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SPECTRAL AND SCATTERING THEORY FOR PERTURBATIONS OF THE CARLEMAN OPERATOR

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Abstract. The spectral properties of the Carleman operator (the Hankel operator with the kernel $h_0(t) = t^{-1}$) are studied; in particular, an explicit formula for its resolvent is found. Then, perturbations are considered of the Carleman operator H_0 by Hankel operators V with kernels $v(t)$ decaying sufficiently rapidly as $t \to \infty$ and not too singular at $t = 0$. The goal is to develop scattering theory for the pair H_0, $H = H_0 + V$ and to construct an expansion in eigenfunctions of the continuous spectrum of the Hankel operator H. Also, it is proved that, under general assumptions, the singular continuous spectrum of the operator H is empty and that its eigenvalues may accumulate only to the edge points 0 and π in the spectrum of H_0. Simple conditions are found for the finiteness of the total number of eigenvalues of the operator H lying above the (continuous) spectrum of the Carleman operator H_0, and an explicit estimate of this number is obtained. The theory constructed is somewhat analogous to the theory of one-dimensional differential operators.

§1. Introduction

1.1. The Hankel operators H can be defined as integral operators,

$$(Hf)(t) = \int_0^\infty h(t+s)f(s)\,ds, \tag{1.1}$$

in the space $L^2(\mathbb{R}_+)$ with kernels h that depend on the sum of variables only. As was pointed out by Howland in [5], selfadjoint Hankel operators are to a certain extent similar to differential operators. In particular, Hankel operators with continuous spectrum resemble singular differential operators. In terms of this analogy, the Carleman operator H_0 corresponding to the kernel $h_0(t) = t^{-1}$ plays the role of the "free" Schrödinger operator D^2, $D = -id/dx$, in the space $L^2(\mathbb{R})$. The Carleman operator can easily be diagonalized by the Mellin transform.

As far as the theory of Hankel operators is concerned, we refer to the books [8] by Peller and [9] by Power. We also note the paper [4] by J. S. Howland where, in the trace class framework, the structure of the absolutely continuous spectra of Hankel operators was described in terms of their symbols.

Our goal here is to study spectral properties of the Carleman operator $H_0 =: \mathbf{C}$ and of Hankel operators H with kernels $h(t)$ behaving asymptotically like t^{-1} as $t \to \infty$ and $t \to 0$. In particular, we develop the scattering theory for the pairs H_0, H.

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1.2. The first part of the paper (§§2, 3, and 4) is devoted to the study of the Carleman operator $\mathbf{C}$ defined in the space $L^2(\mathbb{R}_+)$ by the relation

$$(\mathbf{C}f)(t) = \int_0^\infty (t+s)^{-1} f(s) d\, s. \tag{1.2}$$

It is easily seen that, for all $k \geq 0$,

$$\int_0^\infty (t+s)^{-1} s^{-1/2 \pm ik}\, ds = \lambda(k) t^{-1/2 \pm ik}, \tag{1.3}$$

where

$$\lambda = \lambda(k) = \frac{\pi}{\cosh(\pi k)}. \tag{1.4}$$

This equation establishes a one-to-one correspondence between the quasimomentum $k \geq 0$ and the energy $\lambda \in (0, \pi]$. It can be solved by the formula

$$k = k(\lambda) = \pi^{-1} \ln \big((\pi + \sqrt{\pi^2 - \lambda^2}) \lambda^{-1} \big) \geq 0. \tag{1.5}$$

Relations (1.3) and (1.4) show that the spectrum of the operator $\mathbf{C}$ is absolutely continuous, coincides with the interval $[0, \pi]$, and has multiplicity two.

The dispersive relation (1.4) plays the same role for the Carleman operator as the relation $\lambda = k^2$ between the energy $\lambda > 0$ and the momentum $k > 0$ for the differential operator D^2 in the space $L^2(\mathbb{R})$. In terms of this analogy, the continuous spectrum eigenfunctions $t^{-1/2 \pm ik}$ of the operator $\mathbf{C}$ play the role of the eigenfunctions $e^{\pm ikx}$ of the operator D^2. The singular points $t = 0$ and $t = \infty$ correspond to the singular points $x = -\infty$ and $x = \infty$.

One of our main results (Theorem 2.3) yields an explicit formula for the resolvent $\mathbf{R}(z) = (\mathbf{C} - zI)^{-1}$ of the operator $\mathbf{C}$. It plays the crucial role in our study of perturbations of the Carleman operator.

In §3 we study the boundary values of $\mathbf{R}(z)$ as z approaches the spectrum $[0, \pi]$ of the operator $\mathbf{C}$ and, in particular, its edge points $z = \pi$ and $z = 0$. It turns out that the singularity of $\mathbf{R}(z)$ as $z \to \pi$ is quite similar to that of the resolvent $(D^2 - zI)^{-1}$ as $z \to 0$. In particular, the operator $\mathbf{C}$ has a resonance at the point $z = \pi$. On the contrary, the singularity of $\mathbf{R}(z)$ as $z \to 0$ is rather unusual and contains an oscillating factor.

Also, in §4 we find the asymptotics of the unitary group $\exp(-i\mathbf{C}T)$ as $T \to \pm\infty$. We show that the functions $(\exp(-i\mathbf{C}T)f)(t)$ are localized for large $|T|$ in exponentially small neighbourhoods of the singular points $t = 0$ and $t = \infty$. This should be compared with the well-known fact that the functions $(\exp(-iD^2T)f)(x)$ "live" in the region where $|x|$ and $|T|$ are of the same order. Our formulas for $\exp(-i\mathbf{C}T)$ (see Theorem 4.2) are somewhat similar to but essentially more complicated than those for the unitary group $\exp(-iD^2T)$.

1.3. In the second part of the paper (§5 and §6), we study perturbations of the Carleman operator $H_0 = \mathbf{C}$ by Hankel operators V,

$$(Vf(t) = \int_0^\infty v(t+s) f(s)\, ds, \tag{1.6}$$

with kernels $v(t)$ decaying faster that t^{-1} as $t \to \infty$ and less singular than t^{-1} as $t \to 0$. Such perturbations of $\mathbf{C}$ play the role of perturbations of the operator D^2 by operators of multiplication by functions $\mathsf{V}(x)$ decaying sufficiently rapidly as $|x| \to \infty$. To a certain extent, §5 can be viewed as a continuation of the paper [5], where the Mourre method was used for the study of Hankel operators. Nevertheless, our approach to this problem is

quite different from that of [5]. Actually, we follow the analogy with the one-dimensional Schrödinger operator (see, e.g., the original paper [3] by L. D. Faddeev or the book [12]).

There is, however, an important difference between one-dimensional Schrödinger operators and Hankel operators. In the first case one can develop the theory relying exclusively on Volterra integral equations, while such a possibility is of course lacking in the second case. In this respect, the theory of Hankel operators is closer to the theory of one-dimensional differential operators of order higher than two, where Fredholm integral equations occur naturally (see [11]). Note that for the operator D^n in $L^2(\mathbb{R})$ the eigenfunctions $e^{\pm ikx}$ are the same as those for the operator D^2 for all $n = 1, 2, \ldots$, but the dispersive relation has the form $\lambda = k^n$. For the Carleman operator, the dispersive relation (1.4) is more complicated.

In §5, we proceed from the results of §3 on the existence of suitable boundary values of the resolvent $R_0(z) = (H_0 - zI)^{-1}$ (the limiting absorption principle for the operator H_0). Then we apply general results of abstract scattering theory (see, e.g., the paper [6] or the book [10]) and establish the limiting absorption principle for the operator $H = H_0 + V$. This allows us to obtain rather a detailed information about eigenfunctions of the continuous spectrum of the operator H and to prove an expansion in eigenfunctions of H. Then we find expressions for the wave operators for the pair H_0, H and for the corresponding scattering matrix in terms of the asymptotics of eigenfunctions of H as $t \to \infty$ and $t \to 0$. The formulas obtained look similar to those for one-dimensional differential operators.

Finally, in §6 we study the discrete spectrum of the operator $H = H_0 + V$ lying above its continuous spectrum $[0, \pi]$. We show that it only consists of a finite number of eigenvalues if the function $v(t)$ decays sufficiently rapidly as $t \to \infty$ and is not too singular as $t \to 0$. Moreover, the operator H necessarily has an eigenvalue larger than π if $V \geq 0$ and $V \neq 0$. These results are similar to those on the negative spectrum of the Schrödinger operator $D^2 + \mathsf{V}(x)$. This is of course quite natural, because the singularities of the resolvents at the corresponding edge points of the continuous spectra are also similar.

On the contrary, the finiteness of the negative spectrum of $H = H_0 + V$ is not determined by the behaviour of $v(t)$ at the singular points $t = 0$ and $t = \infty$. For example, one can construct functions $\eta(x)$ in the Schwartz class (in fact, the Fourier transforms of η even belong to the class $C_0^\infty(\mathbb{R})$) such that for the kernel $v(t) = t^{-1}\eta(\ln t)$, the negative spectrum of H is infinite. Of course, this phenomenon is related to a complicated structure of the "free" resolvent $R_0(z)$ as $z \to 0$. But this is a subject of another paper.

Actually, in §5 and §6 we admit perturbations by sufficiently general integral operators V (not necessarily Hankel operators) with kernels $\mathbf{v}(t, s)$ satisfying some decay assumptions at infinity and some regularity assumptions at the origin.

1.4. As was already mentioned, the spectral and scattering theories of differential operators and of Hankel operators are to a certain extent parallel. It might be of interest to extend this analogy a bit further. As examples, let us mention various trace formulas (cf. the paper [2] by Buslaev and Faddeev, where the Schrödinger operator on the half-axis was considered) and the inverse problem of a reconstruction of the kernel given the corresponding scattering matrix (cf. the paper [3] by Faddeev, where the Schrödinger operator on the whole axis was considered). Note, however, that such more advanced questions might be difficult already for differential operators of order higher than two and even more difficult for Hankel operators. In this respect, we mention the paper [7], where a trace formula (to put it differently, an expression for the perturbation determinant in terms of solutions of the corresponding differential equation) was obtained for differential operators of an arbitrary order. The solution of the inverse problem (see, e.g., the book

[1]) for differential operators of an arbitrary order seems to be in a less satisfactory state than for order two.

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§2. The resolvent of the Carleman operator

Here we calculate the resolvent $\mathbf{R}(z) = (\mathbf{C} - zI)^{-1}$ of the Carleman operator $\mathbf{C}$ defined in the space $L^2(\mathbb{R}_+)$ by relation (1.2).

2.1. We introduce the Mellin transform M,

$$(Mf)(k) = (2\pi)^{-1/2} \int_0^\infty t^{-1/2-ik} f(t)\, dt, \tag{2.1}$$

which is a unitary mapping $M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$. Obviously, we have

$$(M\mathbf{C}f)(k) = (2\pi)^{-1/2} \int_0^\infty ds f(s) \int_0^\infty dt t^{-1/2-ik}(t+s)^{-1} = \lambda(k)(Mf)(k) \tag{2.2}$$

where

$$\lambda(k) = \int_0^\infty t^{-1/2-ik}(t+1)^{-1}\, dt. \tag{2.3}$$

As is well known and will be seen below, this integral is given by formula (1.4). Thus, the spectrum of the operator $\mathbf{C}$ is absolutely continuous, coincides with the interval $[0, \pi]$, and has multiplicity 2.

To calculate the resolvent of the operator $\mathbf{C}$, we need to solve the equation

$$(\mathbf{C} - zI)f = f_0, \quad z \in \mathbb{C} \setminus [0, \pi]. \tag{2.4}$$

Making the Mellin transform and using (2.2), we can equivalently rewrite (2.4) as the equation

$$(\lambda(k) - z)\widetilde{f}(k) = \widetilde{f}_0(k)$$

for the functions $\widetilde{f}_0 = Mf_0$ and $\widetilde{f} = Mf$. In view of (1.4), this equation can be solved by the formula

$$\widetilde{f}(k) = (\lambda(k) - z)^{-1}\widetilde{f}_0(k) = -z^{-1}\widetilde{f}_0(k) - \pi z^{-2}\frac{1}{\cosh(\pi k) - \pi z^{-1}}\widetilde{f}_0(k). \tag{2.5}$$

Now we need to make the inverse Mellin transform. For that, we use the following elementary assertion.

Lemma 2.1. *Let Q be the operator of multiplication in the space $L^2(\mathbb{R})$ by a function $q \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then*

$$(M^*QMf)(t) = \int_0^\infty (ts)^{-1/2}\mathbf{q}(t/s)f(s)\, ds, \tag{2.6}$$

where

$$\mathbf{q}(u) = (2\pi)^{-1} \int_{-\infty}^\infty u^{ik} q(k)\, dk. \tag{2.7}$$

Proof. Interchanging the order of integrations, we see that

$$\begin{aligned}(M^*QMf)(t) &= (2\pi)^{-1} \int_{-\infty}^\infty dk\, t^{-1/2+ik} q(k) \int_0^\infty ds\, f(s) s^{-1/2-ik} \\ &= (2\pi)^{-1} \int_0^\infty ds\, f(s) \Big(\int_{-\infty}^\infty t^{-1/2+ik} s^{-1/2-ik} q(k)\, dk \Big).\end{aligned}$$

This yields formulas (2.6), (2.7). □

This result shows that if ϕ is a bounded function of $\lambda \in [0,\pi]$ such that $\lambda^{-1}\phi(\lambda)$ belongs to $L^1(0,\pi)$, then $\phi(\mathbf{C})$ is an integral operator acting by formula (2.6). Here the function $\mathbf{q}(u)$ is defined by (2.7) with $q(k) = \phi(\frac{\pi}{\cosh(\pi k)})$.

By Lemma 2.1, from (2.5) it follows that

$$(\mathbf{R}(z)f_0)(t) = f(t) = -z^{-1}f_0(t) - z^{-2}\pi^{-1}\int_0^\infty (ts)^{-1/2}\mathcal{I}_z(t/s)f_0(s)\,ds, \tag{2.8}$$

where

$$\mathcal{I}_z(u) = \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{u^{ik}}{\cosh(\pi k) - \pi z^{-1}}\,dk. \tag{2.9}$$

2.2. Let us calculate the integral (2.9). We set $\zeta = \pi z^{-1}$, so that $\zeta \in \mathbb{C}\setminus[1,\infty)$ if $z \in \mathbb{C}\setminus[0,\pi]$. Making the change of variables $p = e^{\pi k}$, we see that

$$\mathcal{I}_z(u) = \int_0^\infty \frac{p^{ix}}{p^2 - 2\zeta p + 1}\,dp, \quad x = \pi^{-1}\ln u. \tag{2.10}$$

We are going to calculate this integral by residues. Observe that the equation $p^2 - 2\zeta p + 1 = 0$ has two roots

$$p_1(\zeta) = \zeta + \sqrt{\zeta^2 - 1}, \quad p_2(\zeta) = \zeta - \sqrt{\zeta^2 - 1}, \tag{2.11}$$

which are different if $\zeta \neq -1$, and that $p_1(\zeta)p_2(\zeta) = 1$. We fix $\arg p$ in the complex plane with the cut along $[0,\infty)$ by the condition $\arg p \in [0, 2\pi]$. Then

$$\arg p_1(\zeta) + \arg p_2(\zeta) = 2\pi,$$

whence

$$\ln p_1(\zeta) + \ln p_2(\zeta) = 2\pi i. \tag{2.12}$$

Consider the contour C_ϱ in $\mathbb{C}\setminus[0,\infty)$ consisting of the interval $[0,\varrho]$ on the upper edge of the cut, the circle $|p| = \varrho$, and the interval $[\varrho, 0]$ on the lower edge of the cut. By the Cauchy theorem, for sufficiently large ϱ we have

$$\int_{C_\varrho} \frac{p^{ix}}{p^2 - 2\zeta p + 1}\,dp = 2\pi i \sum_{j=1}^{2} \operatorname*{Res}_{p=p_j(\zeta)} \frac{p^{ix}}{p^2 - 2\zeta p + 1}. \tag{2.13}$$

Computing the residues, we find that the right-hand side equals

$$2\pi i \frac{p_1(\zeta)^{ix} - p_2(\zeta)^{ix}}{p_1(\zeta) - p_2(\zeta)} = \frac{\pi i}{\sqrt{\zeta^2 - 1}}\big(p_1(\zeta)^{ix} - p_2(\zeta)^{ix}\big). \tag{2.14}$$

Note that this expression does not depend on the choice of the sign of $\sqrt{\zeta^2 - 1}$. On the left-hand side of (2.13), the integral over the lower edge of the cut equals

$$\int_\varrho^0 \frac{(pe^{2\pi i})^{ix}}{p^2 - 2\zeta p + 1}\,dp = -e^{-2\pi x}\int_0^\varrho \frac{p^{ix}}{p^2 - 2\zeta p + 1}\,dp.$$

The integral over the circle $|p| = \varrho$ tends to zero as $\varrho \to \infty$, because $|p^{ix}| = e^{-x\arg p}$ is bounded by 1 for $x \geq 0$ and by $e^{-2\pi x}$ for $x < 0$. Therefore, passing in (2.13) to the limit $\varrho \to \infty$, we obtain an equation for the integral (2.10):

$$(1 - u^{-2})\mathcal{I}_z(u) = \frac{\pi i}{\sqrt{\zeta^2 - 1}}\big(p_1(\zeta)^{ix} - p_2(\zeta)^{ix}\big), \quad \zeta = \pi/z. \tag{2.15}$$

If $\zeta = -1$, then $p_1(\zeta) = p_2(\zeta) = -1$, so that the right-hand sides of (2.13), and hence, of (2.15) should be replaced by

$$2\pi i \operatorname*{Res}_{p=-1} \frac{p^{ix}}{(p+1)^2} = 2\pi i \frac{d}{dp} p^{ix}\Big|_{p=-1} = -2\pi x (e^{\pi i})^{ix-1} = 2\pi x e^{-\pi x}.$$

It follows that

$$\mathcal{I}_{-\pi}(u) = \frac{2\ln u}{u - u^{-1}}.$$

We formulate the result obtained.

Lemma 2.2. *The integral* (2.9) *for* $z \in \mathbb{C} \setminus [0, \pi]$ *is given by the formula*

$$(1 - u^{-2})\mathcal{I}_z(u) = \frac{\pi i}{\sqrt{\zeta^2 - 1}} \big(u^{i/\pi \ln p_1(\zeta)} - u^{i/\pi \ln p_2(\zeta)}\big), \quad \zeta = \pi/z, \tag{2.16}$$

where the numbers $p_j(\zeta)$ *are defined by* (2.11) *and* $\arg p_j(\zeta) \in (0, 2\pi)$.

Consider the particular case where $\zeta = 0$. If we, for example, choose $\sqrt{\zeta^2 - 1} = i$, then $\ln p_1(\zeta) = \pi i/2$, $\ln p_2(\zeta) = 3\pi i/2$, and hence, formula (2.16) for the integral (2.9) yields

$$\int_0^\infty \frac{p^{ix}}{p^2 + 1}\, dp = \frac{\pi}{2\cosh(\pi x/2)}.$$

This is equivalent to expression (1.4) for the integral (2.3).

2.3. To state a formula for the resolvent of the Carleman operator, first we rewrite equation (2.16) in terms of the variable $z = \pi\zeta^{-1}$. Consider the function

$$\varphi(z) = \sqrt{z^2 - \pi^2} \tag{2.17}$$

in the complex plane cut along $[-\pi, \pi]$, and fix its branch by the condition $\varphi(z) > 0$ for $z > \pi$. Observe that the function

$$q(z) = \frac{\pi - i\sqrt{z^2 - \pi^2}}{z}, \quad z \in \mathbb{C} \setminus [-\pi, \pi], \tag{2.18}$$

takes no positive values, which allows us to set $\arg q(z) \in (0, 2\pi)$. With this convention, the function

$$\mathbf{k}(z) = \frac{1}{\pi} \ln q(z) \tag{2.19}$$

is analytic for $z \in \mathbb{C} \setminus [-\pi, \pi]$. Since $\sqrt{\zeta^2 - 1} = iz^{-1}\sqrt{z^2 - \pi^2}$ (this fixes the sign of the left-hand side), we have $p_2(\pi/z) = q(z)$ and $\ln p_2(\pi/z) = \pi\mathbf{k}(z)$. Using (2.12), we also see that $\ln p_1(\pi/z) = -\pi\mathbf{k}(z) + 2\pi i$, whence formula (2.16) can be rewritten as

$$(1 - u^{-2})\mathcal{I}_z(u) = \frac{\pi z}{\sqrt{z^2 - \pi^2}} \big(u^{-2} u^{-i\mathbf{k}(z)} - u^{i\mathbf{k}(z)}\big). \tag{2.20}$$

Putting formulas (2.8) and (2.20) together, we obtain an expression for the resolvent of the Carleman operator.

Theorem 2.3. *Let the function* $\mathbf{k}(z)$ *be defined for* $z \in \mathbb{C} \setminus [-\pi, \pi]$ *by formulas* (2.18), (2.19) *and the condition* $\arg q(z) \in (0, 2\pi)$. *Set*

$$\rho(u; z) = \frac{u^{i\mathbf{k}(z)}}{u^{-2} - 1} + \frac{u^{-i\mathbf{k}(z)}}{u^2 - 1}. \tag{2.21}$$

Then the resolvent $\mathbf{R}(z) = (\mathbf{C} - zI)^{-1}$ *of the operator* (1.2) *admits the representation*

$$\mathbf{R}(z) = -z^{-1}(I + \mathbf{A}(z)) \tag{2.22}$$

where $\mathbf{A}(z)$ *is the integral operator with the kernel*

$$\mathbf{a}(t,s;z) = \frac{1}{\sqrt{z^2-\pi^2}}(ts)^{-1/2}\rho(t/s;z). \tag{2.23}$$

Lemma 2.6 shows that Theorem 2.3 remains true for all $z \in \mathbb{C} \setminus [0,\pi]$.

Now we discuss properties of the function $\rho(u;z)$. First, we observe that the function (2.17) satisfies $\varphi(\bar z) = \overline{\varphi(z)}$, $\varphi(z) < 0$ for $z < -\pi$, and

$$\varphi(\lambda \pm i0) = \pm i\sqrt{\pi^2-\lambda^2}, \quad \lambda \in [-\pi,\pi]. \tag{2.24}$$

In the following assertion we collect the necessary properties of the function $\mathbf{k}(z)$.

Lemma 2.4. 1^0. *The function* $\mathbf{k}(z)$ *is an analytic function of* $z \in \mathbb{C} \setminus [-\pi,\pi]$, *and it satisfies the identity*

$$\mathbf{k}(\bar z) = -\overline{\mathbf{k}(z)}. \tag{2.25}$$

2^0. *The limits of* $\mathbf{k}(z)$ *on the cut exist* (*except for the point* $z=0$), *and*

$$\mathbf{k}(\lambda+i0) = k(|\lambda|)+i, \quad \lambda \in [-\pi,0), \tag{2.26}$$

$$\mathbf{k}(\lambda+i0) = k(\lambda)+2i, \quad \lambda \in (0,\pi], \tag{2.27}$$

where $k(\lambda)$ *is the function* (1.5).

Proof. Let $q(z)$ be the function (2.18). If $z > \pi$, then (2.25) is true because $|q(z)| = 1$ whence $\operatorname{Re}\mathbf{k}(z) = 0$. Then, by analytic continuation, (2.25) extends to all complex z.

From (2.24) it follows that

$$q(\lambda+i0) = \bigl(\pi + \sqrt{\pi^2-\lambda^2}\bigr)\lambda^{-1}. \tag{2.28}$$

Therefore, $q(\lambda+i0) < 0$, so that $\arg q(\lambda+i0) = \pi$ for $\lambda \in [-\pi,0)$, which proves (2.26). If $\lambda \in (0,\pi]$, then $q(\lambda+i0) > 0$. To calculate $\arg q(\lambda+i0)$, we pass from the half-line $\lambda > \pi$ to the upper edge of the cut $(0,\pi)$ around the point $\lambda = \pi$ by a small semicircle lying in the upper half-plane. Observe that $\operatorname{Im} q(\lambda) < 0$ for $\lambda > \pi$ and that $q(z)$ arrives to the positive value (2.28) remaining always in the lower half-plane. Therefore, $\arg q(\lambda+i0) = 2\pi$, which proves (2.27). □

Now we come back to the function $\rho(u;z)$.

Proposition 2.5. *Let the function* $\rho(u;z)$ *be defined for* $u > 0$ *and* $z \in \mathbb{C} \setminus [-\pi,\pi]$ *by formula* (2.21). *Then*:

1^0. *The function* $\rho(u;z)$ *depends analytically on* $z \in \mathbb{C}\setminus[-\pi,\pi]$, *and it is a* C^∞*-function of* $u \in \mathbb{R}_+$; *in particular, we have*

$$\rho(1;z) = -1 - i\mathbf{k}(z).$$

2^0. *The function* $\rho(u;z)$ *satisfies the identities*

$$\rho(u^{-1};z) = \rho(u;z) \tag{2.29}$$

and

$$\rho(u;\bar z) = \overline{\rho(u;z)}. \tag{2.30}$$

3^0. *Set* $\kappa(z) = \pi^{-1}\min\{\arg q(z), 2\pi - \arg q(z)\} > 0$ *for* $z \notin [0,\pi]$ (*note that* $\kappa(z) = 1$ *for* $z \in [-\pi,0)$). *Then* $\rho(u;z) = O(u^{-\kappa(z)})$ *as* $u \to \infty$, $\rho(u;z) = O(u^{\kappa(z)})$ *as* $u \to 0$.

4^0. *For all* $u > 0$, *the limits of* $\rho(u;z)$ *on the cut exist* (*except the point* $z=0$) *and*

$$\rho(u;\lambda+i0) = \frac{u^{ik(|\lambda|)} - u^{-ik(|\lambda|)}}{u^{-1}-u}, \quad \lambda \in [-\pi,0), \tag{2.31}$$

$$\rho(u;\lambda+i0) = \frac{u^{ik(\lambda)}}{1-u^2} + \frac{u^{-ik(\lambda)}}{1-u^{-2}}, \quad \lambda \in (0,\pi]. \tag{2.32}$$

5^0. *The function $\rho(u;z)$ is uniformly bounded in $u \in \mathbb{R}_+$ and z belonging to compact subsets of $\mathbb{C} \setminus \{0\}$, including the values of z on the cut along $[-\pi, \pi]$.*

Proof. Statement 1^0 and identity (2.29) are obvious. Identity (2.30) follows from (2.25). Formulas (2.19) and (2.21) show that

$$|\rho(u;z)| \le \left|\frac{u^{-\pi^{-1}\arg q(z)}}{u^{-2}-1}\right| + \left|\frac{u^{\pi^{-1}\arg q(z)}}{u^{2}-1}\right|.$$

This immediately implies statement 3^0. The representations (2.31) and (2.32) are direct consequences of (2.26) and (2.27), respectively. Statement 5^0 is a direct consequence of the definition (2.21) (see also formulas (2.31) and (2.32)). □

Various properties of the resolvent $\mathbf{R}(z)$ are direct consequences of Proposition 2.5. Putting identities (2.29) and (2.30) together, we see that $\rho(u;\bar z) = \overline{\rho(u^{-1};z)}$, whence $\overline{a(t,s;z)} = a(s,t;\bar z)$, which is consistent with the relation $\mathbf{R}(\bar z) = \mathbf{R}^*(z)$ (the selfadjointness of $\mathbf{C}$).

We define the complex conjugation $\mathcal{C}$ by the relation

$$(\mathcal{C}f)(t) = \overline{f(t)}. \tag{2.33}$$

Identity (2.30) shows that $\overline{a(t,s;z)} = a(t,s;\bar z)$, which is consistent with the relation $\mathcal{C}R(z) = R(\bar z)\mathcal{C}$ (the invariance of $\mathbf{C}$ with respect to the complex conjugation). Thus, we have $a(t,s;z) = a(s,t;z)$, which is actually a consequence of (2.29) solely.

The function $\rho(u;z)$ is analytic for $z \in \mathbb{C} \setminus [-\pi,\pi]$ only, while the resolvent $\mathbf{R}(z)$ and, with it, $\mathbf{A}(z)$ should be analytic for all $z \in \mathbb{C} \setminus [0,\pi]$. To see this directly, we observe that the limits $\mathbf{a}(t,s;\lambda \pm i0)$ exist and

$$\mathbf{a}(t,s;\lambda+i0) = \mathbf{a}(t,s;\lambda-i0), \quad \lambda \in [-\pi, 0).$$

Indeed, combining identities (2.30) and (2.31), we see that $\rho(\lambda+i0) = -\rho(\lambda-i0)$. So, it remains to take (2.24) into account. We state the result obtained.

Lemma 2.6. *The function $\mathbf{a}(t,s;z)$ is analytic for $z \in \mathbb{C} \setminus [0,\pi]$, and*

$$\mathbf{a}(t,s;\lambda) = \frac{2}{\sqrt{\pi^2-\lambda^2}}\frac{\sqrt{ts}}{s^2-t^2}\sin\big(k(|\lambda|)\ln(t/s)\big), \quad \lambda \in (-\pi, 0), \tag{2.34}$$

where $k(|\lambda|)$ is determined by (1.5). *In particular, we have*

$$\mathbf{a}(t,s;-\pi) = \frac{2}{\pi^2}\frac{\sqrt{ts}}{s^2-t^2}\ln(t/s).$$

Finally, we note that, by parts 1^0 and 3^0 of Proposition 2.5, we have

$$\int_0^\infty |\rho(u;z)|u^{-1}\,du < \infty.$$

This estimate allows one to prove directly that the integral operator $\mathbf{A}(z)$ with kernel (2.23) is bounded.

§3. Approaching the continuous spectrum

Now we discuss the boundary values of the resolvent $\mathbf{R}(z) = (\mathbf{C} - zI)^{-1}$ as z approaches the cut along $[0,\pi]$.

3.1. Recall that the kernel of the operator $\mathbf{A}(z)$ is given by formula (2.23). From part 4^0 of Proposition 2.5 (see also identity (2.24)) it follows that the limits

$$\mathbf{a}(t,s;\lambda+i0) = -i\frac{1}{\sqrt{\pi^2-\lambda^2}}(ts)^{-1/2}\rho(t/s;\lambda+i0), \quad \lambda\in(0,\pi), \tag{3.1}$$

exist for all $t,s>0$, where the function $\rho(t/s;\lambda+i0)$ is given by formula (2.32).

Consider the spectral family $\mathbf{E}(\lambda)$ of the operator $\mathbf{C}$. Using the formula

$$2\pi i(d\mathbf{E}(\lambda)f,g)/d\lambda = (\mathbf{R}(\lambda+i0)f,g) - (\mathbf{R}(\lambda-i0)f,g),$$

it is easy to derive the following result from formulas (2.22), (2.32), and (3.1).

Lemma 3.1. *For all $t,s>0$, the integral kernel $e(t,s;\lambda)$ of the operator $\mathbf{E}(\lambda)$ is differentiable in λ, and*

$$\frac{e(t,s;\lambda)}{d\lambda} = \frac{1}{\pi\lambda\sqrt{\pi^2-\lambda^2}}(ts)^{-1/2}\cos\big(k(\lambda)\ln(t/s)\big), \quad \lambda\in(0,\pi).$$

3.2. Here we collect the results that will be used in §5. Let Q be the operator in the space $L^2(\mathbb{R}_+)$ defined by

$$(Qf)(t) = \langle\ln t\rangle f(t), \quad \text{where} \quad \langle\ln t\rangle = (1+|\ln t|^2)^{1/2}. \tag{3.2}$$

In accordance with formula (2.23), from parts 4^0 and 5^0 of Proposition 2.5 it follows that the kernel of the operator $Q^{-\beta}\mathbf{A}(z)Q^{-\beta}$ depends continuously on z, and it is uniformly bounded by [1]

$$C(ts)^{-1/2}\langle\ln t\rangle^{-\beta}\langle\ln s\rangle^{-\beta}.$$

This function belongs to $L^2(\mathbb{R}_+\times\mathbb{R}_+)$ for $\beta>1/2$. Therefore, by the Lebesgue dominated convergence theorem, the operator-valued function $Q^{-\beta}\mathbf{A}(z)Q^{-\beta}$ depends continuously on z in the Hilbert–Schmidt norm. In view of the representation (2.22), this leads to the following assertion.

Proposition 3.2. *For all $\beta>1/2$, the operator-valued function $Q^{-\beta}\mathbf{R}(z)Q^{-\beta}$ depends continuously in the norm of the space $L^2(\mathbb{R}_+)$ on z in the complex plane cut along $[0,\pi]$, as z approaches the cut with the exception of the points 0 and π. Moreover, this function is Hölder continuous with any exponent $\gamma<\beta-1/2$ (and $\gamma\le 1$).*

In the formulas below, λ and k are related by identities (1.4) or (1.5). Relation (2.32) shows that for all $\lambda\in(0,\pi)$ there exist the limits

$$\lim_{u\to\infty} u^{ik}\rho(u;\lambda+i0) = \lim_{u\to 0} u^{-ik}\rho(u;\lambda+i0) = 1.$$

Therefore, again by the Lebesgue dominated convergence theorem, formula (3.1) yields the asymptotics of the function $(\mathbf{A}(\lambda+i0)f)(t)$ as $t\to\infty$ and as $t\to 0$. Taking also the representation (2.22) into account, we can state the following result.

Proposition 3.3. *Suppose that*

$$\int_0^\infty t^{-1/2}|f(t)|\,dt < \infty \tag{3.3}$$

and that $f(t)=o(t^{-1/2})$ as $t\to\infty$ and as $t\to 0$. Then for all $\lambda\in(0,\pi)$

$$\lim_{\substack{t\to\infty\\ t\to 0}} t^{1/2\pm ik}(\mathbf{R}(\lambda+i0)f)(t) = \frac{i}{\lambda\sqrt{\pi^2-\lambda^2}}\int_0^\infty s^{-1/2\pm ik}f(s)\,ds. \tag{3.4}$$

[1] Here and in what follows we denote by C (with various indices) positive constants whose values are of no importance.

3.3. Next, we describe the singularities of the resolvent $\mathbf{R}(z)$ as z approaches the edge point $z=\pi$. Let $\mathbf{k}(z)$ be the function (2.19). By (2.27), we have $\mathbf{k}(\pi)=2i$. Observe that

$$\theta(z) := 2 + i\mathbf{k}(z) = \pi^{-2}\sqrt{z^2-\pi^2} + O(|z-\pi|) \tag{3.5}$$

as $z\to\pi$. In terms of $\theta(z)$, the function (2.21) can be written as

$$\rho(u;z) = \frac{u^{\theta(z)}}{1-u^2} + \frac{u^{-\theta(z)}}{1-u^{-2}}. \tag{3.6}$$

It follows that, for all fixed $u\in\mathbb{R}_+$ and $z\to\pi$,

$$\rho(u;z) = \sum_{n=0}^{\infty}\frac{\theta(z)^n}{n!}\sigma_n(u),$$

where $\sigma_n(u)=\ln^n u$ for even n and

$$\sigma_n(u) = \frac{1+u^2}{1-u^2}\ln^n u \tag{3.7}$$

for odd n. Obviously,

$$|\sigma_n(u)| \le C_n(1+|\ln u|)^n$$

for all n. This yields the following result.

Proposition 3.4. *Let $t,s\in\mathbb{R}_+$ be fixed. As $z\to\pi$, the kernel* (2.23) *admits the expansion in the asymptotic series*

$$\mathbf{a}(t,s;z) = \frac{1}{\sqrt{z^2-\pi^2}}(ts)^{-1/2}\sum_{n=0}^{\infty}\frac{\theta(z)^n}{n!}\sigma_n(t/s). \tag{3.8}$$

Combining Theorem 2.3 and Proposition 3.4, we see that the integral kernel of the resolvent $\mathbf{R}(z)$ has the singularity

$$-\pi^{-1}(z^2-\pi^2)^{-1/2}(ts)^{-1/2}$$

as $z\to\pi$. Thus, the Carleman operator $\mathbf{C}$ has a resonance at the point $z=\pi$. Note that $\psi_0(t)=t^{-1/2}$ is an "eigenfunction" of the operator $\mathbf{C}$ corresponding to the spectral point π. This function satisfies the equation $\mathbf{C}\psi_0=\pi\psi_0$ and "almost belongs" to $L^2(\mathbb{R}_+)$.

3.4. Finally, we study the resolvent $\mathbf{R}(z)$ or, equivalently, the operator-valued function $\mathbf{A}(z)$ as z approaches the edge point $z=0$. For simplicity, we suppose that $z=\lambda<0$. We start with the representation (2.34) for the kernel of the operator $\mathbf{A}(\lambda)$. By the definition (1.5), we have the asymptotic relation

$$k(|\lambda|) = -\pi^{-1}\ln|\lambda| + \pi^{-1}\ln\left(\pi+\sqrt{\pi^2-\lambda^2}\right) = -\pi^{-1}\ln|\lambda| + \pi^{-1}\ln(2\pi) + O(\lambda^2)$$

as $\lambda\to-0$. It follows that, for every fixed $u\in\mathbb{R}_+$,

$$\sin\left(k(|\lambda|)\ln u\right) = -\sin\left(\pi^{-1}\ln(|\lambda|/(2\pi))\ln u\right) + O(\lambda^2).$$

Thus, we obtain the following result.

Proposition 3.5. *Let $t,s\in\mathbb{R}_+$, $t\ne s$, be fixed. As $\lambda\to-0$, the kernel of the operator $\mathbf{A}(\lambda)$ obeys the asymptotic relation*

$$\mathbf{a}(t,s;\lambda) = \frac{2}{\sqrt{\pi^2-\lambda^2}}\frac{\sqrt{ts}}{t^2-s^2}\sin\left(\pi^{-1}\ln(|\lambda|/(2\pi))\ln(t/s)\right) + O(\lambda^2). \tag{3.9}$$

Now formula (2.22) shows that the singularity of the resolvent at the point $z=0$ consists of the singular denominator z^{-1} and of an oscillating term. Using formulas (2.32) and (3.1) we can obtain similar results for $z=\lambda\pm i0$ tending to 0 along the continuous spectrum.

§4. Time-dependent evolution

4.1. Here we study the unitary group $\exp(-i\mathbf{C}T)$. Relation (2.2) and Lemma 2.1 show that

$$\exp(-i\mathbf{C}T) = I + B(T),$$

where

$$(B(T)f)(t) = \int_0^\infty (ts)^{-1/2} b(t/s;T) f(s)\,ds,$$

$$b(u;T) = (2\pi)^{-1} \int_{-\infty}^\infty u^{ik}(e^{-i\lambda(k)T} - 1)\,dk$$

and the function $\lambda(k)$ is defined by formula (1.4). Apparently, the integral $b(u;T)$ cannot be expressed in terms of standard functions.

4.2. However, it is possible to find explicitly the asymptotics of $\exp(-i\mathbf{C}T)$ as $T \to \pm\infty$. By (2.2), we have

$$(e^{-i\mathbf{C}T} f)(t) = (2\pi)^{-1/2} t^{-1/2} \int_{-\infty}^\infty e^{i(k\ln t - \lambda(k)T)} \widetilde{f}(k)\,dk, \quad \widetilde{f} = Mf. \tag{4.1}$$

We apply the stationary phase method to the integral (4.1). The stationary points k are determined by the equation

$$\lambda'(k) = \frac{\ln t}{T} =: -\frac{\pi^2}{2}\tau. \tag{4.2}$$

Obviously, the function

$$\lambda'(k) = -\pi^2 \frac{\sinh(\pi k)}{1 + \sinh^2(\pi k)} \tag{4.3}$$

is odd, it is negative for $k > 0$, and $\lambda'(k) \to 0$ as $k \to \infty$. It has the minimum $-\pi^2/2$ at the point $k_0 = \pi^{-1}\ln(\sqrt{2}+1)$, and $\lambda''(k) < 0$ for $k \in [0, k_0)$, $\lambda''(k) > 0$ for $k > k_0$. It follows that equation (4.2) has no solutions if $|\tau| > 1$, it has two positive solutions $k_1(\tau) < k_0 < k_2(\tau)$ for $\tau \in (0,1)$, and it has two negative solutions $k_2(\tau) < -k_0 < k_1(\tau)$ for $\tau \in (-1,0)$. Clearly, $k_j(-\tau) = -k_j(\tau)$. We introduce the shorthand notation

$$\sigma_j(\tau) = 1 + (-1)^j \sqrt{1-\tau^2}.$$

An easy calculation shows that

$$\sinh(\pi k_j(\tau)) = \sigma_j(\tau)/\tau \tag{4.4}$$

and

$$k_j(\tau) = \pi^{-1} \operatorname{sgn}\tau \ln\big(|\tau|^{-1}(\sigma_j(\tau) + \sqrt{2\sigma_j(\tau)})\big). \tag{4.5}$$

Let $\mathcal{M}$ be the set $\mathbb{R}$ with the points 0, k_0, and $-k_0$ removed. Applying the stationary phase method to the integral (4.1), where $\widetilde{f} \in C_0^\infty(\mathcal{M})$, we see that, for $e^{-\pi^2|T|/2} < t < e^{\pi^2|T|/2}$,

$$(e^{-i\mathbf{C}T} f)(t) = |T|^{-1/2} t^{-1/2} \sum_{j=1}^2 \delta_j e^{-i\omega_j(\tau)T} |\lambda''(k_j(\tau))|^{-1/2} \widetilde{f}(k_j(\tau)) + O(|T|^{-1}). \tag{4.6}$$

Here $\delta_1 = e^{i(\operatorname{sgn} T)\pi/4}$, $\delta_2 = e^{-i(\operatorname{sgn} T)\pi/4}$, and

$$\omega_j(\tau) = \pi^2 k_j(\tau)\tau/2 + \lambda(k_j(\tau)). \tag{4.7}$$

Note that $\tau \in (-1,0)$ for $t \in (1, e^{\pi^2|T|/2})$ and $\tau \in (0,1)$ for $t \in (e^{-\pi^2|T|/2}, 1)$. If $|\tau| \geq 1$, that is, $t \geq e^{\pi^2|T|/2}$ or $t \leq e^{-\pi^2|T|/2}$, then integration by parts shows that the integral

(4.1) decays faster than any power of $(|\ln t| + |T|)^{-1}$. Using formulas (4.4) and (4.5), it is easy to calculate

$$\lambda(k_j(\tau)) = 2\pi|\tau|\frac{\sigma_j(\tau) + \sqrt{2\sigma_j(\tau)}}{(\sigma_j(\tau) + \sqrt{2\sigma_j(\tau)})^2 + \tau^2} \tag{4.8}$$

and

$$\lambda''(k_j(\tau)) = (-1)^j \pi^3 |\tau| \sqrt{\frac{1-\tau^2}{2\sigma_j(\tau)}}. \tag{4.9}$$

We formulate the result obtained.

Lemma 4.1. *Suppose that $\widetilde{f} \in C_0^\infty(\mathcal{M})$. If $t \geq e^{\pi^2|T|/2}$ or $t \leq e^{-\pi^2|T|/2}$, then, for all N, we have the estimates*

$$|(e^{-i\mathbf{C}T} f)(t)| \leq C_N(|\ln t| + |T|)^{-N}.$$

If $t \in (e^{-\pi^2|T|/2}, e^{\pi^2|T|/2})$, then the asymptotics of the function $(e^{-i\mathbf{C}T} f)(t)$ is given by formula (4.6)*, where τ and $k_j(\tau)$ are defined by relations* (4.2) *and* (4.5)*. The functions $\omega_j(\tau)$, $\lambda(k_j(\tau))$, and $\lambda''(k_j(\tau))$ are determined by identities* (4.7), (4.8), *and* (4.9), *respectively.*

4.3. Now we set

$$(U_j(T)f)(t) = \chi_T(t)|T|^{-1/2} t^{-1/2} \delta_j e^{-i\omega_j(\tau)T} |\lambda''(k_j(\tau))|^{-1/2} \widetilde{f}(k_j(\tau)), \tag{4.10}$$

where χ_T is the characteristic function of the interval $(e^{-\pi^2|T|/2}, e^{\pi^2|T|/2})$ and τ is related to t and T by (4.2). We calculate

$$\begin{aligned} \|U_j(T)f\|^2 &= |T|^{-1} \int_{e^{-\pi^2|T|/2}}^{e^{\pi^2|T|/2}} t^{-1} |\lambda''(k_j(\tau))|^{-1} |\widetilde{f}(k_j(\tau))|^2 \, dt \\ &= \frac{\pi^2}{2} \int_{-1}^{1} |\lambda''(k_j(\tau))|^{-1} |\widetilde{f}(k_j(\tau))|^2 \, d\tau. \end{aligned} \tag{4.11}$$

By (4.2), we have $\lambda'(k_j(\tau)) = -\pi^2\tau/2$. Differentiating this, we see that $\lambda''(k_j(\tau))k_j'(\tau) = -\pi^2/2$; hence, making the change of variables $k = k_j(\tau)$ in (4.11), we find

$$\|U_j(T)f\|^2 = \int_{I_j} |\widetilde{f}(k)|^2 \, dk, \tag{4.12}$$

where $I_1 = (-k_0, k_0)$ and $I_2 = (-\infty, -k_0) \cup (k_0, \infty)$. It follows that

$$\|U_1(T)f\|^2 + \|U_2(T)f\|^2 = \|f\|^2 \tag{4.13}$$

for all T. In particular, the operators $U_j(T)$ are bounded uniformly in T.

Lemma 4.1 implies that

$$\lim_{|T|\to\infty} \big\| \big(e^{-i\mathbf{C}T} - U_1(T) - U_2(T)\big) f \big\| = 0 \tag{4.14}$$

for $\widetilde{f} \in C_0^\infty(\mathcal{M})$. Of course, this relation extends to all $f \in L^2(\mathbb{R}_+)$, which yields the following assertion.

Theorem 4.2. *Let $U_j(T)$, $j = 1, 2$, be the operators defined by formula* (4.10)*. Then for all $f \in L^2(\mathbb{R}_+)$ relation* (4.14) *holds true.*

Note that the operators $U(T) = U_1(T) + U_2(T)$ are not unitary, but $\|U(T)f\| \to \|f\|$ for all $f \in L^2(\mathbb{R}_+)$ as $|T| \to \infty$. This fact follows from Theorem 4.2. It is equivalent to the relation

$$\lim_{|T|\to\infty} \operatorname{Re}\big(U_1(T)f, U_2(T)f\big) = 0, \tag{4.15}$$

which can be verified directly with the help of (4.10).

§5. Scattering theory

5.1. Our goal now is to develop scattering theory for perturbations of the Carleman operator $H_0 = \mathbf{C}$ by selfadjoint operators V satisfying the condition

$$Q^\alpha V Q^\alpha \in \mathfrak{S}_\infty \tag{5.1}$$

for some $\alpha > 1/2$. Here the operator Q is defined by formula (3.2), and $\mathfrak{S}_\infty$ is the class of compact operators. For example, condition (5.1) is fulfilled if V is an integral operator of the form

$$(Vf)(t) = \int_0^\infty \mathbf{v}(t,s) f(s)\, ds \tag{5.2}$$

with kernel $\mathbf{v}(t,s)$ such that

$$\int_0^\infty \int_0^\infty |\mathbf{v}(t,s)|^2 \langle \ln t\rangle^{2\alpha} \langle \ln s\rangle^{2\alpha}\, dt\, ds < \infty. \tag{5.3}$$

Then the operator $Q^\alpha V Q^\alpha$ belongs to the Hilbert-Schmidt class.

In particular, if $\mathbf{v}(t,s) = v(t+s)$, then V is a Hankel operator acting by formula (1.6). Since

$$\int_0^t \langle \ln s\rangle^{2\alpha} \langle \ln(t-s)\rangle^{2\alpha}\, ds \le C\langle \ln t\rangle^{4\alpha} t,$$

condition (5.3) is satisfied for a Hankel operator V if

$$\int_0^\infty |v(t)|^2 \langle \ln t\rangle^{4\alpha} t\, dt < \infty. \tag{5.4}$$

The main results of scattering theory for the pair of the operators $H_0 = \mathbf{C}$ and $H = H_0 + V$ are collected in the following assertion.

Theorem 5.1. *Suppose that assumption* (5.1) *is satisfied for some* $\alpha > 1/2$. *Then*:

1^0. *The strong limits*

$$\operatorname*{s-lim}_{T\to\pm\infty} e^{iHT} e^{-iH_0 T} =: W_\pm, \tag{5.5}$$

known as the wave operators, exist.

2^0. *The wave operators enjoy the intertwining property* $HW_\pm = W_\pm H_0$ *and are isometric.*

3^0. *The wave operators are complete*:

$$\operatorname{Ran} W_\pm = \mathcal{H}^{(\mathrm{ac})} \tag{5.6}$$

where $\mathcal{H}^{(\mathrm{ac})}$ *is the absolutely continuous subspace of the operator* H.

Corollary 5.2. *The absolutely continuous spectrum of the operator* H *has multiplicity* 2 *and coincides with the interval* $[0,\pi]$.

Theorem 5.1 can be proved with the help of the stationary scattering theory. Denote $R_0(z) = (H_0 - zI)^{-1}$ (of course, $R_0(z) = \mathbf{R}(z)$ in the notation of §2) and $R(z) = (H - zI)^{-1}$, and recall the resolvent identity

$$R(z) - R_0(z) = -R(z) V R_0(z) = -R_0(z) V R(z). \tag{5.7}$$

Set $G_0(z) = Q^{-\beta} R_0(z) Q^{-\beta}$ and $G(z) = Q^{-\beta} R(z) Q^{-\beta}$. From (5.7) it follows that

$$\big(I + G_0(z)K\big) G(z) = G_0(z), \tag{5.8}$$

where $K = Q^\beta V Q^\beta \in \mathfrak{S}_\infty$ for all $\beta \le \alpha$. Viewing (5.8) as an equation for the operator-valued function $G(z)$ and using Proposition 3.2 (the limiting absorption principle for the operator H_0), one can prove (see, e.g., Theorem 7.3 in Chapter 4 of [10]) a similar

statement for the operator H. To formulate the precise result, we introduce the set $\mathcal{N}$ of $\lambda \in (0,\pi)$ where at least one of the equations

$$f + G_0(\lambda \pm i0)Kf = 0$$

has a nontrivial solution.

Theorem 5.3. *Suppose that assumption* (5.1) *is satisfied for some* $\alpha > 1/2$. *Then the set* $\mathcal{N}$ *is closed and is of Lebesgue measure zero. The singular spectrum of the operator* H *is contained in the set* $\mathcal{N}$. *The operator-valued function* $G(z) = Q^{-\beta}R(z)Q^{-\beta}$, *where* $\beta > 1/2$, *is Hölder continuous in* z *with any exponent* $\gamma < \min\{\alpha,\beta\} - 1/2$ (*and* $\gamma \le 1$) *if* $\pm \operatorname{Im} z \ge 0$ *and* $\operatorname{Re} z \in (0,\pi) \setminus \mathcal{N}$.

Theorem 5.1 can be deduced from Theorem 5.3 (for the details, see, e.g., Theorems 6.4 and 6.5 in Chapter 4 of [10]).

If $\beta > 1$, then the operator-valued function $G_0(z)$ is Hölder continuous with an exponent $\gamma > 1/2$. This makes it possible to show (see, e.g., Theorems 7.9 and 7.10 in Chapter 4 of [10]) that the singular set $\mathcal{N}$ coincides with the point spectrum $\operatorname{spec}_p H$ of H. We arrive at to the following result.

Theorem 5.4. *Suppose that assumption* (5.1) *is satisfied for some* $\alpha > 1$. *Then the operator* $H = H_0 + V$ *has no singular continuous spectrum. The eigenvalues of* H *distinct from the points* 0 *and* π *have finite multiplicities and can accumulate to these points only.*

5.2. Standard formulas of stationary scattering theory allow us to obtain an expansion over eigenfunctions of the continuous spectrum of the operator H. For $k > 0$, we set

$$\psi_1^{(0)}(t;k) = t^{-1/2+ik}, \quad \psi_2^{(0)}(t;k) = t^{-1/2-ik},$$

and define eigenfunctions $\psi_j^{(\pm)}(t,k)$ of the operator H by the formula

$$\psi_j^{(\pm)}(k) = \psi_j^{(0)}(k) - R(\lambda(k) \mp i0)V\psi_j^{(0)}(k), \quad j = 1,2, \ \ \lambda(k) \notin \operatorname{spec}_p H, \tag{5.9}$$

where $\lambda = \lambda(k) \in (0,\pi)$ and $k \in \mathbb{R}_+$ are related as in (1.4). Theorems 5.3 and 5.4 imply that

$$\langle \ln t\rangle^{-\beta}\psi_j(t,k) \in L^2(\mathbb{R}_+), \quad \beta > 1/2, \tag{5.10}$$

and these functions depend continuously (actually, Hölder continuously) on $k > 0$ in the norm of this space. Obviously, we have $H\psi_j^{(\pm)}(k) = \lambda(k)\psi_j^{(\pm)}(k)$.

Set

$$(\Psi_0 f)(k) = \big((Mf)(k), (Mf)(-k)\big)^\top, \tag{5.11}$$

where M is the Mellin transform defined by (2.1). Clearly, the mapping $\Psi_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+;\mathbb{C}^2)$ is unitary and, in accordance with (2.2), it diagonalizes the operator H_0:

$$(\Psi_0 H_0 f)(k) = \lambda(k)(\Psi_0 f)(k), \quad f \in L^2(\mathbb{R}_+). \tag{5.12}$$

Now we construct a diagonalization of the operator H.

Theorem 5.5. *Define operators* $\Psi_\pm$ *by the relation*

$$(\Psi_\pm f)(k) = (2\pi)^{-1/2}\bigg(\int_0^\infty \overline{\psi_1^{(\pm)}(t;k)}f(t)\,dt, \int_0^\infty \overline{\psi_2^{(\pm)}(t;k)}f(t)\,dt\bigg)^\top \tag{5.13}$$

on functions f *such that* $\langle \ln t\rangle^\beta f \in L^2(\mathbb{R}_+)$ *for some* $\beta > 1/2$. *Then* $\Psi_\pm f \in L^2(\mathbb{R}_+;\mathbb{C}^2)$, *and the operators* $\Psi_\pm$ *extend to bounded operators* $\Psi_\pm : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+;\mathbb{C}^2)$. *They satisfy the relations*

$$\Psi_\pm^*\Psi_\pm = I - E^{(\mathrm{p})}, \quad \Psi_\pm\Psi_\pm^* = I, \tag{5.14}$$

where $E^{(\mathrm{p})}$ is the orthogonal projection on the subspace $\mathcal{H}^{(\mathrm{p})}$ spanned by the eigenvectors of the operator H. The operators $\Psi_\pm$ diagonalize H, that is,

$$(\Psi_\pm Hf)(k) = \lambda(k)(\Psi_\pm f)(k), \quad f \in L^2(\mathbb{R}_+), \tag{5.15}$$

and are related to the wave operators $W_\pm$ by the formula

$$W_\pm = \Psi_\pm^* \Psi_0.$$

5.3. In terms of the wave operators (5.5), the scattering operator is defined by the formula $\mathbf{S} = W_+^* W_-$. By Theorem 5.1, the scattering operator commutes with H_0, that is, $\mathbf{S}H_0 = H_0\mathbf{S}$, and it is unitary.

To define the scattering matrix, we use the diagonalization (5.12) of the operator H_0. Since the operators $\mathbf{S}$ and H_0 commute, we have

$$(\Psi_0 \mathbf{S} f)(k) = S(k)(\Psi_0 f)(k), \quad k > 0,$$

where the (2×2) matrix

$$S(k) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix} \tag{5.16}$$

is known as the scattering matrix. The matrices $S(k)$ are unitary for all $k \in \mathbb{R}_+$, because the operator $\mathbf{S}$ is unitary. Observe that we parametrize the scattering matrix $S(k)$ by the quasi-momentum k.

The scattering matrix can be expressed in terms of the boundary values of the resolvent $R(z)$. Set

$$\Gamma_0(k)f = \sqrt{\gamma(k)}\big((Mf)(k), (Mf)(-k)\big)^\top, \tag{5.17}$$

where

$$\gamma(k) = |\lambda'(k)|^{-1} = \frac{\cosh^2(\pi k)}{\pi^2 \sinh(\pi k)}. \tag{5.18}$$

Then

$$\int_0^\pi \|\Gamma_0(k(\lambda))f\|^2_{\mathbb{C}^2}\, d\lambda = \int_{-\infty}^\infty |(Mf)(k)|^2\, dk = \|f\|^2.$$

Using, e.g., Theorem 5.5.3 and Proposition 7.4.1 in [10], we can state the following result.

Proposition 5.6. *Suppose that assumption* (5.1)*, is satisfied for some $\alpha > 1$. Then the scattering matrix admits the representation*

$$S(k) = I - 2\pi i \Gamma_0(k)\big(V - VR(\lambda(k) + i0)V\big)\Gamma_0^*(k), \quad \lambda(k) \notin \operatorname{spec}_p H. \tag{5.19}$$

Observe that the right-hand side of (5.19) can be written as a combination of bounded operators.

Formulas (5.13)–(5.15) mean that the functions $\psi_j^{(+)}(t;k)$ and $\psi_j^{(-)}(t;k)$, where $j = 1, 2$, give two "bases" in the "eigenspace" of H corresponding to the "eigenvalue" $\lambda(k)$. Since the absolutely continuous spectrum of H has multiplicity 2, it is natural to expect that the functions $\psi_j^{(-)}(t;k)$, $j = 1, 2$, are linear combinations of the functions $\psi_1^{(+)}(t;k)$ and $\psi_2^{(+)}(t;k)$ (and *vice versa*). It turns out that the link between these two "bases" is given by the elements of scattering matrix (5.16). The proof of the following assertion is very similar to the corresponding result of [11].

Proposition 5.7. *Suppose that assumption* (5.1) *is satisfied for some $\alpha > 1$. Then*

$$\begin{aligned} \psi_1^{(-)}(t;k) &= s_{11}(k)\psi_1^{(+)}(t;k) + s_{21}(k)\psi_2^{(+)}(t;k), \\ \psi_2^{(-)}(t;k) &= s_{12}(k)\psi_1^{(+)}(t;k) + s_{22}(k)\psi_2^{(+)}(t;k). \end{aligned} \tag{5.20}$$

Proof. From the resolvent identity (5.7) and formula (5.19) for the scattering matrix, it can be deduced that

$$\big(I - R(\lambda(k) + i0)V\big)\Gamma_0^*(k) = \big(I - R(\lambda(k) - i0)V\big)\Gamma_0^*(k)S(k). \tag{5.21}$$

Let $b = (b_1, b_2)^\top \in \mathbb{C}^2$. Relation (5.17) implies

$$(\Gamma_0^*(k)b)(t) = \sqrt{\gamma(k)}(2\pi)^{-1/2}(b_1 t^{-1/2+ik} + b_2 t^{-1/2-ik}).$$

Hence, by the definition (5.9),

$$((I - R(\lambda(k) \pm i0)V)\Gamma_0^*(k)b)(t) = \sqrt{\gamma(k)}(2\pi)^{-1/2}(b_1\psi_1^{(\mp)}(t;k) + b_2\psi_2^{(\mp)}(t;k)). \tag{5.22}$$

Thus, by (5.21),

$$b_1\psi_1^{(-)}(t;k)+b_2\psi_2^{(-)}(t;k) = (s_{11}(k)b_1+s_{12}(k)b_2)\psi_1^{(+)}(t;k)+(s_{21}(k)b_1+s_{22}(k)b_2)\psi_2^{(+)}(t;k).$$

Comparing the coefficients of b_1 and b_2, we arrive at formulas (5.20). □

Using the unitarity of $S(k)$, we can rewrite formulas (5.20) as

$$\psi_1^{(+)}(t;k) = \overline{s_{11}(k)}\psi_1^{(-)}(t;k) + \overline{s_{12}(k)}\psi_2^{(-)}(t;k),$$
$$\psi_2^{(+)}(t;k) = \overline{s_{21}(k)}\psi_1^{(-)}(t;k) + \overline{s_{22}(k)}\psi_2^{(-)}(t;k).$$

5.4. The representation (5.19) of the scattering matrix $S(k)$ can be rewritten in terms of the eigenfunctions $\psi_j(t;k) := \psi_j^{(-)}(t;k)$, $j = 1, 2$, of the operator H.

Proposition 5.8. *Suppose that assumption* (5.1) *is satisfied for some* $\alpha > 1$. *Then the elements of the scattering matrix* (5.16) *are given by the formulas*

$$s_{11}(k) = 1 - i\gamma(k)\int_0^\infty t^{-1/2-ik}(V\psi_1(k))(t)\,dt, \tag{5.23}$$

$$s_{12}(k) = -i\gamma(k)\int_0^\infty t^{-1/2+ik}(V\psi_1(k))(t)\,dt, \tag{5.24}$$

$$s_{21}(k) = -i\gamma(k)\int_0^\infty t^{-1/2-ik}(V\psi_2(k))(t)\,dt, \tag{5.25}$$

$$s_{22}(k) = 1 - i\gamma(k)\int_0^\infty t^{-1/2+ik}(V\psi_2(k))(t)\,dt, \tag{5.26}$$

where the coefficient $\gamma(k)$ *is defined by* (5.18).

Proof. Again, let $b = (b_1, b_2)^\top$. The definition (5.17) and formula (5.22) show that

$$2\pi i\Gamma_0(k)V(I - R(\lambda(k) + i0)V)\Gamma_0^*(k)b = i\gamma(k)\big(F_+(k)b, F_-(k)b\big)^\top,$$

where we have used the notation

$$F_\pm(k)b = \int_0^\infty t^{-1/2\mp ik}\big(V(b_1\psi_1(k) + b_2\psi_2(k))\big)(t)\,dt. \tag{5.27}$$

Now (5.19) implies that

$$S(k)b = b - i\gamma(k)\big(F_+(k)b, F_-(k)b\big)^\top.$$

In view of (5.27), this formula for the matrix $S(k)$ is equivalent to formulas (5.23)–(5.26) for its entries. □

Note that the matrices $S(k)$ depend continuously (actually, Hölder continuously) on $k > 0$ (away from the point spectrum of H).

Our next goal is to find asymptotics of the functions $\psi_j(t;k)$ as $t \to \infty$ and as $t \to 0$. We proceed from the Lippmann–Schwinger equation

$$\psi_j(k) = \psi_j^{(0)}(k) - R_0(\lambda(k) + i0)V\psi_j(k), \quad j = 1, 2, \tag{5.28}$$

for the functions $\psi_j(k)$. Recall that this equation is an immediate consequence of the resolvent identity (5.7) and the definition (5.9) of $\psi_j(k)$.

Proposition 5.9. *Suppose that assumption* (5.1) *is satisfied for some* $\alpha > 1$, *and that*

$$(VQ^\alpha g)(t) = o(t^{-1/2}), \quad g \in L^2(\mathbb{R}_+), \tag{5.29}$$

as $t \to \infty$ *and as* $t \to 0$ *for some* $\alpha > 1/2$. *Then*

$$\begin{aligned} &\psi_1(t;k) = s_{11}(k)t^{-1/2+ik} + o(t^{-1/2}), \quad t \to 0, \\ &\psi_1(t;k) = t^{-1/2+ik} + s_{12}(k)t^{-1/2-ik} + o(t^{-1/2}), \quad t \to \infty, \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} &\psi_2(t;k) = t^{-1/2-ik} + s_{21}(k)t^{-1/2+ik} + o(t^{-1/2}), \quad t \to 0, \\ &\psi_2(t;k) = s_{22}(k)t^{-1/2-ik} + o(t^{-1/2}), \quad t \to \infty, \end{aligned} \tag{5.31}$$

where the asymptotic coefficients are the entries of scattering matrix (5.16).

Proof. Set $f_j(t) = (V\psi_j(k))(t)$. Equation (5.28) shows that we only need to find asymptotics of the functions $(R_0(\lambda(k) + i0)f_j)(t)$ as $t \to \infty$ and as $t \to 0$. Combining assumption (5.1) for $\alpha > 1/2$ and relation (5.10), we see that the functions $f_j(t)$ obey condition (3.3). Moreover, $f_j(t) = o(t^{-1/2})$ if condition (5.29) is satisfied. Therefore, applying Proposition 3.3 to functions f_j, we see that

$$\big(R_0(\lambda(k) + i0)V\psi_j(k)\big)(t) = i\gamma(k)t^{-1/2+ik}\int_0^\infty s^{-1/2-ik}\big(V\psi_j(k)\big)(s)\,ds + o(t^{-1/2})$$

as $t \to 0$, and

$$\big(R_0(\lambda(k) + i0)V\psi_j(k)\big)(t) = i\gamma(k)t^{-1/2-ik}\int_0^\infty s^{-1/2+ik}\big(V\psi_j(k)\big)(s)\,ds + o(t^{-1/2})$$

as $t \to \infty$. Substituting these asymptotic relations in the right-hand side of equation (5.28) for $\psi_j(t;k)$ and using (5.23)–(5.26), we get formulas (5.30) and (5.31). □

Note that for the integral operators (5.2) relation (5.29) is true if

$$\int_0^\infty |v(t,s)|^2\langle \ln s\rangle^{2\alpha}\,ds = o(t^{-1}) \tag{5.32}$$

as $t \to \infty$ and $t \to 0$. For the Hankel operators (1.6), assumption (5.4) implies both conditions (5.3) and (5.32).

Like for the one-dimensional Schrödinger equation (see [3]), the asymptotic relations (5.30) and (5.31) can be interpreted in the following way. The solution $\psi_1(t;k)$ describes a wave propagating from $t = \infty$ to $t = 0$. By (5.30), we observe the transmitted part $s_{11}(k)t^{-1/2+ik}$ going to 0 and the reflected part $s_{12}(k)t^{-1/2-ik}$ going back to ∞. In the same way, the solution $\psi_2(t;k)$ describes a wave propagating from $t = 0$ to $t = \infty$. It is natural to call $s_{11}(k)$, $s_{22}(k)$ the transmission coefficients and to call $s_{11}(k)$, $s_{22}(k)$ the reflection coefficients. The squares $|s_{jl}(k)|^2$ give probabilities of the corresponding processes. As might be expected, $|s_{11}(k)|^2 + |s_{12}(k)|^2 = 1$ and $|s_{21}(k)|^2 + |s_{22}(k)|^2 = 1$.

5.5. We always suppose that the operators V are selfadjoint, which corresponds to the condition

$$\mathbf{v}(t,s) = \overline{\mathbf{v}(s,t)} \tag{5.33}$$

for the integral operators (5.2). Let the complex conjugation $\mathcal{C}$ be defined by (2.33). Assume additionally that the operator V commutes with $\mathcal{C}$, that is,

$$\mathcal{C}V = V\mathcal{C}, \tag{5.34}$$

whence $\mathcal{C}H = H\mathcal{C}$. In terms of kernels, this means that $\mathbf{v}(t,s)$ is a real function; the selfadjointness condition (5.33) shows that this is equivalent to the identity

$$\mathbf{v}(t,s) = \mathbf{v}(s,t). \tag{5.35}$$

For the selfadjoint Hankel operators (1.6), this condition is always satisfied.

Under assumption (5.34), the wave operators (5.5) obey the identity $\mathcal{C}W_\pm = W_\mp\mathcal{C}$, and hence

$$\mathcal{C}\mathbf{S} = \mathbf{S}^*\mathcal{C}. \tag{5.36}$$

Set $\mathcal{J}(b_1,b_2)^\top = (b_2,b_1)^\top$. Since the operator (5.11) satisfies the identity $\mathcal{C}\Psi_0 = \mathcal{J}\Psi_0\mathcal{C}$, from (5.36) it follows that

$$\mathcal{J}\mathcal{C}S(k) = S^*(k)\mathcal{J}\mathcal{C}$$

for all $k > 0$. It is easily seen that the last identity is equivalent to

$$s_{11}(k) = s_{22}(k), \quad k > 0. \tag{5.37}$$

Alternatively, identity (5.37) can be deduced from (5.19) if we take into account that $\mathcal{C}R(z) = R(\bar z)\mathcal{C}$.

Observe also that under assumption (5.34), the eigenfunctions (5.9) of the operator H are linked by the relations $\overline{\psi_1^{(+)}(t;k)} = \psi_2^{(-)}(t;k)$ and $\overline{\psi_2^{(+)}(t;k)} = \psi_1^{(-)}(t;k)$.

§6. The discrete spectrum above the continuous spectrum

Here we study the spectrum of the perturbed Carleman operator $H = H_0 + V$ lying above the point $\lambda = \pi$. We prove that, under natural conditions on the kernel $v(t)$, it consists of a finite number of eigenvalues. We also show that for $V > 0$ the operator H has at least one eigenvalue larger than π. In this section we suppose that for the spectral parameter we have $\lambda \geq \pi$.

6.1. In accordance with to Proposition 3.4, the nature of the singularity of the free resolvent $R_0(z)$ at the point $z = \pi$ is the same as that of the resolvent of the operator D^2 acting in the space $L^2(\mathbb{R})$ at the point $z = 0$. So, one can expect that the results on the spectrum of the perturbed Carleman operator $H = H_0 + V$ above the point π are qualitatively similar to those on the negative spectrum of the Schrödinger operator $D^2 + \mathsf{V}(x)$. Here we verify this conjecture.

We proceed from the Birman–Schwinger principle, which in our case is formulated as follows.

Proposition 6.1. *Let H_0 be a bounded selfadjoint operator such that $H_0 \leq \pi$. Suppose that $V \geq 0$ and $V \in \mathfrak{S}_\infty$. Then the total number $N(\lambda)$ of eigenvalues of the operator $H = H_0 + V$ that are larger*[2] *than $\lambda > \pi$ equals the total number of eigenvalues of the operator $B(\lambda) = V^{1/2}(\lambda - H_0)^{-1}V^{1/2}$ that are larger than 1.*

[2]Larger is strictly larger.

6.2. Formula (2.22) shows that

$$B(\lambda) = \lambda^{-1}\big(V + V^{1/2}\mathbf{A}(\lambda)V^{1/2}\big). \tag{6.1}$$

Recall that the integral kernel $\mathbf{a}(t,s;\lambda)$ of the operator $\mathbf{A}(\lambda)$ is given by formula (2.23), where $\rho(u;\lambda)$ is the function (3.6). For $\lambda \geq \pi$, the function (2.19) equals

$$\mathbf{k}(\lambda) = \pi^{-1} i \arg\big(\pi - i\sqrt{\lambda^2 - \pi^2}\big),$$

whence the function (3.5) is equal to

$$\theta(\lambda) = \pi^{-1} \arctan\big(\pi^{-1}\sqrt{\lambda^2 - \pi^2}\big) \in [0, 1/2).$$

The expansion (3.8) shows that the kernel $\mathbf{a}(t,s;\lambda)$ is singular as $\lambda \to \pi$. Distinguishing the first (singular) term, we can rewrite formula (2.23) as

$$\mathbf{a}(t,s;\lambda) = \frac{1}{\sqrt{\lambda^2 - \pi^2}}(ts)^{-1/2} + \widetilde{\mathbf{a}}(t,s;\lambda), \tag{6.2}$$

where

$$\widetilde{\mathbf{a}}(t,s;\lambda) = \frac{1}{\sqrt{\lambda^2 - \pi^2}}(ts)^{-1/2}\widetilde{\rho}(t/s;\lambda)\big) \tag{6.3}$$

and

$$\widetilde{\rho}(u;\lambda) = \rho(u;\lambda) - 1 = \frac{u^{\theta(\lambda)} - 1}{1 - u^2} + \frac{u^{-\theta(\lambda)} - 1}{1 - u^{-2}}. \tag{6.4}$$

Let $\widetilde{\mathbf{A}}(\lambda)$ be the integral operator with the kernel $\widetilde{\mathbf{a}}(t,s;\lambda)$. We show that it has a limit $\widetilde{\mathbf{A}}(\pi) =: \widetilde{A}$ as $\lambda \to \pi$. We denote by $\|A\|_2$ the norm of an operator $A \in \mathfrak{S}_2$ in the Hilbert–Schmidt class $\mathfrak{S}_2$.

Lemma 6.2. *Let $\widetilde{A}$ be the integral operator with the kernel*

$$\widetilde{a}(t,s) = \pi^{-2}(ts)^{-1/2}\sigma_1(t/s), \tag{6.5}$$

where the function σ_1 is defined by formula (3.7). *Then for all $\alpha > 3/2$, we have*

$$\lim_{\lambda\to\pi} \|Q^{-\alpha}\big(\widetilde{\mathbf{A}}(\lambda) - \widetilde{A}\big)Q^{-\alpha}\|_2 = 0. \tag{6.6}$$

Proof. We must check that

$$\lim_{\lambda\to\pi} \int_0^\infty \int_0^\infty \langle \ln t\rangle^{-2\alpha} |\widetilde{\mathbf{a}}(t,s;\lambda) - \widetilde{a}(t,s)|^2 \langle \ln s\rangle^{-2\alpha}\, dt\, ds = 0. \tag{6.7}$$

Since

$$\lim_{\lambda\to\pi} \frac{\widetilde{\rho}(u;\lambda)}{\sqrt{\lambda^2 - \pi^2}} = \pi^{-2} \lim_{\lambda\to\pi} \big(\theta^{-1}(\lambda)\widetilde{\rho}(u;\lambda)\big) = \pi^{-2}\sigma_1(u),$$

the integrand in (6.7) tends to zero for all $t > 0$ and $s > 0$. We shall show that

$$|\widetilde{\mathbf{a}}(t,s;\lambda)| \leq C(ts)^{-1/2}(1 + |\ln(t/s)|), \tag{6.8}$$

where C does not depend on $\lambda \geq \pi$. Then the integrand in (6.7) is bounded by the function

$$C_1(ts)^{-1}\langle \ln t\rangle^{-2\alpha}\langle \ln s\rangle^{-2\alpha}\big(\langle \ln t\rangle^2 + \langle \ln s\rangle^2\big),$$

which belongs to $L^1(\mathbb{R}^2)$ if $2\alpha > 3$. By the Lebesgue dominated convergence theorem, this implies (6.7).

In accordance with (6.3), for the proof of (6.8) we need to check that the function (6.4) satisfies

$$\widetilde{\rho}(u;\lambda) \leq C\theta(\lambda)(1 + |\ln u|), \quad u \in \mathbb{R}_+. \tag{6.9}$$

Since $\widetilde{\rho}(u^{-1};\lambda)=\widetilde{\rho}(u;\lambda)$, it suffices to consider $u\ge 1$. If $\theta\ln u\le 1$, we use the inequalities

$$\left|\frac{u^{\pm\theta}-1}{1-u^{\pm 2}}\right|\le C\theta\frac{\ln u}{|1-u^{\pm 2}|}\le C_1\theta(1+\ln u),\quad u\ge 1.$$

If $\theta\ln u\ge 1$ (and thus $u\ge e^2$), then the absolute value of the first term on the right-hand side of (6.4) is estimated by $u^\theta(u^2-1)^{-1}$, which is bounded by a constant because $\theta<1/2$. The absolute value of the second term on the right-hand side of (6.4) is estimated by $(1-u^{-2})^{-1}\le(1-e^{-4})^{-1}$. This concludes the proof of estimate (6.9) and, with it, of relation (6.7). □

We return to the operator (6.1). Using (6.2), we see that

$$B(\lambda)=\frac{1}{\lambda\sqrt{\lambda^2-\pi^2}}(\cdot,w)w+\widetilde{B}(\lambda),\tag{6.10}$$

where $\psi_0(t)=t^{-1/2}$, $w=V^{1/2}\psi_0$, and

$$\widetilde{B}(\lambda)=\lambda^{-1}\bigl(V+V^{1/2}\widetilde{\mathbf{A}}(\lambda)V^{1/2}\bigr).$$

Observe that under assumption (5.1) the operator $V^{1/2}Q^\alpha$ is bounded, and hence $w\in L^2(\mathbb{R}_+)$ if $\alpha>1/2$. Consider the operator $\widetilde{B}(\lambda)$. The following assertion is a direct consequence of Lemma 6.2.

Lemma 6.3. *Suppose that $V\ge 0$, $V\in\mathfrak{S}_2$, and that* (5.1) *is true for some $\alpha>3/2$. Then the operator $\widetilde{B}(\lambda)$ has the limit*

$$\widetilde{B}(\pi)=\pi^{-1}\bigl(V+V^{1/2}\widetilde{A}V^{1/2}\bigr)\tag{6.11}$$

in the Hilbert–Schmidt norm as $\lambda\to\pi$.

6.3. Now we return to the study of the discrete spectrum of the operator H. Let $\widetilde{n}(\lambda)$ (and $n(\lambda)$) be the total number of eigenvalues of the operator $\widetilde{B}(\lambda)$ (respectively, $B(\lambda)$) that are larger than 1. By (6.10), we have

$$\widetilde{n}(\lambda)\le n(\lambda)\le\widetilde{n}(\lambda)+1.$$

Therefore, from Proposition 6.1 it follows that

$$\widetilde{n}(\lambda)\le N(\lambda)\le\widetilde{n}(\lambda)+1.$$

Since $\widetilde{n}(\lambda)\le\|\widetilde{B}(\lambda)\|_2^2$, Lemma 6.3 implies that the total number $N=N(\pi)$ of eigenvalues of the operator H lying above the point π satisfies the bound

$$N\le\|\widetilde{B}(\pi)\|_2^2+1.$$

In accordance wit (6.11), we have

$$\|\widetilde{B}(\pi)\|_2\le\pi^{-1}\bigl(\|V\|_2+\|Q^\alpha VQ^\alpha\|\|Q^{-\alpha}\widetilde{A}Q^{-\alpha}\|_2\bigr).$$

Using (6.5), we find that

$$\|Q^{-\alpha}\widetilde{A}Q^{-\alpha}\|_2^2=\pi^{-4}\int_0^\infty\int_0^\infty\langle\ln t\rangle^{-2\alpha}(ts)^{-1}\sigma_1(t/s)^2\langle\ln s\rangle^{-2\alpha}\,dt\,ds=:\gamma_\alpha^2.\tag{6.12}$$

This yields the bound

$$N\le\pi^{-2}\bigl(\|V\|_2+\gamma_\alpha\|Q^\alpha VQ^\alpha\|\bigr)^2+1.\tag{6.13}$$

Observe further that, by the representation (6.10) in the case where $w\ne 0$, the operator $B(\lambda)$ has an eigenvalue that tends to $+\infty$ as $\lambda\to\pi$. Therefore, Proposition 6.1

shows that the operator H has at least one eigenvalue above the point $\lambda = \pi$. Note also that

$$\|w\|^2 = (V\psi_0, \psi_0) = \int_0^\infty \int_0^\infty \mathbf{v}(t,s)(ts)^{-1/2}\,dt\,ds. \tag{6.14}$$

In particular, in the Hankel case $\mathbf{v}(t,s) = v(t+s)$, we have

$$\|w\|^2 = \pi \int_0^\infty v(t)\,dt.$$

Since a nonnegative Hankel operator V has a necessarily nonnegative kernel $v(t)$, if $w = 0$, then $V = 0$, so that $w \neq 0$ if V is nontrivial.

We summarize the results obtained.

Theorem 6.4. *Suppose that $V \geq 0$, $V \in \mathfrak{S}_2$, and that* (5.1) *is true for some $\alpha > 3/2$. Then the total number N of eigenvalues of the operator H that are larger than π is finite and satisfies the bound* (6.13)*, where the constant γ_α is defined by* (6.12) *and* (3.7)*.*

Suppose additionally that the integral (6.14) *is not zero* (*or that $V \neq 0$ in the Hankel case*)*. Then the operator H has at least one eigenvalue larger than π.*

If V has a negative part, then of course this can only diminish the number N. In this case, V on the right-hand side of (6.13) should be replaced by $V_+ = (V + |V|)/2$.

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