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REMARKS ON HILBERT IDENTITIES, ISOMETRIC EMBEDDINGS, AND INVARIANT CUBATURE

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Abstract. In 2004, Victoir developed a method to construct cubature formulas with various combinatorial objects. Motivated by this, the authors generalize Victoir's method with yet another combinatorial object, called the regular t-wise balanced design. Many cubature formulas of small indices with few points are provided, which are used to update Shatalov's table (2001) of isometric embeddings in small-dimensional Banach spaces, as well as to improve some classical Hilbert identities. A famous theorem of Bajnok (2007) on Euclidean designs invariant under the Weyl group of Lie type B is extended to all finite irreducible reflection groups. A short proof of the Bajnok theorem is presented in terms of Hilbert identities.

§1. Introduction

Let p be a positive integer such that $p \neq \infty$. The m-dimensional Euclidean space $\mathbb{R}^m$ is a Banach space l_p^m endowed with the norm

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}.$$

Given two spaces l_p^m and l_q^n, a classical problem in Banach space theory asks when there is an $\mathbb{R}$-linear map $F : l_p^m \to l_q^n$ such that

$$\|F(x)\|_q = \|x\|_p$$

for every $x \in l_p^m$. Such a map is called an *isometric embedding from l_p^m to l_q^n*. To exclude trivial cases, we assume that $n \geq m \geq 2$ and $p \neq q$. It is known [22, Theorem 1.1] that if $p, q \neq \infty$ and an isometric embedding from l_p^m to l_q^n exists, then $p = 2$ and q is an even integer. Throughout this paper we only consider the case where $p = 2$ and q is even, and fix the notations p, q, m, n.

Isometric embeddings are closely related to representations of $\left(\sum_{i=1}^m x_i^2\right)^{q/2}$ as a sum of qth powers of linear forms with positive real coefficients. Such representations originally stem from a work of Hilbert on Waring's problem [16], and are therefore called *Hilbert identities* [25]. Hilbert solved Waring's problem, showing on the way that there exist isometric embeddings $l_2^m \to l_q^n$ with n depending on m and q. Several alternative proofs of Hilbert's theorem are known; for example, see [6, 7] and the references therein.

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But most of them, including the original proof by Hilbert, involve nonconstructive arguments in analysis, and do not give any explicit constructions of embeddings.[1] Thus, publications with explicit embeddings continued to appear.

Isometric embeddings are also related to a certain object in numerical analysis. Let Ω be a subset of $\mathbb{R}^m$ on which a normalized measure μ is defined. A finite subset X of Ω with a positive weight w is called a *cubature formula of index* q if

$$\int_\Omega f(x)\mu\,(\mathrm{d}x) = \sum_{x\in X} w(x)f(x) \tag{1.1}$$

for every $f \in \mathrm{Hom}_q(\Omega)$, where $\mathrm{Hom}_q(\Omega)$ is the space of all homogeneous polynomials of degree q restricted to Ω. Lyubich and Vaserstein [22] and Reznick [27] proved the equivalence between an embedding $l_2^m \to l_q^n$ and an n-point cubature formula of index q for the surface measure ρ on the $(m-1)$-dimensional unit sphere S^{m-1}.

Many papers are devoted to the construction of spherical cubature formulas. There are two classical approaches. One employs orbits of finite subgroups of the orthogonal group $O(m)$ acting on S^{m-1} [33], and the other takes a "product" of several lower-dimensional cubatures [34]. Cubature formulas that are studied in the context of numerical analysis and related areas, are often of degree type. Victoir [35] developed a novel technique to construct degree-type cubature for integrals with special symmetry. His idea is as follows. Given a cubature formula invariant under the Weyl group of Lie type B, one eliminates some specified points of the formula by using combinatorial objects such as t-designs and orthogonal arrays. With this method, Victoir found many cubature formulas of small degrees with few points in general dimensional spaces.

In this paper, we have several important aims. First, we generalize the Victoir method to a special class of block designs, called *regular t-wise balanced designs*. The concept of a regular t-wise balanced design has been substantiated by applications in statistics [8, 10, 18], however, it seems that there is insufficient evidence to support it from other mathematical aspects. To find a new meaning of this concept, as well as to let researchers in many areas of mathematics know it, are among our important aims in this paper. On the other hand, Bajnok [1, Theorem 3] proved that Euclidean designs, a generalization of the spherical cubatures, that are invariant under the Weyl group of Lie type B have degree at most 7. We further discuss the Bajnok theorem both from a combinatorial and analytic point of view.

This paper is organized as follows. In §2 we review some basic facts and notions, and explain the Victoir method in detail. In §3 we generalize the Victoir method to regular t-wise balanced designs. In §4, we give general-dimensional index-four and -six cubature formulas, together with some additional examples of index-six cubature formulas that improve Shatalov's table [32, Theorem 4.7.20] of isometric embeddings $l_2^m \to l_6^n$. In §5, we generalize the Bajnok theorem for all finite irreducible reflection groups, and thereby classify the spherical cubature formulas with a certain geometric meaning. In §6, some of the cubatures constructed in §4 and §5 are translated into Hilbert identities, in order to improve classical identities as those by Schur [6] and Reznick [27]. An extremely short proof of the Bajnok theorem is given in terms of Hilbert identities.

§2. Preliminaries

2.1. Isometric embeddings and Hilbert identities. Lyubich and Vaserstein [22] and Reznick [27] observed a close relationship between Hilbert identities, isometric embeddings, and spherical cubature formulas.

[1] Bruce Reznick kindly told us that Stridsberg's proof (1912) is constructive, provided we know how to compute the roots of Hermite's polynomials.

Theorem 2.1. *The following statements are equivalent.*

(i) *There exists a cubature formula of index q on S^{m-1} with n points.*

(ii) *There exists an isometric embedding $l_2^m \to l_q^n$.*

(iii) *There exist n vectors $r_1, \dots, r_n \in \mathbb{R}^m$ such that, for any $x \in \mathbb{R}^m$,*

$$\langle x, x\rangle^{\frac{q}{2}} = \sum_{i=1}^{n} \langle x, r_i\rangle^q.$$

We explain Theorem 2.1 in detail for further arguments in the following sections. Assume that points $x_1, \dots, x_n \in S^{m-1}$ and weights $w_1, \dots, w_n$ form a cubature of index q on S^{m-1}. Let $\langle x, y\rangle^q \in \mathrm{Hom}_q(\mathbb{R}^m)$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Then

$$\sum_{i=1}^{n} w_i \langle x, x_i\rangle^q = \int_{S^{m-1}} \langle x, y\rangle^q \rho\,(\mathrm{d}y) = \langle x, x\rangle^{\frac{q}{2}} c_q,$$

where

$$c_q = \int_{S^{m-1}} y_1^q \rho\,(\mathrm{d}y), \quad y = (y_1, \dots, y_m).$$

This is, equivalently,

$$\langle x, x\rangle^{\frac{q}{2}} = \sum_{i=1}^{n} \langle x, r_i\rangle^q,$$

where $r_i = \sqrt[q]{w_i/c_q}\, x_i$. This polynomial identity is further transformed as follows:

$$\langle x, x\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \langle x, r_i\rangle^q \right)^{\frac{1}{q}},$$

which implies that the mapping

$$x \mapsto (\langle x, r_1\rangle, \dots, \langle x, r_n\rangle)$$

is an isometric embedding $l_2^m \to l_q^n$.

By the early fundamental works of Hilbert [16], there is a positive integer $N(m, q)$ such that for any $n \geq N(m, q)$ an isometric embedding $l_2^m \to l_q^n$ exists. It is known (cf. [27]) that

$$\binom{m + \frac{q}{2} - 1}{m - 1} \leq N(m, q) \leq \binom{m + q - 1}{m - 1}. \tag{2.1}$$

The lower- and the upper-bound part of (2.1) mean the dimension of $\mathrm{Hom}_{q/2}(\mathbb{R}^m)$ and $\mathrm{Hom}_q(\mathbb{R}^m)$, respectively.

2.2. Cubature formulas. Let $\Omega \subset \mathbb{R}^m$, and let μ be a normalized measure on Ω such that Ω and μ are both invariant under the group $O(m)$. We assume that polynomials are integrable up to sufficiently large degrees and put

$$\mathcal{I}[f] = \int_{\Omega} f(x) \mu\,(\mathrm{d}x).$$

Let X be a finite set in $\mathbb{R}^m$ with a positive weight w. The pair (X, w) is called a *cubature formula of degree q for $\mathcal{I}$* if

$$\mathcal{I}[f] = \sum_{x \in X} w(x) f(x)$$

for every $f \in \mathcal{P}_q(\Omega)$, where $\mathcal{P}_q(\Omega)$ denotes the space of all polynomials of degree at most q restricted to Ω. In particular, a spherical cubature is called a *spherical design* if w is a constant weight.

A subset X of $\mathbb{R}^m$ is said to be *antipodal* if it is partitioned into $\widetilde{X}$ and $-\widetilde{X}$, namely, $X = \widetilde{X} \cup (-\widetilde{X})$ and $\widetilde{X} \cap (-\widetilde{X}) = \varnothing$. A cubature formula (X, w) is *centrally symmetric* if X is antipodal and $w(x) = w(-x)$ for any $x \in X$. In the following, we mention the relationship among degree-type and index-type spherical cubature formulas.

Proposition 2.2 (see [22, Proposition 4.3])**.** *. Let X be an antipodal finite subset of S^{m-1}. Then X is a centrally symmetric cubature formula on S^{m-1} of degree $q+1$ with $2n$ points if and only if $\widetilde{X}$ is a cubature formula on S^{m-1} of index q with n points.*

We are interested in the following type of integrals:

$$\int_{\mathbb{R}^m} f(x) W(\|x\|_2)\,\mathrm{d}x, \tag{2.2}$$

where W is a density function on $\mathbb{R}^m$. Such integrals are often considered in the context of analysis; for example, see [38].

Proposition 2.3. *If points $x_1, \dots, x_n$ and weights $w_1, \dots, w_n$ form a cubature formula of index q for* (2.2)*, then the points $x_i/\|x_i\|_2$ and the weights $\frac{\|x_i\|_2^q w_i}{\int_0^\infty r^{q+m-1} W(r)\,\mathrm{d}r}$ form a cubature formula of index q on S^{m-1}. Conversely, if $x_1, \dots, x_n$ and $w_1, \dots, w_n$ form a cubature formula of index q on S^{m-1}, then the points x_i and the weights $w_i \int_0^\infty r^{q+m-1} W(r)\,\mathrm{d}r$ form a cubature formula of index q for* (2.2)*.*

Proof. The result follows by observing that for any $f \in \mathrm{Hom}_q(\mathbb{R}^m)$ we have

$$\begin{aligned}\int_{\mathbb{R}^m} f(x) W(\|x\|_2)\,\mathrm{d}x &= \int_0^\infty \left(\int_{S^{m-1}} f(rx)\rho(\mathrm{d}x) \right) r^{m-1} W(r)\,\mathrm{d}r \\ &= \int_0^\infty r^{q+m-1} W(r)\,\mathrm{d}r \int_{S^{m-1}} f(x)\rho\,(\mathrm{d}x).\end{aligned}$$

□

Remark 2.4. By Proposition 2.3, in order to construct spherical cubature formulas we may find those for any integral of the form (2.2). For example, one may think of Gaussian integrals. Such cubature formulas are of particular interest in probability theory [21] and algebraic combinatorics [2]. Moreover, the m-dimensional Gaussian integral can be represented simply as the m-fold product of one-dimensional Gaussian integrals, which is convenient for explaining Victoir's method.

The following proposition is often used in §3 and §4.

Proposition 2.5. *Let X be an antipodal finite subset of $\mathbb{R}^m$. Let $\widetilde{w}, w$ be weight functions on $\widetilde{X}$ and X, respectively, such that $w(x) = w(-x) = 2\widetilde{w}(x)$ for any $x \in \widetilde{X}$. Then $(\widetilde{X}, \widetilde{w})$ is a cubature formula of index q for* (2.2) *if and only if (X, w) is a centrally symmetric cubature formula of index q for* (2.2)*.*

2.3. The Sobolev theorem. Let G be a finite subgroup of $O(m)$, and let $f \in \mathcal{P}_t(\mathbb{R}^m)$. We define the action of $\sigma \in G$ on f as follows:

$$(\sigma f)(x) = f(x^{\sigma^{-1}}), \quad x \in \mathbb{R}^m.$$

A polynomial f is said to be *G-invariant* if $\sigma f = f$ for every $\sigma \in G$. We denote the set of G-invariant polynomials in $\mathcal{P}_t(\mathbb{R}^m)$ and $\mathrm{Harm}_t(\mathbb{R}^m)$ by $\mathcal{P}_t(\mathbb{R}^m)^G$ and $\mathrm{Harm}_t(\mathbb{R}^m)^G$, respectively, where $\mathrm{Harm}_t(\mathbb{R}^m)$ is the subspace of $\mathcal{P}_t(\mathbb{R}^m)$ of harmonic homogeneous polynomials of degree t.

A cubature formula is said to be *G-invariant* if the domain and measure of the integral are invariant under G, the points form a union of G-orbits $z_1^G, \dots, z_e^G$, and $w(x) = w(x')$ for any $x, x' \in z_i^G$; the orbits $z_1^G, \dots, z_e^G$ and weights $w_1, \dots, w_e$ are said to *generate* the formula.

Theorem 2.6 (see [33]). *With the above setup, a G-invariant cubature formula is of degree t if and only if it is exact for every polynomial $f \in \mathcal{P}_t(\mathbb{R}^m)^G$.*

Theorem 2.6 is known as the Sobolev theorem, which is at the core of the Victoir method, as will be seen in the next subsection.

The concept of Euclidean designs was introduced by Neumaier and Seidel [23] as a generalization of spherical cubature formulas. Let X be a finite set in $\mathbb{R}^m$, and let $\{\|x\|_2 \mid x \in X\} = \{r_1, \dots, r_p\}$. Let S_i^{m-1} be the sphere of radius r_i centered at the origin, let $S = \cup_{i=1}^p S_i^{m-1}$, and let $X_i = X \cap S_i^{m-1}$. To each S_i, the surface measure ρ_i is assigned. Let $|S_i^{m-1}| = \int_{S_i^{m-1}} \rho_i\,(\mathrm{d}x)$, where $\frac{1}{|S_i^{m-1}|}\int_{S_i^{m-1}} f(x)\rho_i\,(\mathrm{d}x) = f(0)$ if $S_i^{m-1} = \{0\}$.

Definition 2.7 (see [23]). With the above setup, X is a *Euclidean t-design of* $\mathbb{R}^m$ if

$$\sum_{i=1}^{p} \frac{\sum_{x\in X_i} w(x)}{|S_i^{m-1}|} \int_{S_i^{m-1}} f(x)\rho_i\,(\mathrm{d}x) = \sum_{x\in X} w(x)f(x) \tag{2.3}$$

for every polynomial $f \in \mathcal{P}_t(S)$.

As is readily seen by the definition, the Euclidean designs can be viewed as cubature formulas on multiple concentric spheres.

The following is a variation of the Sobolev theorem for Euclidean designs, which generalizes the well-known theorem of Neumaier and Seidel [23].

Theorem 2.8 (see [24]). *Let G be a subgroup of $O(m)$. Let $X = \cup_{k=1}^M r_k x_k^G$, where $x_k \in S^{m-1}$ and $r_k > 0$. Then the following statements are equivalent:*

(i) *X is a G-invariant Euclidean t-design of $\mathbb{R}^m$;*

(ii) *$\sum_{x\in X} w(x)\|x\|^{2j}\varphi(x) = 0$ for any $\varphi \in \mathrm{Harm}_l(\mathbb{R}^m)^G$, $1 \le l \le t$, $0 \le j \le \lfloor \frac{t-l}{2} \rfloor$.*

Hereafter, let G be an irreducible reflection group in $\mathbb{R}^m$. Such groups are classified completely [4]. Let integers $1 = d_1 \le d_2 \le \cdots \le d_m$ be the exponents of G (see [4, Chapter V, §6]).

Theorem 2.9 (see [9]). *Let G be a finite irreducible reflection group. Let*

$$q_i = \dim(\mathrm{Harm}_i(\mathbb{R}^m)^G).$$

Then

$$\sum_{i=0}^{\infty} q_i \lambda^i = \prod_{i=2}^{m} \frac{1}{1-\lambda^{1+d_i}}.$$

In particular, for any $x \in \mathbb{R}^m$, the orbit x^G is a spherical d_2-design in S^{m-1}.

Let $\alpha_1, \dots, \alpha_m$ be the fundamental roots of a reflection group G. The *corner vectors* $v_1, \dots, v_m$ are defined by $v_i \perp \alpha_j$ if and only if $i \ne j$. We may assume that $\|v_k\|_2 = 1$. We consider the set

$$\mathcal{X}(G, J) = \bigcup_{k\in J} r_k v_k^G,$$

where $J \subset \{1, 2, \dots, m\}$ and $r_k > 0$. Let R denote the set of all r_k.

Theorem 2.10 (Bajnok [1, Theorem 3]). *Let $m \ge 2$ be an integer. Then there is no choice of R, J, and w for which $(\mathcal{X}(B_m, J), w)$ is a Euclidean 8-design of $\mathbb{R}^m$.*

Similar results are known for the groups A_{m-1}, D_m [24]. In §5, we generalize these results, and determine the maximal degree of invariant Euclidean designs for all irreducible reflection groups.

2.4. The Victoir method.

2.4.1. **Combinatorial tools.** Let K be a set of positive integers $k_1, \dots, k_\ell$. A pair of v elements of V and subsets $\mathcal{B}$ of V of cardinalities from K is called a *t-wise balanced design*, denoted by t-(v, K, λ), if every t elements of V occur exactly λ times in $\mathcal{B}$. The elements of V and $\mathcal{B}$ are called *points* and *blocks*. In particular, if K is a singleton, say $K = \{k\}$, a t-wise balanced design is called a *t-design*, and is denoted by t-(v, k, λ). In this paper we only consider designs without repeated blocks.

It is well known (cf. [19]) that for $0 \le t' \le t$ and a subset $T' \subset V$ of t' elements, the number of blocks of a t-(v, k, λ) design containing T' is given by

$$\lambda \frac{\binom{v-t'}{t-t'}}{\binom{k-t'}{t-t'}} = \frac{(v-t')(v-t'-1)\dots(v-t+1)}{(k-t')(k-t'-1)\dots(k-t+1)}, \tag{2.4}$$

not depending on the choice of T'. For each $0 \le t' \le t$, a t-design is also a t'-design. In general, t-wise balanced designs do not necessarily have this property; see §3 for the details.

Let $(V, \mathcal{B})$ be a t-wise balanced design with v points and b blocks. The incidence matrix M of the design $(V, \mathcal{B})$ is a zero-one matrix of size $v \times b$ which has a row for each point and a column for each block, and for $x \in V$ and $B \in \mathcal{B}$, the (x, B)-entry takes 1 if and only if $x \in B$. Given real numbers α, β, let $\mathbf{v}_l(\alpha, \beta)$ be a v-dimensional vector such that the first l coordinates are α and the remaining $v - l$ coordinates are β. For example, $\mathbf{v}_l(\alpha, 0)^{B_v}$ means the vertices of a generalized hyperoctahedron that is inscribed in the $(v-1)$-dimensional sphere of radius $\sqrt{l\alpha^2}$ [1]. With the matrix M, we associate a *generalized incidence matrix with parameters* α, β by defining $I_{\alpha,\beta} = \beta J_{v,b} + (\alpha - \beta)M$, where $\alpha \ne \beta$ and $J_{v,b}$ is the all-one matrix of size $v \times b$.

An $(N \times l)$-matrix with entries ± 1 is called an *orthogonal array with strength t, constraints l, and index λ* if in every t columns, each of the 2^t ordered combinations of elements ± 1 appears in exactly λ rows. We denote this matrix by $OA(N, l, 2, t)$. We do not put λ in the notation, because $\lambda = N/2^t$ by the definition. When $l \le t$, we allow *trivial OA*, namely, the $(2^l \times l)$-matrix such that every 2^l ordered combinations of elements ± 1 appears in exactly one row.

2.4.2. **Victoir's method.** The group B_m contains two special subgroups: the subgroup L of all transpositions of coordinates in $\mathbb{R}^m$ and the subgroup $\acute{L}$ of all sign changes, which is isomorphic to the elementary Abelian 2-group $(\mathbb{Z}/2\mathbb{Z})^m$. It turns out that $|y^{\acute{L}}| = 2^{|\mathrm{wt}(y)|}$, where $\mathrm{wt}(y)$ is the number of nonzero coordinates of a vector y.

We denote by $\mathcal{I}$ the Gaussian integral

$$\mathcal{I}[f] = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} f((x_1^2, \dots, x_m^2)) \exp\Big(-\frac{\|x\|_2^2}{2}\Big)\, \mathrm{d}x_1 \dots \mathrm{d}x_m.$$

This is equivalent to the integral $\acute{\mathcal{I}}$ on the first orthant $\mathbb{R}_+^m$:

$$\acute{\mathcal{I}}[f] = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}_+^m} f((x_1, \dots, x_m)) \exp\Big(-\frac{\|x\|_1}{2}\Big)\Big(\prod_{i=1}^m x_i\Big)^{-1/2} \mathrm{d}x_1 \dots \mathrm{d}x_m.$$

Let

$$x^2 = (x_1^2, \dots, x_m^2)$$

for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, and let

$$\sqrt{x} = \big(\sqrt{x_1}, \dots, \sqrt{x_m}\big)$$

for $x = (x_1, \dots, x_m) \in \mathbb{R}_+^m$.

Proposition 2.11 (cf. [35, 39]). *If $z_1^{\acute{L}}, \ldots, z_e^{\acute{L}}$ and $w_1, \ldots, w_e$ generate an $\acute{L}$-invariant cubature formula of degree q for $\mathcal{I}$, then $z_1^2, \ldots, z_e^2$ and $w_1 2^{\mathrm{wt}(z_1)}, \ldots, w_e 2^{\mathrm{wt}(z_e)}$ form a cubature formula of degree $q/2$ for $\acute{\mathcal{I}}$. Conversely, if $z_1, \ldots, z_e$ and $w_1, \ldots, w_e$ form a cubature formula of degree $q/2$ for $\acute{\mathcal{I}}$, then $\sqrt{z_1}^{\acute{L}}, \ldots, \sqrt{z_e}^{\acute{L}}$ and $w_1/2^{\mathrm{wt}(z_1)}, \ldots, w_e/2^{\mathrm{wt}(z_e)}$ generate a cubature formula of degree q for $\mathcal{I}$.*

The following theorem is due to Victoir [35, Subsection 4.4].

Theorem 2.12. (i) *Assume that there exists a cubature formula of degree $q/2$ for $\acute{\mathcal{I}}$ of the form*

$$\acute{\mathcal{I}}[f] = \frac{w}{\binom{m}{k}} \sum_{x \in \mathbf{v}_k(\alpha,\beta)^L} f(x) + \sum_{i=1}^{M} \frac{w_i}{|x_i^L|} \sum_{x \in x_i^L} f(x),$$

and a $q/2$-design with m points and b blocks of size k. Let X be the columns of a generalized incidence matrix with parameters α, β. Then

$$\acute{\mathcal{I}}[f] = \frac{w}{b} \sum_{x \in X} f(x) + \sum_{i=1}^{M} \frac{w_i}{|x_i^L|} \sum_{x \in x_i^L} f(x)$$

is a cubature formula of degree $q/2$.

(ii) *Assume that there exists an $\acute{L}$-invariant cubature formula of degree q for $\mathcal{I}$ of the form*

$$\mathcal{I}[f] = \sum_{i=1}^{M} \frac{\lambda_i}{2^{\mathrm{wt}(x_i)}} \sum_{x \in x_i^{\acute{L}}} f(x),$$

and $OA(|X_i|, \mathrm{wt}(x_i), 2, q)$ with rows X_i for $i = 1, \ldots, M$. Then

$$\mathcal{I}[f] = \sum_{i=1}^{M} \frac{\lambda_i}{|X_i|} \sum_{x \in X_i} f(x)$$

is a cubature formula of degree q.

The Victoir method was originally written in a more general setting, namely, the integrals considered there were not restricted to Gaussian integrals. In this paper, however, we take only Gaussian integrals because Victoir's ideas can be fully understood with Gaussian integrals.

§3. Generalizing the Victoir method

In this section we generalize the Victoir method with a strengthening of the concept of t-wise balanced designs. We use the notations $B_m, L, \acute{L}, \mathcal{I}, \acute{\mathcal{I}}, \mathbf{v}_i(\cdot,\cdot), \mathrm{wt}(\cdot)$ that are defined in Subsection 2.4.

A t-wise balanced design $(V, \mathcal{B})$ is said to be *regular* if for each $0 \le t' \le t$ and each t'-subset T' of V, the number of blocks containing T' does not depend on the choice of T'; see [8]. As was noted in Subsection 2.4, any t-design possesses this property, but t-wise balanced designs may fail to do so. When $t = 2$, this concept is equivalent to that of *equireplicate* 2-wise balanced designs [10].

Let $\mathcal{B}$ be the set of blocks of a regular t-(v, K, λ) design, where $K = \{k_1, \ldots, k_f\}$. Let $\mathcal{B}_i = \{B \in \mathcal{B} \mid |B| = k_i\}$. Let $y_i \in \mathbb{R}^m$ with $\mathrm{wt}(y_i) = k_i$, and $y_K = \{y_1, \ldots, y_f\}$. We define the following discrete measure:

$$\delta_{y_K,L} := \sum_{i=1}^{f} \frac{|\mathcal{B}_i|}{|\mathcal{B}|\binom{m}{k_i}} \sum_{x \in y_i^L} \delta_x.$$

Proposition 3.1. *Assume that there exists a regular t-$(m, \{k_i \mid 1 \le i \le f\}, \lambda)$ design $(V, \mathcal{B})$. Let X be the columns of a generalized incidence matrix with parameters α, β with $\alpha \neq \beta$. Let $y_1, \dots, y_f \in X$ be such that $\mathrm{wt}(y_i) = k_i$. Then*

$$\int_{\bigcup_{i=1}^f y_i^L} f(x)\delta_{y_K,L}\,(\mathrm{d}x) = \frac{1}{|\mathcal{B}|}\sum_{x\in X} f(x)$$

for every $f \in \mathcal{P}_t(\bigcup_{i=1}^f y_i^L)$.

Proof. By changing variables $x_i \to (x_i - \beta)/(\alpha - \beta)$, there is no loss of generality in assuming that $\alpha = 1, \beta = 0$. Then for any $e_1, \dots, e_m \ge 0$, we have

$$\int_{\bigcup_{i=1}^f y_i^L} f(x_1^{e_1}, \dots, x_m^{e_m})\delta_{y_K,L}\,(\mathrm{d}x) = \int_{\bigcup_{i=1}^f y_i^L} f(x_1, \dots, x_m)\delta_{y_K,L}\,(\mathrm{d}x).$$

Permuting the rows of an incidence matrix also gives another t-wise balanced design with the same parameters $m, k_1, \dots, k_f, \lambda$. Thus, it suffices to show that

$$\int_{\bigcup_{i=1}^f y_i^L} f(x)\delta_{y_K,L}\,(\mathrm{d}x) = \frac{1}{|\mathcal{B}|}\sum_{x\in X} f(x)$$

for the monomials $f(x) = \prod_{i=1}^j x_i, 1 \le j \le t$. For this, we count the pairs $(T', B) \in \binom{V}{t'} \times \mathcal{B}$, $T' \subset B$ in two ways:

$$\begin{aligned}
\lambda'\binom{m}{t'} &= \sum_{T'\in\binom{V}{t'}}\ \sum_{T'\subset B\in\mathcal{B}} 1 \\
&= \sum_{B\in\mathcal{B}}\ \sum_{\substack{T'\subset B\\ T'\in\binom{V}{t'}}} 1 \\
&= \sum_{i=1}^f \sum_{B\in\mathcal{B}_i}\ \sum_{\substack{T'\subset B\\ T'\in\binom{V}{t'}}} 1 \\
&= \sum_{i=1}^f |\mathcal{B}_i|\binom{k_i}{t'},
\end{aligned}$$

where regularity is used to show the first identity. Thus, for $f(x) = \prod_{i=1}^{t'} x_i$ we have

$$\sum_{y\in X} f(y) = \lambda' = \sum_{i=1}^f \frac{|\mathcal{B}_i|\binom{k_i}{t'}}{\binom{m}{t'}}.$$

This is further transformed to

$$\begin{aligned}
\sum_{i=1}^f \frac{|\mathcal{B}_i|}{\binom{m}{k_i}}\cdot\frac{\binom{m}{k_i}\binom{k_i}{t'}}{\binom{m}{t'}} &= \sum_{i=1}^f \frac{|\mathcal{B}_i|}{\binom{m}{k_i}}\cdot\binom{m-t'}{k_i-t'} \\
&= \sum_{i=1}^f \frac{|\mathcal{B}_i|}{\binom{m}{k_i}}\sum_{x\in y_i^L} f(x) \\
&= |\mathcal{B}|\cdot\int_{\bigcup_{i=1}^f y_i^L} f(x)\delta_{y_K,L}\,(\mathrm{d}x).
\end{aligned}$$

□

Remark 3.2. In a combinatorial framework (cf. [31]), some researchers regard t-wise balanced designs as cubature on "discrete spheres". However, there are only a few publications where the regularity of designs is mentioned. Victoir seems to be the first who employed combinatorial t-designs to reduce the size of cubature for usual continuous integrals.

The following generalizes Theorem 2.12 (i) and motivates the study of regular t-wise balanced designs both in a combinatorial and analytic way.

Theorem 3.3. *Assume that there exists a regular $q/2$-wise balanced design with m points and b_i blocks of size k_i, $i = 1, \dots, e$. Moreover, assume that there exists a cubature formula of degree $q/2$ (or index $q/2$) for $\acute{\mathcal{I}}$ of the form*

$$\acute{\mathcal{I}}[f] = c\Bigg(\sum_{i=1}^{e} \frac{b_i}{\binom{m}{k_i} b} \sum_{x \in \mathbf{v}_{k_i}(\alpha,\beta)^L} f(x)\Bigg) + \sum_{i=2}^{M} \frac{w_i}{|x_i^L|} \sum_{x \in x_i^L} f(x)$$

where b is the total number of blocks of the design and c is a positive number. Let X be the columns of a generalized incidence matrix with parameters α, β. Then

$$\acute{\mathcal{I}}[f] = \frac{c}{b} \sum_{x \in X} f(x) + \sum_{i=2}^{M} \frac{w_i}{|x_i^L|} \sum_{x \in x_i^L} f(x)$$

is a cubature formula of degree $q/2$ (or index $q/2$).

The following proposition is often used in §4.

Proposition 3.4. *Assume there exists a t-(v, k, λ) design. Then:*

(i) *There exists a regular t-$(v-1, \{k, k-1\}, \lambda)$ design with $\lambda \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$ blocks of size $k-1$ and $\frac{(v-k)\lambda}{k} \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$ blocks of size k.*

(ii) *Let X be the columns of an incidence matrix of the design given in* (i), *and let $y_1 = \mathbf{v}_k(1, 0)$, $y_2 = \mathbf{v}_{k-1}(1, 0)$. Then for every $f \in \mathcal{P}_t(y_1^L \cup y_2^L)$,*

$$\sum_{x \in y_1^L \cup y_2^L} f(x) = \frac{\binom{k-1}{t-1}\binom{v-1}{k-1}}{\lambda \binom{v-1}{t-1}} \sum_{x \in X} f(x).$$

Proof. (i) Let $(V, \mathcal{B})$ be a t-(v, k, λ) design, and let $x \in V$. We consider the incidence structure $(V', \mathcal{B}')$, where

$$V' = V \setminus \{x\}, \quad \mathcal{B}' = \{B \in \mathcal{B} \mid x \notin B\} \cup \{B \setminus \{x\} \mid x \in B \in \mathcal{B}\}.$$

Then $(V', \mathcal{B}')$ is a regular t-wise balanced design with the parameters determined by (2.4). Assertion (ii) follows from (i) and Proposition 3.1. □

We close this section with some remarks on regular t-wise balanced designs. First, as far as the authors know, there are only a few general results on the existence of regular t-wise balanced designs for $t \geq 3$. Some examples are known, most of which are obtained trivially, like Proposition 3.4 (i). The second author and Reinhard Laue searched for regular 3-, 4- and 5-wise balanced designs with Discreta, a sophisticated program to compute designs, and found many designs with small parameters, some of which are summarized in Table 1.

We believe that there will be further nontrivial regular t-wise balanced designs. However, in this paper, such thorough discussions are omitted and left for future work.

A natural problem is to find a good bound for the number of blocks of a t-wise balanced design. Recently, Ziqing Xiang [37] derived a Fisher-type bound for regular

TABLE 1. Some new regular t-wise balanced designs

Parameters	Groups
$3\text{-}(25, \{6, 10\}, 4)$	$AGL(1, 25)$
$4\text{-}(27, \{5, 8\}, 5)$	$ASL(3, 3)$
$5\text{-}(33, \{6, 7\}, 10)$	$P\Gamma L(2, 32)$
$5\text{-}(33, \{6, 8\}, 20)$	$P\Gamma L(2, 32)$
$5\text{-}(33, \{6, 9\}, 15)$	$P\Gamma L(2, 32)$
$5\text{-}(33, \{7, 10\}, 42)$	$P\Gamma L(2, 32)$
$5\text{-}(55, \{6, 5\}, 5)$	$C_2 \times P\Gamma L(2, 27)$

t-wise balanced designs. Namely, he showed that if there is a regular $2e$-wise balanced design $(V, \mathcal{B})$ with f distinct sizes of blocks, then

$$|\mathcal{B}| \geq \sum_{i=0}^{f-1} \binom{|V|}{e-i}.$$

This bound is sharp when $t = 2$ and $f = 2$, by a result of Woodal [36]. Moreover, for $t = 4$ and $f = 2$, a tight example can be constructed from the usual tight 4-design that corresponds to the Johnson scheme. Without regularity, no good bounds seem to be known.[2]

§4. CUBATURE ARISING FROM VICTOIR'S METHOD AND ITS GENERALIZATION

In this section many cubature formulas are constructed by Victoir's method and its generalization formulated in §3.

4.1. Index-four cubature. There are many publications on the existence of index-four cubature in small-dimensional spaces that are not minimal but have few points; see, e.g., [26, 34]. In general-dimensional cases, however, it seems that explicit constructions of good cubature formulas are not sufficiently known.[3] Therefore, the following theorem by Shatalov [32] is very important.

Theorem 4.1.

(i) [32, Theorem 4.4.9]. *Assume that, for given m, n, and q, there exists a cubature formula of index q with n points on S^{m-1}. Then for any $M \geq m$, there exists a cubature formula of index q with $((q+2)/2)^{M-m}n$ points on S^{M-1}.*

(ii) [32, Corollary 4.4.12]. *There exists an index-four cubature formula on S^{m-1} with n points when*

$$m = 2^{2l} + s, \quad n = 2^{2l} \cdot 3^s \cdot (2^{2l-1} + 1), \quad l \geq 1, \ \ s \geq 0, \tag{4.1}$$

$$m = 2l + 2 + s, \quad n = 3^{s+1} \cdot ((l+1)^2 + 1), \quad l \text{ is a prime power}, \ \ s \geq 0. \tag{4.2}$$

Remark 4.2. For $s = 0$, Theorem 4.1 is a theorem of König [20]. The family (4.1) improves the upper-bound part of (2.1) if s is fixed and m is sufficiently large, or $s = 1, 2$. A similar conclusion is valid for (4.2).

We can construct cubature formulas in general-dimensional spaces that improve Shatalov's families.

[2]Eiichi Bannai kindly told us detailed information on bounds for regular t-wise balanced designs through email conversation.

[3]Oksana Shatalov and Yuan Xu kindly told us this information.

Theorem 4.3.

(i) *Let $l \geq 2$ and m be integers. Assume that*

$$\ell = \begin{cases} 4l-1 & \text{if} \quad 2^{2l-1} \leq m \leq 2^{2l}; \\ 4l+1 & \text{if} \quad 2^{2l} < m < 2^{2l+1}. \end{cases}$$

Then there is an integer n with $2^{\ell-1}+m < n \leq 2^{\ell}+m$ for which an index-four cubature with n points on S^{m-1} exists.

(ii) *Let $l \geq 2, l'$, and m be integers. Assume that $m \in \{3^{l'+2}-2, 2 \cdot 9^{l'+1}-2\}$, and*

$$\ell = \begin{cases} 4l-1 & \text{if} \quad 2^{2l-1} \leq (m+2)/3 \leq 2^{2l}; \\ 4l+1 & \text{if} \quad 2^{2l} < (m+2)/3 < 2^{2l+1}. \end{cases}$$

Then there is an integer n with $2^{\ell-1}m < n \leq 2^{\ell}m$ for which an index-four cubature with n points on S^{m-1} exists.

The following lemma is employed, the proof of which is easy and so omitted.

Lemma 4.4. *The following is an m-dimensional index-two cubature formula for $\acute{\mathcal{I}}$.*

(i) *For $m \geq 3$,*

$$\acute{\mathcal{I}}[f] = \frac{1}{2m} \sum_{x \in \mathbf{v}_1(\sqrt{4m},0)^L} f(x) + \frac{1}{2} \sum_{x \in \mathbf{v}_m(\sqrt{2},0)^L} f(x).$$

(ii) *For $m \equiv 1 \pmod 3$,*

$$\acute{\mathcal{I}}[f] = \frac{1}{\binom{m}{(m+2)/3}} \sum_{x \in \mathbf{v}_{(m+2)/3}\left(\sqrt{\frac{9m}{m+2}},0\right)^L} f(x).$$

More B_m-invariant cubature formulas can be obtained systematically by using the Sobolev theorem.

Proof of Theorem 4.3.

(i) Take an $OA(2^{4l}, m, 2, 4)$ if $2^{2l-1} \leq m \leq 2^{2l}$, and an $OA(2^{4l+2}, m, 2, 4)$ if $2^{2l} < m < 2^{2l+1}$. These OAs are constructed from an $OA(2^{4l}, 2^{2l}, 2, 4)$ and an $OA(2^{4l+2}, 2^{2l+1}-1, 2, 4)$, which are dual to the Kerdock code and to the BCH code over $\mathbb{F}_2$ (cf. [15, p. 102, p. 94]) respectively, where $0, 1 \in \mathbb{F}_2$ are replaced by $-1, 1$. Hence, using Theorem 2.12 (ii), Lemma 4.4 (i), and Proposition 2.11, we get an index-four cubature formula for $\mathcal{I}$ with at most $2^{\ell+1}+2m$ points. The Kerdock OA has central symmetry (cf. [20]). The BCH OA is also centrally symmetric because it is linear. The result follows by Propositions 2.3 and 3.4.

(ii) The existence of a 2-$(m, (m+2)/3, (m+2)/9)$ design with m blocks is known [17]. So, by Theorem 2.12 (i) and Lemma 4.4 (ii), we obtain an index-two cubature formula for $\acute{\mathcal{I}}$ with m points. In accordance with Proposition 2.11, the resulting cubature is equivalent to an $\acute{L}$-invariant cubature of index 4 with $2^{(m+2)/3}m$. Applying Theorem 2.12 (ii) to this formula and the OA given in the proof of Theorem 4.3 (i), we get an index-four cubature formula for $\mathcal{I}$ with at most $2^{\ell+1}m$ points. Since the Kerdock and BCH OA have central symmetry, the result follows by Propositions 3.4 and 2.3. □

More general-dimensional index-four cubature formulas with $O(m^2)$ or $O(m^3)$ points can be obtained by using suitable OA, 2-designs, and regular pairwise balanced designs.

Remark 4.5.

(i) Theorem 4.3 improves Theorem 4.1 for many values of m. If $2^{2l-1} \leq m \leq 2^{2l}$, the family of Theorem 4.3 (i) comes from centrally symmetric cubature formulas by "halving"

opposite row-vectors of OA. The underlying symmetric cubature formulas were found by Victoir [35, Subsection 5.3].

(ii) Theorem 4.3 does not mention the exact number of points of the constructed cubature. When $m = 2^{2l}$ in Theorem 4.3 (i), the underlying OA is the Kerdock OA and no two distinct rows coincide. So, the resulting cubature has exactly $2^{4l-1} + 2^{2l}$ points, which is equivalent to König's family.

(iii) By Proposition 2.11, the L-invariant formula of Lemma 4.4 (i) is equivalent to the degree-five cubature of Stroud [34]. Moreover the formula in (ii) corresponds to Kürschák's identity in number theory (see §6).

4.2. Index-six cubature. Shatalov [32, Theorem 4.7.20] compiled the known index-six cubature formulas with few points in small-dimensional spheres to get Table 2 (strictly speaking, a part of the original).

TABLE 2. Index-six cubature on S^{m-1} with n points

No	1	2	3	4	5	6	7	8	9	10	11	12	13
m	3	4	5	6	7	8	9	10	11	16	17	18	23
n	11	23	41	63	113	120	480	1920	7680	2160	8640	34650	2300

Nos. 1, 2, 4 are (respectively) in [27, 11, 9].[4] Nos. 3, 5 are in [34], and No. 6 is in [5]. To complete Table 2, Shatalov applied Theorem 4.1 (i) to one of the above formulas. For example, No. 7 has 4 times as many points as No. 6 does. According to Shatalov, Table 2 had not been updated so far, and the existence of general-dimensional index-six cubature formulas with few points is not fully known.

Two families of general-dimensional cubature formulas that improve the upper-bound part of (2.1) can be given.

Theorem 4.6. *Let Q be a prime power such that $Q \equiv 1 \pmod 6, Q \neq 25$. Let $m \in \{Q+1, Q\}$, and let l be an integer with $l \geq 3, 2^{2l-2} < m \leq 2^{2l}$. Then there is an integer $n \leq 2^{6l-2}(3Q(Q+1)+1)+m$ for which an index-six cubature formula with n points on S^{m-1} exists.*

Lemma 4.7. *The following is an m-dimensional index-three cubature formula for $\acute{\mathcal{I}}$.*

(i) *For $m \equiv 2 \pmod 6$ and $8 \leq m$,*

$$\acute{\mathcal{I}}[f] = \frac{1}{3} \sum_{x \in \mathbf{v}_m \left(\sqrt[3]{\frac{12}{5}}, 0\right)^L} f(x) + \frac{1}{3m} \sum_{x \in \mathbf{v}_1 \left(\sqrt[3]{\frac{216m}{m+4}}, 0\right)^L} f(x) + \frac{1}{3\binom{m}{(m+10)/6}} \sum_{x \in \mathbf{v}_{(m+10)/6} \left(\sqrt[3]{\frac{1296m(m-1)}{(m+4)(m+10)}}, 0\right)^L} f(x).$$

(ii) *For $m \equiv 1 \pmod 6$ and $7 \leq m$,*

$$\acute{\mathcal{I}}[f] = \frac{1}{3} \sum_{x \in \mathbf{v}_m \left(\sqrt[3]{\frac{9}{5}}, 0\right)^L} f(x) + \frac{1}{3m} \sum_{x \in \mathbf{v}_1 \left(\frac{1}{3m}, 0\right)^L} f(x) + \frac{1}{3\binom{m+1}{(m+11)/6}} \sum_{\substack{x \in \mathbf{v}_{(m+11)/6} \left(\sqrt[3]{\frac{1296m(m+1)}{(m+5)(m+11)}}, 0\right)^L \\ \cup \mathbf{v}_{(m+5)/6} \left(\sqrt[3]{\frac{1296m(m+1)}{(m+5)(m+11)}}, 0\right)^L}} f(x).$$

[4] The existence of a 23-point cubature formula of index 6 on S^3 is not covered in [32, Theorem 4.7.20].

Proof of Theorem 4.6. First, we consider the case where $m = Q + 1$. There exists a 3-$(Q+1, (Q+11)/6, (Q+5)(Q+11)/72)$ design (cf. [19]), which has $3Q(Q+1)$ blocks by (2.4). By Theorem 2.12 (i) and Lemma 4.7 (i), we obtain an index-three cubature for $\acute{\mathcal{I}}$ with $1 + (Q+1) + 3Q(Q+1)$ points. By Proposition 2.11, this is equivalent to an $\acute{L}$-invariant cubature with $2^{Q+1} + 2(Q+1) + 2^{(Q+11)/6} \cdot 3Q(Q+1)$ points. By applying Theorem 2.12 (ii) to an $OA(2^{6l-1}, Q+1, 2, 7)$ and an $OA(2^{6l-1}, (Q+11)/6, 2, 7)$ that are subarrays of the dual $OA(2^{6l-1}, 2^{2l}, 2, 7)$ of the Delsarte–Goethals code (cf. [15, p. 103]), we obtain an index-six formula for $\mathcal{I}$ with at most $2^{6l-1} \cdot (1 + 3Q(Q+1)) + 2(Q+1)$ points. Note that the $OA(2^{6l-1}, 2^{2l}, 2, 7)$ has central symmetry. In fact, the Delsarte–Goethals code can be constructed by applying the Gray-code mapping $0 \mapsto 00, 1 \mapsto 01, 2 \mapsto 11, 3 \mapsto 10$ to linear, cyclic codes over $\mathbb{Z}_4$. Replacing $0, 1 \in \mathbb{F}_2$ by ± 1 implies the central symmetry of the OA. Thus, the result follows by Propositions 3.4 and 2.3. Similar arguments work when $m = Q$; replace the above L-invariant formula by that of Lemma 4.7 (ii). By Proposition 3.4 (ii) the above 3-design can be reduced to a regular 3-wise balanced design with Q points and $3Q(Q+1)$ blocks. Using Theorem 3.3, we obtain an index-three cubature formula for $\acute{\mathcal{I}}$ with $1 + 3Q(Q+1) + Q$ points. Now the claim follows by the same argument as in the case where $m = Q + 1$. □

Remark 4.8. The family of Theorem 4.6 has $O(m^5)$ points, improving the upper-bound part of (2.1). More general-dimensional index-six formulas with $O(m^5)$ points may be obtained by using known infinite families of 3-designs [19].

Two more interesting cubature formulas can be given.

Example 4.9. The following is a 7-dimensional index-three cubature formula for $\acute{\mathcal{I}}$:

$$\acute{\mathcal{I}}[f] = \frac{1}{140} \sum_{x \in \mathbf{v}_4(\sqrt[3]{28},0)^L \cup \mathbf{v}_3(\sqrt[3]{28},0)^L} f(x) + \frac{1}{14} \sum_{x \in \mathbf{v}_1(\sqrt[3]{112},0)^L} f(x). \tag{4.3}$$

A 3-$(8, 4, 1)$ design exists (cf. [19]), and a regular 3-$(7, \{4, 3\}, 1)$ design with 7 blocks of sizes 4 and 3 also exists by Proposition 3.4 (i). Let X be the columns of an incidence matrix of the 3-wise balanced design. By Proposition 3.4 (ii),

$$\sum_{x \in \mathbf{v}_4(\sqrt[3]{28},0)^L \cup \mathbf{v}_3(\sqrt[3]{28},0)^L} f(x) = 5 \sum_{x \in X} f(x) \tag{4.4}$$

for every $f \in \mathcal{P}_3$. Hence, by (4.3), (4.4), and Proposition 2.11, we get the following index-six cubature for $\mathcal{I}$:

$$\mathcal{I}[f] = \frac{1}{448} \sum_{x \in (\sqrt[6]{28} \cdot X_1)^{\acute{L}}} f(x) + \frac{1}{224} \sum_{x \in (\sqrt[6]{28} \cdot X_2)^{\acute{L}}} f(x) + \frac{1}{28} \sum_{x \in \mathbf{v}_1(\sqrt[3]{112},0)^{B_m}} f(x) \tag{4.5}$$

where $X_1 = \{x \in X \mid \mathrm{wt}(x) = 4\}, X_2 = \{x \in X \mid \mathrm{wt}(x) = 3\}$. This is reduced to a 91-point formula of index 6 on S^6, by Propositions 2.3 and 3.4.

Example 4.10. The following is a 9-dimensional index-three cubature formula for $\acute{\mathcal{I}}$:

$$\acute{\mathcal{I}}[f] = \frac{1}{3} \sum_{x \in \mathbf{v}_9(1,0)^L} f(x) + \frac{1}{630} \sum_{x \in \mathbf{v}_4(\sqrt[3]{60},0)^L \cup \mathbf{v}_3(\sqrt[3]{60},0)^L} f(x) + \frac{1}{27} \sum_{x \in \mathbf{v}_1(\sqrt[3]{180},0)^L} f(x).$$

The existence of a 3-$(10, 4, 1)$ design (cf. [19]) implies that of a regular 3-$(9, \{4, 3\}, 1)$ design with 12 blocks of size 3 and 18 blocks of size 4. In the same way as in Example 4.9, a 457-point formula on S^8 is obtained.

Remark 4.11.

(i) The formula No. 5 of Table 2 implies that $N(7,6) \leq 113$. Example 4.9 improves this to

$$N(7,6) \leq 91. \tag{4.6}$$

The lower-bound part of (2.1) shows that $84 \leq N(7,6)$. The authors do not know of the existence of a cubature with fewer points than the 91-point formula on S^6. It should also be noted that no spherical 84-point index-six cubature on S^6 can exist, by Theorem 1 of [3].

(ii) The formula No. 7 of Table 2 implies that $N(9,6) \leq 480$. Example 4.10 improves this to

$$N(9,6) \leq 457. \tag{4.7}$$

The fundamental roots of the group B_m are $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, m-1$, and $\alpha_m = \sqrt{2}e_m$, where $e_1, \ldots, e_m$ are the standard basis vectors in $\mathbb{R}^m$ [4]. The corner vectors are $v_i = (1/\sqrt{i}, \ldots, 1/\sqrt{i}, 0, \ldots, 0)$ for $i = 1, \ldots, m$. We note that all B_m-invariant cubature formulas of indices $4, 6$ given in §4 consist of the orbits of the corner vectors. By Bajnok's theorem, in order to find higher-index spherical cubature formulas, we must take at least one orbit of points that are not corner vectors; see, e.g., [29] for a simple construction of higher-index formulas on spheres.

A refinement of Bajnok's theorem is proved in the next section.

§5. The maximum strength of invariant Euclidean designs

We use the notations R, J, α_i, v_i, and $\mathcal{X}(G,J)$ defined in Subsection 2.3. Our aim in this section is to prove the following theorem.

Theorem 5.1. *Let G be a finite irreducible reflection group in $\mathbb{R}^m$ with $m \geq 2$. Then there is no choice of R, J, and a weight w for which $(\mathcal{X}(G,J), w)$ is a Euclidean t-design of $\mathbb{R}^m$ in the following cases*:

(i) $t \geq 6$ *if* $G = A_{m-1}$;
(ii) $t \geq 8$ *if* $G = B_m, D_m$;
(iii) $t \geq 10$ *if* $G = E_6$;
(iv) $t \geq 12$ *if* $G = F_4, H_3, E_7$;
(v) $t \geq 16$ *if* $G = E_8$;
(vi) $t \geq 24$ *if* $G = H_4$.

The following lemma plays an important role in the proof of this theorem.

Lemma 5.2. *Let G be a subgroup of $O(m)$, and $X = \{x_1, \ldots, x_M\}$ a subset of S^{m-1}. Let $\{f_{i,k}\}_{k=1}^{m_i}$ be a basis of $\mathrm{Harm}_{2i}(\mathbb{R}^m)^G$, where $m_i = \dim(\mathrm{Harm}_{2i}(\mathbb{R}^m)^G)$. Let V_i be the space $\mathrm{Span}_{\mathbb{R}}\left\{(f_{i,k}(x_1), \ldots, f_{i,k}(x_M)) \mid k = 1, \ldots, m_i\right\} \subset \mathbb{R}^X$. Suppose there is $v \in \sum_{i=1}^{s} V_i$ such that all entries of v are positive. Then there is no choice of radii r_i and a weight w for which $\left(\sum_{i=1}^{M} r_i x_i^G, w\right)$ is a Euclidean $2s$-design.*

Proof. Since $X \subset S^{m-1}$, we can write

$$\begin{aligned} v &= \sum_{i=1}^{s}\sum_{k=1}^{m_i} a_{i,k}\big(f_{i,k}(x_1), \ldots, f_{i,k}(x_M)\big) \\ &= \sum_{i=1}^{s}\sum_{k=1}^{m_i} a_{i,k}\big(\|x_1\|_2^{2s-2i} f_{i,k}(x_1), \ldots, \|x_n\|_2^{2s-2i} f_{i,k}(x_M)\big), \end{aligned}$$

where the $a_{i,k}$ are real numbers. Let $f(x) := \sum_{i=1}^{s}\sum_{k=1}^{m_i} a_{i,k}\|x\|_2^{2s-2i} f_{i,k}(x)$. Then $f \in \sum_{2i+2j=2s, i\geq 1, j\geq 0} \|x\|_2^{2j}\,\mathrm{Harm}_{2i}(\mathbb{R}^m)^G$, and f satisfies $f(x_i) > 0$ for each $i = 1, \dots, M$. It remains to observe that $f(r_i x_i^g) = r_i^{2s} f(x_i^g) = r_i^{2s} f(x_i) > 0$ for $i = 1, \dots, M$ and $g \in G$. □

Remark 5.3. Under our assumptions in Lemma 5.2, no subset of $\{rx^g \mid g \in G, x \in X, r > 0\}$ can form a Euclidean $2s$-design. In particular, for any subgroup H of G, $\left(\sum_{i=1}^{M} r_i x_i^H, w\right)$ is not a Euclidean $2s$-design for any radii r_i and weight w.

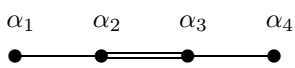
The proof of Theorem 5.1 is divided into some cases. The following notation is used. For a finite irreducible reflection group G, v_i denotes the corner vector normalized by $(v_i, \alpha_i) = 1$, $v_i' := v_i/\sqrt{(v_i, v_i)}$, and $N_i := |v_i^G|$. Let e_i be the column vector with the ith entry 1 and the others 0. Define

$$\mathrm{sym}(f) := \frac{1}{|(S_m)_f|}\sum_{g\in S_m} f(x^g), \quad (S_m)_f := \{g \in S_m \mid f(x^g) = f(x)\}$$

for an m-variable polynomial f, where S_m is the symmetric group of m elements. Let $p_i := x_2^2 + x_3^2 + \cdots + x_{i+1}^2$ for $i \geq 2$. The polynomials h_i in the following subsections are harmonic.

5.1. Group F_4.

Dynkin diagram:



Exponents: 1, 5, 7, 11.

Fundamental roots:

$$\alpha_1 := {}^te_1 - {}^te_2, \alpha_2 := {}^te_2 - {}^te_3, \alpha_3 := {}^te_4, \alpha_4 := \frac{-{}^te_1 - {}^te_2 - {}^te_3 + {}^te_4}{2}.$$

Corner Vectors:

$$v_1 = {}^te_1 + {}^te_4, v_2 = {}^te_1 + {}^te_2 + 2{}^te_4, v_3 = {}^te_1 + {}^te_2 + {}^te_3 + 3\ {}^te_4, v_4 = 2{}^te_4.$$

Size of Orbit: $N_1 = 24$, $N_2 = 96$, $N_3 = 96$, $N_4 = 24$.

Harmonic Molien series:

$$\frac{1}{(1-t^6)(1-t^8)(1-t^{12})} = 1 + t^6 + t^8 + 2t^{12} + t^{14} + \dots.$$

G-invariant harmonic polynomials.

For $i = 6, 8, 12$, $\mathrm{Harm}_i(\mathbb{R}^4)^{F_4}$ is spanned by the following polynomials.

1. *Degree* 6:

$$f_6 := \mathrm{sym}(x_1^6) - 5\,\mathrm{sym}(x_1^4x_2^2) + 30\,\mathrm{sym}(x_1^2x_2^2x_3^2).$$

2. *Degree* 8:

$$f_8 := \mathrm{sym}(x_1^8) - \tfrac{28}{3}\,\mathrm{sym}(x_1^6x_2^2) + \tfrac{98}{3}\,\mathrm{sym}(x_1^4x_2^4) - 28\,\mathrm{sym}(x_1^4x_2^2x_3^2) + 504x_1^2x_2^2x_3^2x_4^2.$$

3. *Degree* 12:

$$\begin{aligned} f_{12,1} := \mathrm{sym}(x_1^{12}) &- 22\,\mathrm{sym}(x_1^{10}x_2^2) + 79\,\mathrm{sym}(x_1^8x_2^4) \\ &+ 258\,\mathrm{sym}(x_1^8x_2^2x_3^2) - 116\,\mathrm{sym}(x_1^6x_2^6) - 236\,\mathrm{sym}(x_1^6x_2^4x_3^2) \\ &- 4392\,\mathrm{sym}(x_1^6x_2^2x_3^2x_4^2) + 570\,\mathrm{sym}(x_1^4x_2^4x_3^4) + 3660\,\mathrm{sym}(x_1^4x_2^4x_3^2x_4^2), \\ f_{12,2} := \mathrm{sym}(x_1^{12}) &- 22\,\mathrm{sym}(x_1^{10}x_2^2) + \tfrac{133}{2}\,\mathrm{sym}(x_1^8x_2^4) \\ &+ \tfrac{591}{2}\,\mathrm{sym}(x_1^8x_2^2x_3^2) - \tfrac{157}{2}\,\mathrm{sym}(x_1^6x_2^6) - \tfrac{1369}{4}\,\mathrm{sym}(x_1^6x_2^4x_3^2) \\ &- 4167\,\mathrm{sym}(x_1^6x_2^2x_3^2x_4^2) + \tfrac{2265}{2}\,\mathrm{sym}(x_1^4x_2^4x_3^4) + \tfrac{6945}{2}\,\mathrm{sym}(x_1^4x_2^4x_3^2x_4^2). \end{aligned}$$

Substitute v_k for G-invariant harmonic polynomials.

1. *Degree* 6:

$$u_6 := \big[f_6(v_1'), f_6(v_2'), f_6(v_3'), f_6(v_4')\big] = \big[-1, -\tfrac{1}{9}, \tfrac{1}{9}, 1\big].$$

2. *Degree* 8:

$$u_8 := [f_8(v_1'), f_8(v_2'), f_8(v_3'), f_8(v_4')] = \big[1, -\tfrac{13}{27}, -\tfrac{13}{27}, 1\big].$$

3. *Degree* 12:

$$u_{12,1} := [f_{12,1}(v_1'), f_{12,1}(v_2'), f_{12,1}(v_3'), f_{12,1}(v_4')] = \big[0, \tfrac{128}{243}, -\tfrac{25}{243}, 1\big],$$
$$u_{12,2} := [f_{12,2}(v_1'), f_{12,2}(v_2'), f_{12,2}(v_3'), f_{12,2}(v_4')] = \big[\tfrac{25}{128}, \tfrac{1751}{3456}, 0, 1\big].$$

Proposition 5.4. *There is no choice of R, J, and w for which $(\mathcal{X}(F_4, J), w)$ is a Euclidean 12-design.*

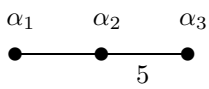
Proof. Since

$$-u_{12,1} + 2u_{12,2} = \big[\tfrac{25}{64}, \tfrac{7567}{15552}, \tfrac{25}{243}, 1\big],$$

the claim follows from Lemma 5.2. □

5.2. Group H_3.

Dynkin diagram:



Exponents: 1, 5, 9.

Fundamental roots:

$$\alpha_1 := -{}^te_1 + {}^te_2,$$
$$\alpha_2 := -{}^te_2 + {}^te_3,$$
$$\alpha_3 := \frac{(1+\sqrt{2}+\sqrt{5}-\sqrt{10})({}^te_1 + {}^te_2) - (2-\sqrt{2}+2\sqrt{5}+\sqrt{10}){}^te_3}{6}.$$

Corner Vectors:

$$v_1 = \frac{-(3\sqrt{2}+\sqrt{10}+8){}^te_1 - (3\sqrt{2}+\sqrt{10}-4)({}^te_2 + {}^te_3)}{12},$$
$$v_2 = \frac{-(3\sqrt{2}+\sqrt{10}+2)({}^te_1 + {}^te_2) - (3\sqrt{2}+\sqrt{10}-4){}^te_3}{6},$$
$$v_3 = -\frac{(\sqrt{2}+\sqrt{10})({}^te_1 + {}^te_2 + {}^te_3)}{4}.$$

Size of Orbit: $N_1 = 12,\ N_2 = 30,\ N_3 = 20.$

Harmonic Molien series:

$$\frac{1}{(1-t^6)(1-t^{10})} = 1 + t^6 + t^{10} + t^{12} + t^{16} + t^{18} + t^{20} + \dots.$$

G-invariant harmonic polynomials.

For $i = 6,\ 10,\ 12$, $\mathrm{Harm}_i(\mathbb{R}^3)^{H_3}$ is spanned by the following polynomials.

1. *Degree* 6:

$$f_6 := 2\,\mathrm{sym}(x_1^6) + 21\,\mathrm{sym}(x_1^5x_2) - 15\,\mathrm{sym}(x_1^4x_2^2) + 21\sqrt{10}\,\mathrm{sym}(x_1^4x_2x_3)$$
$$- (70 - 7\sqrt{10})\,\mathrm{sym}(x_1^3x_2^3) - 21\sqrt{10}\,\mathrm{sym}(x_1^3x_2^2x_3) + 180x_1^2x_2^2x_3^2.$$

2. *Degree* 10:

$$f_{10} := \sum_{g\in H_3} h_{10}(x^g),$$

where

$$h_{10}(x) := 256x_1^{10} - 5760x_1^8p_2 + 20160x_1^6p_2^2 - 16800x_1^4p_2^3 + 3150x_1^2p_2^4 - 63p_2^5.$$

3. *Degree* 12:

$$f_{12} := \sum_{g\in H_3} h_{12}(x^g),$$

where

$$\begin{aligned} h_{12}(x) := {} & 1024x_1^{12} - 33792x_1^{10}p_2 + 190080x_1^8p_2^2 \\ & - 295680x_1^6p_2^3 + 138600x_1^4p_2^4 - 16632x_1^2p_2^5 + 231p_2^6. \end{aligned}$$

Substitute v_k for G-invariant harmonic polynomials.

1. *Degree* 6:

$$u_6 := [f_6(v_1'), f_6(v_2'), f_6(v_3')] = \left[\tfrac{14\sqrt{10}-4}{5}, \tfrac{-7\sqrt{10}+2}{8}, \tfrac{-14\sqrt{10}+4}{9}\right].$$

2. *Degree* 10:

$$\begin{aligned} u_{10} &:= [f_{10}(v_1'), f_{10}(v_2'), f_{10}(v_3')] \\ &= \left[-\tfrac{43124224\sqrt{10}+49637120}{98415}, \tfrac{8422700\sqrt{10}+9694750}{19683}, -\tfrac{1078105600\sqrt{10}+1240928000}{1594323}\right]. \end{aligned}$$

3. *Degree* 12:

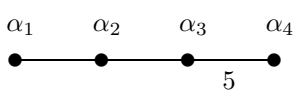
$$\begin{aligned} u_{12} &:= [f_{12}(v_1'), f_{12}(v_2'), f_{12}(v_3')] \\ &= \left[\tfrac{191679488\sqrt{10}-6897476096}{492075}, \tfrac{10856846\sqrt{10}-390677357}{39366}, \tfrac{-191679488\sqrt{10}+6897476096}{14348907}\right]. \end{aligned}$$

Proposition 5.5. *There is no choice of R, J, and w for which $(\mathcal{X}(H_3, J), w)$ is a Euclidean 12-design.*

Proof. There is $u \in \mathrm{Span}_{\mathbb{R}}\{u_6, u_{10}, u_{12}\}$ all whose entries are positive, because the vectors u_6, u_{10}, u_{12} are linearly independent. The result follows by Lemma 5.2. □

5.3. Group H_4.

Dynkin diagram:



Exponents: 1, 11, 19, 29.

Fundamental roots:

$$\alpha_1 := -{}^te_1 + {}^te_2,\ \alpha_2 := -{}^te_2 + {}^te_3,\ \alpha_3 := -{}^te_3 + {}^te_4,\ \alpha_4 := \frac{{}^te_1 + {}^te_2 + {}^te_3 + \sqrt{5}\ {}^te_4}{2}.$$

Corner Vectors:

$$\begin{aligned} v_1 &= \frac{(\sqrt{5}-1)\,{}^te_1 + (\sqrt{5}+3)({}^te_2 + {}^te_3 - {}^te_4)}{4}, \\ v_2 &= \frac{(\sqrt{5}+1)({}^te_1 + {}^te_2) + (\sqrt{5}+3)({}^te_3 - {}^te_4)}{2}, \\ v_3 &= \frac{(3\sqrt{5}+5)({}^te_1 + {}^te_2 + {}^te_3) - 3(\sqrt{5}+3){}^te_4}{4}, \\ v_4 &= \frac{(\sqrt{5}+3)({}^te_1 + {}^te_2 + {}^te_3 - {}^te_4)}{2}. \end{aligned}$$

Size of Orbit: $N_1 = 120, N_2 = 720, N_3 = 1200, N_4 = 600.$

Harmonic Molien series:

$$\frac{1}{(1-t^{12})(1-t^{20})(1-t^{30})} = 1 + t^{12} + t^{20} + t^{24} + t^{30} + \dots.$$

G-invariant harmonic polynomials.

For $i = 12, 20, 24$, $\mathrm{Harm}_i(\mathbb{R}^4)^{H_4}$ is spanned by the following polynomials.

1. *Degree* 12:

$$f_{12} := \sum_{g \in H_4} h_{12}(x^g),$$

where

$$h_{12}(x) := 13x_1^{12} - 286x_1^{10}p_3 + 1287x_1^8p_3^2 - 1716x_1^6p_3^3 + 715x_1^4p_3^4 - 78x_1^2p_3^5 + p_3^6.$$

2. *Degree* 20:

$$f_{20} := \sum_{g \in H_4} h_{20}(x^g),$$

where

$$\begin{aligned} h_{20}(x) := {} & 21x_1^{20} - 1330x_1^{18}p_3 + 20349x_1^{16}p_3^2 - 116280x_1^{14}p_3^3 + 293930x_1^{12}p_3^4 \\ & - 352716x_1^{10}p_3^5 + 203490x_1^8p_3^6 - 54264x_1^6p_3^7 + 5985x_1^4p_3^8 - 210x_1^2p_3^9 + p_3^{10}. \end{aligned}$$

3. *Degree* 24:

$$f_{24} := \sum_{g \in H_4} h_{24}(x^g),$$

where

$$\begin{aligned} h_{24}(x) := {} & x_1^{24} - 92x_1^{22}p_3 + \tfrac{10626}{5}x_1^{20}p_3^2 - 19228x_1^{18}p_3^3 + 81719x_1^{16}p_3^4 \\ & - 178296x_1^{14}p_3^5 + 208012x_1^{12}p_3^6 - \tfrac{653752}{5}x_1^{10}p_3^7 + 43263x_1^8p_3^8 \\ & - 7084x_1^6p_3^9 + 506x_1^4p_3^{10} - 12x_1^2p_3^{11} + \tfrac{1}{25}p_3^{12}. \end{aligned}$$

Substitute v_k for G-invariant harmonic polynomials.

1. *Degree* 12:

$$u_{12} := [f_{12}(v_1'), f_{12}(v_2'), f_{12}(v_3'), f_{12}(v_4')] = \big[-4500, 540, \tfrac{32500}{27}, \tfrac{5625}{4}\big].$$

2. *Degree* 20:

$$u_{20} := [f_{20}(v_1'), f_{20}(v_2'), f_{20}(v_3'), f_{20}(v_4')] = \big[6975, -\tfrac{58869}{25}, \tfrac{4035425}{2187}, \tfrac{216225}{64}\big].$$

3. *Degree* 24:

$$u_{24} := [f_{24}(v_1'), f_{24}(v_2'), f_{24}(v_3'), f_{24}(v_4')] = \big[-\tfrac{2367}{16}, -\tfrac{4689027}{50000}, \tfrac{416329}{104976}, \tfrac{622521}{16384}\big].$$

Proposition 5.6. *There is no choice of R, J, and w for which $(\mathcal{X}(H_4, J), w)$ is a Euclidean 24-design.*

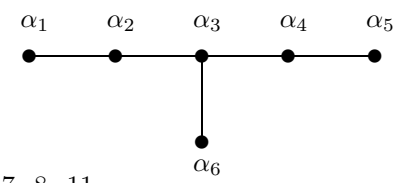
Proof. Since

$$u_{20} - 30u_{24} = \big[\tfrac{91305}{8}, \tfrac{2293281}{5000}, \tfrac{30201755}{17496}, \tfrac{18338985}{8192}\big],$$

this proposition follows from Lemma 5.2. □

5.4. Group E_6.

Dynkin diagram:



Exponents: 1, 4, 5, 7, 8, 11.

Fundamental roots:

$$\alpha_1 := {}^te_1 - {}^te_2,\ \alpha_2 := {}^te_2 - {}^te_3,\ \alpha_3 := {}^te_3 - {}^te_4,\ \alpha_4 := {}^te_4 - {}^te_5,$$

$$\alpha_5 := {}^te_5 - {}^te_6,\ \alpha_6 := \frac{(-3+\sqrt{3})({}^te_1 + {}^te_2 + {}^te_3) + (3+\sqrt{3})({}^te_4 + {}^te_5 + {}^te_6)}{6}.$$

Corner Vectors:

$$v_1 = \frac{(\sqrt{3}+5)\,{}^te_1 + (\sqrt{3}-1)({}^te_2 + {}^te_3 + {}^te_4 + {}^te_5 + {}^te_6)}{6},$$
$$v_2 = \frac{(\sqrt{3}+2)({}^te_1 + {}^te_2) + (\sqrt{3}-1)({}^te_3 + {}^te_4 + {}^te_5 + {}^te_6)}{3},$$
$$v_3 = \frac{(\sqrt{3}+1)({}^te_1 + {}^te_2 + {}^te_3) + (\sqrt{3}-1)({}^te_4 + {}^te_5 + {}^te_6)}{2},$$
$$v_4 = \frac{(\sqrt{3}+1)({}^te_1 + {}^te_2 + {}^te_3 + {}^te_4) + (\sqrt{3}-2)({}^te_5 + {}^te_6)}{3},$$
$$v_5 = \frac{(\sqrt{3}+1)({}^te_1 + {}^te_2 + {}^te_3 + {}^te_4 + {}^te_5) + (\sqrt{3}-5){}^te_6}{6},$$
$$v_6 = \frac{\sqrt{3}({}^te_1 + {}^te_2 + {}^te_3 + {}^te_4 + {}^te_5 + {}^te_6)}{3}.$$

Size of Orbit: $N_1 = 27, N_2 = 216, N_3 = 720, N_4 = 216, N_5 = 27, N_6 = 72.$
Harmonic Molien series:

$$\frac{1}{(1-t^5)(1-t^6)(1-t^8)(1-t^9)(1-t^{12})} = 1 + t^5 + t^6 + t^8 + t^9 + t^{10} + \dots.$$

G-invariant harmonic polynomials.

For $i = 5,\ 6,\ 8,\ 9,\ 10$, $\mathrm{Harm}_i(\mathbb{R}^6)^{E_6}$ is spanned by the following polynomials.

1. *Degree* 5:

$$\begin{aligned} f_5 := \mathrm{sym}(x_1^5) + \mathrm{sym}(x_1^4x_2) - 2\,\mathrm{sym}(x_1^3x_2^2) + \mathrm{sym}(x_1^3x_2x_3) \\ - 3\,\mathrm{sym}(x_1^2x_2x_3x_4) + 24\,\mathrm{sym}(x_1x_2x_3x_4x_5). \end{aligned}$$

2. *Degree* 6:

$$\begin{aligned} f_6 := \mathrm{sym}(x_1^6) + \tfrac{3}{2}\,\mathrm{sym}(x_1^5x_2) + -3\,\mathrm{sym}(x_1^4x_2^2) + \tfrac{15}{14}\,\mathrm{sym}(x_1^4x_2x_3) \\ + \tfrac{5}{7}\,\mathrm{sym}(x_1^3x_2^3) - \tfrac{30}{7}\,\mathrm{sym}(x_1^3x_2^2x_3) + \tfrac{30}{7}\,\mathrm{sym}(x_1^3x_2x_3x_4) + 9\,\mathrm{sym}(x_1^2x_2^2x_3^2) \\ + \tfrac{45}{7}\,\mathrm{sym}(x_1^2x_2^2x_3x_4) - \tfrac{180}{7}\,\mathrm{sym}(x_1^2x_2x_3x_4x_5) + \tfrac{180}{7}x_1x_2x_3x_4x_5x_6. \end{aligned}$$

3. *Degree* 8:

$$f_8 := \sum_{g\in E_6} h_8(x^g),$$

where

$$h_8(x) := x_1^8 - \tfrac{28}{5}x_1^6p_5 + 6x_1^4p_5^2 - \tfrac{4}{3}x_1^2p_5^3 + \tfrac{1}{33}p_5^4.$$

4. *Degree* 9:

$$f_9 := \sum_{g\in E_6} h_9(x^g),$$

where

$$\begin{aligned} h_9(x) := \mathrm{sym}(x_1^9) - \tfrac{36}{5}\,\mathrm{sym}(x_1^7x_2^2) + \tfrac{126}{5}\,\mathrm{sym}(x_1^5x_2^4) - 63\,\mathrm{sym}(x_1^4x_2^3x_3^2) \\ + 63\,\mathrm{sym}(x_1^4x_2^2x_3^2x_4) + 252\,\mathrm{sym}(x_1^3x_2^2x_3^2x_4^2) - 945\,\mathrm{sym}(x_1^2x_2^2x_3^2x_4^2x_5). \end{aligned}$$

5. *Degree* 10:

$$f_{10} := \sum_{g\in E_6} h_{10}(x^g),$$

where

$$h_{10}(x) := x_1^{10} - 9x_1^8p_5 + 18x_1^6p_5^2 - 10x_1^4p_5^3 + \tfrac{15}{11}x_1^2p_5^4 - \tfrac{3}{143}p_5^5.$$

Substitute v_k for G-invariant harmonic polynomials.

1. *Degree* 5:

$$u_5 := [f_5(v_1'), f_5(v_2'), f_5(v_3'), f_5(v_4'), f_5(v_5'), f_5(v_6')] = \left[\tfrac{3\sqrt{3}}{4}, \tfrac{6\sqrt{30}}{125}, 0, -\tfrac{6\sqrt{30}}{125}, -\tfrac{3\sqrt{3}}{4}, 0\right].$$

2. *Degree* 6:

$$u_6 := [f_6(v_1'), f_6(v_2'), f_6(v_3'), f_6(v_4'), f_6(v_5'), f_6(v_6')] = \left[\tfrac{81}{56}, -\tfrac{81}{700}, -\tfrac{9}{28}, -\tfrac{81}{700}, \tfrac{81}{56}, -\tfrac{27}{28}\right].$$

3. *Degree* 8:

$$\begin{aligned} u_8 &:= [f_8(v_1'), f_8(v_2'), f_8(v_3'), f_8(v_4'), f_8(v_5'), f_8(v_6')] \\ &= \left[800, -\tfrac{6784}{25}, -\tfrac{640}{9}, -\tfrac{6784}{25}, 800, \tfrac{3200}{3}\right]. \end{aligned}$$

4. *Degree* 9:

$$\begin{aligned} u_9 &:= [f_9(v_1'), f_9(v_2'), f_9(v_3'), f_9(v_4'), f_9(v_5'), f_9(v_6')] \\ &= \left[2065\sqrt{3}, -\tfrac{185024\sqrt{30}}{625}, 0, \tfrac{185024\sqrt{30}}{625}, -2065\sqrt{3}, 0\right]. \end{aligned}$$

5. *Degree* 10:

$$\begin{aligned} u_{10} &:= [f_{10}(v_1'), f_{10}(v_2'), f_{10}(v_3'), f_{10}(v_4'), f_{10}(v_5'), f_{10}(v_6')] \\ &= \left[\tfrac{11520}{13}, \tfrac{423936}{1625}, \tfrac{51200}{351}, \tfrac{423936}{1625}, \tfrac{11520}{13}, -\tfrac{10240}{39}\right]. \end{aligned}$$

Proposition 5.7. *There is no choice of R, J, and w for which $(\mathcal{X}(E_6, J), w)$ is a Euclidean* 10*-design.*

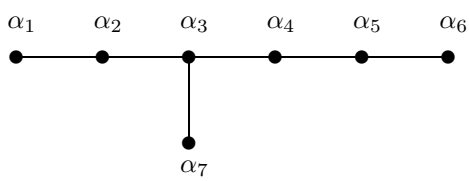
Proof. Since

$$u_{10} + u_8 = \left[\tfrac{11745}{2816}, \tfrac{13527}{220000}, \tfrac{387}{1760}, \tfrac{13527}{220000}, \tfrac{11745}{2816}, \tfrac{621}{352}\right],$$

this proposition follows from Lemma 5.2. □

5.5. Group E_7.

Dynkin diagram:



Exponents: 1, 5, 7, 9, 11, 13, 17.

Fundamental roots:

$$\alpha_1 := {}^te_1 - {}^te_2,\ \ \alpha_2 := {}^te_2 - {}^te_3,\ \ \alpha_3 := {}^te_3 - {}^te_4,\ \ \alpha_4 := {}^te_4 - {}^te_5,\ \ \alpha_5 := {}^te_5 - {}^te_6,$$

$$\alpha_6 := {}^te_6 - {}^te_7,\ \ \alpha_7 := \frac{(-4+\sqrt{2})({}^te_1 + {}^te_2 + {}^te_3) + (3+\sqrt{2})({}^te_4 + {}^te_5 + {}^te_6 + {}^te_7)}{7}.$$

Corner Vectors:

$$v_1 = \frac{(6+2\sqrt{2})^t e_1 + (-1+2\sqrt{2})({}^t e_2 + {}^t e_3 + {}^t e_4 + {}^t e_5 + {}^t e_6 + {}^t e_7)}{7},$$
$$v_2 = \frac{(5+4\sqrt{2})({}^t e_1 + {}^t e_2) + (-2+4\sqrt{2})({}^t e_3 + {}^t e_4 + {}^t e_5 + {}^t e_6 + {}^t e_7)}{7},$$
$$v_3 = \frac{(4+6\sqrt{2})({}^t e_1 + {}^t e_2 + {}^t e_3) + (-3+6\sqrt{2})({}^t e_4 + {}^t e_5 + {}^t e_6 + {}^t e_7)}{7},$$
$$v_4 = \frac{(6+9\sqrt{2})({}^t e_1 + {}^t e_2 + {}^t e_3 + {}^t e_4) + (-8+9\sqrt{2})({}^t e_5 + {}^t e_6 + {}^t e_7)}{14},$$
$$v_5 = \frac{(2+3\sqrt{2})({}^t e_1 + {}^t e_2 + {}^t e_3 + {}^t e_4 + {}^t e_5) + (-5+3\sqrt{2})({}^t e_6 + {}^t e_7)}{7},$$
$$v_6 = \frac{(-2-3\sqrt{2})({}^t e_1 + {}^t e_2 + {}^t e_3 + {}^t e_4 + {}^t e_5 + {}^t e_6) + (12-3\sqrt{2})^t e_7}{14},$$
$$v_7 = \frac{{}^t e_1 + {}^t e_2 + {}^t e_3 + {}^t e_4 + {}^t e_5 + {}^t e_6 + {}^t e_7}{\sqrt{2}}.$$

Size of Orbit:

$$N_1 = 126,\ N_2 = 2016,\ N_3 = 10080,\ N_4 = 4032,\ N_5 = 756,\ N_6 = 56,\ N_7 = 576.$$

Harmonic Molien series:

$$\frac{1}{(1-t^6)(1-t^8)(1-t^{10})(1-t^{12})(1-t^{14})(1-t^{18})} = 1 + t^6 + t^8 + t^{10} + 2t^{12} + \dots.$$

G-invariant harmonic polynomials.

For $i = 6, 8, 10, 12$, $\mathrm{Harm}_i(\mathbb{R}^7)^{E_7}$ is spanned by the following polynomials.

1. *Degree* 6:

$$f_6 := \sum_{g \in E_7} h_6(x^g),$$

where

$$h_6(x) := 32x_1^6 - 80x_1^4 p_6 + 30x_1^2 p_6^2 - p_6^3.$$

2. *Degree* 8:

$$f_8 := \sum_{g \in E_7} h_8(x^g),$$

where

$$h_8(x) := 384x_1^8 - 1792x_1^6 p_6 + 1680x_1^4 p_6^2 - 336x_1^2 p_6^3 + 7p_6^4.$$

3. *Degree* 10:

$$f_{10} := \sum_{g \in E_7} h_{10}(x^g),$$

where

$$h_{10}(x) := 256x_1^{10} - 1920x_1^8 p_6 + 3360x_1^6 p_6^2 - 1680x_1^4 p_6^3 + 210x_1^2 p_6^4 - 3p_6^5.$$

4. *Degree* 12:

$$f_{12,1} := \sum_{g \in E_7} h_{12,1}(x^g), \quad f_{12,2} := \sum_{g \in E_7} h_{12,2}(x^g),$$

where

$$\begin{aligned} h_{12,1}(x) := {} & 4096x_1^{12} - 45056x_1^{10} p_6 + 126720x_1^8 p_6^2 \\ & - 118272x_1^6 p_6^3 + 36960x_1^4 p_6^4 - 3168x_1^2 p_6^5 + 33p_6^6, \\ h_{12,2}(x) := {} & x_1 x_2 \big(2048x_1^{10} - 14080x_1^8 p_6 + 25344x_1^6 p_6^2 - 14784x_1^4 p_6^3 + 2640x_1^2 p_6^4 - 99p_6^5\big). \end{aligned}$$

Substitute v_k for G-invariant harmonic polynomials.

1. *Degree* 6:

$$\begin{aligned} u_6 &:= [f_6(v_1'), f_6(v_2'), f_6(v_3'), f_6(v_4'), f_6(v_5'), f_6(v_6'), f_6(v_7')] \\ &= \Big[\tfrac{-7700659200+9488793600\sqrt{2}}{16807}, \tfrac{-427814400+527155200\sqrt{2}}{2401}, \tfrac{-1818211200+2240409600\sqrt{2}}{16807}, \\ &\quad \tfrac{-547602432+674758656\sqrt{2}}{16807}, \tfrac{2887747200-3558297600\sqrt{2}}{16807}, \tfrac{20535091200-25303449600\sqrt{2}}{16807}, \\ &\quad \tfrac{-123210547200+151820697600\sqrt{2}}{823543}\Big]. \end{aligned}$$

2. *Degree* 8:

$$\begin{aligned} u_8 &:= [f_8(v_1'), f_8(v_2'), f_8(v_3'), f_8(v_4'), f_8(v_5'), f_8(v_6'), f_8(v_7')] \\ &= \Big[\tfrac{6579988992000-5480856576000\sqrt{2}}{823543}, \tfrac{731109888000-608984064000\sqrt{2}}{823543}, \\ &\quad \tfrac{-3527605209600+2938348108800\sqrt{2}}{823543}, \tfrac{-3134999199744+2611323666432\sqrt{2}}{823543}, \\ &\quad \tfrac{-1809496972800+1507235558400\sqrt{2}}{823543}, \tfrac{3509327462400-2923123507200\sqrt{2}}{823543}, \\ &\quad \tfrac{-115807806259200+96463075737600\sqrt{2}}{40353607}\Big]. \end{aligned}$$

3. *Degree* 10:

$$\begin{aligned} u_{10} &:= [f_{10}(v_1'), f_{10}(v_2'), f_{10}(v_3'), f_{10}(v_4'), f_{10}(v_5'), f_{10}(v_6'), f_{10}(v_7')] \\ &= \Big[\tfrac{-6428624451840+415928908800\sqrt{2}}{5764801}, \tfrac{357145802880-23107161600\sqrt{2}}{823543}, \\ &\quad \tfrac{2388412556760-154529143200\sqrt{2}}{5764801}, \tfrac{-73143460429824+4732346695680\sqrt{2}}{720600125}, \\ &\quad \tfrac{-7433097022440+480917800800\sqrt{2}}{5764801}, \tfrac{30476441845760-1971811123200\sqrt{2}}{5764801}, \\ &\quad \tfrac{3291455719342080-212955601305600\sqrt{2}}{1977326743}\Big]. \end{aligned}$$

4. *Degree* 12:

$$\begin{aligned} u_{12,1} &:= [f_{12,1}(v_1'), f_{12,1}(v_2'), f_{12,1}(v_3'), f_{12,1}(v_4'), f_{12,1}(v_5'), f_{12,1}(v_6'), f_{12,1}(v_7')] \\ &= \Big[\tfrac{27363005574796800+17942314142016000\sqrt{2}}{1977326743}, \tfrac{-760132890073600-689327933184000\sqrt{2}}{282475249}, \\ &\quad \tfrac{-513174301527400-792264524693400\sqrt{2}}{1977326743}, \tfrac{-14026148038967296-72141536776421376\sqrt{2}}{49433168575}, \\ &\quad \tfrac{3931481294451000-4960153279164600\sqrt{2}}{1977326743}, \tfrac{-12979679661260800-37832882828083200\sqrt{2}}{1977326743}, \\ &\quad \tfrac{24975708064081182720 0+28489814673278238720 0\sqrt{2}}{33232930569601}\Big]. \end{aligned}$$

$$\begin{aligned} u_{12,2} &:= [f_{12,2}(v_1'), f_{12,2}(v_2'), f_{12,2}(v_3'), f_{12,2}(v_4'), f_{12,2}(v_5'), f_{12,2}(v_6'), f_{12,2}(v_7')] \\ &= \Big[\tfrac{-2419675164360000-1489162193640000\sqrt{2}}{1977326743}, \tfrac{113867977056000+18727597152000\sqrt{2}}{282475249}, \\ &\quad \tfrac{156757191916575-26126480038725\sqrt{2}}{1977326743}, \tfrac{16622260339703808-6701937797136384\sqrt{2}}{49433168575}, \\ &\quad \tfrac{1494452243214675-1107985853945025\sqrt{2}}{1977326743}, \tfrac{8313279170969600-2771226582220800\sqrt{2}}{1977326743}, \\ &\quad \tfrac{-516863878334078976 00+773622407142604800\sqrt{2}}{33232930569601}\Big]. \end{aligned}$$

Proposition 5.8. *There is no choice of R, J, and w for which $(\mathcal{X}(E_7, J), w)$ is a Euclidean* 12*-design.*

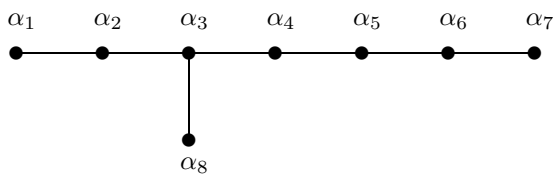
Proof. Since

$$-2u_{12,1} - 25u_{12,2} + u_{10} = [2.86443 \times 10^6, 256489., 513956., 989894., 2.86352 \times 10^6, 1.64917 \times 10^7, 293023.],$$

this proposition follows from Lemma 5.2. □

5.6. Group E_8.

Dynkin diagram:



Exponents: 1, 7, 11, 13, 17, 19, 23, 29.

Fundamental roots:

$$\alpha_1 := {}^te_1 - {}^te_2,\ \alpha_2 := {}^te_2 - {}^te_3,\ \alpha_3 := {}^te_3 - {}^te_4,\ \alpha_4 := {}^te_4 - {}^te_5,\ \alpha_5 := {}^te_5 - {}^te_6,$$
$$\alpha_6 := {}^te_6 - {}^te_7,\ \alpha_7 := {}^te_7 - {}^te_8,\ \alpha_8 := \frac{-{}^te_1 - {}^te_2 - {}^te_3 + {}^te_4 + {}^te_5 + {}^te_6 + {}^te_7 + {}^te_8}{2}.$$

Corner Vectors:

$$\begin{aligned}
v_1 &= \frac{3\,{}^te_1 + {}^te_2 + {}^te_3 + {}^te_4 + {}^te_5 + {}^te_6 + {}^te_7 + {}^te_8}{2},\\
v_2 &= 2\,{}^te_1 + 2\,{}^te_2 + {}^te_3 + {}^te_4 + {}^te_5 + {}^te_6 + {}^te_7 + {}^te_8,\\
v_3 &= \frac{5\,{}^te_1 + 5\,{}^te_2 + 5\,{}^te_3 + 3\,{}^te_4 + 3\,{}^te_5 + 3\,{}^te_6 + 3\,{}^te_7 + 3\,{}^te_8}{2},\\
v_4 &= 2\,{}^te_1 + 2\,{}^te_2 + 2\,{}^te_3 + 2\,{}^te_4 + {}^te_5 + {}^te_6 + {}^te_7 + {}^te_8,\\
v_5 &= \frac{3\,{}^te_1 + 3\,{}^te_2 + 3\,{}^te_3 + 3\,{}^te_4 + 3\,{}^te_5 + {}^te_6 + {}^te_7 + {}^te_8}{2},\\
v_6 &= -{}^te_1 - {}^te_2 - {}^te_3 - {}^te_4 - {}^te_5 - {}^te_6,\\
v_7 &= \frac{-{}^te_1 - {}^te_2 - {}^te_3 - {}^te_4 - {}^te_5 - {}^te_6 - {}^te_7 + {}^te_8}{2},\\
v_8 &= {}^te_1 + {}^te_2 + {}^te_3 + {}^te_4 + {}^te_5 + {}^te_6 + {}^te_7 + {}^te_8.
\end{aligned}$$

Size of Orbit:

$$N_1 = 2160,\quad N_2 = 69120,\quad N_3 = 483840,\quad N_4 = 241920,$$
$$N_5 = 60480,\quad N_6 = 6720,\quad N_7 = 240,\quad N_8 = 17280.$$

Harmonic Molien series:

$$\frac{1}{(1-t^8)(1-t^{12})(1-t^{14})(1-t^{18})(1-t^{20})(1-t^{24})(1-t^{30})} = 1 + t^8 + t^{12} + t^{14} + t^{16} + t^{18} + 2t^{20} + \dots.$$

G-invariant harmonic polynomials.

For $i = 8, 12, 14, 16$, $\mathrm{Harm}_8(\mathbb{R}^8)^{E_8}$ is spanned by the following.

1. *Degree* 8:

$$f_8 := \sum_{g \in E_8} h_8(x^g),$$

where

$$h_8(x) := 429x_1^8 - 1716x_1^6p_7 + 1430x_1^4p_7^2 - 260x_1^2p_7^3 + 5p_7^4.$$

2. *Degree* 12:

$$f_{12} := \sum_{g \in E_8} h_{12}(x^g),$$

where

$$h_{12}(x) := 1547x_1^{12} - 14586x_1^{10}p_7 + 36465x_1^8p_7^2 - 30940x_1^6p_7^3 + 8925x_1^4p_7^4 - 714x_1^2p_7^5 + 7p_7^6.$$

3. *Degree* 14:

$$f_{14} := \sum_{g \in E_8} h_{14}(x^g),$$

where

$$\begin{aligned} h_{14}(x) := {} & 969x_1^{14} - 12597x_1^{12}p_7 + 46189x_1^{10}p_7^2 - 62985x_1^8p_7^3 \\ & + 33915x_1^6p_7^4 - 6783x_1^4p_7^5 + 399x_1^2p_7^6 - 3p_7^7. \end{aligned}$$

4. *Degree* 16:

$$f_{16} := \sum_{g \in E_8} h_{16}(x^g),$$

where

$$\begin{aligned} h_{16}(x) := {} & 6783x_1^{16} - 116280x_1^{14}p_7 + 587860x_1^{12}p_7^2 - 1175720x_1^{10}p_7^3 \\ & + 1017450x_1^8p_7^4 - 379848x_1^6p_7^5 + 55860x_1^4p_7^6 - 2520x_1^2p_7^7 + 15p_7^8. \end{aligned}$$

Substitute v_k for G-invariant harmonic polynomials.

1. *Degree* 8:

$$\begin{aligned} u_8 &:= [f_8(v_1'), f_8(v_2'), f_8(v_3'), f_8(v_4'), f_8(v_5'), f_8(v_6'), f_8(v_7'), f_8(v_8')] \\ &= [174182400, \tfrac{4926873600}{49}, 82059264, 62705664, \\ &\qquad 19353600, -116121600, -1045094400, 97977600]. \end{aligned}$$

2. *Degree* 12:

$$\begin{aligned} u_{12} &:= [f_{12}(v_1'), f_{12}(v_2'), f_{12}(v_3'), f_{12}(v_4'), f_{12}(v_5'), f_{12}(v_6'), f_{12}(v_7'), f_{12}(v_8')] \\ &= [1680315840, \tfrac{15655887360}{49}, \tfrac{14950365696}{125}, -\tfrac{2608490304}{125}, \\ &\qquad -275607360, -734952960, 4480842240, 148777965]. \end{aligned}$$

3. *Degree* 14:

$$\begin{aligned} u_{14} &:= [f_{14}(v_1'), f_{14}(v_2'), f_{14}(v_3'), f_{14}(v_4'), f_{14}(v_5'), f_{14}(v_6'), f_{14}(v_7'), f_{14}(v_8')] \\ &= [1207483200, -\tfrac{567924825600}{16807}, -\tfrac{2009165312}{15}, -\tfrac{671799744}{5}, \\ &\qquad -\tfrac{253422400}{3}, 184307200, -2634508800, -293294925]. \end{aligned}$$

4. *Degree* 16:

$$\begin{aligned} u_{16} &:= [f_{16}(v_1'), f_{16}(v_2'), f_{16}(v_3'), f_{16}(v_4'), f_{16}(v_5'), f_{16}(v_6'), f_{16}(v_7'), f_{16}(v_8')] \\ &= [1490121360, -\tfrac{393199971840}{2401}, \tfrac{3287394820358656}{1265625}, \tfrac{36512571016971}{62500}, \\ &\qquad \tfrac{1232569520}{3}, 2075906560, 7529034240, -\tfrac{9749511135}{16}]. \end{aligned}$$

Proposition 5.9. *There is no choice of R, J, and w for which $(\mathcal{X}(E_8, J), w)$ is a Euclidean* 16*-design.*

Proof. Since

$$\begin{aligned} u_{16} - 3u_{14} + 2u_{12} = [1228303440, \tfrac{9691313402880}{16807}, \tfrac{4098709695302656}{1265625}, \tfrac{59096571112971}{62500}, \\ \tfrac{339192560}{3}, 53079040, 24394245120, \tfrac{9089540145}{16}], \end{aligned}$$

this proposition follows from Lemma 5.2. □

Now, we are ready to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. Case (1) is in Theorem 2.10, and cases (2), (3) in [24]. Thus, the theorem follows from Propositions 5.4–5.9. □

The following result, together with Theorem 5.1, determines the maximal degree of the spherical cubature formulas $(\mathcal{X}(G,J),w)$ for all irreducible reflection groups G.

Theorem 5.10.

(i) *An F_4-invariant cubature of degree* 11 *that consists of the orbits of corner vectors is classified by*:

$$w_1=\tfrac{13-960w_4}{960},\quad w_2=\tfrac{3(-1+192w_4)}{256},\quad w_3=\tfrac{3(1-120w_4)}{160},\quad \tfrac{1}{192}\le w_4\le\tfrac{1}{120}.$$

(ii) *An H_3-invariant cubature of degree* 11 *that consists of the orbits of corner vectors is classified by*:

$$w_1=\tfrac{125}{5544},\quad w_2=\tfrac{64}{3465},\quad w_3=\tfrac{27}{3080}.$$

(iii) *An H_4-invariant cubature of degree* 23 *that consists of the orbits of corner vectors is classified by*:

$$\begin{aligned}&w_1=\tfrac{368-9625w_4}{315392},\quad w_2=\tfrac{125(16+5625w_4)}{2359296},\\&w_3=-\tfrac{6561(16-51975w_4)}{504627200},\quad 0\le w_4\le\tfrac{16}{51975}.\end{aligned}$$

(iv) *An E_6-invariant cubature of degree* 9 *that consists of the orbits of corner vectors is classified by*:

$$\begin{aligned}&w_1=\tfrac{2(1-96w_6)}{729},\quad w_2=\tfrac{125(1+1200w_6)}{186624},\quad w_3=\tfrac{1}{1280}-\tfrac{9w_6}{16},\\&w_4=\tfrac{125(1+1200w_6)}{186624},\quad w_5=\tfrac{2(1-96w_6)}{729},\quad 0\le w_6\le\tfrac{1}{720}.\end{aligned}$$

(v) *An E_7-invariant cubature of degree* 11 *that consists of the orbits of corner vectors is classified by the following two types of weights*:

$$\begin{aligned}(1)\quad & w_1=-\tfrac{4(-296924467+966078461040w_2+107900687895w_3+95875084800w_7)}{610410794301},\\&w_4=-\tfrac{625(-945994+3215011030w_2+24066363475w_3+1769169600w_7)}{4340698981696},\\&w_5=\tfrac{8(34900936+247702641648w_2+1231161574335w_3+182083866624w_7)}{1831232382903},\\&w_6=-\tfrac{27(-32430307+60983896974w_2+30607311735w_3+25518620160w_7)}{542587372712},\\&0\le w_7\le-\tfrac{2401(-394+1339030w_2+10023475w_3)}{1769169600},\\&0\le w_3<-\tfrac{2(-197+669515w_2)}{10023475},\quad 0\le w_2\le\tfrac{197}{669515}.\\(2)\quad & w_1=-\tfrac{4(-211+686070w_2)}{440055},\ w_3=-\tfrac{2(-197+669515w_2)}{10023475},\\&w_5=\tfrac{16(1231+1230075w_2)}{54126765},\ w_6=-\tfrac{351(-71+129360w_2)}{16037560},\\&0\le w_2\le\tfrac{197}{669515},\ w_4=0,\ w_7=0.\end{aligned}$$

(vi) *An E_8-invariant cubature of degree* 15 *that consists of the orbits of corner vectors is classified by the nonnegative solutions w_i of the system of equations*

$$u_8{}^tv=0,\quad u_{12}{}^tv=0,\quad u_{14}{}^tv=0,\quad \sum_{i=1}^{8}N_iw_i=1,\tag{5.1}$$

where $v=(N_1w_1,\dots,N_8w_8)$, and the u_i, N_i are as defined in Subsection 5.6. *The precise solutions of* (5.1) *are referred to the Appendix.*

Remark 5.11. The H_3-invariant cubature of Theorem 5.10 (i) was constructed by Goethals and Seidel [9, p. 214] who found, moreover, a spherical cubature of degree 15 by taking the orbits of v_1', v_2', v_3', plus one more orbit; for example, see [12] for further information on the existence of three-dimensional spherical cubature formulas. It is also interesting to note that the formula given in Theorem 5.10 (vi) is equivalent to a 26400-point cubature of degree 15 which comes from shells of the Korkin–Zorotalev lattice [13]. In [9, p. 214], Goethals and Seidel found a spherical cubature of degree 19 that consists of the H_4-orbits of the zeros of an invariant harmonic homogeneous polynomial of degree 12. Salihov [28] found another H_4-invariant cubature of degree 19 by taking the union of the 120-cell and the 600-cell. Motivated by this, the authors searched three and four H_4-orbits of the corner vectors, and found the higher-degree cubature of Theorem 5.10 (ii).

§6. Hilbert identities and cubature formulas

As was explained in §2, a cubature formula of index q on S^{m-1} with n points exists if and only if there are n vectors $r_1, \dots, r_n \in \mathbb{R}^m$ such that

$$\sum_{i=1}^{n} \langle x, r_i \rangle^q = \langle x, x \rangle^{\frac{q}{2}} \tag{6.1}$$

for every $x \in \mathbb{R}^m$. Identity (6.1) yields a representation of $(\sum_{i=1}^m x_i^2)^{q/2}$ as a sum of qth powers of real linear forms with positive real coefficients. Such a representation is called a *Hilbert identity* [25]. Various aesthetic meanings of Hilbert identities were discussed in a famous paper by Reznick, see [27].

Many Hilbert identities can be obtained with the help of the cubature formulas that were constructed in §4 and §5. In particular, some of the resulting identities involve sums of qth powers of rational linear forms with positive rational coefficients. Such *rational representations* were used not only in studying Waring's problem [6, pp. 717–725], but also in the work of Schmid on real holomorphy rings [30]. An aesthetic meaning of rational representations would be stated as follows.[5] We would take all coefficients $\{a_i\}$ that appear in a formula, and consider the field created by adjoining them, and then look at its dimension $[\mathbb{Q}(\{a_i\}) : \mathbb{Q}]$. With this measure, the "best formulas" would only involve rationals, and the minimum value occurs if the coefficients are already in $\mathbb{Q}$.

It is well known (going back to Hilbert [16]) that

$$\int_{S^{m-1}} y_1^q \rho\,(\mathrm{d}y) = \frac{(q-1)!!(m-2)!!}{(m+q-2)!!}. \tag{6.2}$$

This is certainly a rational number. All cubature formulas given in §4 have rational weights, and points in orbits are of the form $(\sqrt[q]{a}, \dots, \sqrt[q]{a}, 0, \dots, 0)^{B_m}$ with rational a. Thus, by Proposition 2.3, we can obtain many rational representations.

For example, the 91-point cubature of Example 4.9 is translated into the following rational representation, which Reznick [27] was not able to find.

Theorem 6.1.

$$\begin{aligned} 120\Big(\sum_{i=1}^{7} x_i^2\Big)^3 &= \sum_{56} (x_i \pm x_{i+2} \pm x_{i+3} \pm x_{i+4})^6 \\ &\quad + 2\sum_{28} (x_i \pm x_{i+2} \pm x_{i+3})^6 + \sum_{7} (2x_i)^6 \end{aligned} \tag{6.3}$$

where the indices on the right are taken cyclic modulo 7 *and all possible combinations of signs occur in summation.*

[5] This was suggested by Bruce Reznick through email conversation.

Remark 6.2. Reznick [27, p. 112] translated an index-six cubature on S^6 found by Stroud in 1967 into the following beautiful representation:

$$960\left(\sum_{i=1}^{7} x_i^2\right)^3 = 2\sum_{7}(2x_i)^6 + \sum_{2\cdot\binom{7}{2}}(2x_i \pm 2x_j)^6 + \sum_{2^6}(x_1 \pm \cdots \pm x_7)^6, \tag{6.4}$$

where on the right all possible combinations of signs and pairs of the 7 variables $x_1, \ldots, x_7$ occur in the second summation. Identity (6.3) improves Reznick's representation. Namely, (6.3) has fewer number of sixth powers than (6.4).

More rational representations are available. For example, look at the following Kürschák's representation:

$$2^k\binom{3k}{k}\left(\sum_{i=1}^{3k+1} x_i^2\right)^2 = \sum(x_{i_1} \pm x_{i_2} \pm \cdots \pm x_{i_{k+1}})^4$$

where on the right all possible combinations of signs and $(k+1)$-subsets of the $3k+1$ variables $x_1, \ldots, x_{3k+1}$ occur [6, p. 723]. This corresponds to the cubature of Lemma 4.4 (ii), which, by Theorem 4.3, reduces to many rational representations involving much fewer number of fourth powers.

We give yet another interesting Hilbert identity, though it is not always rational.

Theorem 6.3.

$$\begin{aligned}\left(\sum_{i=1}^{4} x_i^2\right)^5 &= \tfrac{1}{2520}\sum_{4}(2x_i)^{10} + \tfrac{1}{2520}\sum_{8}(x_1 \pm x_2 \pm x_3 \pm x_4)^{10} \\ &+ \tfrac{1-120a}{272160}\sum_{32}(3x_i \pm x_j \pm x_k \pm x_l)^{10} + \tfrac{1-120a}{272160}\sum_{16}(2x_i \pm 2x_j \pm 2x_k)^{10} \\ &+ \tfrac{192a-1}{68040}\sum_{48}(2x_i \pm x_j \pm x_k)^{10} + \tfrac{12-960a}{630}\sum_{12}(x_i \pm x_j)^{10},\end{aligned} \tag{6.5}$$

where $\frac{1}{192} \le a \le \frac{1}{120}$. *In particular, if a is rational, then so is the corresponding identity.*

Proof. The cubature of Theorem 5.10 (1) is centrally symmetric, which reduces to the half-size formula of index 10. The result then follows by (6.1) and (6.2). □

Identity (6.5) unifies the following well-known identity by I. Schur (cf. [6, p. 721]).

Corollary 6.4.

$$\begin{aligned}22680\left(\sum_{i=1}^{4} x_i^2\right)^5 &= 9\sum_{4}(2x_i)^{10} + 9\sum_{8}(x_1 \pm x_2 \pm x_3 \pm x_4)^{10} \\ &+ \sum_{48}(2x_i \pm x_j \pm x_k)^{10} + 180\sum_{12}(x_i \pm x_j)^{10}.\end{aligned} \tag{6.6}$$

Proof. Take $a = 1/120$ in (6.5). □

Remark 6.5. Some classical identities as such by Lucus (1876) and Liouville (1859), are often picked up for an introduction in the study of Hilbert identities [6]. It is well known (see, e.g., [14, 27]) that Liouville's and Lucas's identities are closely related to each other by a linear change and provide essentially the same cubature on S^3. The Hurwitz identity

$$\begin{aligned}5040\left(\sum_{i=1}^{4} x_i^2\right)^4 &= 6\sum_{4}(2x_i)^8 + 6\sum_{8}(x_1 \pm x_2 \pm x_3 \pm x_4)^8 \\ &+ \sum_{48}(2x_i \pm x_j \pm x_k)^8 + 60\sum_{12}(x_i \pm x_j)^8\end{aligned}$$

is also well known [6, p. 721]. It is interesting to note that Hurwitz's and Schur's identities are the same in terms of spherical cubature, i.e., the corresponding formulas have the same weights and points. In [14, 27], this observation was not remarked, though the relationship between Liouville's and Lucas's identities was mentioned.

The story so far implies how powerful the cubature approach is to construct Hilbert identities. In turn, we look at an advantage of translating spherical cubature into Hilbert identities.

Theorem 6.6. *Let $m \geq 2$ be an integer. Then $\left(\sum_{i=1}^m x_i^2\right)^4$ cannot be represented as an $\mathbb{R}$-linear combination of $(a_1x_1 + \cdots + a_mx_m)^8$ with $a_i \in \{0, -1, 1\}$.*

Proof. The ratio of the coefficients of $x_1^6x_2^2$ and $x_1^4x_2^4$ is $(2:3)$ in $(\sum_{i=1}^n x_i^2)^4$. But it is $(2:5)$ in any form $(a_1x_1 + \cdots + a_nx_n)^8$ with $a_i \in \{0, \pm 1\}$, $0 \notin \{a_1, a_2\}$. □

Corollary 6.7. *Let $m \geq 2$, and let G be a subgroup of B_m. Then there exists no G-invariant Euclidean 8-design of $\mathbb{R}^m$ that consists of the orbits of the form*

$$(1, \ldots, 1, 0, \ldots, 0)^G.$$

Proof. Restricting (2.3) to homogeneous polynomials of degree 8 implies the existence of a cubature formula of index 8 on S^{m-1}, by suitably rescaling points and weights. The result then follows by Theorem 6.6. □

A variation of Corollary 6.7 is valid for all irreducible reflection groups. Namely, Theorem 5.1 can be proved even if each irreducible reflection group is replaced by its subgroup.

Remark 6.8.

(i) Corollary 6.7 is the Bajnok theorem for $G = B_m$, and case (3) of Theorem 5.1 for $G = D_m$. It is also interesting to note that Theorem 6.6 states that the Bajnok theorem is valid even if negative coefficients are allowed.

(ii) To prove Theorem 2.10, Bajnok used the Sobolev theorem implicitly. The approach based on the Sobolev theorem is of theoretic interest, but it basically requires tedious calculations on invariant harmonic homogeneous polynomials. In summary, the original proof of Bajnok requires a few pages [1, §2 and Proposition 15] and seems to be involved. Whereas, the present proof is short and simple, because it only involves elementary counting techniques. The Bajnok theorem is well known in algebra and combinatorics; however, it is not fully recognized in numerical analysis, though it can be used to determine the maximal degree of a symmetric cubature on the simplex [39], which is traditionally studied in the context of numerical analysis.[6] The authors expect that the new proof will make researchers in many fields more familiar with the Bajnok theorem.

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[6]The second author learned this fact form Yuan Xu. In [29], we proved a variation of the Bajnok theorem for cubature formulas on the simplex, particularly intended for researchers in numerical analysis.

Appendix A. Classification of E_8-invariant cubature

An E_8-invariant cubature of degree 15 that consists of the orbits of corner vectors is classified by the following 27 types of weights:

$$\begin{aligned}
w_1 &= \frac{23}{504000} - \frac{4288512w_2}{823543} - \frac{258048w_3}{15625} - \frac{70224w_4}{15625} - \frac{15w_8}{128},\\
w_5 &= \frac{3}{224000} - \frac{1244160w_2}{823543} - \frac{171008w_3}{15625} - \frac{79704w_4}{15625} - \frac{243w_8}{512},\\
w_6 &= \frac{9}{896000} + \frac{4193208w_2}{823543} + \frac{507384w_3}{15625} + \frac{180792w_4}{15625} + \frac{3645w_8}{2048},\\
w_7 &= \frac{67}{672000} - \frac{2465280w_2}{823543} - \frac{290304w_3}{15625} - \frac{94752w_4}{15625} - \frac{603w_8}{512},
\end{aligned}$$

and

(1) $w_4 = 0$, $0 \le w_2 \le \frac{12588443}{1449551462400}$, $0 \le w_3 < \frac{44118375-4976640000000w_2}{36053104984064}$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-72106209968128w_3}{3126889828125}$,

(2) $w_4 = 0$, $0 \le w_2 \le \frac{12588443}{1449551462400}$, $w_3 = \frac{44118375-4976640000000w_2}{36053104984064}$, $w_8 = 0$,

(3) $w_4 = 0$, $\frac{12588443}{1449551462400} < w_2 < \frac{117649}{13436928000}$,
$0 \le w_3 \le \frac{44118375-5038848000000w_2}{14396954722304}$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-72106209968128w_3}{3126889828125}$,

(4) $w_4 = 0$, $\frac{12588443}{1449551462400} < w_2 < \frac{117649}{13436928000}$,
$\frac{44118375-5038848000000w_2}{14396954722304} < w_3 < \frac{338240875-38596608000000w_2}{122407847460864}$,
$0 \le w_8 \le \frac{676481750-77193216000000w_2-244815694921728w_3}{1737161015625}$,

(5) $w_4 = 0$, $\frac{12588443}{1449551462400} < w_2 < \frac{117649}{13436928000}$,
$w_3 = \frac{338240875-38596608000000w_2}{122407847460864}$, $w_8 = 0$,

(6) $w_4 = 0$, $w_2 = \frac{117649}{13436928000}$, $w_3 = 0$, $0 \le w_8 \le \frac{2}{5740875}$,

(7) $w_4 = 0$, $w_2 = \frac{117649}{13436928000}$, $0 < w_3 < \frac{625}{252829237248}$,
$0 \le w_8 \le \frac{1250-505658474496w_3}{3588046875}$,

(8) $w_4 = 0$, $w_2 = \frac{117649}{13436928000}$, $w_3 = \frac{625}{252829237248}$, $w_8 = 0$,

(9) $w_4 = 0$, $\frac{117649}{13436928000} < w_2 < \frac{2705927}{308772864000}$,
$0 \le w_3 < \frac{338240875-38596608000000w_2}{122407847460864}$,
$0 \le w_8 \le \frac{676481750-77193216000000w_2-244815694921728w_3}{1737161015625}$,

(10) $w_4 = 0$, $\frac{117649}{13436928000} < w_2 < \frac{2705927}{308772864000}$, $w_3 = \frac{338240875-38596608000000w_2}{122407847460864}$, $w_8 = 0$,

(11) $w_4 = 0$, $w_2 = \frac{2705927}{308772864000}$, $w_3 = 0$, $w_8 = 0$,

(12) $0 < w_4 < \frac{125}{3092173056}$, $0 \le w_2 < \frac{62942215+7929414597888w_4}{7247757312000}$,
$0 \le w_3 < \frac{44118375-4976640000000w_2-16803755845632w_4}{36053104984064}$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-72106209968128w_3-33607511691264w_4}{3126889828125}$,

(13) $0 < w_4 < \frac{125}{3092173056}$, $0 \le w_2 < \frac{62942215+7929414597888w_4}{7247757312000}$,
$w_3 = \frac{44118375-4976640000000w_2-16803755845632w_4}{36053104984064}$, $w_8 = 0$,

(14) $0 < w_4 < \frac{125}{3092173056}$, $w_2 = \frac{62942215+7929414597888w_4}{7247757312000}$,
$0 \le w_3 < \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-72106209968128w_3-33607511691264w_4}{3126889828125}$,

(15) $0 < w_4 < \frac{125}{3092173056}$, $w_2 = \frac{62942215+7929414597888w_4}{7247757312000}$,
$w_3 = \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$, $w_8 = 0$,

(16) $0 < w_4 < \frac{125}{3092173056}$,
$\frac{62942215+7929414597888w_4}{7247757312000} < w_2 < \frac{14706125-1123879249584w_4}{1679616000000}$,
$0 \le w_3 \le \frac{44118375-5038848000000w_2-3371637748752w_4}{14396954722304}$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-72106209968128w_3-33607511691264w_4}{3126889828125}$,

(17) $0 < w_4 < \frac{125}{3092173056}$,
$\frac{62942215+7929414597888w_4}{7247757312000} < w_2 < \frac{14706125-1123879249584w_4}{1679616000000}$,
$\frac{375}{122372096} - \frac{307546875w_2}{878720381} - \frac{255879w_4}{1092608} < w_3 < \frac{2875}{1040449536} - \frac{5453125w_2}{17294403} - \frac{209w_4}{768}$,
$0 \le w_8 \le \frac{676481750-77193216000000w_2-244815694921728w_3-66623021144064w_4}{1737161015625}$,

(18) $0 < w_4 < \frac{125}{3092173056}$,
$\frac{62942215+7929414597888w_4}{7247757312000} < w_2 < \frac{14706125-1123879249584w_4}{1679616000000}$,
$w_3 = \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$, $w_8 = 0$,

(19) $0 < w_4 < \frac{125}{3092173056}$, $w_2 = \frac{14706125-1123879249584w_4}{1679616000000}$, $w_3 = 0$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-33607511691264w_4}{3126889828125}$,

(20) $0 < w_4 < \frac{125}{3092173056}$, $w_2 = \frac{14706125-1123879249584w_4}{1679616000000}$,
$0 < w_3 < \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$,
$0 \le w_8 \le \frac{676481750-77193216000000w_2-244815694921728w_3-66623021144064w_4}{1737161015625}$,

(21) $0 < w_4 < \frac{125}{3092173056}$, $w_2 = \frac{14706125-1123879249584w_4}{1679616000000}$,
$w_3 = \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$, $w_8 = 0$,

(22) $0 < w_4 < \frac{125}{3092173056}$,
$\frac{14706125-1123879249584w_4}{1679616000000} < w_2 < \frac{338240875-33311510572032w_4}{38596608000000}$,
$0 \le w_3 < \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$,
$0 \le w_8 \le \frac{676481750-77193216000000w_2-244815694921728w_3-66623021144064w_4}{1737161015625}$,

(23) $0 < w_4 < \frac{125}{3092173056}$,
$\frac{14706125-1123879249584w_4}{1679616000000} < w_2 < \frac{338240875-33311510572032w_4}{38596608000000}$,
$w_3 = \frac{338240875-38596608000000w_2-33311510572032w_4}{122407847460864}$, $w_8 = 0$,

(24) $0 < w_4 < \frac{125}{3092173056}$, $w_2 = \frac{338240875-33311510572032w_4}{38596608000000}$, $w_3 = 0$, $w_8 = 0$,

(25) $\frac{125}{3092173056} \le w_4 \le \frac{125}{47609856}$, $0 \le w_2 < \frac{14706125-5601251948544w_4}{1658880000000}$,
$0 \le w_3 < \frac{44118375-4976640000000w_2-16803755845632w_4}{36053104984064}$,
$0 \le w_8 \le \frac{88236750-9953280000000w_2-72106209968128w_3-33607511691264w_4}{3126889828125}$,

(26) $\frac{125}{3092173056} \le w_4 \le \frac{125}{47609856}$, $0 \le w_2 < \frac{14706125-5601251948544w_4}{1658880000000}$,
$w_3 = \frac{44118375-4976640000000w_2-16803755845632w_4}{36053104984064}$, $w_8 = 0$,

(27) $\frac{125}{3092173056} \le w_4 \le \frac{125}{47609856}$, $w_2 = \frac{14706125-5601251948544w_4}{1658880000000}$, $w_3 = 0$, $w_8 = 0$.

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