum_j \int_{|\theta|=1} \int_0^{(\lambda_j(x,\theta)/\lambda)^{1/2}} \rho^{n-1}\, d\rho\, dS(\theta) \\ &= \frac{1}{n\lambda^{n/2}} \int_{|\theta|=1} \sum_j \lambda_j(x,\theta)^{n/2}\, dS(\theta). \end{aligned}$$ □

Remark 3.3. Formula (3.4) for ν can also be written in the form

$$\begin{aligned} \nu(x) = \frac{1}{n(2\pi)^n} \int_{|\theta|=1} \mathrm{tr}\big[\big((\star Q(\beta_{k+1}(x), \alpha, \xi))^{-1/2} \star \big(\gamma(x)^+ \beta_k(x)^{-1} \gamma(x) \\ + (-1)^{k(n-k+1)} \gamma(x)^+ (\xi \wedge Q(\beta_k(x), \xi)^{-1} (\xi \wedge \gamma(x)\,.\,))\big) \\ \times (\star Q(\beta_{k+1}(x), \alpha, \xi))^{-1/2}\big)^{n/2}\big] dS(\theta). \end{aligned} \tag{3.5}$$

3.2. "Simple" algebraic problem. Now, we consider the case where $\gamma(x) = \beta_k(x)$. Then problem (3.1) has kernel of the form

$$h = i\xi \wedge \psi, \quad \psi \in \mathcal{F}^{k-1} \,:\, \xi \wedge \star\psi = 0. \tag{3.6}$$

If a form h is orthogonal to the kernel (3.6) in the metric of the right-hand side of (3.1), then

$$\xi \wedge \psi \wedge \big((-1)^k \xi \wedge \beta_{k+1}(\xi \wedge h) + (-1)^{(k-1)(n-k)} \alpha \star (\xi \wedge \star(\xi \wedge \star h))\big) = 0$$
$$\Rightarrow \quad \xi \wedge \psi \wedge \star(\xi \wedge \star(\xi \wedge \star h)) = 0, \quad \psi \in \mathcal{F}^{k-1}.$$

On the other hand,

$$\zeta \wedge \star(\xi \wedge \star(\xi \wedge \star h)) = 0 \ \text{ for all } \ \zeta \in \mathcal{F}^k \ \text{ with } \ \xi \wedge \star\zeta = 0.$$

Consequently, for such h we have $\xi \wedge \star(\xi \wedge \star h) = 0$, whence $\xi \wedge \star h = 0$. Thus, if $\gamma(x) = \beta_k(x)$, then the nonzero spectrum of problem (3.1) does not depend on α and coincides with the nonzero spectrum of the problem

$$\begin{cases} \beta_k(x)h - i\beta_k(x)(\xi \wedge \psi) = (-1)^k \lambda \xi \wedge \beta_{k+1}(x)(\xi \wedge h), \\ \xi \wedge \star h = 0, \quad i\xi \wedge \beta_k(x)(\xi \wedge \psi) = \xi \wedge \beta_k(x)h, \quad \xi \wedge \star\psi = 0. \end{cases} \tag{3.7}$$

Next, if a form h is a solution of (3.7), then the form $h_0 = h - i\xi \wedge \psi$ satisfies the equation

$$\beta_k(x)h_0 = (-1)^k \lambda \xi \wedge \beta_{k+1}(x)(\xi \wedge h_0). \tag{3.8}$$

Conversely, let h_0 be a solution of (3.8). We find a $(k-1)$-form ψ from the equations

$$-i\xi \wedge \star(\xi \wedge \psi) = \xi \wedge \star h_0, \quad \xi \wedge \star\psi = 0,$$

and put $h = h_0 + i\xi \wedge \psi$. Then $\xi \wedge \star h = 0$, and

$$\xi \wedge \beta_k(x)h = i\xi \wedge \beta_k(x)(\xi \wedge \psi)$$

because $\xi \wedge \beta_k(x)h_0 = 0$ by (3.8). So, h satisfies (3.7). We have obtained the following result.

Lemma 3.4. *For $\gamma(x) = \beta_k(x)$, the nonzero spectrum of problem* (3.1) *coincides* (*with multiplicity taken into account*) *with the nonzero spectrum of problem* (3.8).

Now, let $\{\widetilde{\lambda}_j(x,\xi)\}$ be the eigenvalues of (3.8). We introduce the functions

$$\widetilde{n}(\lambda; x, \xi) = \#\{j \,:\, \widetilde{\lambda}_j(x,\xi) > \lambda\}, \ \ \lambda > 0, \quad \widetilde{\nu}(\lambda; x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widetilde{n}(\lambda; x, \xi)\, d^n\xi.$$

Lemma 3.4 ensures that, if $\gamma(x) = \beta_k(x)$, we have $\nu(\lambda; x) = \widetilde{\nu}(\lambda; x)$. Consequently, $\widetilde{\nu}(\lambda; x) = \lambda^{-n/2}\widetilde{\nu}(x)$ by Lemma 3.2, and

$$\widetilde{\nu}(x) = \frac{1}{n(2\pi)^n} \int_{|\theta|=1} \sum_j \widetilde{\lambda}_j(x,\theta)^{n/2}\, dS(\theta). \tag{3.9}$$

3.3. The main result.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Suppose that $0 \le k \le n-1$, $\beta_k \in M_+^k(\Omega)$, and $\beta_{k+1} \in M_+^{k+1}(\Omega)$. Let $\mathcal{M}_k(\beta_k, \beta_{k+1})$ and $M_k(\beta_k, \beta_{k+1})$ be the operators defined, respectively, in* §1 *and* §2. *Then the counting function for their positive eigenvalues obeys the asymptotic law*

$$N(\lambda, \mathcal{M}_k(\beta_k, \beta_{k+1})) = N(\lambda, M_k(\beta_k, \beta_{k+1})) \sim \lambda^n \int_\Omega \widetilde{\nu}(x)\, d^n x, \quad \lambda \to +\infty, \tag{3.10}$$

where $\widetilde{\nu}$ was defined by (3.9).

Remark 3.6. Consider the case of constant coefficients: $\beta_k(x) = \star_k$, $\beta_{k+1}(x) = \star_{k+1}$. The right-hand side of equation (3.8) vanishes on the subspace $\{h_0 \in \mathcal{F}^k : \xi \wedge h_0 = 0\}$. The subspace $\{h_0 \in \mathcal{F}^k : \xi \wedge \star h_0 = 0\}$ is orthogonal to it and is an eigenspace, it corresponds to the eigenvalue $\lambda = |\xi|^2$ of multiplicity $\binom{n-1}{k}$. In this case

$$\widetilde{\nu}(x) = \frac{\binom{n-1}{k}}{n(2\pi)^n} \int_{|\theta|=1} dS(\theta)$$

and

$$N(\lambda, M_k(\star_k, \star_{k+1})) \sim \frac{\varkappa_n \binom{n-1}{k} \operatorname{meas} \Omega}{(2\pi)^n} \lambda^n, \quad \lambda \to +\infty,$$

where $\varkappa_n = \operatorname{meas}\{\xi \in R^n : |\xi| < 1\}$ is the measure of the unit ball in $\mathbb{R}^n$.

3.4. Examples.

The case of $n = 1$, $k = 0$. We may assume that $\Omega = (0, 1)$. Let $\beta_0, \beta_1, \beta_0^{-1}, \beta_1^{-1} \in L_\infty(0, 1)$. In this case, the Maxwell operator M_0 is defined in $L_2(\beta_0) \oplus L_2(\beta_1^{-1})$ on the domain $\operatorname{Dom} M_0 = \mathring{W}_{2,1}(0, 1) \oplus W_{2,1}(0, 1)$ by the formula

$$M_0 \begin{pmatrix} \omega \\ \theta \end{pmatrix} = \begin{pmatrix} -\beta_0^{-1}\theta' \\ \beta_1\omega' \end{pmatrix}.$$

The asymptotic formula turns into

$$N(\lambda, M_0) \sim \frac{\lambda}{\pi} \int_0^1 \sqrt{\frac{\beta_0(x)}{\beta_1(x)}}\, dx, \quad \lambda \to +\infty. \tag{3.11}$$

The case of $k = 0$ *and an arbitrary* n. In this case, $\beta_0(x)$ can be identified with a positive scalar function and $\beta_1(x)$ can be viewed as a positive definite $(n \times n)$-matrix-valued function; $\beta_0, \beta_0^{-1} \in L_\infty(\Omega)$, $\beta_1, \beta_1^{-1} \in L_\infty(\Omega, \operatorname{Mat}(n))$. The operator $R_0^* R_0$ coincides with the operator

$$A_D = -\beta_0(x)^{-1} \operatorname{div}(\beta_1(x)\nabla\cdot)$$

of the Dirichlet problem in the space $L_2(\Omega, \beta_0(x) d^n x)$. By (3.10), the asymptotics (2.2) turns into

$$N(\lambda, A_D) \sim \frac{\lambda^{n/2}\varkappa_n}{(2\pi)^n} \int_\Omega \frac{\beta_0(x)^{n/2}\, d^n x}{(\det \beta_1(x))^{1/2}}, \quad \lambda \to +\infty. \tag{3.12}$$

The case of $k = n - 1$. Let $\beta_n, \beta_n^{-1} \in L_\infty(\Omega)$ and $\beta_{n-1}, \beta_{n-1}^{-1} \in L_\infty(\Omega, \operatorname{Mat}(n))$. Then $R_{n-1}^* R_{n-1}$ coincides with the operator

$$A_N = (-1)^n \beta_n(x) \operatorname{div}(\beta_{n-1}(x)^{-1}\nabla\cdot)$$

of the Neumann problem in $L_2(\Omega, \beta_n(x)^{-1} d^n x)$. The asymptotics (3.10) turns into

$$N(\lambda, A_N) \sim \frac{\lambda^{n/2}\varkappa_n}{(2\pi)^n} \int_\Omega \frac{(\det \beta_{n-1}(x))^{1/2}\, d^n x}{\beta_n(x)^{n/2}}, \quad \lambda \to +\infty. \tag{3.13}$$

Formulas (3.11)–(3.13) are well known, see, e.g., [8] and the references therein.

The case of $n = 3$, $k = 1$. This is the "physical" Maxwell operator (0.1)–(0.3) with $\beta_1 = \varepsilon$, $\beta_2 = \mu^{-1}$. The asymptotics (3.10) turns into

$$N(\lambda, M) \sim \frac{\lambda^3}{24\pi^3} \int_\Omega \int_{|\xi|=1} \left(\Lambda_1(x, \xi)^{-3/2} + \Lambda_2(x, \xi)^{-3/2}\right) dS(\xi)\, d^3 x, \quad \lambda \to +\infty, \tag{3.14}$$

where the $\Lambda_{1,2}$ are the nonzero eigenvalues of the problem

$$\left[\xi, \mu(x)^{-1}[h, \xi]\right] = \Lambda\varepsilon(x)h, \quad \xi, h \in \mathbb{C}^3, \ \Lambda \in \mathbb{C}, \tag{3.15}$$

$[\,.\,,\,.\,]$ is the vector product in $\mathbb{R}^3$. Equation (3.15) differs from (3.8) by the substitution $\Lambda = \lambda^{-1}$. For $\xi \in \mathbb{R}^3 \setminus \{0\}$, it has precisely one eigenvector $h = \xi$ that corresponds to

the eigenvalue $\Lambda = 0$, therefore two eigenvalues $\Lambda_{1,2}$ occur in (3.14). Formula (3.14) is the main result of [5].

§4. Decomposition theorem for the space F^k_{el}

4.1. Poincaré lemma. We recall that a domain $U \subset \mathbb{R}^n$ is said to be

- starlike relative to a point $x_0 \in U$ if an arbitrary ray emanating from x_0 hits the boundary ∂U only once;
- starlike relative to a ball $B \subset U$ if U is starlike relative to an arbitrary point $x_0 \in B$.

The following facts are well known.

Lemma 4.1. a) *If a domain U is starlike relative to a ball, then it has Lipschitz boundary.*

b) *If Ω is a domain with Lipschitz boundary, then there exists a finite covering $\{U_i\}_{i=1}^N$ and a partition of unity $\{\zeta_i\}_{i=1}^N$ such that*

- $\Omega = \cup_{i=1}^N U_i$ *and each U_i starlike relative to a ball;*
- $\zeta_i \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\zeta_i \subset \bar{U}_i$, *and* $\sum_{i=1}^N \zeta_i(x) = 1$ *in* Ω.

We denote by $\mathring{W}_{p,1}(\Omega, \mathcal{F}^l)$ the space of functions whose values are l-forms all components of which belong to $\mathring{W}_{p,1}(\Omega)$. The following theorem was proved in [24, Proposition 11.2].

Theorem 4.2. *Let $U \subset \mathbb{R}^n$ be a bounded domain starlike relative to some ball, and let $1 < p < \infty$. Then there exist continuous linear operators*

$$S^{(k)} : L^k_p(U) \to \mathring{W}_{p,1}(U, \mathcal{F}^{k-1}), \quad k = 1, \dots, n,$$

such that, for every $\psi \in L^k_p(U)$ with $d\psi \in L^{k+1}_p(U)$, $j^*\psi = 0$, we have

$$\psi = S^{(k+1)}(d\psi) + d(S^{(k)}\psi), \quad k = 0, \dots, n-1, \tag{4.1}$$

where $S^{(0)} = 0$ by definition.

Remark 4.3. If the form ψ is closed, $d\psi = 0$, then (4.1) implies that it is exact. So, the claim is a generalization of the Poincaré lemma. It is important that the Sobolev norms of the forms in question can be controlled. The proof of these norm estimates is based on the Calderón–Zygmund theory [14].

Remark 4.4. It should be noted that for $k = n$ an analog of Theorem 4.2 holds true under the additional condition $\int_\Omega \psi = 0$ and without the first summand on the right in (4.1). This result was obtained in [12] in the following form: if $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying a cone condition and $f \in L_p(\Omega)$ is a scalar functions $1 < p < \infty$ and $\int_\Omega f(x)\, d^n x = 0$, then there exists a vector-valued function $v \in \mathring{W}_{p,1}(\Omega, \mathbb{R}^n)$ such that

$$\operatorname{div} v(x) = f(x) \tag{4.2}$$

and $\|v\|_{W_{p,1}} \le C\|f\|_{L_p}$. See also [16, 13] about the solvability of equation (4.2) in various function classes.

4.2. The space $\mathcal{W}^k_p$. In this subsection, we study the space of k-forms

$$\{\psi \in \mathcal{W}^k_p : j^*\psi = 0\}. \tag{4.3}$$

Clearly, it includes the spaces $\mathring{W}_{p,1}(\Omega, \mathcal{F}^k)$ and $\{d\varphi : \varphi \in \mathring{W}_{p,1}(\Omega, \mathcal{F}^{k-1})\}$. It turns out that (4.3) coincides with the linear sum of these spaces,

$$\{\psi \in \mathcal{W}^k_p : j^*\psi = 0\} = \mathring{W}_{p,1}(\Omega, \mathcal{F}^k) + d\mathring{W}_{p,1}(\Omega, \mathcal{F}^{k-1})$$

(surely, this is not a direct sum).

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Suppose that $1 \le k \le n-1$ and $1 < p < \infty$. Then there exist linear operators*

$$X : \{\psi \in \mathcal{W}_p^k : j^*\psi = 0\} \to \mathring{W}_{p,1}(\Omega, \mathcal{F}^k),$$
$$Z : \{\psi \in \mathcal{W}_p^k : j^*\psi = 0\} \to \mathring{W}_{p,1}(\Omega, \mathcal{F}^{k-1})$$

such that

$$X\psi + d(Z\psi) = \psi \quad \text{for all} \quad \psi \in \mathcal{W}_p^k \quad \text{with} \quad j^*\psi = 0,$$

and

$$\|X\psi\|_{W_{p,1}} + \|Z\psi\|_{W_{p,1}} \le C\left(\|d\psi\|_{L_p} + \|\psi\|_{L_p}\right).$$

Remark 4.6. a) For $k = 0$ a similar statement is trivial:

$$\{\psi \in \mathcal{W}_p^0 : j^*\psi = 0\} = \mathring{W}_{p,1}(\Omega)$$

(here $X = I$, $Z = 0$).

b) For $k = n$ this is also true and not interesting:

$$L_p^n(\Omega) = \mathring{W}_{p,1}(\Omega, \mathcal{F}^n) + d\mathring{W}_{p,1}(\Omega, \mathcal{F}^{n-1}).$$

For the role of the first summand $X\psi$, it suffices to take any n-form in $\mathring{W}_{p,1}$ with the same integral: $\int_\Omega X\psi = \int_\Omega \psi$; then it remains to apply Remark 4.4 to the difference $(\psi - X\psi)$.

b) For $n = 3$, $k = 1$, $p = 2$, Theorem 4.5 was obtained in [9].

In the proof of Theorem 4.5, we shall use the following easy statement (see also [15, Lemma 7.4]).

Lemma 4.7. *Let Ω be a domain with Lipschitz boundary, and let $\psi \in \mathcal{W}_p^k(\Omega)$, $j_\Omega^*\psi = 0$. Suppose that $U \subset \Omega$, $\zeta \in C^1(\bar{\Omega})$, and* $\operatorname{supp}\zeta \subset \bar{U}$. *Then*

$$(\zeta\psi)\big|_U \in \mathcal{W}_p^k(U), \quad j_U^*(\zeta\psi) = 0.$$

Proof of Theorem 4.5. Let $\{U_i\}_{i=1}^N$ and $\{\zeta_i\}_{i=1}^N$ be the covering of Ω and the corresponding partition of unity mentioned in Lemma 4.1, b). Suppose that $\psi \in \mathcal{W}_p^k(\Omega)$ and $j_\Omega^*\psi = 0$. By Lemma 4.7 ,

$$(\zeta_i\psi)\big|_{U_i} \in \mathcal{W}_p^k(U_i), \quad j_{U_i}^*(\zeta_i\psi) = 0, \quad i = 1, \dots, N.$$

We denote by $S_i^{(k)}$ the operators given by Theorem 4.2 when it is applied to the domains U_i. Put

$$X\psi = \sum_{i=1}^N S_i^{(k+1)} d(\zeta_i\psi), \quad Z\psi = \sum_{i=1}^N S_i^{(k)}(\zeta_i\psi).$$

Then

$$X\psi \in \mathring{W}_{p,1}(\Omega, \mathcal{F}^k), \qquad Z\psi \in \mathring{W}_{p,1}(\Omega, \mathcal{F}^{k-1}),$$
$$X\psi + d(Z\psi) = \sum_{i=1}^N \left(S_i^{(k+1)} d(\zeta_i\psi) + dS_i^{(k)}(\zeta_i\psi)\right) = \sum_{i=1}^N \zeta_i\psi = \psi.$$

Finally, by the continuity of the operators $S_i^{(k)}$, we have

$$\|Z\psi\|_{W_{p,1}} \le C\sum_{i=1}^N \|\zeta_i\psi\|_{L_p} \le C\|\psi\|_{L_p},$$
$$\|X\psi\|_{W_{p,1}} \le C\sum_{i=1}^N \|d(\zeta_i\psi)\|_{L_p} \le C\left(\|d\psi\|_{L_p} + \|\psi\|_{L_p}\right). \qquad \square$$

4.3. Description of the space $F^k_{\mathrm{el}}(\star_k)$. The following fact is a consequence of Theorem 4.5.

Theorem 4.8. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary and $0 \le k \le n-1$. Then*

$$F^k_{\mathrm{el}}(\star_k) = \mathring{W}_{2,1}(\Omega, \mathcal{F}^k) + E^k_{\mathrm{el}}(\star_k),$$

moreover, there exist continuous linear operators

$$X : F^k_{\mathrm{el}}(\star_k) \to \mathring{W}_{2,1}(\Omega, \mathcal{F}^k), \quad Y : F^k_{\mathrm{el}}(\star_k) \to E^k_{\mathrm{el}}(\star_k)$$

such that

$$X\psi + Y\psi = \psi, \quad \psi \in F^k_{\mathrm{el}}(\star_k). \tag{4.4}$$

Proof. For $k = 0$ the claim is trivial because $F^0_{\mathrm{el}}(\star_k) = \mathring{W}_{2,1}(\Omega)$ ($X = I$, $Y = 0$). For $k \ge 1$, we take the operator X from Theorem 4.5 with $p = 2$ for the role of "X" in the present statement and put $Y = d \circ Z$. Theorem 4.5 implies (4.4) directly and shows that X is continuous. It follows that

$$Y\psi = \psi - X\psi \in F^k_{\mathrm{el}}(\star_k) \quad \Rightarrow \quad Y\psi \in E^k_{\mathrm{el}}(\star_k)$$

and

$$\|Y\psi\|_{F^k_{\mathrm{el}}(\star_k)} \le \|X\psi\|_{F^k_{\mathrm{el}}(\star_k)} + \|\psi\|_{F^k_{\mathrm{el}}(\star_k)} \le C\|\psi\|_{F^k_{\mathrm{el}}(\star_k)}. \qquad \square$$

Remark 4.9. $\ker X \subset E^k_{\mathrm{el}}(\star_k) \subset G^k_0$.

Theorem 4.8 implies that the embedding $F^k_{\mathrm{el}}(\star_k) \subset L^k_2$ is compact. This property is well known (see [25]), we give a proof only for the reader's convenience. It is also well known that the embedding $F^k_{\mathrm{el}}(\beta_k) \subset L^k_2$ is compact for every $\beta_k \in M^k_+(\Omega)$. Note that, subsequently, we shall prove in fact a stronger property of the embedding $F^k_{\mathrm{el}}(\star_k) \subset L^k_2$, specifically, the power-type estimate $s_m \le Cm^{-1/n}$ for its singular numbers.

Corollary 4.10. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary and $0 \le k \le n-1$. Then the embedding $F^k_{\mathrm{el}}(\star_k) \subset L^k_2$ is compact. For $k = 0$, the Lipschitz property of the boundary is redundant.*

Proof. Let a sequence $\{\omega_m\}_{m=1}^\infty \subset F^k_{\mathrm{el}}(\star_k)$ converge weakly to zero in $F^k_{\mathrm{el}}(\star_k)$. It suffices to prove that $\omega_m \to 0$ strongly in L^k_2. We apply Theorem 4.8 to every term of the sequence: $\omega_m = X\omega_m + Y\omega_m$. Then the $X\omega_m$ converge weakly to zero in $\mathring{W}_{2,1}(\Omega, \mathcal{F}^k)$ and, therefore, $X\omega_m \to 0$ in L^k_2. Since $Y = d \circ Z$, where Z is the operator from Theorem 4.5, for the second summand we have

$$\|Y\omega_m\|^2_{L^k_2(\star_k)} = (-1)^{k(n-k)+k+1} \int_\Omega d(\star Y\omega_m) \wedge \overline{Z\omega_m}.$$

By Theorem 4.5, $Z\omega_m \to 0$ strongly in L^{k-1}_2. At the same time, the sequence of the factors $\{d(\star Y\omega_m)\}$ is bounded in L^{n-k+1}_2 by the continuity of the operator Y. $\square$

Corollary 4.11. *The space $K^k(\star_k)$ is finite-dimensional.*

Proof. On $K^k(\star_k)$, the norm $\|\,.\,\|_{F^k(\star_k)}$ coincides with $\|\,.\,\|_{L^k_2(\star_k)}$. Since the embedding $F^k_{\mathrm{el}}(\star_k) \subset L^k_2(\star_k)$ is compact, it follows that $\dim K^k(\star_k) < \infty$. $\square$

Remark 4.12. If the domain Ω is starlike relative to a ball, then $K^k(\star_k) = \{0\}$, $k = 0, \dots, n-1$. Indeed, this is obvious for $k = 0$. If $k \ge 1$, we use Theorem 4.2. Let $\varphi \in K^k(\star_k)$. Then $\varphi = d(S^{(k)}\varphi)$, whence

$$\|\varphi\|^2_{L^k_2(\star_k)} = \int_\Omega d(S^{(k)}\varphi) \wedge \star_k\overline{\varphi} = (-1)^k \int_\Omega S^{(k)}\varphi \wedge d(\star_k\overline{\varphi}) = 0.$$

§5. Compact operators

In this section we list the properties to be used below of singular numbers of compact operators (see, e.g., [8] and [6]).

5.1. The functionals d_q. Let S be a positive compact operator in a Hilbert space $\mathcal{H}$. We denote by $\lambda_k(S)$ its eigenvalues counted with multiplicities. For a compact operator T acting from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{N}$, let $s_k(T) = \lambda_k(T^*T)^{1/2}$ be its singular numbers, and let $n(s,T)$ denote the distribution function for the singular numbers,

$$n(s,T) = \#\{k : s_k(T) > s\}, \quad s > 0.$$

An equivalent definition is based on the minimax properties of the singular numbers:

$$n(s,T) = \max \dim L, \quad L \text{ runs through the subspaces of } \mathcal{H} \text{ with } \|Tx\|_{\mathcal{N}} > s\|x\|_{\mathcal{H}} \text{ for all } x \in L \setminus \{0\}.$$

If $\mathcal{H} = \mathcal{N}$ and T is a positive operator, we have $s_k(T) = \lambda_k(T)$ and

$$n(s,T) = \max \dim L, \ L \text{ runs through the subspaces of } \mathcal{H} \text{ with } (Tx,x) > s\|x\|^2 \text{ for all } x \in L \setminus \{0\}. \tag{5.1}$$

If $T = T^*$ and $2T_\pm = |T| \pm T$, then

$$n(s,T) = n(s,T_+) + n(s,T_-). \tag{5.2}$$

For $q \geq 1$, for compact operators $T : \mathcal{H} \to \mathcal{N}$ we introduce the asymptotic functionals

$$\Delta_q(T) = \limsup_{s\to 0} s^q n(s,T), \quad \delta_q(T) = \liminf_{s\to 0} s^q n(s,T), \tag{5.3}$$

which are permitted to take zero or infinite values. In what follows, we denote by $d_q(T)$ any of the functionals (5.3). We list some of their properties. We have

$$d_q(T^*) = d_q(T) \tag{5.4}$$

and

$$\left|d_q(T_1)^{1/(q+1)} - d_q(T_2)^{1/(q+1)}\right| \leq \Delta_q(T_1 - T_2)^{1/(q+1)}. \tag{5.5}$$

Lemma 5.1. *Let $\mathcal{H}$, $\mathcal{N}_1$, and $\mathcal{N}_2$ be Hilbert spaces, T_1 a bounded operator from $\mathcal{N}_1$ to $\mathcal{N}_2$, and T_2 a compact operator from $\mathcal{H}$ to $\mathcal{N}_1$. Then*

$$\Delta_q(T_1T_2) \leq \|T_1\|^q \Delta_q(T_2). \tag{5.6}$$

If T_1 is compact and $\Delta_q(T_2) < \infty$, then $\Delta_q(T_1T_2) = 0$.

Along with (5.6), we have the inequality

$$\Delta_q(T_1T_2) \leq 2\Delta_{2q}(T_1)^{1/2}\Delta_{2q}(T_2)^{1/2}. \tag{5.7}$$

Lemma 5.2. *Let T, T_1, T_2 be selfadjoint operators in a Hilbert space $\mathcal{H}$. Suppose that T_1, T_2 are positive and*

$$|(Tx,x)| \leq (T_1x,x)^{1/2}(T_2x,x)^{1/2}, \quad x \in \mathcal{H}.$$

Then

$$\Delta_q(T) \leq 4\Delta_q(T_1)^{1/2}\Delta_q(T_2)^{1/2}. \tag{5.8}$$

The next statement was proved in [7] (see also [4]).

Lemma 5.3. *Let T_1, T_2 be nonnegative compact operators. Then*

$$\Delta_{2q}\left(T_1^{1/2} - T_2^{1/2}\right) \leq C\Delta_q(T_1 - T_2).$$

5.2. Quadratic forms. Suppose we are given two nonnegative quadratic forms a and b on a space $\mathcal{H}$, moreover, b determines a scalar product on $\mathcal{H}$. Consider the ratio

$$\frac{a[u]}{b[u]}. \tag{5.9}$$

We denote by $n(\lambda, (5.9))$ the counting function for the maxima of this ratio. In other words, $n(\lambda, (5.9)) = n(\lambda, T)$, where T is the compact operator determined by the form a in the Hilbert space with the scalar product b. The expressions $\delta_q((5.9))$, $\Delta_q((5.9))$, and $d_q((5.9))$ have similar meaning.

Lemma 5.4. *Let $S = S^*$ be a compact operator on a Hilbert space $\mathcal{H}$. We introduce the quadratic form $b[u, v] = (u, v) + (Su, v)$ and suppose that $b[u] > 0$ for $u \neq 0$. Then $b[u, v]$ determines a scalar product on $\mathcal{H}$, moreover, the new norm is equivalent to the initial one. Let a be the quadratic form of some compact operator on $\mathcal{H}$. Then for the ratios*

$$a[u]/\|u\|^2 \tag{5.10}$$

and

$$a[u]/b[u] \tag{5.11}$$

we have $d_q((5.10)) = d_q((5.11))$.

Below, all these statements will be employed for $q = n/2$ and $q = n$.

§6. Ratios of quadratic forms

6.1. The projection Π. For $k \geq 1$, we introduce the orthogonal projection from $L_2^k(\beta_k)$ onto G_0^k:

$$\begin{gathered} \Pi(\beta_k)\omega = d\varphi, \quad \text{where} \quad \varphi \in \mathcal{W}_2^{k-1}, \quad j^*\varphi = 0, \quad \text{and} \\ \int_\Omega d\psi \wedge \beta_k \omega = \int_\Omega d\psi \wedge \beta_k d\varphi, \quad \text{for all} \quad \psi \in \mathcal{W}_2^{k-1} \text{ with } j^*\psi = 0; \end{gathered} \tag{6.1}$$

for $k = 0$ we define $\Pi(\beta_0) = 0$. Note that the k-form $d\varphi$ is uniquely determined by (6.1) though φ itself is defined up to a summand in $\ker d_{k-1} = G_0^{k-1} \oplus_\star K^{k-1}(\star)$.

Lemma 6.1. *The operator $I - \Pi(\star_k)$ takes bijectively $\Phi_{\mathrm{el}}^k(\beta_k)$ onto $\Phi_{\mathrm{el}}^k(\star_k)$ and $K^k(\beta_k)$ onto $K^k(\star_k)$. The inverse mapping is provided by the operator $I - \Pi(\beta_k)$.*

Proof. Suppose $\omega \in \Phi_{\mathrm{el}}^k(\beta_k)$, $d\varphi = \Pi(\star_k)\omega \in G_0^k$. Then $d(\star_k(\omega - d\varphi)) = 0$ by the definition (6.1), and $d(\omega - d\varphi) = d\omega \in L_2^k$. Consequently, the operator $I - \Pi(\star_k)$ takes $\Phi_{\mathrm{el}}^k(\beta_k)$ to $\Phi_{\mathrm{el}}^k(\star_k)$. Similarly, the operator $I - \Pi(\beta_k)$ takes $\Phi_{\mathrm{el}}^k(\star_k)$ to $\Phi_{\mathrm{el}}^k(\beta_k)$. Now, let $\Pi(\star_k)\omega = d\varphi$ and $\Pi(\beta_k)(\omega - d\varphi) = d\psi \in G_0^k$. Then

$$(I - \Pi(\beta_k))\,(I - \Pi(\star_k))\,\omega = \omega - d\varphi - d\psi \in \Phi_{\mathrm{el}}^k(\beta_k),$$

whence

$$d(\varphi + \psi) \in G_0^k \cap \Phi_{\mathrm{el}}^k(\beta_k) = \{0\}.$$

Consequently, $d(\varphi + \psi) = 0$ and $(I - \Pi(\beta_k))(I - \Pi(\star_k))\omega = \omega$.

Finally, if $\omega \in K^k(\beta_k)$, then $d(\omega - d\varphi) = 0$ and $\omega - d\varphi \in K^k(\star_k)$. □

Corollary 6.2. *The space $K^k(\beta_k)$ is finite-dimensional.*

Proof. We have $\dim K^k(\beta_k) = \dim K^k(\star_k) < \infty$ by Corollary 4.11. □

6.2. The form K_{β_k}. By Remark 2.2, to describe the spectrum of the Maxwell operator M_k, it suffices to consider the spectrum of $R_k^* R_k$. It is more convenient to treat the inverse of the last operator on the orthogonal complement to the kernel of R_k. It corresponds to the following ratio of quadratic forms:

$$\frac{\int_\Omega \omega \wedge \beta_k \overline{\omega}}{\int_\Omega d\omega \wedge \beta_{k+1}(\overline{d\omega})}, \quad \omega \in \Phi^k_{\mathrm{el}}(\beta_k) \ominus_{\beta_k} K^k(\beta_k). \tag{6.2}$$

We have

$$n(\lambda, (6.2)) = N(\lambda^{-1}, R_k^* R_k). \tag{6.3}$$

On $L_2^k(\beta_k)$ we introduce the bounded symmetric quadratic form

$$\begin{aligned} K_{\beta_k}[\omega_1, \omega_2] &= \int_\Omega (\omega_1 - \Pi(\beta_k)\omega_1) \wedge \beta_k \big(\overline{\omega_2 - \Pi(\beta_k)\omega_2}\big) \\ &= \int_\Omega \omega_1 \wedge \beta_k \overline{\omega_2} - \int_\Omega \Pi(\beta_k)\omega_1 \wedge \beta_k \overline{\Pi(\beta_k)\omega_2}. \end{aligned}$$

The second expression for $K_{\beta_k}[\omega_1, \omega_2]$ is deduced from the first with the help of (6.1). Clearly, $K_{\beta_k}[\omega] \geq 0$ and

$$K_{\beta_k}[\theta, \omega] = 0 \ \text{ for } \ \theta \in G_0^k, \omega \in L_2^k(\beta_k). \tag{6.4}$$

If $\omega_1 - \omega_2 \in G_0^k$, then

$$K_{\beta_k}[\omega_1] = K_{\beta_k}[\omega_1, \omega_2] = K_{\beta_k}[\omega_2]. \tag{6.5}$$

Now, we introduce the following ratio of quadratic forms:

$$\frac{K_{\beta_k}[\eta]}{\int_\Omega d\eta \wedge \beta_{k+1} \overline{d\eta}}, \quad \eta \in \Phi^k_{\mathrm{el}}(\star_k) \ : \int_\Omega \eta \wedge \beta_k \psi = 0, \quad \psi \in K^k(\beta_k). \tag{6.6}$$

Lemma 6.3. *We have* $n(\lambda, (6.2)) = n(\lambda, (6.6))$.

Proof. Let $\eta \in \Phi^k_{\mathrm{el}}(\star_k)$, and let $\omega = (I - \Pi(\beta_k))\eta = \eta - d\varphi$. Then $\omega \in \Phi^k_{\mathrm{el}}(\beta_k)$ by Lemma 6.1, and $K_{\beta_k}[\eta] = \int_\Omega \omega \wedge \beta_k \overline{\omega}$ by the definition of the form K_{β_k}. It is also clear that $d\omega = d\eta$, consequently, the denominators in (6.2) and (6.6) coincide. Finally, we have

$$\int_\Omega d\varphi \wedge \beta_k \psi = (-1)^k \int_\Omega \varphi \wedge \ d(\beta_k \psi) = 0, \quad \psi \in K^k(\beta_k),$$

so that the orthogonality conditions in (6.2) transform to those in (6.6). □

It is convenient to extend the domain of (6.6). Consider the ratio

$$\frac{K_{\beta_k}[\eta]}{\int_\Omega d\eta \wedge \beta_{k+1} \overline{d\eta} + d(\star\eta) \wedge \star\overline{d(\star\eta)}}, \ \eta \in F^k_{\mathrm{el}}(\star_k) \ \text{ and } \int_\Omega \eta \wedge \beta_k \psi = 0 \ \text{ for all } \ \psi \in K^k(\beta_k). \tag{6.7}$$

Lemma 6.4. *We have* $n(\lambda, (6.6)) = n(\lambda, (6.7))$.

Proof. We use the decomposition (2.1). If $\eta \in E^k_{\mathrm{el}}(\star_k)$, the condition

$$\int_\Omega \eta \wedge \beta_k \psi = 0, \quad \psi \in K^k(\beta_k),$$

is fulfilled automatically; then $K_{\beta_k}[\eta] = 0$ by (6.4).

On the space $\Phi^k_{\mathrm{el}}(\star_k)$, the ratio (6.7) coincides with (6.6) because then $d(\star_k \eta) = 0$. □

We observe that we can plug any factor $\alpha > 0$ in the denominator of (6.7) before the term $d(\star\eta) \wedge \star\overline{d(\star\eta)}$. This will not influence the spectrum.

6.3. Proof of Theorem 3.5. In Chapter II, we shall prove the following result.

Theorem 6.5. *Let $\alpha > 0$. For the ratio of quadratic forms*

$$\frac{K_{\beta_k}[\theta]}{\int_\Omega d\theta \wedge \beta_{k+1}\overline{d\theta} + \alpha\left(d(\star\theta)\wedge\star d(\overline{\star\theta}) + \theta\wedge\beta_k\bar\theta\right)}, \quad \theta \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k), \tag{6.8}$$

we have the following asymptotics:

$$n(\lambda, (6.8), \alpha) \sim \lambda^{-n/2}\int_\Omega \widetilde{\nu}(x)\, d^n x = \int_\Omega \widetilde{\nu}(\lambda^{-1}, x)\, d^n x, \tag{6.9}$$

where the functions $\widetilde{\nu}(x)$ and $\widetilde{\nu}(\lambda^{-1}, x)$ are as defined in Subsection 3.2 and do not depend on α.

To deduce Theorem 3.5 from Theorem 6.5, we consider yet another ratio of quadratic forms:

$$\frac{K_{\beta_k}[\theta]}{\int_\Omega d\theta \wedge \beta_{k+1}\overline{d\theta} + d(\star\theta)\wedge\star d(\overline{\star\theta}) + \theta\wedge\beta_k\bar\theta}, \quad \theta \in F^k_{\mathrm{el}}(\star_k). \tag{6.10}$$

Lemma 6.6. *The following limit exists:*

$$\lim_{\lambda\to 0}\lambda^{n/2} n(\lambda, (6.10)) = \int_\Omega \widetilde{\nu}(x)\, d^n x.$$

Proof. From the minimax principle and the inclusion $\mathring{W}_{2,1}(\Omega, \mathcal{F}^k) \subset F^k_{\mathrm{el}}(\star_k)$ it follows that $n(\lambda, (6.10)) \geq n(\lambda, (6.8), 1)$, whence

$$\liminf_{\lambda\to 0}\lambda^{n/2} n(\lambda, (6.10)) \geq \int_\Omega \widetilde{\nu}(x)\, d^n x.$$

Conversely, let L be a subspace of $F^k_{\mathrm{el}}(\star_k)$ with $\dim L = n(\lambda, (6.10))$ and

$$K_{\beta_k}[\theta] > \lambda\int_\Omega d\theta \wedge \beta_{k+1}\overline{d\theta} + d(\star\theta)\wedge\star d(\overline{\star\theta}) + \theta\wedge\beta_k\bar\theta, \quad \theta \in L\setminus\{0\}.$$

We put $L_0 = XL$, where X is the operator from Theorem 4.8. Then $L_0 \subset \mathring{W}_{2,1}(\Omega, \mathcal{F}^k)$ and $\dim L_0 = \dim L$ by Remark 4.9 and (6.4). Let $\eta = X\theta \neq 0$. We have $d\eta = d\theta$ and $K_{\beta_k}[\eta] = K_{\beta_k}[\theta]$ by (6.5). Consequently,

$$\begin{aligned}
&\int_\Omega d\eta \wedge \beta_{k+1}\,\overline{d\eta} + \alpha\left(d(\star\eta)\wedge\star d(\overline{\star\eta}) + \eta\wedge\beta_k\bar\eta\right)\\
&\quad\leq \int_\Omega d\theta \wedge \beta_{k+1}\overline{d\theta} + C\alpha\|\theta\|^2_{F^k(\star_k)}\\
&\quad\leq (1+\widetilde{C}\alpha)\int_\Omega d\theta \wedge \beta_{k+1}\overline{d\theta} + d(\star\theta)\wedge\star d(\overline{\star\theta}) + \theta\wedge\beta_k\bar\theta\\
&\quad\leq \frac{1+\widetilde{C}\alpha}{\lambda}K_{\beta_k}[\theta] = \frac{1+\widetilde{C}\alpha}{\lambda}K_{\beta_k}[\eta].
\end{aligned}$$

Therefore, $n(\lambda, (6.10)) \leq n(\frac{\lambda}{1+\widetilde{C}\alpha}, (6.8), \alpha)$, whence

$$\limsup_{\lambda\to+0}\lambda^{n/2} n(\lambda, (6.10)) \leq (1+\widetilde{C}\alpha)^{n/2}\limsup_{\lambda\to+0}\lambda^{n/2} n(\lambda, (6.8), \alpha) \leq (1+\widetilde{C}\alpha)^{n/2}\int_\Omega \widetilde{\nu}(x)\, d^n x$$

by Theorem 6.5. Passing to the limit as $\alpha \to 0$, we obtain

$$\limsup \lambda^{n/2} n(\lambda, (6.10)) \leq \int_\Omega \widetilde{\nu}(x)\, d^n x. \qquad \square$$

Proof of Theorem 3.5. We consider the ratio (6.10) on k-forms orthogonal to $K^k(\beta_k)$:

$$\begin{gathered}\frac{K_{\beta_k}[\eta]}{\int_\Omega d\eta\wedge\beta_{k+1}\overline{d\eta}+d(\star\eta)\wedge\star d(\overline{\star\eta})+\eta\wedge\beta_k\overline{\eta}},\\ \eta\in F^k_{\mathrm{el}}(\star_k)\ :\ \int_\Omega \eta\wedge\beta_k\psi=0\ \text{ for all }\ \psi\in K^k(\beta_k).\end{gathered}\tag{6.11}$$

We have

$$n(\lambda,(6.10))-\dim K^k(\beta_k)\le n(\lambda,(6.11))\le n(\lambda,(6.10)).$$

The dimension of $K^k(\beta_k)$ is finite by Corollary 6.2. Consequently,

$$n(\lambda,(6.11))\sim n(\lambda,(6.10)),\quad \lambda\to+\infty.$$

Next, the form $\int_\Omega \eta\wedge\beta_k\overline{\eta}$ is compact with respect to the denominator of (6.11) by Corollary 4.10. Applying Lemma 5.4 and 6.6, we obtain

$$\lim_{\lambda\to0}\lambda^{n/2}n(\lambda,(6.7))=\lim_{\lambda\to0}\lambda^{n/2}n(\lambda,(6.11))=\int_\Omega\widetilde{\nu}(x)\,d^nx.$$

Lemmas 6.3 and 6.4 imply a similar identity for the ratio (6.2):

$$\lim_{\lambda\to0}\lambda^{n/2}n(\lambda,(6.2))=\int_\Omega\widetilde{\nu}(x)\,d^nx.$$

Finally, formulas (6.3) and (2.2) show that

$$\lim_{\lambda\to\infty}\lambda^{-n}N(\lambda,M_k(\beta_k,\beta_{k+1}))=\int_\Omega\widetilde{\nu}(x)\,d^nx.\qquad\square$$

It remains to prove Theorem 6.5.

Chapter II

§7. Generalization of the claim

7.1. Statement of the result. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, and let $0\le k\le n-1$. For $k>1$ we assume that the boundary $\partial\Omega$ is Lipschitz. Let $\gamma\in L_p(\Omega,\mathrm{Lin}_k)$, where p is a fixed number, $p>n$, and let $\beta_k\in M^k_+(\Omega)$. For $k\ge1$, we introduce the operator

$$\Pi(\beta_k,\gamma)\ :\ \mathring{W}_{2,1}(\Omega,\mathcal{F}^k)\to G^k_0,\quad \Pi(\beta_k,\gamma)\omega=d\varphi,$$

where the $(k-1)$-form φ is determined by the system

$$\begin{cases}\int_\Omega d\psi\wedge\beta_k\,d\varphi=\int_\Omega d\psi\wedge\gamma\omega\ \text{ for all }\ \psi\in\mathcal{W}^{k-1}_2\ \text{ with }\ j^*\psi=0,\\ j^*\varphi=0,\quad d(\star\varphi)=0,\quad \varphi\perp_{\star_{k-1}}K^{k-1}(\star_{k-1}).\end{cases}\tag{7.1}$$

Corollary 4.10 shows that

$$\|\varphi\|_{F^{k-1}(\star_{k-1})}\le C\|d\varphi\|_{L_2}\le C\|\beta_k^{-1}\|^{-1}_{L_\infty}\|\gamma\omega\|_{L_2}.\tag{7.2}$$

Also, clearly,

$$\|\gamma\omega-\beta_k d\varphi\|_{L_2}\le C(\beta_k)\|\gamma\omega\|_{L_2}.\tag{7.3}$$

We put $\Pi(\beta_0,\gamma)=0$ for $k=0$.

Now, we define the quadratic form

$$\begin{aligned}L_{\beta_k,\gamma}[\omega]&=\int_\Omega\gamma\omega\wedge\beta_k^{-1}(\overline{\gamma\omega})-d\varphi\wedge\beta_k\,\overline{d\varphi}\\ &=\int_\Omega(\gamma\omega-\beta_k\,d\varphi)\wedge\beta_k^{-1}\left(\overline{\gamma\omega}-\beta_k\overline{d\varphi}\right),\quad \omega\in\mathring{W}_{2,1}(\Omega,\mathcal{F}^k),\end{aligned}\tag{7.4}$$

for $k \geq 1$, and $L_{\beta_0,\gamma}[\omega] = \int_\Omega \gamma\omega \wedge \beta_0^{-1}(\overline{\gamma\omega})$ for $k = 0$. We recall that, by definition, the inverse mapping β_k^{-1} is positive definite (see Subsection 1.2). Note that, for $\gamma = \beta_k$, the operator Π and the form L coincide with the corresponding projection and form introduced in §6:

$$\Pi(\beta_k, \beta_k) = \Pi(\beta_k), \quad L_{\beta_k,\beta_k} = K_{\beta_k}.$$

Next, let $\alpha > 0$, $\beta_{k+1} \in M_+^{k+1}(\Omega)$. We introduce the quadratic form

$$A_{\beta_{k+1}}[\omega] = \int_\Omega d\omega \wedge \beta_{k+1}\overline{d\omega} + \alpha\, d(\star\omega) \wedge \star d(\overline{\star\omega}), \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k). \tag{7.5}$$

In this chapter, we study the spectrum of the ratio

$$\frac{L_{\beta_k,\gamma}[\omega]}{A_{\beta_{k+1}}[\omega]}, \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k). \tag{7.6}$$

Our purpose is the proof of the following statement.

Theorem 7.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Suppose that*

$$\alpha > 0, \quad 0 \leq k \leq n-1, \quad \beta_k \in M_+^k(\Omega), \quad \beta_{k+1} \in M_+^{k+1}(\Omega), \quad \gamma \in L_p(\Omega, \mathrm{Lin}_k), \quad p > n;$$

the forms $L_{\beta_k,\gamma}$ and $A_{\beta_{k+1}}$ have been defined above. Then the counting function for the ratio (7.6) *obeys the asymptotics*

$$n(\lambda, (7.6)) \sim \lambda^{-n/2} \int_\Omega \nu(x)\, d^n x = (2\pi)^{-n} \int_\Omega \int_{\mathbb{R}^n} n(\lambda, (3.1), x, \xi)\, d^n x\, d^n \xi, \ \ \lambda \to +0. \tag{7.7}$$

Here $n(\lambda, (3.1), x, \xi)$ is the number of eigenvalues of problem (3.1) *that are greater than λ; the function ν was defined in Lemma* 3.2. *For $k = 0$, 1 the Lipschitz property of the boundary $\partial\Omega$ is redundant.*

Theorem 6.5 is a consequence of Theorem 7.1. Indeed, put $\gamma(x) = \beta_k(x)$. Then $L_{\beta_k,\gamma} = K_{\beta_k}$ and the numerators of the ratios (7.6) and (6.8) coincide. Since the denominators differ by a relatively compact summand, we have $d_{n/2}((7.6)) = d_{n/2}((6.8))$ by Lemma 5.4. Next,

$$\delta_{n/2}((7.6)) = \Delta_{n/2}((7.6)) = \lim_{\lambda \to 0} \lambda^{n/2} n(\lambda, (7.6)) = \int_\Omega \nu(x)\, d^n x,$$

consequently, the limit

$$\lim_{\lambda \to 0} \lambda^{n/2} n(\lambda, (6.8)) = \int_\Omega \nu(x)\, d^n x.$$

exists. Finally, $\nu(x) = \widetilde{\nu}(x)$ for $\gamma(x) = \beta_k(x)$ by Lemma 3.4.

Remark 7.2. The proof of the more general Theorem 7.1 turns out to be simpler than that of Theorem 6.5 because the parameters β_k and γ are independent.

Remark 7.3. It is clear from (3.5) that the integral on the right in (7.7) is continuous under variation of the coefficients β_k, β_{k+1}, γ in the L_{2n}-norm.

Remark 7.4. For $n = 3$, $k = 1$ and an *arbitrary* bounded domain, Theorem 7.1 was proved in [3]. The same result can be found in the Thesis [1] in a more general form. However, [1] is hardly accessible, therefore a direct proof was given in [3]. In [1], problems with constraints were treated: the quadratic form in the numerator involves a summand determined by a certain differential equation. In our case, the constraint is the system (7.1) for finding the $(k-1)$-form φ by a given k-form ω. Spectral asymptotics were found in [1] for such ratios of quadratic forms. However, in [1] it was assumed that the constraint is uniformly elliptic, i.e., the symbol of the quadratic form on the left in the first equation in (7.1) is positive definite. This is so for $k = 1$ because φ is a scalar

function in $\mathring{W}_{2,1}$ and $\|\varphi\|_{W_{2,1}} \le C\|d\varphi\|_{L_2}$. But for $k > 1$ this is no longer true. Thus, Theorem 7.1 is not covered by the results of [1]. Therefore, we shall give a complete proof of Theorem 7.1, following the same pattern of [1] and [3].

7.2. Periodic problem. Let $\mathbb{T}$ be the torus (the cube in $\mathbb{R}^n$ with opposite faces identified), and let

$$\beta_k \in M_+^k(\mathbb{T}), \quad \beta_{k+1} \in M_+^{k+1}(\mathbb{T}), \quad \gamma \in L_p(\mathbb{T}, \mathrm{Lin}_k), \quad p > n, \ \ \alpha > 0.$$

We put

$$\widetilde{A}_{\beta_{k+1}}[\widetilde{\omega}] = \int_{\mathbb{T}} d\widetilde{\omega} \wedge \beta_{k+1}\overline{d\widetilde{\omega}} + \alpha\, d(\star\widetilde{\omega}) \wedge \star d(\overline{\star\widetilde{\omega}}), \quad \widetilde{\omega} \in W_{2,1}(\mathbb{T}, \mathcal{F}^k). \tag{7.8}$$

Next, we introduce the operator $\widetilde{\Pi}(\beta_k, \gamma)\widetilde{\omega} = d\widetilde{\varphi}$, where the form $\widetilde{\varphi}$ is determined by the system

$$\begin{cases} \int_{\mathbb{T}} d\widetilde{\psi} \wedge \beta_k\, d\widetilde{\varphi} = \int_{\mathbb{T}} d\widetilde{\psi} \wedge \gamma\widetilde{\omega}, \quad \widetilde{\psi} \in \mathcal{W}_2^{k-1}(\mathbb{T}), \\ d(\star\widetilde{\varphi}) = 0, \quad \widetilde{\varphi} \perp_{\star_{k-1}} K^{k-1}(\star_{k-1}), \end{cases} \tag{7.9}$$

for $k \ge 1$, and $\widetilde{\Pi}(\beta_0, \gamma) = 0$ for $k = 0$. Also, we introduce the quadratic form

$$\begin{aligned} \widetilde{L}_{\beta_k,\gamma}[\widetilde{\omega}] &= \int_{\mathbb{T}} \gamma\widetilde{\omega} \wedge \beta_k^{-1}(\overline{\gamma\widetilde{\omega}}) - d\widetilde{\varphi} \wedge \beta_k\, \overline{d\widetilde{\varphi}} \\ &= \int_{\mathbb{T}} (\gamma\widetilde{\omega} - \beta_k\, d\widetilde{\varphi}) \wedge \beta_k^{-1}\big(\overline{\gamma\widetilde{\omega}}\big), \quad \widetilde{\omega} \in W_{2,1}(\mathbb{T}, \mathcal{F}^k), \end{aligned} \tag{7.10}$$

for $k \ge 1$ and $\widetilde{L}_{\beta_0,\gamma}[\widetilde{\omega}] = \int_\Omega \gamma\widetilde{\omega} \wedge \beta_0^{-1}(\overline{\gamma\widetilde{\omega}})$ for $k = 0$, and the ratio

$$\frac{\widetilde{L}_{\beta_k,\gamma}[\widetilde{\omega}]}{\widetilde{A}_{\beta_{k+1}}[\widetilde{\omega}]}, \quad \widetilde{\omega} \in \widetilde{W}, \tag{7.11}$$

where we have denoted

$$\widetilde{W} = \{\widetilde{\omega} \in W_{2,1}(\mathbb{T}, \mathcal{F}^k) \,:\, \widetilde{\omega} \perp_{\star_k} K^k(\mathbb{T}, \star_k)\}.$$

Note that, in the periodic case, the space $K^k(\mathbb{T}, \star_k)$ consists of constants only, so that

$$\dim K^k(\mathbb{T}, \star_k) = \dim \mathcal{F}^k = \binom{n}{k}. \tag{7.12}$$

In the next section we shall show that, to prove Theorem 7.1, it suffices to establish the spectral asymptotics for the ratio (7.11).

§8. Reduction to the periodic case

8.1. Continuity in γ. In this subsection, we fix the coefficients β_k and β_{k+1} and prove the continuity of the functionals $d_{n/2}((7.6))$ and $d_{n/2}((7.11))$ with respect to $\gamma \in L_p$, $p > n$. The continuity in β_k and β_{k+1} will be established in Subsection 10.2.

It is well known that the ratio

$$\frac{\|\gamma\omega\|^2}{\|\omega\|^2_{W_{2,1}}}, \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k) \text{ or } \omega \in W_{2,1}(\mathbb{T}, \mathcal{F}^k), \tag{8.1}$$

satisfies the estimate

$$\Delta_{n/2}((8.1)) \le C\|\gamma\|^n_{L_p}, \tag{8.2}$$

see, e.g., [8, Chapter 4]. It should be noted that for $n > 2$ inequality (8.2) holds true also for $p = n$ (the Rosenblum–Lieb–Cviekel estimate), but an arbitrary $p < \infty$ fits our purposes.

Suppose that $\gamma_1, \gamma_2 \in L_p(\Omega, \mathrm{Lin}_k)$, $\omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k)$, and φ_1, φ_2 are the corresponding solutions of system (7.1). Consider the difference

$$\begin{aligned} L[\omega] &= L_{\beta_k,\gamma_1}[\omega] - L_{\beta_k,\gamma_2}[\omega] \\ &= \int_\Omega (\gamma_1-\gamma_2)\omega \wedge \beta_k^{-1}(\overline{\gamma_1\omega}) + \gamma_2\omega \wedge \beta_k^{-1}\big(\overline{(\gamma_1-\gamma_2)\omega}\big) \\ &\qquad - \int_\Omega (d\varphi_1 - d\varphi_2)\wedge \beta_k \overline{d\varphi_1} + d\varphi_2 \wedge \beta_k \left(\overline{d\varphi_1 - d\varphi_2}\right). \end{aligned}$$

By (7.2), we have

$$|L[\omega]| \le C\left(\|\beta_k\|_{L_\infty}, \|\beta_k^{-1}\|_{L_\infty}\right)\left(\|\gamma_1\omega\|_{L_2} + \|\gamma_2\omega\|_{L_2}\right)\|(\gamma_1-\gamma_2)\omega\|_{L_2}.$$

Let T_1 and T_2 be the operators generated by the forms L_{β_k,γ_1}, and L_{β_k,γ_2} in the space $\mathring{W}_{2,1}(\Omega, \mathcal{F}^k)$ with the norm $A_{\beta_{k+1}}$. Then Lemma 5.2 and (8.2) imply

$$\Delta_{n/2}(T_1 - T_2) \le C\left(\|\gamma_1\|_{L_p} + \|\gamma_2\|_{L_p}\right)^{n/2}\|\gamma_1-\gamma_2\|_{L_p}^{n/2},$$
$$C = C\left(\alpha, \|\beta_k\|_{L_\infty}, \|\beta_k^{-1}\|_{L_\infty}, \|\beta_{k+1}^{-1}\|_{L_\infty}\right).$$

A similar inequality for the problem on the torus is established in the same way. Together with (5.5), these estimates yield the following statements.

Lemma 8.1. *The functionals $d_{n/2}((7.6))^{2/(n+2)}$ and $d_{n/2}((7.11))^{2/(n+2)}$ are Hölder continuous with the exponent $n/(n+2)$ with respect to variation of γ in the L_p-norm. Moreover, the constant in the corresponding inequality only depends on Ω, $\mathbb{T}$, and the quantities α, $\|\beta_k\|_{L_\infty}$, $\|\beta_k^{-1}\|_{L_\infty}$, and $\|\beta_{k+1}^{-1}\|_{L_\infty}$.*

8.2. Reduction to the cube. An upper estimate. Let $\Omega \subset Q \subset \mathbb{R}^n$, where Q is an open cube; for $k > 1$ we assume that the boundary of Ω is Lipschitz. Suppose that $\beta_k \in M_+^k(Q)$, $\beta_{k+1} \in M_+^{k+1}(Q)$, $\gamma \in L_p(Q, \mathrm{Lin}_k)$, and $p > n$. Let $\mathbb{T}$ be the torus arising from Q by identification of opposite faces.

Lemma 8.2. *Let $k \ge 1$. For a form $\omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k)$, find a $(k-1)$-form φ from system (7.1). Denote by $\widetilde{\omega}$ the extension of ω to $\mathbb{T} \setminus \Omega$ by zero and find $\widetilde{\varphi}$ from system (7.9). Then for the ratio*

$$\frac{\|\varphi\|_{L_2}^2 + \|\widetilde{\varphi}\|_{L_2}^2}{\|\omega\|_{W_{2,1}}^2}, \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k), \tag{8.3}$$

we have $\Delta_{n/2}((8.3)) = 0$.

Proof. First, we consider the ratio

$$\frac{\|\varphi\|_{F^{k-1}(\star_{k-1})}^2 + \|\widetilde{\varphi}\|_{F^{k-1}(\star_{k-1})}^2}{\|\omega\|_{W_{2,1}}^2}, \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k). \tag{8.4}$$

From (7.2) and (8.2) it follows that $\Delta_{n/2}((8.4)) < \infty$.

Next, in the case of the torus, the embedding $F^{k-1}(\mathbb{T}) \subset L_2^{k-1}$ is compact. In the case of a domain and $k = 1$, the embedding $F_{\mathrm{el}}^0(\star_0) = \mathring{W}_{2,1}(\Omega) \subset L_2^0$ is compact whenever Ω is bounded. For $k > 1$, the embedding $F_{\mathrm{el}}^{k-1} \subset L_2^{k-1}$ is compact by Corollary 4.10. Therefore, the claim follows from Lemma 5.1. □

We denote by A_Ω, L_Ω the quadratic forms (7.5), (7.4), and by $\widetilde{A}_{\mathbb{T}}$, $\widetilde{L}_{\mathbb{T}}$ the quadratic forms (7.8), (7.10). Also, we shall indicate the set in the notation for the functionals $d_{n/2}$. Consider yet another auxiliary ratio

$$\frac{\widetilde{L}_{\mathbb{T}}[\omega]}{\widetilde{A}_{\mathbb{T}}[\omega]}, \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k), \tag{8.5}$$

where the k-form ω is assumed to be extended by zero to $\mathbb{T} \setminus \Omega$.

Lemma 8.3. *Let* $\operatorname{supp}\gamma \subset \Omega$. *Then* $d_{n/2}((7.6)) = d_{n/2}((8.5))$.

Proof. If $k = 0$, then the ratios (7.6) and (8.5) coincide. Let $k \geq 1$. Consider the difference of the ratios in question:

$$\frac{\widetilde{L}_{\mathbb{T}}[\omega] - L_{\Omega}[\omega]}{A_{\Omega}[\omega]}, \quad \omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k). \tag{8.6}$$

We have

$$\begin{aligned}\widetilde{L}_{\mathbb{T}}[\omega] - L_{\Omega}[\omega] &= \int_{\Omega} d\varphi \wedge \beta_k \, \overline{d\varphi} - \int_{\mathbb{T}} d\widetilde{\varphi} \wedge \beta_k \, \overline{d\widetilde{\varphi}} \\ &= \int_{\mathbb{T}} d(\varphi - \widetilde{\varphi}) \wedge \gamma\bar{\omega} = \operatorname{Re} \int_{\mathbb{T}} d(\varphi - \widetilde{\varphi}) \wedge \gamma\bar{\omega},\end{aligned}$$

where the $(k-1)$-forms φ and $\widetilde{\varphi}$ are defined by (7.1) and (7.9), respectively. Since $\operatorname{supp}\gamma$ is closed and Ω is open, there exists a real function $\zeta \in C_0^{\infty}(\Omega)$ such that $\zeta|_{\operatorname{supp}\gamma} = 1$. Then $\zeta\gamma = \gamma$ and

$$\int_{\Omega} d\varphi \wedge \gamma\bar{\omega} = \int_{\Omega} (d(\zeta\varphi) - d\zeta \wedge \varphi) \wedge \gamma\bar{\omega} = \int_{\Omega} d(\zeta\varphi) \wedge \beta_k \, \overline{d\widetilde{\varphi}} - d\zeta \wedge \varphi \wedge \gamma\bar{\omega}.$$

Similarly,

$$\int_{\Omega} d\widetilde{\varphi} \wedge \gamma\bar{\omega} = \int_{\Omega} d(\zeta\widetilde{\varphi}) \wedge \beta_k \, \overline{d\varphi} - d\zeta \wedge \widetilde{\varphi} \wedge \gamma\bar{\omega}.$$

Since β_k is real (see Subsection 1.2), we have

$$\operatorname{Re} \int_{\Omega} \zeta \, d\varphi \wedge \beta_k \, \overline{d\widetilde{\varphi}} = \operatorname{Re} \int_{\Omega} \zeta \, d\widetilde{\varphi} \wedge \beta_k \, \overline{d\varphi},$$

whence

$$\widetilde{L}_{\mathbb{T}}[\omega] - L_{\Omega}[\omega] = \operatorname{Re} \int_{\Omega} d\zeta \wedge \varphi \wedge \left(\beta_k \, \overline{d\widetilde{\varphi}} - \gamma\bar{\omega}\right) - \operatorname{Re} \int_{\Omega} d\zeta \wedge \widetilde{\varphi} \wedge \left(\beta_k \, \overline{d\varphi} - \gamma\bar{\omega}\right).$$

Consequently, taking into account (7.3) and a similar inequality in the periodic case, we obtain

$$\left|\widetilde{L}_{\mathbb{T}}[\omega] - L_{\Omega}[\omega]\right| \leq C \left(\|\varphi\|_{L_2} + \|\widetilde{\varphi}\|_{L_2}\right) \|\gamma\omega\|_{L_2}.$$

This estimate, Lemma 8.2, inequality (8.2), and Lemma 5.2 imply the relation $\Delta_{n/2}((8.6)) = 0$. Now, the claim of the lemma follows from (5.5). □

By the variational principle and (7.12), we see that

$$n(\lambda, (8.5)) \leq n(\lambda, (7.11)) + \binom{n}{k},$$

whence $d_{n/2}((8.5)) \leq d_{n/2}((7.11))$. Thus, we have proved the inequality

$$d_{n/2}((7.6)) \leq d_{n/2}((7.11)). \tag{8.7}$$

8.3. Reduction to the cube. A lower estimate. To prove the inequality reverse to (8.7), we need the following fact.

Lemma 8.4. *Suppose* $k \geq 1$, $\omega \in \mathring{W}_{2,1}(\Omega, \mathcal{F}^k)$, *and* φ , $\widetilde{\varphi}$ *are defined by* (7.1) *and* (7.9). *Then*

$$\int_{\Omega} d\varphi \wedge \beta_k \, d\varphi \leq \int_{\mathbb{T}} d\widetilde{\varphi} \wedge \beta_k \, d\widetilde{\varphi}.$$

Proof. By the definition of the $(k-1)$-forms φ and $\widetilde{\varphi}$, we have

$$\int_\Omega d\varphi \wedge \beta_k\, d\varphi = \int_{\mathbb{T}} d\varphi \wedge \gamma\omega = \int_{\mathbb{T}} d\varphi \wedge \beta_k\, d\widetilde{\varphi}.$$

Since β_k is positive definite, we obtain

$$\int_\Omega d\varphi \wedge \beta_k\, d\varphi = \int_{\mathbb{T}} d\varphi \wedge \beta_k\, d\widetilde{\varphi} \le \left(\int_\Omega d\varphi \wedge \beta_k\, d\varphi\right)^{1/2} \left(\int_{\mathbb{T}} d\widetilde{\varphi} \wedge \beta_k\, d\widetilde{\varphi}\right)^{1/2},$$

and the required inequality follows. □

Lemma 8.5. *Let real functions $\zeta \in C_0^\infty(Q)$ and $\eta \in C^\infty(\mathbb{T})$ satisfy $\zeta^2 + \eta^2 = 1$. Then the asymptotics for the counting function of the ratio*

$$\frac{\widetilde{L}_{\beta_k,\gamma}[\omega]}{\widetilde{A}_{\beta_{k+1}}[\zeta\omega] + \widetilde{A}_{\beta_{k+1}}[\eta\omega]}, \quad \omega \in \widetilde{W}, \tag{8.8}$$

coincides with that for the ratio (7.11), $d_{n/2}((7.11)) = d_{n/2}((8.8))$.

Proof. Let $\omega \in W_{2,1}(\mathbb{T}, \mathcal{F}^k)$. We have

$$\begin{aligned}\widetilde{A}_{\beta_{k+1}}[\omega] =& \int_{\mathbb{T}} \zeta d\omega \wedge \beta_{k+1}\zeta d\bar{\omega} + \alpha\zeta d(\star\omega) \wedge \star\zeta d(\star\bar{\omega}) + \eta d\omega \wedge \beta_{k+1}\eta d\bar{\omega} + \alpha\eta d(\star\omega) \wedge \star\eta d(\star\bar{\omega}) \\ =& \widetilde{A}_{\beta_{k+1}}[\zeta\omega] + \widetilde{A}_{\beta_{k+1}}[\eta\omega] + \Phi[\omega],\end{aligned}$$

where Φ is a quadratic form involving only first powers of the first derivatives of ω (consequently, Φ is compact with respect to the quadratic form $\widetilde{A}_{\beta_{k+1}}$).

We show that

$$\widetilde{A}_{\beta_{k+1}}[\zeta\omega] + \widetilde{A}_{\beta_{k+1}}[\eta\omega] > 0 \ \text{ for all } \ \omega \in \widetilde{W},\ \omega \ne 0.$$

Indeed, if $\widetilde{A}_{\beta_{k+1}}[\zeta\omega] + \widetilde{A}_{\beta_{k+1}}[\eta\omega] = 0$, then $\zeta\omega, \eta\omega \in K^k(\mathbb{T}, \star_k)$. Moreover, $\zeta\omega \in K^k(\operatorname{int} Q, \star_k)$, whence $\zeta\omega = 0$ by Remark 4.12. It follows that $\omega = \eta\omega$ (if $\zeta(x) \ne 0$, then $\omega(x) = 0$, but if $\zeta(x) = 0$, then $\eta(x) = 1$). Consequently, $\eta\omega \perp_{\star_k} K^k(\mathbb{T}, \star_k)$, thus, $\omega = \eta\omega = 0$.

Now, the identity $d_{n/2}((7.11)) = d_{n/2}((8.8))$ follows from Lemma 5.4. □

Theorem 8.6. *Let* $\operatorname{supp}\gamma \subset \Omega \subset Q$. *Then*

$$d_{n/2}((7.6), \Omega) = d_{n/2}((7.11), \mathbb{T}).$$

Proof. Suppose that $\zeta \in C_0^\infty(\Omega)$, $\eta \in C^\infty(\mathbb{T})$, $\zeta^2 + \eta^2 = 1$, $\zeta\big|_{\operatorname{supp}\gamma} = 1$. Then $\zeta\gamma = \gamma$, $\widetilde{\Pi}\omega = \widetilde{\Pi}(\zeta\omega)$, $\widetilde{L}[\omega] = \widetilde{L}[\zeta\omega]$. Taking Lemma 8.4 into account, for $\omega \in W_{2,1}(\mathbb{T}, \mathcal{F}^k)$ we obtain

$$\begin{aligned}\widetilde{L}[\omega] &= \int_{\mathbb{T}} \gamma\omega \wedge \beta_k^{-1}(\overline{\gamma\omega}) - \widetilde{\Pi}(\zeta\omega) \wedge \beta_k \overline{\widetilde{\Pi}(\zeta\omega)} \\ &\le \int_Q \gamma\zeta\omega \wedge \beta_k^{-1}(\overline{\gamma\zeta\omega}) - \Pi(\zeta\omega) \wedge \beta_k \overline{\Pi(\zeta\omega)} = L[\zeta\omega].\end{aligned} \tag{8.9}$$

Let $\mathcal{L}$ be a subspace of $W_{2,1}(\mathbb{T}, \mathcal{F}^k)$ with $\dim \mathcal{L} = n(\lambda, (8.8))$, and let

$$\widetilde{L}[\omega] > \lambda\left(\widetilde{A}[\zeta\omega] + \widetilde{A}[\eta\omega]\right), \quad \omega \in \mathcal{L} \setminus \{0\}. \tag{8.10}$$

If $\omega \in \mathcal{L} \setminus \{0\}$, then $\zeta\omega \ne 0$, because otherwise $\widetilde{L}[\omega] = 0$. Therefore, $\dim(\zeta\mathcal{L}) = \dim \mathcal{L}$. By (8.9) and (8.10), we obtain

$$L[\theta] > \lambda A[\theta], \quad \theta \in \zeta\mathcal{L} \setminus \{0\}.$$

Consequently, $n(\lambda,(8.8)) \le n(\lambda,(7.6))$. Applying Lemma 8.5, we arrive at the inequality $d_{n/2}((7.11)) \le d_{n/2}((7.6))$. The reverse inequality (8.7) has already been proved. □

§9. Estimates of singular numbers

We shall need an analog of Ladyzhenskaya's second principal inequality (see [21]) for differential forms.

Theorem 9.1. *Let $\beta \in M^l_+(\mathbb{T}) \cap \mathrm{Lip}(\mathbb{T},\mathrm{Lin}_l)$, and let $\alpha > 0$. Suppose that a form $\omega \in W_{2,1}(\mathbb{T},\mathcal{F}^{l-1})$ satisfies the equation*

$$(-1)^l d(\beta d\omega) + (-1)^{l(n-l-1)+1}\alpha \star d(\star d(\star\omega)) = \theta \in L_2^{n-l+1}(\mathbb{T}). \tag{9.1}$$

Then $\omega \in W_{2,2}(\mathbb{T},\mathcal{F}^l)$ and we have

$$\|\omega\|_{W_{2,2}} \le C\max(\alpha^{-1},\|\beta^{-1}\|_{L_\infty})\|\theta\|_{L_2} + C(\alpha,\beta)\|\omega\|_{W_{2,1}}. \tag{9.2}$$

Proof. Inequality (1.5) shows that (9.1) is a Legendre–Hadamard elliptic system. It is well known that such systems are strongly resolvable with the estimate

$$\|\omega\|_{W_{2,2}} \le C\|\theta\|_{L_2} + C\|\omega\|_{W_{2,1}} \tag{9.3}$$

(see, e.g., [19] or [23]). If we trace the dependence of constants in the proof of inequality (9.3) in [23, Chapter 4], we can observe that the coefficient of $\|\theta\|_{L_2}$ depends on the coefficients α, β of system (9.1) only via the ellipticity constants. □

9.1. Estimate of singular numbers (for β_k).

Lemma 9.2. *Suppose that $p < \infty$, $\beta_l^{(1)},\beta_l^{(2)} \in M^l_+(\mathbb{T})$. Then there exists a form $\widehat{\beta}_l \in M^l_+(\mathbb{T}) \cap \mathrm{Lip}(\mathbb{T},\mathrm{Lin}_l)$ such that*

$$\|\beta_l^{(1)} - \widehat{\beta}_l\|_{L_p} + \|\widehat{\beta}_l - \beta_l^{(2)}\|_{L_p} \le 2\|\beta_l^{(1)} - \beta_l^{(2)}\|_{L_p}, \quad \|(\widehat{\beta}_l)^{-1}\|_{L_\infty} \le \|(\beta_l^{(1)})^{-1}\|_{L_\infty}.$$

Proof. It suffices to take a mollification of $\widehat{\beta}_l$ for $\beta_l^{(1)}$. □

Lemma 9.3. *Suppose that $\beta_k \in M^k_+(\mathbb{T}) \cap \mathrm{Lip}(\mathbb{T},\mathrm{Lin}_k)$, $\gamma \in \mathrm{Lip}(\mathbb{T},\mathrm{Lin}_k)$. Let the operator $\widetilde{\Pi}(\beta_k,\gamma)$ be defined by (7.9), and let $\gamma' \in L_p(\mathbb{T},\mathrm{Lin}_k)$ with $n < p < \infty$. Then the ratio*

$$\frac{\|\gamma'\widetilde{\Pi}(\beta_k,\gamma)\omega\|^2_{L_2}}{\|\omega\|^2_{W_{2,1}}}, \quad \omega \in W_{2,1}(\mathbb{T},\mathcal{F}^k), \tag{9.4}$$

satisfies the estimate

$$\Delta_{n/2}((9.4)) \le C(\gamma)\|\beta_k^{-1}\|^n_{L_\infty}\|\gamma'\|^n_{L_p}.$$

Proof. The consecutive maxima of the ratio (9.4) do not change if the supplementary condition of orthogonality to the kernel of $\widetilde{\Pi}$ is imposed on the form ω:

$$\frac{\|\gamma' d\varphi\|^2_{L_2}}{\|\omega\|^2_{W_{2,1}}}, \quad \omega \in W_{2,1}(\mathbb{T},\mathcal{F}^k), \quad \omega \perp \ker\widetilde{\Pi}(\beta_k,\gamma), \tag{9.5}$$

where $d\varphi = \widetilde{\Pi}(\beta_k,\gamma)\omega$. The spectrum of the ratio (9.5) is estimated in terms of the spectrum of the ratio

$$\frac{\|\gamma' d\varphi\|^2_{L_2}}{\|d\varphi\|^2_{W_{2,1}}}, \quad d\varphi \in \mathrm{Im}\,\widetilde{\Pi}(\beta_k,\gamma). \tag{9.6}$$

Indeed, Theorem 9.1 applied to the form φ yields

$$\begin{aligned}\|d\varphi\|_{W_{2,1}} &\le C\|\beta_k^{-1}\|_{L_\infty}\|d(\gamma\omega)\|_{L_2} + C(\beta_k)\|\varphi\|_{W_{2,1}}\\ &\le C(\gamma)\|\beta_k^{-1}\|_{L_\infty}\|\omega\|_{W_{2,1}} + C(\beta_k)\|\varphi\|_{W_{2,1}}.\end{aligned}$$

Consequently,

$$\frac{\|\gamma' d\varphi\|^2_{L_2}}{\|d\varphi\|^2_{W_{2,1}}} \ge C(\gamma)\|\beta_k^{-1}\|^{-2}_{L_\infty} \frac{\|\gamma' d\varphi\|^2_{L_2}}{\|\omega\|^2_{W_{2,1}} + C\|\varphi\|^2_{W_{2,1}}}.$$

$$\Delta_{n/2}((9.5)) \le C(\gamma)\|\beta_k^{-1}\|^n_{L_\infty}\Delta_{n/2}((9.6)).$$

Now, the claim follows from (8.2). □

Lemma 9.4. *Let $\beta_k^{(1)}, \beta_k^{(2)} \in M_+^k(\mathbb{T})$, and let*

$$b = \max\left(\|(\beta_k^{(1)})^{-1}\|_{L_\infty}, \|(\beta_k^{(2)})^{-1}\|_{L_\infty}\right).$$

Suppose that $\gamma \in \mathrm{Lip}(\mathbb{T}, \mathrm{Lin}_k)$. Then the ratio

$$\frac{\|\widetilde{\Pi}(\beta_k^{(1)}, \gamma)\omega - \widetilde{\Pi}(\beta_k^{(2)}, \gamma)\omega\|^2_{L_2}}{\|\omega\|^2_{W_{2,1}}}, \quad \omega \in W_{2,1}(\mathbb{T}, \mathcal{F}^k), \tag{9.7}$$

obeys the estimate $\Delta_{n/2}((9.7)) \le C(\gamma)b^{2n}\|\beta_k^{(1)} - \beta_k^{(2)}\|^n_{L_p}$. Here the operators $\widetilde{\Pi}$ are defined by (7.9) and $n < p < \infty$.

Proof. By Lemma 9.2, there is no loss of generality in assuming that $\beta_k^{(2)} \in \mathrm{Lip}(\mathbb{T}, \mathrm{Lin}_k)$.

Denoting $\widetilde{\Pi}(\beta_k^{(j)}, \gamma)\omega = d\varphi_j$, $j = 1, 2$, we have

$$\int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge \beta_k^{(1)}\, \overline{d\varphi_1} = \int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge \gamma\bar{\omega} = \int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge \beta_k^{(2)} \overline{d\varphi_2},$$

whence

$$\int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge \beta_k^{(1)}\, \overline{d(\varphi_1 - \varphi_2)} = \int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge (\beta_k^{(2)} - \beta_k^{(1)})\, \overline{d\varphi_2}$$

and $\|d(\varphi_1 - \varphi_2)\|_{L_2} \le b\|(\beta_k^{(1)} - \beta_k^{(2)})d\varphi_2\|_{L_2}$. Therefore, instead of (9.7), we can consider the ratio

$$\frac{\|(\beta_k^{(1)} - \beta_k^{(2)})d\varphi_2\|^2_{L_2}}{\|\omega\|^2_{W_{2,1}}}, \quad \omega \in W_{2,1}(\mathbb{T}, \mathcal{F}^k); \tag{9.8}$$

furthermore, $\Delta_{n/2}((9.7)) \le b^n \Delta_{n/2}((9.8))$. By Lemma 9.3 with $\gamma' = \beta_k^{(1)} - \beta_k^{(2)}$, we see that

$$\Delta_{n/2}((9.8)) \le C(\gamma)b^n\|\beta_k^{(1)} - \beta_k^{(2)}\|^n_{L_p}.$$ □

9.2. Estimate of singular numbers (for β_{k+1}).

Lemma 9.5. *Suppose that $\alpha > 0$ and $\beta_{k+1} \in M_+^{k+1}(\mathbb{T})$. Let $\widetilde{A}_{\beta_{k+1}}$ be the quadratic form (7.8), and let $\mathcal{A}$ be the corresponding selfadjoint operator in the space $L_2(\mathbb{T}, \mathcal{F}^k) \ominus_{\star_k} \{\mathrm{const}\}$. Then*

$$\Delta_n(\mathcal{A}^{-1/2}) \le C \max\left(\alpha^{-n}, \|\beta_{k+1}^{-1}\|^n_{L_\infty}\right).$$

Proof. The definition of the functionals Δ shows that $\Delta_n(\mathcal{A}^{-1/2}) = \Delta_{n/2}(\mathcal{A}^{-1})$. In its turn, the functional $\Delta_{n/2}(\mathcal{A}^{-1})$ coincides with the functional $\Delta_{n/2}$ for the ratio

$$\frac{\|\omega\|^2_{L_2}}{A_{\beta_{k+1}}[\omega]}, \quad \omega \in \widetilde{W}. \tag{9.9}$$

Next,

$$\|\omega\|^2_{W_{2,1}} \le C\left(\|d\omega\|^2_{L_2} + \|d(\star\omega)\|^2_{L_2}\right) \le C \max\left(\alpha^{-1}, \|\beta_{k+1}^{-1}\|_{L_\infty}\right) A_{\beta_{k+1}}[\omega], \quad \omega \in \widetilde{W}.$$

Consequently, for the ratio

$$\frac{\|\omega\|_{L_2}^2}{\|\omega\|_{W_{2,1}}^2}, \quad \omega \in \widetilde{W}, \tag{9.10}$$

we have $\Delta_{n/2}((9.9)) \le C \max\left(\alpha^{-n}, \|\beta_{k+1}^{-1}\|_{L_\infty}^n\right) \Delta_{n/2}((9.10))$. To estimate $\Delta_{n/2}((9.10))$, it suffices to refer to (8.2). □

Lemma 9.6. *Under the assumptions of the preceding lemma, we have* $\Delta_n(d\mathcal{A}^{-1}) \le C\left(\alpha, \|\beta_{k+1}^{-1}\|_{L_\infty}\right)$.

Proof. The operator $d\mathcal{A}^{-1/2}$ is bounded on L_2 because

$$\|d\mathcal{A}^{-1/2}\theta\|_{L_2}^2 \le \|\beta_{k+1}^{-1}\|_{L_\infty} \widetilde{A}_{\beta_{k+1}}\left[\mathcal{A}^{-1/2}\theta\right] = \|\beta_{k+1}^{-1}\|_{L_\infty}\|\theta\|_{L_2}^2.$$

Consequently, by (5.6), we obtain $\Delta_n(d\mathcal{A}^{-1}) \le \|\beta_{k+1}^{-1}\|_{L_\infty}^{n/2}\Delta_n(\mathcal{A}^{-1/2})$. □

Lemma 9.7. *Suppose that* $\alpha > 0$, $\beta_{k+1} \in \mathrm{Lip}(\mathbb{T}, \mathrm{Lin}_k)$, *Let* $\widetilde{A}_{\beta_{k+1}}$ *be the quadratic form* (7.8) *and* $\mathcal{A}$ *the corresponding selfadjoint operator,*

$$\mathcal{A}\omega = (-1)^{k+1} d(\beta_{k+1} d\omega) + (-1)^{(k+1)(n-k)+1}\alpha \star d(\star d(\star\omega)),$$
$$\mathrm{Dom}\,\mathcal{A} = W_{2,2}(\mathbb{T}, \mathcal{F}^k).$$

Let $\rho \in L_p(\mathbb{T}, \mathrm{Lin}_k)$ *with* $p > n$. *Then the ratio*

$$\frac{\|\rho d\omega\|_{L_2}^2}{\|\mathcal{A}\omega\|_{L_2}^2}, \quad \omega \in W_{2,2}(\mathbb{T}, \mathcal{F}^k), \ \ \omega \perp_{\star_k} K^k(\mathbb{T}, \star_k), \tag{9.11}$$

obeys the estimate

$$\Delta_{n/2}((9.11)) \le C\|\beta_{k+1}^{-1}\|_{L_\infty}^n \|\rho\|_{L_p}^n. \tag{9.12}$$

Proof. We use the decomposition $L_2^k(\mathbb{T}) = \widetilde{G}^k \oplus_{\star_k} \widetilde{J}^k$, where

$$\widetilde{G}^k = \{d\varphi \,:\, \varphi \in \mathcal{W}_2^{k-1}(\mathbb{T})\}, \quad \widetilde{J}^k = \{\omega \in L_2^k(\mathbb{T}) \,:\, d(\star\omega) = 0\}.$$

The numerator of (9.11) vanishes on $\widetilde{G}$, so it suffices to consider the ratio

$$\frac{\|\rho\theta\|_{L_2}^2}{\|\theta\|_{W_{2,1}}^2}, \quad \theta = d\omega, \ \ \omega \in W_{2,2}(\mathbb{T}, \mathcal{F}^k) \cap \widetilde{J}, \ \ \omega \perp_{\star_k} K^k(\mathbb{T}, \star_k). \tag{9.13}$$

By Theorem 9.1 (combined with the identity $d(\star\omega) = 0$) and Lemma 5.4, we have

$$\Delta_{n/2}((9.11)) \le C\|\beta_{k+1}^{-1}\|_{L_\infty}^n \Delta_{n/2}((9.13)).$$

Extending the domain of (9.13) up to $\theta \in W_{2,1}(\mathbb{T}, \mathcal{F}^k)$ and applying (8.2), we arrive at (9.12). □

Lemma 9.8. *Let* $\mathcal{A}$ *and* $\widehat{\mathcal{A}}$ *be the operators in* $L_2(\mathbb{T}, \mathcal{F}^k) \ominus_{\star_k} K^k(\mathbb{T}, \star_k)$ *corresponding to the quadratic forms* $\widetilde{A}_{\beta_{k+1}}$, $\widetilde{A}_{\widehat{\beta}_{k+1}}$, *where* $\beta_{k+1}, \widehat{\beta}_{k+1} \in M_+^{k+1}(\mathbb{T})$, *and moreover,* $\widehat{\beta}_{k+1} \in \mathrm{Lip}(\mathbb{T}, \mathrm{Lin}_k)$. *Let* $p > n$. *Then*

$$\Delta_{n/2}\big(\mathcal{A}^{-1} - \widehat{\mathcal{A}}^{-1}\big) \le C\big(\alpha, \|\beta_{k+1}^{-1}\|_{L_\infty}, \|\widehat{\beta}_{k+1}^{-1}\|_{L_\infty}\big)\|\beta_{k+1} - \widehat{\beta}_{k+1}\|_{L_p}^{n/2}.$$

Proof. Put $\omega = \mathcal{A}^{-1}\theta$, $\widehat{\omega} = \widehat{\mathcal{A}}^{-1}\widehat{\theta}$, where $\theta, \widehat{\theta} \perp \{\mathrm{const}\}$. Then

$$(\mathcal{A}^{-1}\theta, \widehat{\theta}) - (\theta, \widehat{\mathcal{A}}^{-1}\widehat{\theta}) = (\omega, \widehat{\mathcal{A}}\widehat{\omega}) - (\mathcal{A}\omega, \widehat{\omega}) = \int_{\mathbb{T}} d\omega \wedge (\widehat{\beta}_{k+1} - \beta_{k+1})\,\overline{d\widehat{\omega}}.$$

Consequently,

$$\mathcal{A}^{-1} - \widehat{\mathcal{A}}^{-1} = (-1)^{(k+1)(n-k-1)}\left(d\mathcal{A}^{-1}\right)^* \star_{n-k-1} (\widehat{\beta}_{k+1} - \beta_{k+1}) d\widehat{\mathcal{A}}^{-1},$$

whence by (5.4) and (5.7) we deduce that

$$\Delta_{n/2}\big(\mathcal{A}^{-1}-\widehat{\mathcal{A}}^{-1}\big)\le 2\Delta_n\big(d\mathcal{A}^{-1}\big)^{1/2}\Delta_n\big((\widehat{\beta}_{k+1}-\beta_{k+1})d\widehat{\mathcal{A}}^{-1}\big)^{1/2}.$$

By Lemma 9.6, $\Delta_n(d\mathcal{A}^{-1})\le C\left(\alpha,\|\beta_{k+1}^{-1}\|_{L_\infty}\right)$. Next, put $\rho=\widehat{\beta}_{k+1}-\beta_{k+1}$. We have

$$\begin{aligned}\Delta_n(\rho d\widehat{\mathcal{A}}^{-1})&=\Delta_{n/2}\big((\rho d\widehat{\mathcal{A}}^{-1})^*\rho d\widehat{\mathcal{A}}^{-1}\big)=\Delta_{n/2}\left(\frac{\|\rho d\widehat{\mathcal{A}}^{-1}\theta\|_{L_2}^2}{\|\theta\|_{L_2}^2}\right)\\&=\Delta_{n/2}\left(\frac{\|\rho d\omega\|_{L_2}^2}{\|\widehat{\mathcal{A}}\omega\|_{L_2}^2}\right)\le C\|\widehat{\beta}_{k+1}^{-1}\|_{L_\infty}^n\|\widehat{\beta}_{k+1}-\beta_{k+1}\|_{L_p}^n\end{aligned}$$

by Lemma 9.7. □

Applying Lemmas 9.2 and 5.3, we arrive at the following statement.

Corollary 9.9. *Suppose that $\beta_{k+1}^{(1)},\beta_{k+1}^{(2)}\in M_+^{k+1}(\mathbb{T})$, let $\mathcal{A}_1$, $\mathcal{A}_2$ be the corresponding operators, and let $n<p<\infty$. Then*

$$\Delta_n\big(\mathcal{A}_1^{-1/2}-\mathcal{A}_2^{-1/2}\big)\le C\big(\alpha,\|(\beta_{k+1}^{(1)})^{-1}\|_{L_\infty},\|(\beta_{k+1}^{(2)})^{-1}\|_{L_\infty}\big)\|\beta_{k+1}^{(1)}-\beta_{k+1}^{(2)}\|_{L_p}^{n/2}.$$

§10. Proof of Theorem 7.1

10.1. Model problem. Let $\beta_k,\gamma\in\mathrm{Lin}_k$, $\beta_{k+1}\in\mathrm{Lin}_{k+1}$ be constant coefficients. Denote by $\mathcal{L}$ and $\mathcal{A}$ the operators corresponding to the quadratic forms $\widetilde{L}$ and $\widetilde{A}$. The spectrum of the ratio (7.11) coincides with that of the equation $\mathcal{L}\omega=\lambda\mathcal{A}\omega$, in other words, with the spectrum of the system

$$\begin{cases}(-1)^{k(n-k)}\gamma^+\beta_k^{-1}\left(\gamma\omega-\beta_k d\varphi\right)\\\quad=\lambda\left((-1)^{k(n-k+1)+1}d(\beta_{k+1}d\omega)+(-1)^{n-k-1}\alpha\star d(\star d(\star\omega))\right),\\d(\beta_k d\varphi)=d(\gamma\omega),\quad d(\star\varphi)=0.\end{cases}\tag{10.1}$$

For definiteness, we assume that $\mathbb{T}$ is the torus obtained by identification of opposite faces for the cube $[0,l]^n$. Then the eigenfunctions of system (10.1) have the form

$$\omega(x)=he^{\frac{2\pi imx}{l}},\quad\varphi(x)=\psi e^{\frac{2\pi imx}{l}},$$

where $h\in\mathcal{F}^k$, $\psi\in\mathcal{F}^{k-1}$, $m\in\mathbb{Z}^n\setminus\{0\}$. We introduce the 1-form

$$\mu=\frac{2\pi}{l}\sum_{j=1}^n m_j dx^j.\tag{10.2}$$

Then

$$d\omega=i\mu\wedge he^{\frac{2\pi imx}{l}},\quad d\varphi=i\mu\wedge\psi e^{\frac{2\pi imx}{l}}.$$

Substituting ω and φ in (10.1) and observing that $\beta_k^{-1}\beta_k=(-1)^{k(n-k)}$ (see Subsection 1.2), we obtain

$$\begin{cases}(-1)^{k(n-k)}\gamma^+\beta_k^{-1}\gamma h-i\gamma^+(\mu\wedge\psi)\\\quad=\lambda\left((-1)^{k(n-k+1)}\mu\wedge\beta_{k+1}(\mu\wedge h)+(-1)^{n-k}\alpha\star(\mu\wedge\star(\mu\wedge\star h))\right),\\-\mu\wedge\beta_k(\mu\wedge\psi)=i\mu\wedge\gamma h,\quad\mu\wedge\star\psi=0.\end{cases}\tag{10.3}$$

For constant coefficients β_k, β_{k+1}, and γ, system (10.3) coincides with (3.1) after changing ξ for μ. Lemma 3.1 shows that the eigenvalues of system (10.1) are nonnegative. For the number of eigenvalues of problem (10.1) that exceed a positive number λ, we have

$$n(\lambda,(10.1))=\sum_j\#\big\{m\in\mathbb{Z}^n\setminus\{0\}\,:\,\lambda_j((3.1),\mu)>\lambda\big\},\tag{10.4}$$

where $\lambda_j((3.1),\mu)$ are the eigenvalues of problem (3.1) (with constant β_k, β_{k+1}, and γ), and μ is related to m by (10.2).

Lemma 10.1. *Let $\lambda \in C(\mathbb{R}^n \setminus \{0\})$ be a function homogeneous of degree (-2), $\lambda(t\xi) = t^{-2}\lambda(\xi)$. Then*

$$\#\{m \in \mathbb{Z}^n \setminus \{0\} \,:\, \lambda(m) > \tau\} \sim \operatorname{meas}\{\xi \in \mathbb{R}^n \,:\, \lambda(\xi) > \tau\}, \quad \tau \to +0.$$

Proof. Clearly,

$$\operatorname{meas}\{\xi \in \mathbb{R}^n \,:\, \lambda(\xi) > \tau\} = \tau^{-n/2} \operatorname{meas}\{\xi \in \mathbb{R}^n \,:\, \lambda(\xi) > 1\}.$$

On the other hand,

$$\begin{aligned}\tau^{n/2}&\#\{m \in \mathbb{Z}^n \setminus \{0\} \,:\, \lambda(m) > \tau\}\\ &= \tau^{n/2}\#\{\widehat{m} \in (\sqrt{\tau}\,\mathbb{Z})^n \setminus \{0\} \,:\, \lambda(\widehat{m}) > 1\} \underset{\tau\to+0}{\longrightarrow} \operatorname{meas}\{\xi \in \mathbb{R}^n \,:\, \lambda(\xi) > 1\}. \qquad \square\end{aligned}$$

By Lemma 3.1, the functions $\lambda_j((3.1),\xi)$ are homogeneous of degree (-2) in ξ; therefore, formula (10.4) implies

$$\begin{aligned} n(\lambda,(10.1)) &= \sum_j \#\left\{m \in \mathbb{Z}^n \setminus \{0\} \,:\, \lambda_j((3.1),m) > \frac{4\pi^2\lambda}{l^2}\right\}\\ &\sim \sum_j \operatorname{meas}\left\{\xi \in \mathbb{R}^n \,:\, \lambda_j((3.1),\xi) > \frac{4\pi^2\lambda}{l^2}\right\}\\ &= \left(\frac{l}{2\pi}\right)^n \int_{\mathbb{R}^n} n(\lambda,(3.1),\xi)\, d^n\xi. \end{aligned}$$

We have proved the following result.

Theorem 10.2. *Let $\mathbb{T}$ be the torus obtained by identification of opposite faces of the cube $[0,l]^n$. Suppose that the coefficients $\beta_k, \gamma \in \operatorname{Lin}_k$, $\beta_{k+1} \in \operatorname{Lin}_{k+1}$ do not depend on x, and $\beta_k \in M_+^k$, $\beta_{k+1} \in M_+^{k+1}$. Then*

$$n(\lambda,(7.11)) \sim \left(\frac{l}{2\pi}\right)^n \int_{\mathbb{R}^n} n(\lambda,(3.1),\xi)\, d^n\xi, \quad \lambda \to +0.$$

10.2. Continuity with respect to β_k, β_{k+1}.

Theorem 10.3. *Let $\gamma \in \operatorname{Lip}(\mathbb{T},\operatorname{Lin}_k)$, and let $p > n$. The functionals $d_{n/2}((7.11))$ are continuous with respect to variation of the coefficients β_k, β_{k+1} in the norm of $L_p(\mathbb{T})$.*

Proof. Let $\mathcal{L}$ and $\mathcal{A}$ be the operators corresponding to quadratic forms $\widetilde{L}$ and $\widetilde{A}$. The spectrum of the ratio (7.11) coincides with the spectrum of the operator $\mathcal{A}^{-1/2}\mathcal{L}\mathcal{A}^{-1/2}$ in the space $L_2(\mathbb{T},\mathcal{F}^k)\ominus_{\star_k} K^k(\mathbb{T},\star_k)$. For two pairs of coefficients $\beta_k^{(1)}$, $\beta_k^{(2)}$, $\beta_{k+1}^{(1)}$, $\beta_{k+1}^{(2)}$, we denote the corresponding operators by $\mathcal{A}_1$, $\mathcal{L}_1$, $\mathcal{A}_2$, $\mathcal{L}_2$, and consider the difference

$$\begin{aligned}&\mathcal{A}_1^{-1/2}\mathcal{L}_1\mathcal{A}_1^{-1/2} - \mathcal{A}_2^{-1/2}\mathcal{L}_2\mathcal{A}_2^{-1/2}\\ &\quad= \big(\mathcal{A}_1^{-1/2} - \mathcal{A}_2^{-1/2}\big)\mathcal{L}_1\mathcal{A}_1^{-1/2} + \mathcal{A}_2^{-1/2}(\mathcal{L}_1 - \mathcal{L}_2)\mathcal{A}_2^{-1/2} + \mathcal{A}_2^{-1/2}\mathcal{L}_1\big(\mathcal{A}_1^{-1/2} - \mathcal{A}_2^{-1/2}\big)\\ &\quad=: T_1 + T_2 + T_3.\end{aligned}$$

In this proof, the estimational constants may depend on γ and the numbers

$$\alpha, \quad \|(\beta_k^{(j)})^{-1}\|_{L_\infty}, \quad \|(\beta_{k+1}^{(j)})^{-1}\|_{L_\infty}, \quad j = 1,2.$$

From (5.6), (5.7), Lemma 9.5, and Corollary 9.9, it follows that

$$\Delta_{n/2}(T_1) \le 2\Delta_n\big(\mathcal{A}_1^{-1/2} - \mathcal{A}_2^{-1/2}\big)^{1/2}\|\mathcal{L}_1\|^{n/2}\Delta_n\big(\mathcal{A}_1^{-1/2}\big)^{1/2} \le C\|\beta_{k+1}^{(1)} - \beta_{k+1}^{(2)}\|_{L_p}^{n/4}.$$

Similarly,

$$\Delta_{n/2}(T_3) \le C\|\beta_{k+1}^{(1)} - \beta_{k+1}^{(2)}\|_{L_p}^{n/4}.$$

The spectrum of T_2 coincides with the spectrum of the ratio

$$\frac{\widetilde{L}_{\beta_k^{(1)},\gamma}[\omega] - \widetilde{L}_{\beta_k^{(2)},\gamma}[\omega]}{\widetilde{A}_{\beta_{k+1}^{(2)}}[\omega]}, \quad \omega \in \widetilde{W}. \tag{10.5}$$

We have

$$\widetilde{L}_{\beta_k^{(1)},\gamma}[\omega] - \widetilde{L}_{\beta_k^{(2)},\gamma}[\omega] = \int_{\mathbb{T}} \gamma\omega \wedge \big((\beta_k^{(1)})^{-1} - (\beta_k^{(2)})^{-1}\big)(\overline{\gamma\omega}) + \int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge \overline{\gamma\omega}.$$

The inequality

$$\Big| \int_{\mathbb{T}} \gamma\omega \wedge \big((\beta_k^{(1)})^{-1} - (\beta_k^{(2)})^{-1}\big)(\overline{\gamma\omega}) \Big| \le \|\gamma\omega\|_{L_2} \big\|\big((\beta_k^{(1)})^{-1} - (\beta_k^{(2)})^{-1}\big)(\gamma\omega)\big\|_{L_2},$$

Lemma 5.2, and estimate (8.2) show that the ratio

$$\frac{\int_{\mathbb{T}} \gamma\omega \wedge \big((\beta_k^{(1)})^{-1} - (\beta_k^{(2)})^{-1}\big)(\overline{\gamma\omega})}{\widetilde{A}_{\beta_{k+1}^{(2)}}[\omega]}, \quad \omega \in \widetilde{W}, \tag{10.6}$$

obeys the estimate

$$\Delta_{n/2}((10.6)) \le C\big\|\big((\beta_k^{(1)})^{-1} - (\beta_k^{(2)})^{-1}\big)\big\|_{L_p}^{n/2} \le C\big\|\beta_k^{(1)} - \beta_k^{(2)}\big\|_{L_p}^{n/2}.$$

Similarly, applying Lemmas 5.2 and 9.4 and formula (8.2), we see that the ratio

$$\frac{\int_{\mathbb{T}} d(\varphi_1 - \varphi_2) \wedge \overline{\gamma\omega}}{\widetilde{A}_2[\omega]}, \quad \omega \in \widetilde{W}, \tag{10.7}$$

obeys the estimate

$$\Delta_{n/2}((10.7)) \le C\big\|\beta_k^{(1)} - \beta_k^{(2)}\big\|_{L_p}^{n/2}.$$

Consequently,

$$\Delta_{n/2}(T_2) = \Delta_{n/2}((10.5)) \le C\big\|\beta_k^{(1)} - \beta_k^{(2)}\big\|_{L_p}^{n/2}.$$

It remains to refer to (5.5). □

This theorem and Lemma 8.1 imply the following statement.

Corollary 10.4. *The functionals $d_{n/2}((7.11))$ are continuous relative to simultaneous variation of the coefficients γ, β_k, β_{k+1} in the $L_p(\mathbb{T})$-norm for $p > n$.*

10.3. Reduction to the case of constant coefficients.

Lemma 10.5. *Let Q, Q_j, $j = 1, \dots, M$, be open cubes in $\mathbb{R}^n$ with*

$$Q_i \cap Q_j = \varnothing \text{ for } i \ne j, \quad \bar{Q} = \cup_{j=1}^M \bar{Q}_j.$$

Denote by $\mathbb{T}$, $\mathbb{T}_j$ the corresponding tori. Suppose that $\beta_k \in M_+^k(Q)$, $\beta_{k+1} \in M_+^{k+1}(Q)$, $\gamma \in L_p(Q, \mathrm{Lin}_k)$, $p > n$. Then

$$d_{n/2}((7.11), \mathbb{T}) = \sum_{j=1}^M d_{n/2}((7.11), \mathbb{T}_j).$$

Proof. Put $\Xi = \cup_{j=1}^{M} Q_j$. By Lemma 8.1 we may assume that $\operatorname{supp}\gamma \subset \Xi$. For the Dirichlet problem, we have

$$n(\lambda, (7.6), \Xi) = \sum_{j=1}^{M} n(\lambda, (7.6), Q_j).$$

Finally, Theorem 8.6 shows that

$$d_{n/2}((7.6), \Xi) = d_{n/2}((7.11), \mathbb{T}), \quad d_{n/2}((7.6), Q_j) = d_{n/2}((7.11), \mathbb{T}_j). \qquad \square$$

Proof of Theorem 7.1. Let Q be a cube including the bounded set Ω, and let $\mathbb{T}$ be the corresponding torus. By Lemma 8.1, we may assume that $\operatorname{supp}\gamma \subset \Omega$. We extend the coefficients β_k, β_{k+1}, and γ to $Q \setminus \Omega$, namely, γ is extended by zero, and β_k and β_{k+1} are extended with preservation of the upper and lower estimates. By Theorem 8.6, it suffices to prove that

$$\lambda^{n/2} n(\lambda, (7.11), \mathbb{T}) \sim (2\pi)^{-n} \lambda^{n/2} \int_Q \int_{\mathbb{R}^n} n(\lambda, (3.1), x, \xi)\, d^n\xi\, d^n x, \quad \lambda \to 0, \tag{10.8}$$

(recall that, by Lemma 3.1, if $\gamma(x) = 0$, then all eigenvalues of Problem (3.1) vanish, so $n(\lambda, (3.1), x, \xi) = 0$ for such x). By Remark 7.3 and Corollary 10.4, the two sides of (10.8) are continuous in the coefficients β_k, β_{k+1}, γ. Thus, it suffices to prove (10.8) for piecewise constant β_k, β_{k+1}, and γ. Let Q_j, $j = 1, \dots, M$ be open cubes such that

$$\bar{Q} = \cup_{j=1}^{M} \bar{Q}_j, \quad Q_i \cap Q_j = \varnothing \ \text{ for } \ i \neq j,$$

and β_k, β_{k+1}, γ are constant on each Q_j. Denote by $\mathbb{T}$, $\mathbb{T}_j$ the corresponding tori. Theorem 10.2 and Lemma 3.2 imply the relation

$$d_{n/2}((7.11), \mathbb{T}_j) = \lim_{\lambda \to +0} \lambda^{n/2} n(\lambda, (7.11), \mathbb{T}_j) = |Q_j| \nu(x), \quad x \in Q_j,$$

where the function ν is defined by (3.4). Finally, we use Lemma 10.5:

$$\delta_{n/2}((7.11), \mathbb{T}) = \Delta_{n/2}((7.11), \mathbb{T}) = \int_Q \nu(x)\, d^n x,$$

Applying Lemma 3.2 once again, we arrive at (10.8). $\square$

References

[1] A. B. Alekseev, *Spectral asymptotics of elliptic boundary value problems with solvable constraints*, Kand. diss., Leningrad. Gos. Univ., Leningrad, 1977. (Russian)

[2] A. B. Alekseev and M. Sh. Birman, *Asymptotic behavior of the spectrum of elliptic boundary value problems with solvable constraints*, Dokl. Akad. Nauk SSSR **230** (1976), no. 3, 505–507; English transl., Soviet Math. Dokl. **17** (1976), no. 5, 1319–1322. MR0420027 (54:8044)

[3] A. B. Alekseev, M. Sh. Birman, and N. D. Filonov, *Spectrum asymptotics for one "nonsmooth" variational problem with solvable constraint*, Algebra i Analiz **18** (2006), no. 5, 1–22; English transl., St. Petersburg Math. J. **18** (2007), no. 5, 681–697. MR2301038 (2008b:35193)

[4] M. Sh. Birman, L. S. Koplienko, and M. Z. Solomyak, *Estimates of the spectrum of a difference of fractional powers of selfadjoint operators*, Izv. Vyssh. Uchebn. Zaved. Mat. **1975**, no. 3, 3–10; English transl., Soviet Math. (Iz. VUZ) **19** (1975), no. 3, 1–6 (1976). MR0385597 (52:6458)

[5] M. Sh. Birman and N. D.Filonov, *Weyl asymptotics of the spectrum of the Maxwell operator with non-smooth coefficients in Lipschitz domains*, Nonlinear Equations and Spectral Theory, Amer. Math. Soc. Transl. Ser. 2, vol. 220, Amer. Math. Soc., Providence, RI, 2007, pp. 27–44. MR2343605 (2009a:35243)

[6] M. Sh. Birman and M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert space*, 2nd ed., Lan', St. Petersburg, 2010; English transl. of 1st ed., Math. Appl. (Soviet Series), D. Reidel Publ. Co., Dordrecht, 1987. MR1192782 (93g:47001)

[7] ———, *Spectral asymptotics of nonsmooth elliptic operators.* II, Trudy Moskov. Mat. Obshch. **28** (1973), 3–34; English transl., Transl. Moscow Math. Soc. **28** (1973), 1–32 (1975) MR0364898 (51:1152)

[8] ———, *Quantitative analysis in Sobolev's imbedding theorems and applications to spectral theory*, Tenth Mathematical Schood (Summer School, Kaciveli/Nal′chik, 1972), Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1974, pp. 5–189. (Russian) MR0482138 (58:2224)

[9] ———, *The Maxwell operator in domains with a nonsmooth boundary*, Sibirsk. Mat. Zh. **28** (1987), no. 1, 23–36; English transl., Siberian Math. J. **28** (1987), no. 1, 12–24. MR0886850 (88g:35095)

[10] ———, *Weyl asymptotics of the spectrum of the Maxwell operator for domains with a Lipschitz boundary*, Vestnik Leningrad. Univ. Ser. 1 **1987**, vyp. 3, 23–28; English transl., Vestnik Leningrad Univ. Math. **20** (1987), no. 3, 15–21. MR0928156 (89h:35253)

[11] ———, *The selfadjoint Maxwell operator in arbitrary domains*, Algebra i Analiz **1** (1989), no. 1, 96–110; English transl., Leningrad Math. J. **1** (1990), no. 1, 99–115. MR1015335 (91e:35197)

[12] M. E. Bogovskiĭ, *Solutions of some problems of vector analysis, associated with the operators* div *and* grad, Trudy Sem. S. L. Soboleva **1980**, no. 1, 5–40. (Russian) MR0631691 (82m:26014)

[13] J. Bourgain and H. Brezis, *On the equation* div $Y = f$ *and application to control of phases*, J. Amer. Math. Soc. **16** (2003), 393–426. MR1949165 (2004d:35032)

[14] A. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309. MR0084633 (18:894a)

[15] M. N. Demchenko and N. D. Filonov, *Spectral asymptotics of the Maxwell operator on Lipschitz manifolds with boundary*, Spectral Theory of Differential Operators, Amer. Math. Soc. Transl. Ser. 2, vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 73–90. MR2509776 (2010k:35335)

[16] N. Filonov, *On a bounded solution of the equation* div $u = f$ *in plane domains*, Tr. S.-Peterburg. Mat. Obshch. **6** (1998), 231–245; English transl., Amer. Math. Soc. transl. Ser. 2, vol. 199. Proc. St. Petersburg Math. Soc. vol. 6, Amer. Math. Soc., Providence, RI, 2000, pp. 199–207. MR1768334; MR1766023 (2002b:35168)

[17] K. O. Friedrichs, *Differential forms on Riemannian manifolds*, Comm. Pure Appl. Math. **8** (1955), 551–590. MR0087763 (19:407a)

[18] V. M. Gol′dshteĭn, V. I. Kuz′minov, and I. A. Shvedov, *Differential forms on a Lipschitz manifold*, Sibirsk. Mat. Zh. **23** (1982), no. 2, 16–30; English transl., Siberian Math. J. **23** (1982), no. 2, 151–161. MR0652220 (83j:58004)

[19] L. Hörmander, *Analysis of linear partial differential operators*. III. Pseudodifferential operators, Grundlehren Math. Wiss., Bd. 274, Springer-Verlag, Berlin, 1985. MR0781536 (87d:35002a)

[20] B. Jawerth and M. Mitrea, *Higher-dimensional electromagnetic scattering theory on* C^1 *and Lipschitz domains*, Amer. J. Math. **117** (1995), 929–963. MR1342836 (96h:35143)

[21] O. A. Ladyzhenskaya and N. N. Ural′tseva, *Linear and quasilinear equations of elliptic type*, 2nd ed., Nauka, Moscow, 1973; English transl. of 1st ed., *Linear and quasilinear elliptic equations*, Acad. Press, New York–London, 1968. MR0509265 (58:23009); MR0244627 (39:5941)

[22] R. Leis, *Initial-boundary value problems in mathematical physics*, B. G. Teubner, Stutgart; J. Wiley and Sons, Ltd., Chichester, 1986. MR0841971 (87h:35003)

[23] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge Univ. Press, Cambridge, 2000. MR1742312 (2001a:35051)

[24] M. Mitrea, *Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds*, Duke Math. J. **125** (2004), no. 3, 467–547. MR2166752 (2007g:35246)

[25] R. Picard, *An elementary proof for a compact imbedding result in generalized electromagnetic theory*, Math. Z. **187** (1984), 151–164. MR0753428 (85k:35212)

[26] R. Picard, N. Weck, and K.-J. Witsch, *Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles*, Analysis (Munich) **21** (2001), 231–263. MR1855908 (2002h:78009)

[27] Yu. G. Safarov, *Asymptotics of the spectrum of the Maxwell operator*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **127** (1983), 169–180; English transl. in J. Soviet Math. **27** (1984), no. 2. MR0702849 (85d:35088)

[28] N. A. Veniamoniv, *Estimate for the remainder in the Weyl asymptotics of the spectrum of the Maxwell operator in Lipschitz domains*, Probl. Mat. anal., No. 47, Novosibirsk, 2010, pp. 43–58; English transl., J. Math. Sci. (N. Y.) **169** (2010), no. 1, 46–63. MR2839003 (2012k:35541)

[29] C. Weber, *Regularity theorems for Maxwell's equations*, Math. Methods Appl. Sci. **3** (1981), 523–536. MR0657071 (83h:35113)

[30] N. Weck, *Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries*, J. Math. Anal. Appl. **46** (1974), 410–437. MR0343771 (49:8511)

[31] H. Weyl, *Über das Spectrum der Hohlraumstrahlung*, J. Reine Angew. Math. **141** (1912), 163–181.
[32] ———, *Die natürlichen Randwertaufgaben im Aussenraum für Strahlungsfelder beliebiger Dimension und beliebiger Ranges*, Math. Z. **56** (1952), 105–119. MR0054524 (14:933g)

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023; St. Petersburg State University, Universitetskii pr. 28, Staryi Peterhof, St. Petersburg 198504, Russia

E-mail address: filonov@pdmi.ras.ru

Received 21/SEP/2012

Translated by S. KISLYAKOV