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ON STRONG MEANS OF SPHERICAL FOURIER SUMS

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Abstract. The spherical Fourier sums

$$S_n(f,x) = \sum_{\|k\|\le n} \widehat{f}(k)\, e^{ik\cdot x}$$

of a periodic function f in m variables and their strong means

$$H_{n,p}(f,x) = \left(\frac{1}{n}\sum_{j=0}^{n-1} |S_j(f,x)|^p\right)^{\frac{1}{p}} \quad \text{for } p \ge 1$$

are considered. In contrast to the one-dimensional case treated by Hardy and Littlewood, for $m \ge 2$ the norms $\sup_{|f|\le 1} H_{n,p}(f,0)$ are not bounded. The sharp order of growth of these norms is found (the upper and lower bounds differ by a factor depending only on the dimension m).

§1. Introduction

Given a function f integrable on the cube $\mathbb{T}^m = [-\pi,\pi]^m$, the spherical partial sum $S_R(f,x)$ of the Fourier series of f has the form (hereafter $k \in \mathbb{Z}^m$)

$$S_R(f,x) = \sum_{\|k\|\le R} \widehat{f}(k)\, e^{ik\cdot x},$$

where

$$\widehat{f}(k) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(u)\, e^{-ik\cdot u}\, du.$$

This is the convolution of f with the spherical Dirichlet kernel

$$D_R(x) = \frac{1}{(2\pi)^m}\sum_{\|k\|\le R} e^{-ik\cdot x}.$$

The L_1-norm

$$\mathcal{L}_R = \int_{\mathbb{T}^m} |D_R(x)|\, dx = \sup_{|f|\le 1} |S_R(f,0)|$$

of this kernel is called the Lebesgue constant. In the multidimensional case ($m > 1$), the two-sided bound $A_m R^{\frac{m-1}{2}} \le \mathcal{L}_R \le B_m R^{\frac{m-1}{2}}$ is valid for $R \ge 1$ (see [1, 2]).

The strong spherical means of the Fourier series are defined as follows (hereafter $p \ge 1$ and $n \in \mathbb{N}$):

$$H_{n,p}(f,x) = \left(\frac{1}{n}\sum_{j=0}^{n-1} |S_j(f,x)|^p\right)^{\frac{1}{p}} = \frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\left|\sum_{j=0}^{n-1}\varepsilon_j S_j(f,x)\right|.$$

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Here, the supremum is taken over all collections $\varepsilon = \{\varepsilon_0, \dots, \varepsilon_{n-1}\}$ of real numbers satisfying the condition

$$|\varepsilon|_q = \left(|\varepsilon_0|^q + \dots + |\varepsilon_{n-1}|^q\right)^{\frac{1}{q}} \le 1 \ \left(q \text{ is the conjugate exponent of } p\text{: } \frac{1}{p} + \frac{1}{q} = 1\right).$$

Our purpose is to estimate the norms of the corresponding operators, i.e., the quantities

$$H_{n,p} = \sup_{|f|\le 1} H_{n,p}(f,0) = \frac{1}{n^{\frac{1}{p}}} \sup_{|\varepsilon|_q \le 1} \int_{\mathbb{T}^m} \left| \sum_{j=0}^{n-1} \varepsilon_j D_j(x) \right| dx.$$

Note that, by Hölder's inequality, the means $H_{n,p}(f)$, and hence the norms $H_{n,p}$, increase with p.

The notion of strong summability of Fourier series was introduced by Hardy and Littlewood [3] in the one-dimensional case precisely one hundred years ago. They proved that for $m = 1$ and a fixed p, the norms $H_{n,p}$ are bounded.

In the multidimensional case, the situation is different. For $m \ge 3$, the norms $H_{n,p}$ are not bounded, being of order of $n^{\frac{m-1}{2} - \min\{\frac{1}{2}, \frac{1}{p}\}}$ (see [4, 5]). In the two-dimensional case, the results are not complete: this two-sided bound still holds true for any fixed $p > 2$ (see [6]), while for $p \in [1,2]$ only the upper bound $H_{n,p} \le c\sqrt{\log(n+1)}$ was known [7].

§2. The main result

Our purpose in this paper is, on the one hand, to simplify the proofs of the results mentioned above, and, on the other hand, to complete the study of the two-dimensional case, which has turned out to be the most interesting. Thus, we prove that[1]

$$H_{n,p} \asymp \begin{cases} n^{\frac{m-1}{2} - \min\{\frac{1}{2}, \frac{1}{p}\}} & \text{if } m \ge 3,\ p \ge 1; \\ n^{\frac{1}{2} - \frac{1}{p}} \min^{\frac{1}{p}}\left\{\log(n+1), \frac{1}{p-2}\right\} & \text{if } m = 2,\ p \ge 2; \\ \sqrt{\log(n+1)} & \text{if } m = 2,\ p \in [1,2]. \end{cases}$$

Note that in the two-dimensional case, for "large" p (with $p-2$ greater than a fixed positive number) the factor $\min^{\frac{1}{p}}\left\{\log(n+1), \frac{1}{p-2}\right\}$ can be replaced by 1.

§3. Reducing $H_{n,p}$ to a one-dimensional integral

Since the kernel D_R cannot be written in a compact form, we obtain an approximate integral representation for D_R as follows: we replace summation over the ball $B(R) = \{\|y\| \le R\}$ by integration over a set close to this ball (this trick has been known for a long time; see, e.g., [8, the proof of Theorem 2.29 in Chapter V]). For this, we use the identity

$$e^{-iat} = \frac{t/2}{\sin t/2} \int_{a-\frac{1}{2}}^{a+\frac{1}{2}} e^{-ist}\, ds.$$

In the multidimensional case, for the shifted unit cube $Q_k = k + [-\frac{1}{2}, \frac{1}{2}]^m$, at a point $x = (x_1, \dots, x_m)$ we have

$$e^{-ik\cdot x} = \theta(x) \int_{Q_k} e^{-iy\cdot x}\, dy, \quad \text{where} \quad \theta(x) = \frac{x_1/2}{\sin x_1/2} \cdots \frac{x_m/2}{\sin x_m/2}.$$

[1]The notation $\alpha_n \asymp \beta_n$ means that $\alpha_n = O(\beta_n)$ and $\beta_n = O(\alpha_n)$ simultaneously. Instead of $\alpha_n = O(\beta_n)$ we also write $\alpha_n \ll \beta_n$. Hereafter, the constants in the corresponding inequalities valid for all n may depend only on the dimension m.

Setting $T(R)=\bigcup_{\|k\|\le R} Q_k$, we arrive at the identity

$$D_R(x)=\frac{\theta(x)}{(2\pi)^m}\int_{T(R)} e^{-iy\cdot x}\,dy=\frac{\theta(x)}{(2\pi)^m}\biggl(\int_{B(R)} e^{-iy\cdot x}\,dy+\int_{\mathbb{R}^m}\eta_R(y)\,e^{-iy\cdot x}\,dy\biggr),$$

where $\eta_R=\chi_{T(R)}-\chi_{B(R)}$ is the difference of the characteristic functions of the sets $T(R)$ and $B(R)$. Thus, the Dirichlet kernel can be written in terms of the Fourier transforms of the functions $\chi_{B(R)}$ and η_R: $D_R(x)=\theta(x)\bigl(\widehat{\chi}_{B(R)}(x)+\widehat{\eta}_R(x)\bigr)$. Therefore,

$$\sum_{j=0}^{n-1}\varepsilon_j D_j(x)=\theta(x)\biggl(\sum_{j=0}^{n-1}\varepsilon_j\widehat{\chi}_{B(j)}(x)+\widehat{\Delta}_n(x)\biggr),\quad\text{where}\quad \Delta_n=\sum_{j=0}^{n-1}\varepsilon_j\eta_j.$$

The contribution of the error term Δ_n can be estimated with the help of the Plancherel theorem:

$$\int_{\mathbb{T}^m}|\theta(x)\widehat{\Delta}_n(x)|\,dx\ll\|\widehat{\Delta}_n\|_{L_2(\mathbb{R}^m)}=\|\Delta_n\|_{L_2(\mathbb{R}^m)}=\sqrt{\sum_{0\le j,l<n}\varepsilon_j\varepsilon_l\int_{\mathbb{R}^m}\eta_j(y)\eta_l(y)\,dy}.$$

Since $|\eta_j|\le 1$ and the support of η_j lies in the annulus $j-\frac{\sqrt{m}}{2}\le\|y\|\le j+\frac{\sqrt{m}}{2}$, we have

$$\sum_{0\le j,l<n}\varepsilon_j\varepsilon_l\int_{\mathbb{R}^m}\eta_j(y)\eta_l(y)\,dy\ll\sum_{|j-l|\le\sqrt{m}}\frac{\varepsilon_j^2+\varepsilon_l^2}{2}n^{m-1}\ll n^{m-1}\sum_{j=0}^{n-1}\varepsilon_j^2.$$

This gives the following estimate: $\int_{\mathbb{T}^m}|\theta(x)\widehat{\Delta}_n(x)|\,dx\ll n^{\frac{m-1}{2}}|\varepsilon|_2$. Since $|\varepsilon|_2\le|\varepsilon|_q\le 1$ for $p\ge 2$ and $|\varepsilon|_2\le n^{\frac{1}{p}-\frac{1}{2}}|\varepsilon|_q\le n^{\frac{1}{p}-\frac{1}{2}}$ for $p\le 2$, finally we obtain

$$H_{n,p}=\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\int_{\mathbb{T}^m}\theta(x)\biggl|\sum_{j=0}^{n-1}\varepsilon_j\widehat{\chi}_{B(j)}(x)\biggr|\,dx+O\bigl(n^{\varkappa}\bigr),$$

where

$$\varkappa=\frac{m-1}{2}-\min\Bigl\{\frac{1}{p},\frac{1}{2}\Bigr\}.$$

It is known (see [9, Theorem 4.15 in Chapter IV]) that the Fourier transform of the characteristic function of a ball can be written in terms of Bessel functions:

$$\widehat{\chi}_{B(j)}(x)=j^m\widehat{\chi}_{B(1)}(jx)=\Bigl(\frac{j}{2\pi\|x\|}\Bigr)^{\frac{m}{2}}J_{\frac{m}{2}}(j\|x\|).$$

Since $\theta(x)\le\bigl(\frac{\pi}{2}\bigr)^m$, replacing the cube $\mathbb{T}^m$ with the ball of radius $m\pi$ in which it is contained and passing to the spherical coordinates, we obtain an upper bound for $H_{n,p}$:

$$H_{n,p}\ll\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\int_0^{m\pi}\biggl|\sum_{j=0}^{n-1}\varepsilon_j(jt)^{\frac{m}{2}}J_{\frac{m}{2}}(jt)\biggr|\frac{dt}{t}+n^{\varkappa}.$$

To proceed further, we use the following asymptotic formula for $J_{\frac{m}{2}}(r)$ (see, e.g., [9, Lemma 3.11 in Chapter IV]):

$$J_{\frac{m}{2}}(r)=\sqrt{\frac{2}{\pi r}}\cos\Bigl(r-\frac{\pi}{4}(m+1)\Bigr)+O\Bigl(\frac{1}{r^{3/2}}\Bigr)\quad\text{as}\quad r\to+\infty.$$

This formula gives a good description for the behavior of the Bessel function at infinity, but for small r its error term is too large. However, since the Bessel function is bounded on $[0,\infty)$, it follows that the error term is also $O\bigl(\frac{1}{(1+r)\sqrt{r}}\bigr)$ uniformly on $(0,\infty)$. Using

the asymptotic formula for $J_{\frac{m}{2}}$ with the error term refined in this way, we see that its contribution to the upper bound of $H_{n,p}$ obtained above does not exceed

$$\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\sum_{j=0}^{n-1}|\varepsilon_j|\int_0^{m\pi}\frac{(jt)^{\frac{m-1}{2}}}{1+jt}\,\frac{dt}{t}\le\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}|\varepsilon|_1\int_0^{nm\pi}\frac{u^{\frac{m-3}{2}}}{1+u}\,du.$$

Clearly, $|\varepsilon|_1\le n^{\frac{1}{p}}|\varepsilon|_q\le n^{\frac{1}{p}}$, and the integral $\int_0^{nm\pi}\frac{u^{\frac{m-3}{2}}}{1+u}\,du$ is $O\big(n^{\frac{m-3}{2}}\big)$ for $m>3$, $O(\log n)$ for $m=3$, and $O(1)$ for $m=2$. This allows us to rewrite the upper bound for $H_{n,p}$ as follows:

$$H_{n,p}\ll\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\int_0^{m\pi}\left|\sum_{j=0}^{n-1}\varepsilon_j j^{\frac{m-1}{2}}\cos\Big(jt-\frac{\pi}{4}(m+1)\Big)\right|t^{\frac{m-3}{2}}\,dt+n^{\varkappa}.$$

In a similar way, since $\theta(x)\ge 1$, taking a ball contained in $\mathbb{T}^m$, we can obtain a lower bound for $H_{n,p}$. We shall need it only in the two-dimensional case: for $m=2$ we have

$$(*)\qquad H_{n,p}+n^{\frac{1}{2}-\min\{\frac{1}{p},\frac{1}{2}\}}\gg\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\int_0^{\pi}\left|\sum_{j=0}^{n-1}\varepsilon_j\sqrt{j}\cos\Big(jt+\frac{\pi}{4}\Big)\right|\frac{dt}{\sqrt{t}}.$$

Thus, to find the order of $H_{n,p}$, we should estimate the integral norm with the weight $t^{\frac{m-3}{2}}$ of a "random" trigonometric polynomial. In the two-dimensional case, this weight does not lie in the L_2-space, which not only makes the argument more difficult, but also leads to a more complicated asymptotics of $H_{n,p}$.

§4. Estimating $H_{n,p}$ from above

1. The case where $m\ge 3$. Since the norms $H_{n,p}$ grow with p, in order to prove the relation $H_{n,p}\ll n^{\varkappa}=n^{\frac{m}{2}-1}$ for $p\in[1,2]$, it suffices to establish it for $p=2$. Therefore, we assume that $p\ge 2$ throughout this section. We check the relation

$$H_{n,p}\ll n^{\varkappa}=n^{\frac{m-1}{2}-\frac{1}{p}}.$$

For $m\ge 3$, the weight $t^{\frac{m-3}{2}}$ is bounded; hence, the upper bound of $H_{n,p}$ obtained above implies that

$$H_{n,p}\ll\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\int_0^{\pi}\left|\sum_{j=0}^{n-1}\varepsilon_j j^{\frac{m-1}{2}}\cos\Big(jt-\frac{\pi}{4}(m+1)\Big)\right|dt+n^{\varkappa}.$$

Parseval's theorem and the inequality $|\varepsilon|_2\le|\varepsilon|_q\le 1$ (for $q\le 2$) yield a desired bound:

$$H_{n,p}\ll\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\sqrt{\sum_{j=0}^{n-1}\varepsilon_j^2 j^{m-1}}+n^{\varkappa}\le n^{\frac{m-1}{2}-\frac{1}{p}}+n^{\varkappa}=2n^{\varkappa}.$$

2. In the case where $m=2$, we must estimate the integral

$$A_n(\varepsilon)=\int_0^{\pi}\left|\sum_{j=0}^{n-1}\varepsilon_j\sqrt{j}\cos\Big(jt-\frac{3}{4}\pi\Big)\right|\frac{dt}{\sqrt{t}}=\int_0^{\pi}\big|T_n(t)\big|\,\frac{dt}{\sqrt{t}}$$

from above. Parseval's theorem is no longer applicable, because the weight $\frac{1}{\sqrt{t}}$ does not lie in $L_2(0,\pi)$.

For $p\ge 2$, the integral over $(0,\frac{\pi}{n})$ is $O(n^{\frac{1}{p}})$ because $|T_n(t)|\le n^{\frac{1}{2}+\frac{1}{p}}|\varepsilon|_q\le n^{\frac{1}{2}+\frac{1}{p}}$. The integral over $(\frac{\pi}{n},\pi)$ can be estimated via Hölder's inequality:

$$A_n(\varepsilon)\le\|T_n\|_{L_p(0,\pi)}\left(\int_{\frac{\pi}{n}}^{\pi}t^{-\frac{q}{2}}\,dt\right)^{\frac{1}{q}}+O\big(n^{\frac{1}{p}}\big).$$

Elementary calculations show that the integral in this formula is less than $2\pi\frac{1-n^{q-2}}{2-q} \asymp \min\{\log n, \frac{1}{2-q}\}$. The integral norm of the sum T_n can be estimated from above by the l_q norm of its coefficients, thanks to the invertibility of the Hausdorff–Young inequality (see [8, inequality (2.17) in Chapter XII]):

$$\|T_n\|_{L_p(0,\pi)} \ll \left(\sum_{j=0}^{n-1} |\varepsilon_j|^q j^{\frac{q}{2}}\right)^{\frac{1}{q}} \le \sqrt{n}|\varepsilon|_q \le \sqrt{n}.$$

Hence, $A_n(\varepsilon) \ll \sqrt{n}\min^{\frac{1}{q}}\{\log n, \frac{1}{2-q}\} + n^{\frac{1}{p}} \ll \sqrt{n}\min^{\frac{1}{p}}\{\log n, \frac{1}{p-2}\}$. So, for $n>1$,

$$H_{n,p} \ll n^{-\frac{1}{p}} \sup_{|\varepsilon|_q\le 1} A_n(\varepsilon) + n^{\varkappa} \ll n^{\frac{1}{2}-\frac{1}{p}}\min^{\frac{1}{p}}\Big\{\log n, \frac{1}{p-2}\Big\}$$

(because $\varkappa = \frac{1}{2} - \frac{1}{p}$ for $p \ge 2$). In particular, $H_{n,2} \ll \sqrt{\log n}$. This estimate is also true for $H_{n,p}$ with $p \in [1,2]$, because $H_{n,p} \le H_{n,2}$ for such p.

§5. Estimating $H_{n,p}$ from below

Like in the preceding section, the two-dimensional case will be considered separately.

1. The case where $m \ge 3$. If $p \ge 2$, then the argument is easy: it suffices to keep only the term with D_{n-1} (i.e., take $\varepsilon_0 = \varepsilon_1 = \dots = \varepsilon_{n-2} = 0$) in the definition of $H_{n,p}$. Then $H_{n,p} \ge n^{-\frac{1}{p}}\|D_{n-1}\|_{L_1(\mathbb{T}^m)} = n^{-\frac{1}{p}}\mathcal{L}_{n-1}$, and, from the lower bound for the Lebesgue constant it follows that $H_{n,p} \gg n^{\frac{m-1}{2}-\frac{1}{p}}$.

For p in the interval $[1,2]$, we must check that $H_{n,p} \gg n^{\frac{m}{2}-1}$. Since $H_{n,p} \ge H_{n,1}$, it suffices to estimate only $H_{n,1}$. In this case, $q = +\infty$ and

$$H_{n,1} = \frac{1}{n}\sup_{\varepsilon_j=\pm1}\int_{\mathbb{T}^m}\left|\sum_{j=0}^{n-1}\varepsilon_j D_j(x)\right| dx \ge \frac{1}{n}\int_{\mathbb{T}^m}\left(\frac{1}{2^n}\sum_{\varepsilon_j=\pm1}\left|\sum_{j=0}^{n-1}\varepsilon_j D_j(x)\right|\right) dx.$$

Khintchine's inequality yields

$$H_{n,1} \ge \frac{1}{2n}\int_{\mathbb{T}^m}\sqrt{\sum_{j=0}^{n-1} D_j^2(x)}\, dx \ge \frac{1}{2n^{\frac{3}{2}}}\int_{\mathbb{T}^m}\sum_{j=0}^{n-1}\big|D_j(x)\big|\, dx = \frac{1}{2n^{\frac{3}{2}}}\sum_{j=0}^{n-1}\mathcal{L}_j.$$

Again using the estimate for the Lebesgue constants, we see that $H_{n,1} \gg n^{\frac{m}{2}-1}$.

2. The case where $m = 2$ requires greater effort. First, let $p \ge 2$. Then, of course, the argument used for $m \ge 3$ is still valid: $H_{n,p} \ge n^{-\frac{1}{p}}\mathcal{L}_{n-1} \gg n^{\frac{1}{2}-\frac{1}{p}}$. Hence for "large" p (with $p-2$ positive and bounded away from zero), we have $H_{n,p} \asymp n^{\frac{1}{2}-\frac{1}{p}}$. However, as p approaches 2, this lower bound becomes useless. To refine it, we employ inequality $(*)$ established at the end of §3; for $p \ge 2$ this yields

$$H_{n,p} + n^{\frac{1}{2}-\frac{1}{p}} \gg \frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\le 1}\int_0^\pi \cos\Big(nt+\frac{\pi}{4}\Big)\sum_{j=0}^{n-1}\varepsilon_j\sqrt{j}\cos\Big(jt+\frac{\pi}{4}\Big)\frac{dt}{\sqrt{t}}.$$

Interchanging the integral and the sum, we see that

$$H_{n,p} + n^{\frac{1}{2}-\frac{1}{p}} \gg \frac{1}{n^{\frac{1}{p}}}\left(\sum_{j=0}^{n-1}\Big|\sqrt{j}\int_0^\pi \cos\Big(nt+\frac{\pi}{4}\Big)\cos\Big(jt+\frac{\pi}{4}\Big)\frac{dt}{\sqrt{t}}\Big|^p\right)^{\frac{1}{p}}.$$

An easy calculation gives

$$\int_0^\pi \cos\Big(nt+\frac{\pi}{4}\Big)\cos\Big(jt+\frac{\pi}{4}\Big)\frac{dt}{\sqrt{t}} = \frac{1}{2\sqrt{n-j}}\int_0^{(n-j)\pi}\frac{\cos t}{\sqrt{t}}\, dt + O\Big(\frac{1}{\sqrt{n}}\Big).$$

Integrating by parts twice, we conclude that the last integral is bounded away from zero (hereafter $\nu = n - j \in \mathbb{N}$):

$$\int_0^{\nu\pi} \frac{\cos t}{\sqrt{t}}\, dt = \frac{1-(-1)^\nu}{2(\nu\pi)^{3/2}} + \frac{3}{4}\int_0^{\nu\pi} \frac{1-\cos t}{t^{5/2}}\, dt \ge \frac{3}{4}\int_0^{\pi} \frac{1-\cos t}{t^{5/2}}\, dt.$$

Thus, for some $A, B > 0$ we have

$$\int_0^{\pi} \cos\Big(nt + \frac{\pi}{4}\Big)\cos\Big(jt + \frac{\pi}{4}\Big)\frac{dt}{\sqrt{t}} \ge \frac{2A}{\sqrt{n-j}} - \frac{B}{\sqrt{n}} \ge \frac{A}{\sqrt{n-j}},$$

provided that $\frac{j}{n} \ge \tau = 1 - \big(\frac{A}{B}\big)^2$. We may assume that $B > A$. Then $0 < \tau < 1$ and

$$H_{n,p} + n^{\frac{1}{2}-\frac{1}{p}} \gg \frac{1}{n^{\frac{1}{p}}}\Bigg(\sum_{\tau n \le j < n}\Big(\frac{j}{n-j}\Big)^{\frac{p}{2}}\Bigg)^{\frac{1}{p}}.$$

The sum in the last formula can easily be estimated from below:

$$\sum_{\tau n \le j < n}\Big(\frac{j}{n-j}\Big)^{\frac{p}{2}} \ge \sum_{1 \le j \le (1-\tau)n} \frac{(\tau n)^{\frac{p}{2}}}{j^{\frac{p}{2}}} \ge (\tau n)^{\frac{p}{2}} \int_1^{(1-\tau)n} \frac{dt}{t^{\frac{p}{2}}} \gg (\tau n)^{\frac{p}{2}} \min\Big\{\log n, \frac{1}{p-2}\Big\}.$$

Since we may assume that p lies in an arbitrarily small (but fixed) neighborhood of 2, this implies a lower bound for $p \ge 2$:

$$H_{n,p} \gg n^{\frac{1}{2}-\frac{1}{p}} \min^{\frac{1}{p}}\Big\{\log n, \frac{1}{p-2}\Big\}.$$

Now, let $1 \le p \le 2$. Since $H_{n,p}$ grows with p, to prove the inequality $H_{n,p} \gg \sqrt{\log n}$ it suffices to establish it only for $p = 1$. For this, we use inequality $(*)$ with $p = 1$:

$$H_{n,1} + 1 \gg \frac{1}{n}\sup_{\varepsilon_j = \pm 1}\int_0^{\pi}\Bigg|\sum_{j=0}^{n-1}\varepsilon_j\sqrt{j}\cos\Big(jt + \frac{\pi}{4}\Big)\Bigg|\frac{dt}{\sqrt{t}}.$$

In the lemma below we consider a more general sum.

Lemma. *Let $a_1, \dots, a_n$ be nonnegative numbers. Then*

$$\sup_{\varepsilon_j = \pm 1}\int_0^{\pi}\Bigg|\sum_{j=1}^{n}\varepsilon_j a_j\cos\Big(jt + \frac{\pi}{4}\Big)\Bigg|\frac{dt}{\sqrt{t}} \gg \sqrt{\frac{\log n}{n}}\sum_{j=1}^{n} a_j.$$

With $a_j = \sqrt{j}$, this immediately implies the desired relation $H_{n,1} \gg \sqrt{\log n}$.

Proof. Given an arbitrary collection of signs $\sigma = (\sigma_1, \dots, \sigma_n)$, we define a function f_σ by

$$f_\sigma(t) = \sigma_l\sqrt{\frac{n}{l}} \qquad \text{if} \quad t \in \Big[\frac{l-1}{n}, \frac{l}{n}\Big), \quad l = 1, \dots, n,$$
$$f_\sigma(t) = 0 \qquad \text{if} \quad 1 \le t < \pi.$$

Clearly, $|f_\sigma(t)| \le \frac{1}{\sqrt{t}}$ everywhere on $(0, \pi)$, and

$$\int_0^{\pi} f_\sigma(t)\cos\Big(jt + \frac{\pi}{4}\Big)\, dt = \sum_{l=1}^{n}\sigma_l I_l(j), \quad \text{where} \quad I_l(j) = \int_{\frac{l-1}{n}}^{\frac{l}{n}}\sqrt{\frac{n}{l}}\cos\Big(jt + \frac{\pi}{4}\Big)\, dt.$$

We shall need a lower bound for the sums $\sum_l I_l^2(j)$ for $j = 1, \dots, n$. It is easily seen that

$$\sum_{l=1}^{n} I_l^2(j) = \sum_{l=1}^{n}\frac{4n}{j^2 l}\sin^2\frac{j}{2n}\cos^2\Big(j\frac{l-\frac{1}{2}}{n} + \frac{\pi}{4}\Big) \gg \frac{1}{n}\sum_{l=1}^{n}\frac{1-\sin j\frac{2l-1}{n}}{l} \gg \frac{\log n}{n}$$

(at the end, we have used the fact that the partial sums of the series $\sum_l \frac{1}{l}\sin lt$ are uniformly bounded).

By Khintchine's inequality, the quantity $\left|\sum_l \sigma_l I_l(j)\right|$ averaged over all σ is not less than $\frac{1}{2}\sqrt{\sum_l I_l^2(j)}$ and, consequently, not less than const $\sqrt{\frac{\log n}{n}}$. Therefore, the mean value of the sums

$$\sum_{j=1}^{n} a_j \left| \int_0^1 f_\sigma(t) \cos\Big(jt + \frac{\pi}{4}\Big)\, dt \right| = \sum_{j=1}^{n} a_j \left| \sum_{l=1}^{n} \sigma_l I_l(j) \right|$$

is at least const $\sqrt{\frac{\log n}{n}} \sum_j a_j$. Thus, there exists a collection of signs σ such that

$$\sum_{j=1}^{n} a_j \left| \int_0^1 f_\sigma(t) \cos\Big(jt + \frac{\pi}{4}\Big)\, dt \right| \gg \sqrt{\frac{\log n}{n}} \sum_{j=1}^{n} a_j.$$

Let ε_j, for $j = 1, \dots, n$, be the sign of the integral $\int_0^\pi f_\sigma(t) \cos\big(jt + \frac{\pi}{4}\big)\, dt$. With this choice, the last inequality takes the form

$$\int_0^\pi f_\sigma(t) \sum_{j=1}^{n} a_j \varepsilon_j \cos\Big(jt + \frac{\pi}{4}\Big)\, dt \gg \sqrt{\frac{\log n}{n}} \sum_{j=1}^{n} a_j.$$

Since $|f_\sigma(t)| \le \frac{1}{\sqrt{t}}$, this completes the proof. □

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