Алгебра и анализ
Том 25 (2013), № 4

St. Petersburg Math. J.
Vol. 25 (2014), No. 4, Pages 513–527
S 1061-0022(2014)01303-9
Article electronically published on June 5, 2014

ON SUBSPACES GENERATED BY INDEPENDENT FUNCTIONS IN SYMMETRIC SPACES WITH THE KRUGLOV PROPERTY

S. V. ASTASHKIN

Abstract. For a broad class of symmetric spaces X, it is shown that the subspace generated by independent functions f_k $(k = 1, 2, \dots)$ is complemented in X if and only if so is the subspace in a certain symmetric space Z_X^2 on the semiaxis generated by their disjoint shifts $\bar{f}_k(t) = f_k(t-k+1)\chi_{[k-1,k)}(t)$. Moreover, if $\sum_{k=1}^{\infty} m(\operatorname{supp} f_k) \le 1$, then Z_X^2 can be replaced by X itself in the last statement. This result is new even for L_p-spaces. Some consequences are deduced; in particular, it is shown that symmetric spaces enjoy an analog of the well-known Dor–Starbird theorem on the complementability in $L_p[0,1]$ $(1 \le p < \infty)$ of the closed linear span of some independent functions under the assumption that this closed linear span is isomorphic to ℓ_p.

§1. Introduction

It is well known that properties of independent functions and related probabilistic inequalities are an efficient tool for the study of the geometry of function spaces (see, e.g., the monographs [1, 2, 3] and the references therein). Of particular importance is the question as to whether the subspace generated by some system of independent functions is complemented, i.e., admits a bounded projection onto itself. As was shown in [4] (and independently in [1, Theorem 2.b.4(ii)]) in the simplest and most important case of the Rademacher functions $r_k(t) = \operatorname{sign} \sin 2^k \pi t$ $(k = 1, 2, \dots)$ on the interval $[0,1]$, their closed linear span is complemented in a symmetric space X if and only if $G \subset X \subset G'$, where G is the closure of L_∞ in the Orlicz space L_{N_2} constructed by the function $N_2(u) = e^{u^2} - 1$, and G' is its dual (see the next section for the definitions). In particular, the above inclusion is true for $L_p[0,1]$ if and only if $1 < p < \infty$. Soon after that, this result was generalized by Braverman to sequences of independent identically distributed functions (see [5]) and to sequences of independent uniformly bounded functions (see [6]). We mention also the paper [7] by Dor and Starbird, where it was shown that the closed linear span $[f_n]$ in $L_p[0,1]$ $(1 \le p < \infty)$ with independent f_k's is complemented provided it is isomorphic to ℓ_p.

The main result of the present paper is Theorem 1, saying that, for a broad class of symmetric (or rearrangement invariant, r.i. for short) spaces that possess the so-called Kruglov property together with their dual, the subspace generated by certain independent functions f_k $(k = 1, 2, \dots)$ is complemented if and only if so is the subspace generated by their disjoint copies (e.g., by the shifts $\bar{f}_k(t) = f_k(t-k+1)\chi_{[k-1,k)}(t)$) in the r.i. space

2010 *Mathematics Subject Classification.* Primary 46E30.

Key words and phrases. Complemented subspace, independent functions, Rademacher functions, symmetric space, Kruglov property, Boyd indices, lower p-estimate.

The author was supported in part by RFBR (grant nos. 10-01-00077 and 12-01-00198).

Z_X^2 on the semiaxis $[0,\infty)$ that consists of all functions f satisfying

$$\|f\|_{Z_X^2} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_2[1,\infty)} < \infty; \tag{1}$$

here f^* is the left-continuous monotone nonincreasing rearrangement of $|f|$. Under the condition $\sum_{k=1}^\infty m(\operatorname{supp} f_k) \le 1$ (m is the Lebesgue measure), the space Z_X^2 can be replaced by X itself in the last statement (Corollary 2). It should be noted that these results are new already for L_p-spaces.

A close relationship between properties of independent functions and of their disjoint copies was recognized fairly long ago, at least as early as in 1970, in the remarkable paper [8] by Rosenthal about isomorphic classification of subspaces of L_p. In 1989, Johnson and Schechtman (see [9]) extended Rosenthal's inequality to the class of r.i. spaces. They showed that for every r.i. space X on $[0,1]$ there exists a constant $\beta > 0$ such that an arbitrary sequence of independent functions $\{f_k\}_{k=1}^\infty \subset X$ with $\int_0^1 f_k(t)\,dt = 0$ ($k = 1, 2, \dots$) satisfies the inequality

$$\Big\|\sum_{k=1}^n \bar f_k\Big\|_{Z_X^2} \le \beta\Big\|\sum_{k=1}^n f_k\Big\|_X \quad (n \in \mathbb{N}). \tag{2}$$

Moreover, if $X \supset L_p$ for some $p < \infty$, then for some $\alpha > 0$ depending on X only we have the opposite inequality

$$\Big\|\sum_{k=1}^n f_k\Big\|_X \le \alpha\Big\|\sum_{k=1}^n \bar f_k\Big\|_{Z_X^2} \quad (n \in \mathbb{N}). \tag{3}$$

Later, the author and Sukochev (see [10, Theorem 3.1], and also Theorem 25 in the survey [11]) showed that (3) is fulfilled under a less restrictive (and this time sharp) requirement that X should possess the so-called Kruglov property. For instance, the exponential Orlicz spaces L_{N_γ} have this property for $0 < \gamma \le 1$ (surely, the condition "$X \supset L_p$ for some $p < \infty$" fails for these spaces). Inequalities (2) and (3) will play a crucial role both in the proof of Theorem 1 and in that of Theorem 2, the latter being an analog of the Dor–Starbird theorem mentioned above for the r.i. spaces. Theorem 3 is devoted to the important particular case of the Lorentz spaces $L_{p,q}$ ($1 < p < \infty$, $1 \le q \le \infty$). In the final part of the paper, it is shown that the approach exploited here allows us to considerably simplify the proofs of Braverman's theorems about the complementability of subspaces generated by sequences of independent functions that are either uniformly bounded or identically distributed, provided that the space in question possesses the Kruglov property. Also in the final section, yet another description of the Hilbert space L_2 in terms of complemented subspaces is presented (Theorem 5).

Partly, the results of the present paper were announced in [12].

§2. Definitions and notation

Let J be either $[0,1]$ or $(0,\infty)$, and let m be the Lebesgue measure on J. For a measurable function $x = x(t)$ on J, we introduce the distribution function in the way usual in Analysis: $n_x(\tau) := m\{t \in J : |x(t)| > \tau\}$ ($\tau > 0$). Two functions x and y are said to be *equimeasurable* if $n_x(\tau) = n_y(\tau)$ ($\tau > 0$). In particular, for every measurable function x its absolute value is equimeasurable with its monotone decreasing left-continuous rearrangement

$$x^*(t) := \inf\{\tau \ge 0 : n_x(\tau) < t\} \quad (t \in J).$$

A Banach space X of real-valued functions Lebesgue measurable on J is said to be *symmetric* or *rearrangement invariant* (r.i. for short) if 1) $y \in X$ and $\|y\|_X \le \|x\|_X$

whenever $x \in X$ and $|y(t)| \le |x(t)|$; 2) if $x \in X$ and x and y are equimeasurable, then $y \in X$ and $\|y\|_X = \|x\|_X$.

If X and Y are Banach spaces, we write $X \subset Y$ to indicate that X embeds in Y linearly and continuously; then there is a constant $C > 0$ such that $\|x\|_Y \le C\|x\|_X$ for all $x \in X$. Without loss of generality, we assume throughout that $\|\chi_{[0,1]}\|_X = 1$. Then for every r.i. space X we have $L_\infty \subset X \subset L_1$ and $\|x\|_X \le \|x\|_{L_\infty}$ $(x \in L_\infty)$, $\|x\|_{L_1} \le \|x\|_X$ $(x \in X)$, see [13, Theorem 2.4.1].

If X is an r.i. space on J, then its *dual* (or *associated*) space X' consists of all y with

$$\|y\|_{X'} = \sup\left\{ \int_J x(t)y(t)\,dt \,:\, \|x\|_X \le 1 \right\} < \infty.$$

The space X' is also r.i., it embeds continuously in the conjugate space X^*. Next, $X' = X^*$ if and only if X is separable. An arbitrary r.i. space X embeds in its second dual, to be denoted X''. An r.i. space X is said to be *maximal* (or to have the *Fatou property*) if, whenever $x_n \in X$ $(n = 1, 2, \dots)$, $\sup_{n=1,2,\dots} \|x_n\|_X < \infty$, and $x_n \to x$ a.e., we have $x \in X$ and $\|x\|_X \le \limsup_{n\to\infty} \|x_n\|_X$. If an r.i. space X has a similar property under the additional *a priori* assumption that $x \in X$, then X is said to have *order semicontinuous norm*. This is equivalent to the statement that X embeds in X'' isometrically, see [14, Theorem 6.1.6]. A space X is maximal if and only if its embedding in X'' is an isometric surjection, see [14, Theorem 6.1.7]. As in [1], we assume throughout that *all r.i. spaces are separable or maximal*. In particular, the norm of every such space is order semicontinuous.

The most well-known symmetric spaces, namely, L_p, are part of the family of Lorentz spaces $L_{p,q} = L_{p,q}(J)$ $(1 < p < \infty, 1 \le q \le \infty)$ that consist of the functions x measurable on J and such that the following quantity is finite:

$$\|x\|_{p,q} = \begin{cases} \left[\frac{q}{p}\int_J \left(t^{1/p}x^*(t)\right)^q \frac{dt}{t}\right]^{1/q}, & 1 \le q < \infty, \\ \operatorname{ess\,sup}_{t\in J}\left(t^{1/p}x^*(t)\right), & q = \infty. \end{cases}$$

Though the functional $\|\cdot\|_{p,q}$ is not subadditive, it is equivalent to the norm $\|x\|'_{p,q} = \|x^{**}\|_{p,q}$, where $x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds$. It can easily be checked (see [13, Lemma 2.6.5]) that $L_{p,q_1} \subset L_{p,q_2}$ $(1 \le q_1 \le q_2 \le \infty)$ and $L_{p,p} = L_p$.

The Orlicz spaces are an example of another natural generalization of L_p-spaces. Let $M(u)$ be an Orlicz function, i.e., a monotone increasing convex function on $[0,\infty)$ with $M(0) = 0$. The *Orlicz space* $L_M = L_M(J)$ consists of all functions x measurable on J with $\int_J M(|x(t)|/\lambda)\,dt \le 1$ for some $\lambda > 0$. The norm in it is defined to be the infimum of all λ for which the last inequality holds true. Finally, if ψ is a nonnegative concave function on J, then the Lorentz space $\Lambda_\psi = \Lambda_\psi(J)$ consists of all functions x measurable on J and such that

$$\|x\|_{\Lambda_\psi} := \int_J x^*(t)\,d\psi(t) < \infty.$$

An important characteristic of an r.i. space X is its *fundamental function* $\varphi_X(s) := \|\chi_{(0,s)}\|_X$ $(s \in J)$. In particular, $\varphi_{L_{p,q}}(s) = s^{1/p}$, $\varphi_{\Lambda_\psi}(s) = \psi(s)$. The fundamental function of an arbitrary symmetric space is equivalent to its smallest concave majorant $\widetilde{\varphi}_X(t)$ (see [13, Theorem 2.1.1]), more precisely, $\frac{1}{2}\widetilde{\varphi}_X(t) \le \varphi_X(t) \le \widetilde{\varphi}_X(t)$ $(t \in J)$.

In every r.i. space X, the *dilation operator*

$$\sigma_\tau x(t) := x(t/\tau)\cdot\chi_{[0,1]}(t/\tau) \quad \text{if} \quad J = [0,1],$$

and

$$\sigma_\tau x(t) := x(t/\tau) \quad \text{if} \quad J = [0,\infty) \quad (\tau > 0)$$

acts continuously; moreover, $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$ see [13, Theorem 2.4.5]. This operator is used to introduce the upper and lower Boyd indices:

$$\alpha_X = \lim_{\tau\to 0+} \frac{\ln \|\sigma_\tau\|_{X\to X}}{\ln \tau} \quad \text{and} \quad \beta_X = \lim_{\tau\to +\infty} \frac{\ln \|\sigma_\tau\|_{X\to X}}{\ln \tau}.$$

We always have $0 \le \alpha_X \le \beta_X \le 1$, see [13, §2.1, p. 138]. In particular, $\alpha_{L_{p,q}} = \beta_{L_{p,q}} = 1/p$. A symmetric space X is said to satisfy the *lower p-estimate* (respectively, the *upper p-estimate*), where $1 \le p \le \infty$, if there exists a constant $C_X > 0$ such that for every mutually disjoint functions $x_1, \dots, x_n$ in X we have

$$\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p} \le C_X \Big\|\sum_{k=1}^n x_k\Big\| \quad \Big(\text{respectively,} \quad \Big\|\sum_{k=1}^n x_k\Big\| \le C_X \Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p}\Big)$$

(for $p=\infty$, $\big(\sum_{k=1}^n \|x_k\|^p\big)^{1/p}$ is replaced by $\max_{k=1,\dots,n} \|x_k\|$).

The space Z_X^2 defined by (1) (X being an r.i. space on $[0,1]$) was first introduced in [2]. It can easily be shown that the quasinorm on X is equivalent to a certain norm, so that Z_X^2 is an r.i. space on $[0,\infty)$; see [11, §4.1, p. 27]. If $\{f_k\}_{k=1}^\infty$ is a sequence of functions on $[0,1]$, we denote by $\{\bar f_k\}_{k=1}^\infty$ a sequence of pairwise disjoint copies on $[0,\infty)$ (a *copy* is an equimeasurable function), for example $\bar f_k(t) = f_k(t-k+1)\chi_{[k-1,k)}(t)$. See the monographs [13, 1, 15] for more details on r.i. spaces.

Let f be a measurable function (random variable) on $[0,1]$. We denote by $\pi(f)$ the function (random variable) $\sum_{i=1}^N f_i$, where the f_i are mutually independent copies of f, and N is a random variable that has Poisson distribution with parameter 1 and is independent of the sequence $\{f_i\}$. The following property was introduced and studied by Braverman [5]; he used certain probabilistic constructions by Kruglov, see [16].

Definition 1. An r.i. space X on $[0,1]$ has the *Kruglov property* ($X \in \mathbb{K}$) if $\pi(f) \in X$ whenever $f \in X$.

Roughly speaking, $X \in \mathbb{K}$ if X is "rather far away" from L_∞. In particular, this is so if $X \supset L_p$ for some $p < \infty$ and, moreover, if $\alpha_X > 0$, see [5, Theorems 1.2 and 1.3, p. 16]. For example, $L_{p,q} \in \mathbb{K}$ for all $1 < p < \infty$, $1 \le q \le \infty$. Next, if L_{N_γ} is the exponential Orlicz space constructed by the function $N_\gamma(u) = e^{u^\gamma} - 1$, then $L_{N_\gamma} \in \mathbb{K}$ if and only if $0 < \gamma \le 1$, see [5, p. 42]. Consult the monograph [5] and the survey [11] for more information on the Kruglov property and its applications.

In what follows, $\operatorname{supp} f$ stands for the measurable support of f. If f, g are two nonnegative functions (in particular, two quasinorms) defined on a set T, we write $f \asymp g$ ($t \in T$) to signalize that $C^{-1} f(t) \le g(t) \le C f(t)$ with some constant C independent of t.

§3. Auxiliary results

The proofs of the first two lemmas are standard and, therefore, will be omitted.

Lemma 1 (see [7, Fact 2.2])**.** *Let X be a Banach space and Y, E its subspaces, where E is finite-dimensional. Then Y is complemented in X if and only if $Y+E$ is complemented in X.*

Lemma 2. *If X is a separable (maximal) r.i. space on $[0,1]$, then Z_X^2 is a separable (respectively, maximal) r.i. space on $[0,\infty)$. Moreover, $(Z_X^2)' = Z_{X'}^2$.*

Lemma 3. *Suppose that an r.i. space X on $[0,1]$ satisfies the lower p-estimate, $p > 2$. Then there is a constant $B > 0$ such that*

$$\Big(\sum_{k=1}^\infty \|x_k^*\|_X^p\Big)^{1/p} \le B_X \Big\|\sum_{k=1}^\infty x_k\Big\|_{Z_X^2} \tag{4}$$

for every sequence $\{x_k\} \subset Z_X^2$ *with* $\operatorname{supp} x_k \subset [k-1,k]$ $(k=1,2,\dots)$.

Proof. First, if an r.i. space X satisfies the lower p-estimate, then $L_{p,1}[0,1] \subset X$. Indeed, for every $t \in (0,1]$ there exists $n \in \mathbb{N}$ with $1/2 < nt \le 1$. Since $\chi_{(0,tn)} = \sum_{k=1}^n \chi_{(t(k-1),tk)}$, our assumptions imply the following relation for the fundamental function φ_X of X:

$$\varphi_X(tn) = \|\chi_{(0,tn)}\|_X \ge C_X^{-1}\Big(\sum_{k=1}^n \|\chi_{(t(k-1),tk)}\|^p\Big)^{1/p} = C_X^{-1}\varphi_X(t)n^{1/p}.$$

By the choice of n, we obtain

$$\varphi_X(t) \le C_X\varphi(nt)n^{-1/p} \le C_X\varphi(1)2^{1/p}t^{1/p} = C_1 t^{1/p},$$

Since $L_{p,1} = \Lambda_{t^{1/p}}$ isometrically, from [13, Theorem 2.5.5] it follows that

$$L_{p,1}[0,1] \subset \Lambda_{\widetilde{\varphi}_X}[0,1] \subset X \quad \text{and} \quad \|x\|_X \le C_1\|x\|_{L_{p,1}[0,1]} \quad (x \in L_{p,1}[0,1]) \tag{5}$$

($\widetilde{\varphi}_X$ is the smallest concave majorant for φ_X).

Denoting by $Z_X^{p,1}$ the symmetric space of functions $f : [0,\infty) \to \mathbb{R}$ such that

$$\|f\|_{Z_X^{p,1}} = \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_{L_{p,1}[0,\infty)} < \infty,$$

we show that

$$\Big(\sum_{k=1}^\infty \|x_k^*\|_X^p\Big)^{1/p} \le C\Big\|\sum_{k=1}^\infty x_k\Big\|_{Z_X^{p,1}} \tag{6}$$

for every sequence $\{x_k\} \subset Z_X^{p,1}$, $\operatorname{supp} x_k \subset [k-1,k]$ $(k=1,2,\dots)$, with C independent of the sequence. By the definition of $\|\cdot\|_{Z_X^{p,1}}$, we have

$$\Big\|\sum_{k=1}^\infty x_k\Big\|_{Z_X^{p,1}} = \Big\|\Big(\sum_{k=1}^\infty u_k\Big)^*\Big\|_X + \Big\|\sum_{k=1}^\infty v_k\Big\|_{L_{p,1}[0,\infty)},$$

where $x_k = u_k + v_k$, $m(\operatorname{supp}\sum_{k=1}^\infty u_k) \le 1$, $\operatorname{supp} u_k \subset [k-1,k]$, $\operatorname{supp} v_k \subset [k-1,k]$, and u_k and v_k are disjoint for every $k = 1,2,\dots$. Moreover, since both X and $L_{p,1}(0,\infty)$ admit the lower p-estimate (see, e.g., [17] or [18]) and $m(\operatorname{supp} v_k) \le 1$, we obtain

$$\Big\|\Big(\sum_{k=1}^\infty u_k\Big)^*\Big\|_X \ge C_X^{-1}\Big(\sum_{k=1}^\infty \|u_k^*\|_X^p\Big)^{1/p}$$

and

$$\Big\|\sum_{k=1}^\infty v_k\Big\|_{L_{p,1}[0,\infty)} \ge C_{p,1}^{-1}\Big(\sum_{k=1}^\infty \|v_k^*\|_{L_{p,1}[0,1]}^p\Big)^{1/p}.$$

Consequently, by (5), we arrive at

$$\begin{aligned}\Big\|\sum_{k=1}^\infty x_k\Big\|_{Z_X^{p,1}} &\ge \min(C_X^{-1}, C_{p,1}^{-1})\Big(\sum_{k=1}^\infty (\|u_k^*\|_X + \|v_k^*\|_{L_{p,1}[0,1]})^p\Big)^{1/p}\\ &\ge C_1^{-1}\min(C_X^{-1}, C_{p,1}^{-1})\Big(\sum_{k=1}^\infty (\|u_k^*\|_X + \|v_k^*\|_X)^p\Big)^{1/p}\\ &\ge C^{-1}\Big(\sum_{k=1}^\infty \|x_k^*\|_X^p\Big)^{1/p},\end{aligned}$$

and (6) is proved.

Next, since $p > 2$ and $\|\chi_{[0,1]}\|_X = 1$, we see that

$$\begin{aligned}\|f^*\chi_{[1.\infty)}\|_{L_{p,1}[0,\infty)} &:= \frac{1}{p}\int_0^\infty f^*(t+1)t^{1/p-1}\,dt \\ &\le \frac{1}{p}\biggl(f^*(1)p + \biggl(\int_1^\infty f^*(t)^2\,dt\biggr)^{1/2}\biggl(\int_1^\infty t^{2/p-2}\,dt\biggr)^{1/2}\biggr) \\ &\le \|f^*\chi_{[0,1]}\|_X + (p(p-2))^{-1/2}\|f^*\chi_{[1.\infty)}\|_{L_2[1,\infty)}.\end{aligned}$$

By the definition of the norms in $Z_X^{p,1}$ and Z_X^2, it follows that $\|f\|_{Z_X^{p,1}} \le A\|f\|_{Z_X^2}$ $(f \in Z_X^2)$ with some constant $A = A(p) > 0$. Together with (6), this implies (4), and the lemma is proved. □

The proof of the next statement is much the same, so we omit it.

Lemma 4. *Suppose that an r.i. space X admits the lower 2-estimate and $X \supset L_2[0,1]$. Then there exists a constant $B_X > 0$ such that*

$$\biggl(\sum_{k=1}^\infty \|x_k^*\|_X^2\biggr)^{1/2} \le B_X\biggl\|\sum_{k=1}^\infty x_k\biggr\|_{Z_X^2}$$

for every sequence $\{x_k\} \subset Z_X^2$ with $\operatorname{supp} x_k \subset [k-1,k]$ $(k = 1,2,\dots)$.

The last statement in this section shows that, for a subspace generated by disjoint functions in a symmetric space, the property to be complemented is completely determined by their distributions.

Lemma 5. *Suppose that measurable functions g_k, h_k belonging to an r.i. space Z on $[0,\infty)$ are equimeasurable for every $k = 1,2,\dots$ and* $\operatorname{supp} g_k \subset [k-1,k]$, $\operatorname{supp} h_k \subset [k-1,k]$ $(k = 1,2,\dots)$. *Then the subspaces $[g_k]$ and $[h_k]$ are complemented or not in Z simultaneously.*

Proof. Suppose, for example, that $[g_k]$ is complemented in Z, and let Q be a bounded projection from Z onto $[g_k]$. Then (see [13, Theorem 2.2.1] or [15, Proposition 2.7.4]), given $\delta > 0$, for every $k = 1,2,\dots$ there exists a measure preserving mapping $w_k : [k-1,k] \to [k-1,k]$ (one-to-one up to a set of measure 0) and a measurable function $\varepsilon_k : [k-1,k] \to \{\pm 1\}$ such that

$$\|h_k - \varepsilon_k g_k(w_k)\|_Z < \delta 2^{-k} \quad (k = 1,2,\dots). \tag{7}$$

It is easily seen that the mapping $w : (0,\infty) \to (0,\infty)$ defined by

$$w(t) := \sum_{k=1}^\infty w_k(t)\chi_{[k-1,k]}(t)$$

is measure preserving. Consequently, the operator $T_{w,\varepsilon}x(t) := x(w(t))\cdot\varepsilon(t)$, where $\varepsilon(t) := \sum_{k=1}^\infty \varepsilon_k(t)\chi_{[k-1,k]}(t)$ is an isometry of Z, and $P := T_{w,\varepsilon}\cdot Q\cdot T_{w,\varepsilon}^{-1}$ is a bounded projection of Z onto the subspace $[T_{w,\varepsilon}g_k] = [\varepsilon g_k(w)]$. So, this subspace is also complemented in Z.

At the same time, $\sum_{k=1}^\infty \|h_k - \varepsilon g_k(w)\|_Z \le \delta$ by (7). By the stability of the property to be complemented (see, e.g., [19, Theorem 1.3.9]), we can take $\delta > 0$ so small that the last inequality ensures that $[h_k]$ is complemented in Z. □

§4. Main result

Proposition 1. *Let X be an r.i. space on $[0,1]$ with $X \in \mathbb{K}$. If the subspace $[f_k]$ generated by a sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions is complemented in X, then the subspace $[\bar{f}_k]$ generated by their pairwise disjoint copies is complemented in Z_X^2.*

Proof. By Lemma 1, we may and do assume that $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$. Next, it can easily be verified that the subspace

$$\bar{N}(X) := \left\{ y \in Z_X^2 \ : \ \int_{k-1}^{k} y(s)\,ds = 0 \quad \text{for all} \quad k = 1, 2, \dots \right\}$$

is closed in Z_X^2. Moreover, by [13, 2.3.2, p. 111], the linear operator

$$Uy(t) := y(t) - \sum_{k=1}^{\infty} \int_{k-1}^{k} y(s)\,ds \chi_{[k-1,k)}(t) \tag{8}$$

is a bounded projection from Z_X^2 onto $\bar{N}(X)$. So, $\bar{N}(X)$ is complemented in Z_X^2.

Denote by Σ_k the σ-algebra $\sigma(f_k)$ of subsets of $[0, 1]$ generated by the function f_k, and let $\mathbb{E}_k$ be the corresponding conditional expectation operator, i.e., $\mathbb{E}_k f := \mathbb{E}\big(f \mid \Sigma_k\big)$. Putting $y_k(s) := y(s + k - 1)$ $(0 \le s \le 1)$, where $y \in \bar{N}(X)$ is arbitrary, and using the fact that the σ-algebras Σ_k are mutually independent, we see that the functions $\mathbb{E}_k y_k$ are mutually independent and

$$\int_0^1 \mathbb{E}_k y_k(t)\,dt = \int_0^1 y_k(t)\,dt = \int_{k-1}^{k} y(s)\,ds = 0.$$

Applying (3), we obtain

$$\left\| \sum_{k=1}^{n} \mathbb{E}_k y_k \right\|_X \le \alpha \left\| \sum_{k=1}^{n} \big(\mathbb{E}_k y_k\big)(t - k + 1)\chi_{[k-1,k)}(t) \right\|_{Z_X^2} \tag{9}$$

for all $n \in \mathbb{N}$. On the other hand, since the operators $\mathbb{E}_k$ are bounded on $L_p[0, 1]$ $(1 \le p \le \infty)$ (see, e.g., [20, Remark 1.14]), we see that the operator

$$Ty(t) := \sum_{k=1}^{\infty} \big(\mathbb{E}_k y_k\big)(t - k + 1)\chi_{[k-1,k)}(t) \tag{10}$$

is bounded and of norm 1 both on $L_1[0, \infty)$ and $L_\infty[0, \infty)$. By Lemma 2, Z_X^2 is separable or maximal (because so is X). So, it is an interpolation space for the couple $(L_1[0, \infty), L_\infty[0, \infty))$ with constant 1, see [13, Theorem 2.4.9]. Thus, T is bounded with norm 1 on Z_X^2. Now, it is easy to check that the operator

$$Vy(t) := \sum_{k=1}^{\infty} \mathbb{E}_k y_k(t) \quad (0 \le t \le 1) \tag{11}$$

acts boundedly from $\bar{N}(X)$ to X. Indeed, if X (hence, also Z_X^2) is separable, then the series on the right in (10) converges in Z_X^2 for an arbitrary $y \in Z_X^2$. Thus, by (9), the series $\sum_{k=1}^{\infty} \mathbb{E}_k y_k$ converges in X and

$$\|Vy\|_X \le \alpha \|Ty\|_{Z_X^2} \le \alpha \|y\|_{Z_X^2}. \tag{12}$$

Now, suppose that X is maximal. Then, by (9), the norms $\big\| \sum_{k=1}^{n} \mathbb{E}_k y_k \big\|_X$ $(n = 1, 2, \dots)$ are uniformly bounded for every $y \in Z_X^2$. By [21, Corollary 5.2.3], the series on the right in (11) converges a.e. on $[0, 1]$. Since X is maximal, we see that $Vy \in X$, and we obtain (12) again.

Next, by assumption, there exists a bounded projection $R : X \to [f_k]$. Consider the operator $S := LRVU : Z_X^2 \to [\bar{f}_k]$, where the operator $L : [f_k] \to [\bar{f}_k]$ is defined by the formula $L\big(\sum_{k=1}^{\infty} a_k f_k\big) = \sum_{k=1}^{\infty} a_k \bar{f}_k$ $(a_k \in \mathbb{R})$. Since the f_k are independent and $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$, we see that L is bounded from $[f_k]$ to $[\bar{f}_k]$ by (2). Thus, S is also bounded and its image coincides with $[\bar{f}_k]$. Furthermore, since $\mathbb{E}_i f_i = f_i$, we have

$$S\bar{f}_i = LRV\bar{f}_i = LRf_i = Lf_i = \bar{f}_i$$

for every $i \in \mathbb{N}$. As a result, we see that S is a bounded projection from Z_X^2 onto $[\bar{f}_k]$, and the claim follows. $\square$

If we assume additionally that $X' \in \mathbb{K}$, the converse is also true.

Proposition 2. *Suppose that X is an r.i. space on $[0,1]$ with $X \in \mathbb{K}$ and $X' \in \mathbb{K}$. If a sequence $\{f_k\}_{k=1}^\infty \subset X$ of independent functions has the property that the space $[\bar{f}_k]$ is complemented in Z_X^2, then $[f_k]$ is complemented in X.*

Proof. As in the proof of the preceding statement, we may assume that $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$. Next,

$$N(X) := \left\{ f \in X \,:\, \int_0^1 f(s)\,ds = 0 \right\}$$

is a complemented subspace of X. Indeed, putting $Qf(t) := f(t) - \int_0^1 f(s)\,ds$, we have $\|Qf\|_X \le 2\|f\|_X$ because $\|f\|_{L_1} \le \|f\|_X$ $(f \in X)$ by [13, Theorem 2.4.1]. Moreover, $Qf = f$ for $f \in N(X)$.

As before, let $\mathbb{E}_k$ be the conditional expectation operator with respect to the σ-algebra $\sigma(f_k)$ generated on $[0,1]$ by f_k. On $N(X)$, we define the operator

$$Gf(t) := \sum_{k=1}^\infty (\mathbb{E}_k f)(t-k+1)\chi_{[k-1,k)}(t) \quad (t > 0)$$

and show that it is bounded on Z_X^2. By Lemma 2, Z_X^2 is either separable or maximal (together with X). So, it embeds isometrically into its second dual $(Z_X^2)'' = Z_{X''}^2$. Consequently,

$$\begin{aligned} \|Gf\|_{Z_X^2} &= \sup\left\{ \int_0^\infty \sum_{k=1}^\infty \big(\mathbb{E}_k f\big)(s-k+1)\chi_{[k-1,k)}(s) y(s)\,ds \,:\, \|y\|_{Z_{X'}^2} \le 1 \right\} \\ &= \sup\left\{ \sum_{k=1}^\infty \int_0^1 \big(\mathbb{E}_k f\big)(s) y_k(s)\,ds \,:\, \|y\|_{Z_{X'}^2} \le 1 \right\}, \end{aligned} \tag{13}$$

where $y_k(s) := y(s+k-1)$ $(0 \le s \le 1)$. The properties of conditional expectation and the relation $f \in N(X)$ allow us to observe that

$$\begin{aligned} \int_0^1 \big(\mathbb{E}_k f\big)(s) y_k(s)\,ds &= \int_0^1 \mathbb{E}_k\big(\mathbb{E}_k f \cdot y_k\big)(s)\,ds = \int_0^1 \big(\mathbb{E}_k f\big)(s) \cdot \big(\mathbb{E}_k y_k\big)(s)\,ds \\ &= \int_0^1 \mathbb{E}_k\big(f \cdot \mathbb{E}_k y_k\big)(s)\,ds - \int_0^1 \int_0^1 y_k(t)\,dt\, f(s)\,ds \\ &= \int_0^1 f(s) \cdot \left(\big(\mathbb{E}_k y_k\big)(s) - \int_0^1 y_k(t)\,dt \right) ds. \end{aligned}$$

By (13), it follows that

$$\begin{aligned} \|Gf\|_{Z_X^2} &= \sup\left\{ \sum_{k=1}^\infty \int_0^1 f(s)\left(\big(\mathbb{E}_k y_k\big)(s) - \int_0^1 y_k(t)\,dt \right) ds \,:\, \|y\|_{Z_{X'}^2} \le 1 \right\} \\ &\le \|f\|_X \sup\left\{ \left\| \sum_{k=1}^\infty \left(\big(\mathbb{E}_k y_k\big)(s) - \int_0^1 y_k(t)\,dt \right) \right\|_{X'} \,:\, \|y\|_{Z_{X'}^2} \le 1 \right\}. \end{aligned} \tag{14}$$

The functions $g_k(s) := \big(\mathbb{E}_k y_k\big)(s) - \int_0^1 y_k(t)\,dt$ are independent and $\int_0^1 g_k(s)\,ds = 0$ $(k = 1, 2, \dots)$. Therefore, since $X' \in \mathbb{K}$, formula (3) implies

$$\begin{aligned}
&\Big\| \sum_{k=1}^n \Big(\big(\mathbb{E}_k y_k\big)(s) - \int_0^1 y_k(t)\,dt \Big) \Big\|_{X'} \\
&\qquad \le \alpha \Big\| \sum_{k=1}^n \Big(\big(\mathbb{E}_k y_k\big)(s-k+1) - \int_0^1 y_k(t)\,dt \Big) \chi_{[k-1,k)}(s) \Big\|_{Z^2_{X'}} \qquad (15)\\
&\qquad \le \alpha \|Ty\|_{Z^2_{X'}} + \alpha \Big\| \sum_{k=1}^n \int_{k-1}^k y(t)\,dt \chi_{[k-1,k)} \Big\|_{Z^2_{X'}},
\end{aligned}$$

for every $n \in \mathbb{N}$, where the operator T is defined by (10). As in the proof of Proposition 1, we see that $\|Ty\|_{Z^2_{X'}} \le \|y\|_{Z^2_{X'}}$ $(y \in Z^2_{X'})$. Moreover, by [13, 2.3.2, p. 111] it follows that

$$\Big\| \sum_{k=1}^\infty \int_{k-1}^k y(t)\,dt \chi_{[k-1,k)} \Big\|_{Z^2_{X'}} \le \|y\|_{Z^2_{X'}}.$$

Since X' is maximal, by (15) we obtain

$$\Big\| \sum_{k=1}^\infty \Big(\big(\mathbb{E}_k y_k\big)(s) - \int_0^1 y_k(t)\,dt \Big) \Big\|_{X'} \le 2\alpha \|y\|_{Z^2_{X'}} \quad (y \in Z^2_{X'}).$$

Thus, by (14), G acts boundedly from $N(X)$ to Z^2_X.

By assumption, there exists a bounded projection $N : Z^2_X \to [\bar{f}_k]$. Consider the operator $S := MNGQ : X \to [f_k]$, where the operator $M : [\bar{f}_k] \to [f_k]$ is defined by $M\big(\sum_{k=1}^\infty a_k \bar{f}_k\big) = \sum_{k=1}^\infty a_k f_k$ $(a_k \in \mathbb{R})$. Since the f_k are mutually independent, $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ and $X \in \mathbb{K}$, applying (3) once again we conclude that M is bounded from $[\bar{f}_k]$ to $[f_k]$. As a result, S is bounded and its image coincides with $[f_k]$. Moreover, since $\mathbb{E}_k f_i = 0$ for $k \ne i$ and $\mathbb{E}_i f_i = f_i$, $i \in \mathbb{N}$, we see that

$$Sf_i = MNGf_i = MN\big(f_i(t-i+1)\chi_{[i-1,i)}(t)\big) = MN\bar{f}_i = M\bar{f}_i = f_i.$$

Thus, S is a bounded projection of X onto $[f_k]$, and the proof is complete. □

Remark 1. Generally speaking, neither of the conditions $X \in \mathbb{K}$ and $X' \in \mathbb{K}$ can be dropped in the last statement. Indeed, the system of Rademacher functions (see the Introduction) generates a complemented subspace in an r.i. space X if and only if $G \subset X \subset G'$, where G is the closure of L_∞ in the Orlicz space L_{N_2} with $N_2(u) = e^{u^2} - 1$ (see [4] or [1, Theorem 2.b.4(ii)]). At the same time, the sequence of their disjoint copies $\big\{\chi_{[k-1,k-1/2]} - \chi_{[k-1/2,k]}\big\}$ (or the equivalent sequence $\big\{\chi_{[k-1,k]}\big\}$, see Lemma 5) generates a complemented subspace in Z^2_X for every r.i. space X.

Propositions 1 and 2 imply the main result of the paper.

Theorem 1. *Suppose that X is an r.i. space on $[0,1]$ such that $X \in \mathbb{K}$ and $X' \in \mathbb{K}$. Let $\{f_k\}_{k=1}^\infty \subset X$ be an arbitrary sequence of independent functions, and let $\{\bar{f}_k\}_{k=1}^\infty$ be a sequence of their pairwise disjoint copies on $[0,\infty)$. Then the space $[f_k]$ is complemented in X if and only if the space $[\bar{f}_k]$ is complemented in Z^2_X.*

Corollary 1. *Let an r.i. space X satisfy the assumptions of Theorem 1. If $\{f_k\}$ and $\{f'_k\}$ are two sequences of independent functions in X such that f_k and f'_k are equimeasurable for every $k = 1, 2, \dots$, then the spaces $[f_k]$ and $[f'_k]$ are complemented or not in Z^2_X simultaneously.*

Proof. Consider the sequences $\{\bar{f}_k\}$ and $\{\bar{f}'_k\}$ of disjoint copies for $\{f_k\}$ and $\{f'_k\}$, respectively. Then the functions $\bar{f}_k$ and $\bar{f}'_k$ are equimeasurable for every $k = 1, 2, \dots$. By Lemma 5, the subspaces $[\bar{f}_k]$ and $[\bar{f}'_k]$ are complemented or not in Z^2_X simultaneously. Applying Theorem 1, we prove the claim. □

Corollary 2. *Let an r.i. space X satisfy the assumptions of Theorem 1, and let $\{f_k\} \subset X$ be a sequence of independent functions with $\sum_{k=1}^{\infty} m(\operatorname{supp} f_k) \le 1$. Then the spaces $[f_k]$ and $[\bar{f}_k]$ are complemented or not in X simultaneously.*

Proof. It is easily seen that X is complemented in Z^2_X. Next, if $f \in Z^2_X$ is such that $m(\operatorname{supp} f) \le 1$, then $\|f\|_{Z^2_X} = \|f^*\|_X$. Thus, the subspace $[\bar{f}_k]$ is complemented in Z^2_X if and only if it is complemented in X. Applying Theorem 1, we prove the claim. □

§5. On sequences of independent functions equivalent to the standard basis of ℓ_p

We shall prove an r.i. space version of the well-known Dor–Starbird theorem about complemented subspaces in $L_p[0,1]$ generated by independent functions equivalent to the standard basis of ℓ_p.

Theorem 2. *Let $1 \le p < \infty$. Suppose that an r.i. space X admits the lower p-estimate and $\{f_k\} \subset X$ is a sequence of independent functions equivalent to the standard basis of ℓ_p. If* (a) *$p > 2$, or* (b) *$p = 2$ and $X \supset L_2$, or* (c) *$\|f_k \chi_{A_k}\|_X \ge \delta > 0$ $(k = 1, 2, \dots)$ for some sets $A_k \subset [0,1]$ with $\sum_{k=1}^{\infty} m(A_k) < \infty$, then $[f_k]$ is complemented in X.*

Proof. An argument similar to that at the beginning of the proof of Lemma 3 shows that the lower p-estimate for X implies $\|\sigma_t\|_{X\to X} \le Ct^{1/p}$ $(0 < t \le 1)$. Thus, $\alpha_X > 0$, and, in particular, $X \in \mathbb{K}$ (see [5, Theorem 1.3]). Therefore, X satisfies the assumptions of Theorem 1. Without loss of generality, we assume that $\|f_k\|_X = 1$ $(k = 1, 2, \dots)$. Next, assuming first that $\int_0^1 f_k(s)\,ds = 0$ $(k = 1, 2, \dots)$ we consider three cases, depending on a particular assumption mentioned in the theorem.

(a) Let $\bar{f}_k$ be disjoint copies of f_k, $\operatorname{supp} \bar{f}_k \subset [k-1, k]$ $(k = 1, 2, \dots)$. We take some functions $g_k \in Z^2_{X'}$ with $\|g_k\|_{Z^2_{X'}} = 1$, $\operatorname{supp} g_k \subset [k-1, k]$, $\int_0^\infty \bar{f}_k(s) g_k(s)\,ds = 1$ $(k = 1, 2, \dots)$ and define an operator P on Z^2_X by

$$Pf(t) := \sum_{k=1}^{\infty} \int_0^\infty f(s) g_k(s)\,ds \cdot \bar{f}_k(t) \quad (t > 0).$$

By (2), the assumptions of the theorem, and Lemma 3, we obtain

$$\begin{aligned}
\|Pf\|_{Z^2_X} &\le \beta \Big\| \sum_{k=1}^{\infty} \int_0^\infty f(s) g_k(s)\,ds \cdot f_k \Big\|_X \le \beta C \Big(\sum_{k=1}^{\infty} \Big| \int_0^\infty f(s) g_k(s)\,ds \Big|^p \Big)^{1/p} \\
&\le \beta C \Big(\sum_{k=1}^{\infty} \|g_k\|^p_{Z^2_{X'}} \|f \chi_{[k-1,k]}\|^p_{Z^2_X} \Big)^{1/p} \\
&= \beta C \Big(\sum_{k=1}^{\infty} \|(f\chi_{[k-1,k]})^*\|^p_X \Big)^{1/p} \le \beta C C_X \|f\|_{Z^2_X}.
\end{aligned}$$

Therefore, P projects Z^2_X boundedly onto $[\bar{f}_k]$, and it remains to apply Theorem 1.

(b) The proof is the same as in case (a) with the difference that Lemma 4 is used instead of Lemma 3.

(c) Denote $f_k' = f_k\chi_{A_k}$ $(k = 1, 2, \dots)$. Since $\|f_k'\|_X \geq \delta$ by assumption for all $k = 1, 2, \dots$, there exist functions $h_k \in X'$, $\|h_k\|_{X'} \leq \delta^{-1}$, $\operatorname{supp} h_k \subset A_k$, with

$$\int_0^1 f_k(s)h_k(s)\,ds = \int_0^1 f_k'(s)h_k(s)\,ds = 1 \quad (k = 1, 2, \dots).$$

Putting $\bar{f}_k(t) = f_k(t-k+1)\chi_{[k-1,k]}(t)$, $\bar{h}_k(t) = h_k(t-k+1)\chi_{[k-1,k]}(t)$, we see that

$$\int_0^\infty \bar{h}_k(t)\bar{f}_k(t)\,dt = 1 \ \ (k = 1, 2, \dots), \qquad \int_0^\infty \bar{h}_k(t)\bar{f}_i(t)\,dt = 0 \ \ (k \neq i) \tag{16}$$

and

$$\|\bar{h}_k\|_{Z^2_{X'}} = \|h_k\|_{X'} \leq \delta^{-1} \quad (k = 1, 2, \dots). \tag{17}$$

Suppose that $m_0 := \sum_{k=1}^\infty m(A_k) > 1$ (if $m_0 \leq 1$, the arguments simplify). Since $m(\operatorname{supp} h_k) \leq m(A_k)$ $(k = 1, 2, \dots)$ and $\|\sigma_\tau\|_{Y\to Y} \leq \max(1, \tau)$ $(\tau > 0)$ for every symmetric space Y (see [13, Corollary 2.4.1]), the definition of the norm in $Z^2_{X'}$ shows that

$$\Big\|\sum_{k=1}^\infty a_k\bar{h}_k\Big\|_{Z^2_{X'}} \leq m_0\Big\|\sum_{k=1}^\infty a_k\sigma_{m_0^{-1}}\bar{h}_k\Big\|_{Z^2_{X'}} = m_0\Big\|\Big(\sum_{k=1}^\infty a_k\sigma_{m_0^{-1}}\bar{h}_k\Big)^*\Big\|_{X'}.$$

By assumption, X' admits the upper q-estimate with $1/p + 1/q = 1$ (see [1, Proposition 1.f.5]). Since the functions $\sigma_{m_0^{-1}}\bar{h}_k$ are disjoint, applying the inequality

$$\big\|\big(\sigma_{m_0^{-1}}\bar{h}_k\big)^*\big\|_{X'} \leq \|\bar{h}_k^*\|_{X'} \leq \delta^{-1}$$

and (17) we deduce from the preceding inequality that

$$\Big\|\sum_{k=1}^\infty a_k\bar{h}_k\Big\|_{Z^2_{X'}} \leq C_{X'}m_0\Big(\sum_{k=1}^\infty |a_k|^q\big\|\big(\sigma_{m_0^{-1}}\bar{h}_k\big)^*\big\|^q_{X'}\Big)^{1/q} \leq C_{X'}m_0\delta^{-1}\Big(\sum_{k=1}^\infty |a_k|^q\Big)^{1/q} \tag{18}$$

(if $p = 1$, then $\big(\sum_{k=1}^\infty |a_k|^q\big)^{1/q}$ should be replaced by $\sup_{k=1,2,\dots}|a_k|$).

On the other hand, since $X \in \mathbb{K}$, relations (2) and (3) imply

$$\Big\|\sum_{k=1}^\infty a_kf_k\Big\|_X \asymp \Big\|\sum_{k=1}^\infty a_k\bar{f}_k\Big\|_{Z^2_X}, \tag{19}$$

so that, by assumption, the sequence $\{\bar{f}_k\}$ is equivalent in Z^2_X to the standard basis of ℓ_p. Furthermore, by (16), the sequence $\{\bar{h}_k\}$ is biorthogonal to it. By (18) and the well-known criterion for a subspace degenerated by a sequence equivalent to the standard basis of ℓ_p to be complemented (see, e.g., [7, Fact 2.1]), we see that $[\bar{f}_k]$ is complemented in Z^2_X. But then $[f_k]$ is complemented in X by Theorem 1.

In the general case, consider the functions $u_k := f_k - \int_0^1 f_k(s)\,ds$ $(k = 1, 2, \dots)$. They are independent and $\int_0^1 u_k(s)\,ds = 0$ $(k = 1, 2, \dots)$. We show that the sequence $\{u_k\}$ also satisfies the assumptions of the theorem.

By the well-known stability result for bases in [22], to show that this sequence is equivalent to the standard basis of ℓ_p, it suffices to prove that $\sum_{k=1}^\infty \|f_k - u_k\|^q_X < \infty$. But indeed, since $\{f_k\}$ is equivalent to the standard basis of ℓ_p, we have

$$\begin{aligned}\sum_{k=1}^\infty \|f_k - u_k\|^q_X = \sum_{k=1}^\infty \Big|\int_0^1 f_k(s)\,ds\Big|^q &\leq \sup\Big\{\Big(\int_0^1\Big|\sum_{k=1}^\infty a_kf_k(s)\Big|\,ds\Big)^q : \sum_{k=1}^\infty |a_k|^p \leq 1\Big\}\\ &\leq \sup\Big\{\Big\|\sum_{k=1}^\infty a_kf_k(s)\Big\|^q_X : \sum_{k=1}^\infty |a_k|^p \leq 1\Big\} < \infty.\end{aligned}$$

Moreover, if $\{f_k\}$ satisfies (c), then so does $\{u_k\}$ starting with some index, because $\int_0^1 f_k(s)\,ds \to 0$ as $k \to \infty$: the same sets A_k fit with a somewhat smaller $\delta > 0$.

Invoking Lemma 1, we reduce the proof to the case considered above. □

Remark 2. The "almost disjointness" condition for $[f_k] \subset X$ in part (c) of the above theorem is essential if $1 \le p \le 2$. Indeed, it is well known that a sequence of independent p-stable random variables is equivalent in $L_r = L_r[0,1]$ to the standard basis of ℓ_p for $1 \le r < p < 2$ and generates an uncomplemented subspace of L_r (see, e.g., [19, Theorems 6.4.18 and 6.4.21]).

Now, we consider the particular case of the Lorentz spaces $L_{p,q} = L_{p,q}[0,1]$ ($1 < p < \infty$, $1 \le q \le \infty$). It is well known (see, e.g., [23, Theorem 5.1] and [24, Lemma 3.1]) that every sequence of pairwise disjoint functions in $L_{p,q}$ has a subsequence $\{g_k\}$ equivalent to the standard basis of ℓ_p and such that $[g_k]$ is complemented in $L_{p,q}$. Note that $L_{p,q} \in \mathbb{K}$ and $(L_{p,q})' = L_{p',q'} \in \mathbb{K}$ ($1/p + 1/p' = 1$, $1/q + 1/q' = 1$), see the Introduction. By Corollary 2, the subspace $[f_k]$, where the f_k are independent copies of g_k ($k = 1, 2, \dots$), is also complemented in $L_{p,q}$. Moreover, since $L_{p,q}$ obeys the lower $\max(p,q)$-estimate (see [17] or [25, Theorem 3]), the following consequence of Theorem 2 holds.

Theorem 3. *Let $1 < \max(p,q) \le r < \infty$. If $\{f_k\} \subset L_{p,q}$ is a sequence of independent functions equivalent in $L_{p,q}$ to the standard basis of ℓ_r, then the subspace $[f_k]$ is complemented in $L_{p,q}$ provided either $r > 2$, or $p < r = 2$, or $p = q = r = 2$, or $\|f_k \chi_{A_k}\|_{L_{p,q}} \ge \delta > 0$ ($k = 1, 2, \dots$) for some sequence of sets $A_k \subset [0,1]$ with $\sum_{k=1}^\infty m(A_k) < \infty$.*

Since $L_{p,p} = L_p$, we arrive at the following statement.

Corollary 3. *If $1 < p < 2$ and $\{f_k\} \subset L_p$ is a sequence of independent functions equivalent in L_p to the standard basis of ℓ_2, then $[f_k]$ is complemented in L_p.*

In connection with this corollary, we recall the classical result by Kadec and Pelczyński [26, Corollary 1] saying that every subspace of L_p, $p > 2$, isomorphic to ℓ_2, is complemented in L_p. Corollary 3 shows that this statement extends to $1 < p < 2$ if we restrict ourselves to subspaces generated by independent functions. At the same time, it is well known (see [27, Theorem 3.3]) that for every $p \in (1,2)$ the space L_p has an uncomplemented subspace isomorphic to ℓ_2.

Assume that $1 \le p < \infty$ and that $\{f_k\}_{k=1}^\infty$ is a sequence of independent functions in $L_p = L_p[0,1]$ with $\|f_k\|_p = 1$, $\int_0^1 f_k(t)\,dt = 0$ ($k = 1, 2, \dots$) and such that $[f_k]$ is isomorphic to ℓ_p. We show that then $\{f_k\}$ is equivalent in L_p to the standard basis of ℓ_p. Indeed, first, an arbitrary sequence of mean zero independent random variables lying in a symmetric space is an unconditional basic sequence in that space, see [5, Proposition 1.1.4]. Thus, if $p = 1$ or $p = 2$, it suffices to refer to the uniqueness of a normalized unconditional basis in ℓ_1 (see [28]) or in a separable Hilbert space (see [29, Proposition 1.1]).

If $2 < p < \infty$, then Theorem 4 and Lemma 7 in [8] show that either $\{f_k\}$ is equivalent in L_p to the standard basis of ℓ_p or $[f_k]$ contains a subspace isomorphic to ℓ_2; the latter is impossible in our setting. Finally, if $1 < p < 2$, then the arguments at the beginning of the proof of Theorem A in [7] (see p. 168, 169) show that $\{f_k\}$ is equivalent in L_p to the standard basis of ℓ_p.

Since $L_{p,p} = L_p$ ($1 < p < \infty$) and for $p \ne 2$ we have $\|f_k \chi_{A_k}\|_{L_p} \ge \delta > 0$ ($k = 1, 2, \dots$) for some sequence of pairwise disjoint sets $A_k \subset [0,1]$ (see [30, Theorem B]), provided that the sequence $\{f_k\}$ of independent functions is equivalent to the standard basis in ℓ_p, the above arguments together with Theorems 2 and 3 imply the result by Dor and Starbird mentioned above.

Corollary 4 (see [7, Theorem A]). *Suppose that $1 \le p < \infty$ and $\{f_k\}_{k=1}^{\infty}$ is a sequence of independent functions in $L_p = L_p[0,1]$. If the closed linear span $[f_k]$ is isomorphic to ℓ_p, then it is complemented in L_p.*

§6. Complementability of subspaces generated by sequence of independent functions that are either equidistributed or uniformly bounded

(a) The case of equidistributed functions. Theorem 1 and the results of [31] make it possible to characterize the complemented subspaces generated by sequences of independent equidistributed functions in an r.i. space with the Kruglov property.

Theorem 4. *Let X be an r.i. space on $[0,1]$ such that $X \in \mathbb{K}$ and $X' \in \mathbb{K}$. Suppose that functions $f_k \in X$ $(k = 1, 2, \dots)$ are independent and equidistributed, and $\int_0^1 f_1(t)\,dt = 0$. Then the sequence $\{f_k\}$ generates a complemented subspace in X if and only if it is equivalent to the standard basis of ℓ_2.*

Proof. Clearly, the disjoint copies $\bar{f}_k$ of the functions f_k are the corresponding shifts of the function $\bar{f}_1$. Therefore, by [31, Theorem 2.2 and Corollary 2.4], the subspace $[\bar{f}_k]$ is complemented in Z_X^2 if and only if $\{\bar{f}_k\}$ is equivalent in Z_X^2 to the sequence $\{\chi_{[k-1,k]}\}$, which, in its turn, is clearly equivalent to the standard basis of ℓ_2. Since the $f_k \in X$ are mutually independent, $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$, and $X \in \mathbb{K}$, by (2) and (3) relation (19) holds true. Thus, the claim follows directly from Theorem 1. □

The next result is yet another characterization of the Hilbert space L_2 in terms of the complementation property for its subspaces.

Theorem 5. *Let X be an r.i. space on $[0,1]$ with $X \in \mathbb{K}$, $X' \in \mathbb{K}$. The following statements are equivalent*:

(i) *for every two sequences of independent and equidistributed functions $\{f_k\} \subset X$ and $\{f_k'\} \subset X'$ with $\int_0^1 f_1(t)\,dt = \int_0^1 f_1'(t)\,dt = 0$, the subspaces $[f_k]$ and $[f_k']$ are complemented in X and X', respectively*;

(ii) $X = L_2[0,1]$.

Proof. It suffices to prove that (i) implies (ii). By Theorem 1, (i) is equivalent to the fact that for every sequences $\{a_k\} \subset X$ and $\{a_k'\} \subset X'$ of equidistributed functions with $\operatorname{supp} a_k \subset [k-1,k]$ and $\operatorname{supp} a_k' \subset [k-1,k]$, the subspaces $[a_k]$ and $[a_k']$ are complemented in Z_X^2 and $(Z_X^2)' = Z_{X'}^2$, respectively. But then from [31, Theorem 5.4] it follows that $Z_X^2 = L_p[0,\infty)$ for some $1 < p < \infty$. Surely, this implies that $X = L_2[0,1]$. □

(b) The case of uniformly bounded functions.

Theorem 6. *Let X be an r.i. space on $[0,1]$ with $X \in \mathbb{K}$, $X' \in \mathbb{K}$. If $\{f_k\} \subset X$ is a sequence of independent functions with $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ and*

$$M := \sup_k \|f_k\|_{L_\infty} < \infty, \quad \alpha := \inf_k \|f_k\|_{L_2} > 0, \tag{20}$$

then the subspace $[f_k]$ is complemented in X.

Proof. By Theorem 1, it suffices to prove that the subspace $[\bar{f}_k]$ is complemented in Z_X^2. Lemma 5 allows us to assume that $\bar{f}_k \ge 0$. On the space Z_X^2, we consider the operator

$$Pf(t) := \sum_{k=1}^{\infty} \left(\int_{k-1}^{k} \bar{f}_k(s)\,ds \right)^{-1} \int_{k-1}^{k} f(s)\,ds \cdot \bar{f}_k(t), \quad t > 0.$$

Then $P\bar{f}_k = \bar{f}_k$ $(k = 1, 2, \dots)$. Next, since

$$\int_{k-1}^{k} \bar{f}_k(s)\,ds \ge \frac{1}{M}\int_{k-1}^{k} \bar{f}_k^2(s)\,ds \ge \frac{\alpha^2}{M} \quad (k = 1, 2, \dots)$$

by (20), it follows that

$$Pf(t) \le \frac{M}{\alpha^2}\sum_{k=1}^{\infty}\left|\int_{k-1}^{k} f(s)\,ds\right| \cdot \bar{f}_k(t) \le \Big(\frac{M}{\alpha}\Big)^2 \sum_{k=1}^{\infty}\left|\int_{k-1}^{k} f(s)\,ds\right| \cdot \chi_{[k-1,k]}(t).$$

Applying [13, 2.3.2, p. 111] once again, we obtain $\|Pf\|_{Z^2_X} \le \big(M/\alpha\big)^2\|f\|_{Z^2_X}$, and the theorem is proved. □

Remark 3. Theorem 4 and 6 show to what extent the Kruglov property simplifies the study of subspaces generated by either equidistributed or uniformly bounded independent functions, compared to the general case studied earlier by Braverman. In [5] (see Theorem 4.2), a much more involved proof of the claim of Theorem 4 was presented under the assumption $G \subset X \subset G'$, which is weaker than $X \in \mathbb{K}$ and $X' \in \mathbb{K}$. Furthermore, it was shown there that these embeddings and the equivalence (in X) of a sequence of independent equidistributed functions $\{f_k\}$ with $\int_0^1 f_k(t)\,dt = 0$ $(k = 1, 2, \dots)$ to the standard basis of ℓ_2 are consequences of the fact that $[f_k]$ is complemented in X. In [6], a similar and stronger version of Theorem 6 was stated without proof; the sharpness of the condition $G \subset X \subset G'$ follows by considering the Rademacher functions (see [4] or [1, Theorem 2.b.4(ii)]).

In conclusion, we note that a weaker version of the last statement was proved in [5, Theorem 4.1]. In that paper, the L_2-norm was replaced by the L_1-norm in the second expression in (20).

References

[1] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces.* II. *Function spaces*, Springer-Verlag, Berlin–New York, 1979. MR540367 (81c:46001)

[2] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. **19** (1979), no. 217, 298. MR527010 (82j:46025)

[3] B. S. Kashin and A. A. Saakyan, *Ortogonal series*, AFTs, Moscow, 1999. (Russian) MR1845025 (2002b:42047)

[4] V. A. Rodin and E. M. Semenov, *The complementability of a subspace that is generated by the Rademacher system in a symmetric space*, Functional. Anal. i Prilozhen. **13** (1979), no. 2, 91–92. (Russian) MR541648 (80j:46048)

[5] M. Sh. Braverman, *Independent random variables and rearrangement invariant spaces*, Cambridge Univ. Press, Cambridge, 1994. MR1303591 (95j:60027)

[6] ———, *On the complementability of subspaces that are generated by independent functions in a symmetric space*, Functional. Anal. i Prilozhen. **16** (1982), no. 2, 66–67. (Russian) MR659167 (84d:46028)

[7] L. E. Dor and T. Starbird, *Projections of L_p onto subspaces spanned by independent random variables*, Compositio Math. **39** (1979), no. 2, 141–175. MR546965 (82e:46043)

[8] H. P. Rosenthal, *On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables*, Israel. J. Math. **8** (1970), 273–303. MR0271721 (42:6602)

[9] W. B. Johnson and G. Schechtman, *Sums of independent random variables in rearrangement invariant function spaces*, Ann. Probab. **17** (1989), no. 2, 789–808. MR985390 (90h:60045)

[10] S. V. Astashkin and F. A. Sukochev, *Series of zero-mean independent functions in symmetric spaces with the Kruglov property*, Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) **345** (2007), 25–50; English transl., J. Math. Sci. (N.Y.) **148** (2008), no. 6, 795–809. MR2432174 (2010j:60103)

[11] ———, *Independent functions and the geometry of Banach spaces*, Uspekhi Mat. Nauk **65** (2010), no. 6, 3–86; English transl., Russian Math. Surveys **65** (2010), no. 6, 1003–1081. MR2779360 (2012b:46002)

[12] S. V. Astashkin, *On the complementability of subspaces in symmetric spaces with the Kruglov property*, Functional. Anal. i Prilozen. **47** (2013), no. 2, 80–84. MR3113871

[13] S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, Nauka, Moscow, 1978. (Russian) MR506343 (81f:46086)

[14] L. V. Kantorovich and G. P. Akilov, *Functional analysis*, Nauka, Moscow, 1977. (Russian) MR0511615 (58:23465)

[15] C. Bennett and R. Sharpley, *Interpolation of operators*, Acad. Press, Boston, 1988. MR928802 (89e:46001)

[16] V. M. Kruglov, *A remark on the theory of infinitely divisible laws*, Teor. Verojatnost. i Primenen. **15** (1970), 330–336. MR0272018 (42:6899)

[17] J. Creekmore, *Type and cotype in Lorentz $L_{p,q}$ spaces*, Indag. Math. **43** (1981), no. 2, 145–152. MR707247 (84i:46032)

[18] A. Kaminska and L. Maligranda, *Order convexity and concavity in Lorentz spaces $\Lambda_{p,w}$, $0 < p < \infty$*, Studia Math. **160** (2004), no. 3, 267–286. MR2033403 (2005e:46047)

[19] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Grad. Texts in Math. vol. 233, Springer, New York, 2006. MR2192298 (2006h:46005)

[20] R. S. Elliott, *Stochastic calculus and applications*, Appl. Math., vol. 18, Springer-Verlag, New York, 1982. MR678919 (85b:60059)

[21] N. N. Vakhaniya, V. I. Tarieladze, and S. A. Chobanyan, *Probability distributions in Banach spaces*, Nauka, Moscow, 1985. (Russian) MR787803 (86j:60014)

[22] I. T. Gohberg and A. S. Markus, *Stability of bases of Banach and Hilbert spaces*, But. Akad. Štiince RSS Moldoven. **5** (1962), 17–35. (Russian) MR0226365 (37:1955)

[23] T. Figiel, W. B. Johnson, and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure and applications to Lorentz function spaces*, J. Approximation Theory **13** (1975), 297–312. MR0367624 (51:3866)

[24] N. L. Carothers and S. J. Dilworth, *Geometry of Lorentz spaces via interpolation*, Texas Functional Analysis Seminar 1985–1986 (London Notes, Univ. Texas. Austin, TX, 1986, pp. 107–134. MR1017049 (90g:46044)

[25] S. J. Dilworth, *Special Banach lattices and their applications*, Handbook of the geometry of Banach spaces, Vol. 1, North-Holland, Amsterdam, 2001, 497–532. MR1863700 (2003d:46027)

[26] M. I. Kadec and A. Pelczynski, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1962), 161–176. MR0152879 (27:2851)

[27] G. Bennett, L. E. Dor, V. Goodman, W. B. Johnson, and C. M. Newman, *On uncomplemented subspaces of L_p, $1 < p < 2$*, Israel. J. Math. **26** (1977), no. 2, 178–187. MR0435822 (55:8778)

[28] J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications*, Studia Math. **29** (1968), 275–326. MR0231188 (37:6743)

[29] L. Tzafriri, *Uniqueness of structure in Banach spaces*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1635–1669. MR1999605 (2004i:46020)

[30] L. E. Dor, *On projections in L_1*, Ann. of Math.(2) **102** (1975), no. 3, 463–474. MR0420244 (54:8258)

[31] F. L. Hernandez and E. M. Semenov, *Subspaces generated by translations in rearrangement invariant spaces*, J. Funct. Anal. **169** (1999), no. 1, 52–80. MR1726747 (2001f:46041)

Samara State University 1, Academician Pavlov Street, 443011 Samara, Russia
E-mail address: astash@samsu.ru

Received 10/OCT/2012

Translated by S. KISLYAKOV