

## SPECTRUM OF PERIODIC ELLIPTIC OPERATORS WITH DISTANT PERTURBATIONS IN SPACE

A. M. GOLOVINA

**ABSTRACT.** A periodic selfadjoint differential operator of even order and with distant perturbations in a multidimensional space is treated. The role of perturbations is played by arbitrary localized operators. The localization is described by specially chosen weight functions. The behavior of the spectrum of the perturbed operator is studied under the condition that the distance between the domains where the perturbations are localized tends to infinity. It is shown that there exists a simple isolated eigenvalue of the perturbed operator that tends to a simple isolated eigenvalue of the limit operator. Series expansions are obtained for this eigenvalue of the perturbed operator and for the corresponding eigenfunction. Uniform convergence for these series is shown and formulas for their terms are deduced.

### §1. INTRODUCTION

The spectra of operators with distant perturbations have been considered in a series of papers (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). The principal attention has been paid to the asymptotic behavior of eigenvalues and the corresponding eigenfunctions. Also, a fairly extensive study of the periodic Schrödinger operator with potential-type perturbations should be mentioned; see, e.g., [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]). In certain papers, perturbations were defined in a different way, see, e.g., [1, 2, 3, 17]). We briefly describe the results presented in the papers quoted above.

In [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], the Laplace operator with several distant perturbations in  $\mathbb{R}^d$  was considered. In [4, 5, 6, 7, 8], the role of perturbations was played by potentials satisfying various smoothness and decay at infinity assumptions. First terms of the asymptotic expansions for eigenvalues and the corresponding eigenfunctions were calculated. In [8], the number of eigenvalues of the perturbed operator was estimated in terms of the number of eigenvalues of a certain limit operator. In [9, 10, 11], Coulomb potentials were taken for perturbations, complete asymptotic series for eigenvalues and the corresponding eigenfunctions were constructed, and certain estimates for the terms of these series were given. In [12], the role of distant perturbations was played by compactly supported potentials on  $\mathbb{R}^d$  with  $d = 1$  or  $d = 3$ . Complete asymptotic expansions in the form of uniformly convergent series were obtained for eigenvalues and eigenfunctions of the perturbed operator.

The papers [13, 14] were devoted to the behavior of the eigenvalues that arise from the edge of the essential spectrum of the limit operator. Various cases where such eigenvalues exist were studied and the principal terms of the corresponding asymptotic expansions were described. In [15, 16], the Dirac operator in 3-space was treated. In [15], potentials decaying at infinity were employed as perturbations, whereas in [16] the same role was

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played by Coulomb potentials. Principal terms of the asymptotic expansions for eigenvalues and the corresponding eigenfunctions of the perturbed operator were constructed. In [17], a delta-potential was taken for the perturbation. Lower estimates were obtained for the first spectral gaps of the Laplace operator; these estimates can be applied to a distant delta-potential.

In [2], a change of the boundary conditions was treated as a perturbation. The portions of the boundary on which the type of boundary conditions was changed were situated far away from one another. Convergence theorems were proved and the leading terms of asymptotic series for eigenvalues and eigenfunctions were constructed. In [1, 2, 3], the role of perturbations was played by arbitrary localized operators of abstract nature considered in multidimensional space or in an infinite cylinder. The localization property of the perturbations meant that each one was defined on a certain bounded domain. It was proved that eigenvalues and eigenfunctions of the perturbed operator converge to those of the limit operator whatever be the multiplicity of the limit eigenvalue. Leading terms were calculated for the asymptotic expansions of eigenvalues and the corresponding eigenfunctions.

In the present paper, we consider an elliptic operator with distant perturbations in multidimensional space. A nonperturbed operator is a multidimensional periodic matrix selfadjoint operator of even order and of a fairly general form. Perturbations are arbitrary symmetric operators of abstract nature. The main assumption about them is that they are localized. Localization is described by weight functions subject to certain conditions that ensure smoothness and a decay at infinity. However, there are almost no restrictions on the rate of decay. The perturbations treated in the previous papers fit into this pattern as special cases. Even the abstract perturbations in [1, 2, 3] can be included into our considerations if the weight functions are chosen to have compact support. It should be noted that perturbations similar to those in the present paper were considered in [18], but for problems of a different type. For a perturbation, we can take either a differential operator of (high) order not exceeding the order of the nonperturbed operator, or an integral operator, or an operator of finite rank, or a pseudodifferential operator.

In [1, §8, Example 5], a transformation was described that made it possible to reduce a delta-potential to a second order differential operator. Employing this transformation, we can include delta-potentials in our considerations. It should also be noted that in [19, 20] the behavior of the resolvent for the perturbed operator was studied. An explicit formula for that resolvent was deduced and, on that basis, the convergence of the perturbed operator to a certain limit was established.

In this paper, we study the behavior of the spectrum of the perturbed operator as the distance between the domains in which the perturbations are localized tends to infinity. We prove the stability of the essential spectrum under perturbations and the convergence of the eigenvalues of the perturbed operator to eigenvalues of the limit operator. The main result of the paper consists of complete asymptotic expansions for the eigenvalues of the perturbed operator that converge to simple isolated eigenvalues of the limit operator, and also of asymptotics for the corresponding eigenfunctions. Moreover, it is proved that the asymptotic series in question converge uniformly to the corresponding eigenvalue and eigenfunction of the perturbed operator. Explicit formulas are given for the calculation of the coefficients of these series.

The main results are stated in Theorem 2.1 and 2.2. Their proofs are based on two fairly simple methods. The existence and convergence of eigenvalues of perturbed operators are proved by using the uniform resolvent convergence results established in [19, 20]. But complete asymptotic expansions and their convergence are proved by a new original method. The essence is that the equation involving the eigenvalues of

the perturbed operator is reduced to a certain regularly perturbed equation in a special Hilbert space. It turns out that the smallness of the perturbation can be described by two specific small parameters. After that, the adapted Birman–Swinger method described in [21, 22] makes it possible to reduce the problem to the analysis of an operator equation and to the search for zeros of some holomorphic function. The study of this function allows us to obtain representations for the perturbed operator, and for the corresponding eigenfunction, in terms of uniformly convergent series. A fairly simple and elegant method is suggested for the calculation of the coefficients of these series.

We briefly describe the organization of the paper. In the next section, we formulate the problem and state the main results. In §3, we prove the stability of the essential spectrum and the convergence theorem. In §4, we reduce the initial eigenvalue problem for the perturbed operator to the search for zeros of a certain holomorphic function. In §5, we study the zeros of this holomorphic function, deduce series representations for the eigenvalues of the perturbed operator and the corresponding eigenfunctions, and determine the terms of these series.

Finally, we introduce some notation. The symbol  $\mathcal{D}(\cdot)$  stands for the domain of an operator, the symbol  $\|\cdot\|_{Y_1 \rightarrow Y_2}$  denotes the norm of a linear operator acting from a normed space  $Y_1$  to a normed space  $Y_2$ , the symbol  $B_r$  denote the ball centered at zero and of radius  $r$  in  $\mathbb{R}^d$ ; next,  $i$  is the imaginary unit and  $\sigma_{ess}(\cdot)$  is the essential spectrum of an operator.

§2. STATEMENT OF THE PROBLEM AND THE MAIN RESULT

Let  $x = (x_1, \dots, x_d)$  be the Cartesian coordinates in  $\mathbb{R}^d$ ,  $d \geq 1$ , and let  $\Gamma$  be an arbitrary  $d$ -dimensional periodic lattice in  $\mathbb{R}^d$  with elementary cell  $\square$ . In the space  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ , we introduce the operator

$$\mathcal{H}_0 := (-1)^m \sum_{\substack{\beta, \gamma \in \mathbf{Z}_+^d \\ |\beta|=|\gamma|=m}} \frac{\partial^\beta}{\partial x^\beta} A_{\beta\gamma} \frac{\partial^\gamma}{\partial x^\gamma} + \sum_{\substack{\beta \in \mathbf{Z}_+^d \\ |\beta| \leq 2m-1}} A_\beta \frac{\partial^\beta}{\partial x^\beta}$$

with the domain  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ , where  $m \in \mathbb{N}$ , and  $A_{\beta\gamma} \in C^m(\mathbb{R}^d)$ ,  $A_\beta \in C^{m-1}(\mathbb{R}^d)$  are  $\Gamma$ -periodic matrix-valued functions. Here by  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  and  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  we mean Sobolev spaces of  $\mathbb{C}^n$ -valued functions. It is assumed that the operator  $\mathcal{H}_0$  is selfadjoint and satisfies the ellipticity condition

$$\nu \sum_{\substack{\beta \in \mathbf{Z}_+^d \\ |\beta|=m}} |\xi_\beta|^2 \leq \left| \sum_{\substack{\beta, \gamma \in \mathbf{Z}_+^d \\ |\beta|=|\gamma|=m}} (A_{\beta\gamma}(x) \xi_\beta, \xi_\gamma)_{\mathbb{C}^n} \right|,$$

where  $\xi_\beta \in \mathbb{C}^n$  and  $\nu$  is a positive constant independent of  $x$  and  $\xi_\beta$ . This ellipticity condition is borrowed from [19]; it is equivalent to the strong ellipticity condition (1.7) in [23]. Note that this condition does not imply that  $\mathcal{H}_0$  is lower semibounded.

Let  $\varsigma_i = \varsigma_i(r)$  and  $\eta_i = \eta_i(r) \in C^{2m}(\mathbb{R}_+)$ ,  $i = 1, \dots, k$ , be nonnegative functions equal to 1 near zero and satisfying the following conditions.

- (A1) There exists a function  $a \in C^{2m-1}[0, +\infty)$  with

$$\varsigma_i(r) \leq C e^{-\int_0^r a(t) dt},$$

where  $i = 1, \dots, k$  and  $C$  is a constant. The function  $a(r)$  vanishes near zero and is uniformly bounded on  $[0, +\infty)$  together with all its derivatives of order up to  $2m - 1$ . Next,  $\int_0^{+\infty} a(t) dt = +\infty$ .

- (A2) Together with all their derivatives of order up to  $2m$ , the functions  $\eta_i$  tend to zero at infinity.

In the space  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ , we consider a family of arbitrary operators  $\mathcal{L}_i^0, i = 1, \dots, k$ , with domain  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . Suppose that they take  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  to  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  boundedly, but, in general, are not bounded on  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . Denote by  $\mathcal{L}_i, i = 1, \dots, k$ , the operators in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  with domain  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  that act in accordance with the rule

$$(\mathcal{L}_i u)(x) := \varsigma_i(|x|)(\mathcal{L}_i^0 \eta_i(|x|)u)(x).$$

Distant perturbations are defined to be those of the form

$$\sum_{i=1}^k \mathcal{S}(-X_i) \mathcal{L}_i \mathcal{S}(X_i),$$

where  $\mathcal{S}(X_i)$  is the translation operator defined in the following way:

$$(\mathcal{S}(X_i)u)(\cdot) := u(\cdot - X_i);$$

here  $X_i \in \Gamma$  are discrete parameters. We denote by  $X$  the vector  $X = (X_1, \dots, X_k)$  and put  $\tau(X) = \min_{i \neq j} |X_i - X_j|$ . We shall assume that  $\tau(X) \rightarrow \infty$ . Clearly, any two different points  $X_i$  can be taken to one another by finitely many shifts along  $\Gamma$ .

The perturbed operator is considered in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  and is introduced by the formula

$$\mathcal{H}_X := \mathcal{H}_0 + \sum_{i=1}^k \mathcal{S}(-X_i) \mathcal{L}_i \mathcal{S}(X_i), \quad \mathcal{D}(\mathcal{H}_X) = W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n).$$

In  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ , we consider yet another family of operators, namely,  $\mathcal{H}_i := \mathcal{H}_0 + \mathcal{L}_i$  with domain  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . We impose the following assumption.

(A3) *The operators  $\mathcal{H}_i$  and  $\mathcal{H}_X$  are selfadjoint.*

Since  $\mathcal{H}_0$  is also selfadjoint, it follows that the operators  $\mathcal{L}_i$  are symmetric.

In this paper, we study the behavior of the spectrum of the perturbed operator  $\mathcal{H}_X$  in the case where the limit eigenvalue is simple. The main objective is the evolution of the discrete spectrum and the essential spectrum of  $\mathcal{H}_X$  as  $\tau(X) \rightarrow \infty$ .

The first result describes the position of the essential spectrum.

**Theorem 2.1.** *The essential spectra of the operators  $\mathcal{H}_X$  and  $\mathcal{H}_i, i = 1, \dots, k$ , coincide with that of  $\mathcal{H}_0$ .*

Let  $\lambda_0$  be a simple isolated eigenvalue of one of the  $\mathcal{H}_i$  (say, of  $\mathcal{H}_1$ ) that does not belong to the spectra of  $\mathcal{H}_0$  or  $\mathcal{H}_i, i = 2, \dots, k$ , and let  $\psi_0$  be the corresponding eigenfunction normalized in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . We fix a small neighborhood  $U$  of  $\lambda_0$  whose closure contains no other points of the spectra of  $\mathcal{H}_0$  and  $\mathcal{H}_i$ . Denote by  $\mathcal{R}_1(\lambda)$  the reduced resolvent of  $\mathcal{H}_1$  in  $\bar{U}$ . We remind the reader that the reduced resolvent  $\mathcal{R}_1(\lambda)$  (see [24, Chapter I, §5, Subsection 3]) is the holomorphic part of the Laurent series at  $\lambda_0$  for the operator  $(\mathcal{H}_1 - \lambda)^{-1}$ . We put

$$\begin{aligned} \mathcal{R}_j(\lambda) &:= (\mathcal{H}_j - \lambda)^{-1}, \quad j \geq 2, \quad \lambda \in \bar{U}, \\ \varepsilon(X) &:= \max \left\{ \max_{j=2, \dots, k} \left\| \mathcal{L}_j^0 \eta_j(|\cdot|) \mathcal{S}(X_j - X_1) \psi_0 \right\|_{L_2(\mathbb{R}^d, \mathbb{C}^n)}, \right. \\ (2.1) \quad &\left. \max_{\lambda \in \bar{U}} \max_{\substack{p, j=1, \dots, k \\ p \neq j}} \left\| \mathcal{L}_p^0 \eta_p(|\cdot|) \mathcal{S}(X_p - X_j) \mathcal{R}_j(\lambda) \varsigma_j(|\cdot|) \right\|_{L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^n)} \right\}. \end{aligned}$$

**Theorem 2.2.** *For  $\tau(X)$  sufficiently large, there exists a unique eigenvalue  $\lambda_X$  for  $\mathcal{H}_X$  in  $\bar{U}$ , which converges to  $\lambda_0$  as  $\tau(X) \rightarrow \infty$ . The eigenvalue  $\lambda_X$  is simple and isolated, and is represented by the series*

$$(2.2) \quad \lambda_X = \lambda_0 + \sum_{j=2}^{\infty} \Lambda_j(X)$$

convergent for  $\tau(X)$  sufficiently large. A corresponding eigenfunction can be chosen in such a way that it is representable by the series

$$(2.3) \quad \psi_X(x) = \psi_0(x + X_1) + \sum_{p=1}^k \sum_{j=1}^{\infty} \phi_{p,j}(x + X_p, X).$$

convergent in  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  for large  $\tau(X)$ . The series (2.2), (2.3) converge uniformly in  $X$  for sufficiently large  $\tau(X)$ . The terms of these series are determined by the formulas

$$(2.4) \quad \Lambda_j = \sum_{t=2}^k (\mathcal{L}_1 \mathcal{S}(X_1 - X_t) \phi_{t,j-1}, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)},$$

$$(2.5) \quad \phi_{p,j} = \mathcal{R}_p(\lambda_0) \left( \sum_{t=2}^j \Lambda_t \phi_{p,j-t} - \sum_{\substack{t=1 \\ t \neq p}}^k \mathcal{L}_p \mathcal{S}(X_p - X_t) \phi_{t,j-1} \right),$$

where  $p = 1, \dots, k, j \geq 1$ , and we put

$$(2.6) \quad \phi_{1,0} := \psi_0, \quad \phi_{p,0} := 0, \quad p = 2, \dots, k.$$

We have

$$(2.7) \quad |\Lambda_j(X)| \leq C^j \varepsilon^j(X), \quad \|\phi_{p,j}\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} \leq C^j \varepsilon^j(X),$$

where  $C$  is a constant independent of  $p, j$ , and  $X$ . We have

$$\varepsilon(X) \rightarrow 0 \quad \text{as} \quad \tau(X) \rightarrow \infty.$$

We comment on the main results. The stability of the essential spectrum under perturbations and the existence of an eigenvalue of the perturbed operator that converges to an eigenvalue of the limit operator are two facts that were expected. The most nontrivial result is constituted by the expansions (2.2), (2.3), formulas (2.4), (2.5), (2.6), and estimates (2.7). Estimates (2.7) show that, in a sense, the expansions (2.2) and (2.3) are asymptotic series for the eigenvalue  $\lambda_X$  and the corresponding eigenfunction  $\psi_X$  of the perturbed operator. Specifically, the terms of these series are estimated by the asymptotic sequence of the functions  $C^i \varepsilon^i(X)$ . At the same time, the two series converge uniformly to  $\lambda_X$  and  $\psi_X$  for  $\tau(X)$  sufficiently large.

Thus, formulas (2.2) and (2.3) are precise identities for the eigenvalue  $\lambda_X$  and the corresponding eigenfunction  $\psi_X$ . Moreover, in accordance with (2.4), (2.5), and (2.6), in order to calculate the terms of the series (2.2) and (2.3) it suffices to know only the operators  $\mathcal{R}_1(\lambda_0)$  and  $(\mathcal{H}_j - \lambda_0)^{-1}, j = 2, \dots, k$ , because only explicit quantities are involved in the remaining part of these formulas. We emphasize that no similar complete results in arbitrary dimension had been known before, even in the case where distant perturbations are compactly supported potentials.

Examples of nonperturbed and perturbed operators and of weight functions for which the theorem is applicable can be found in §3 of [19]. It suffices only to require that non-perturbed operators be selfadjoint and that perturbations be symmetric. To be specific, we write out the formulas for an eigenvalue and eigenfunction of the operator

$$(2.8) \quad \mathcal{H}_\ell = -\frac{d^2}{dx^2} + V_-(\cdot + \ell) + V_+(\cdot - \ell) \quad \text{in} \quad L_2(\mathbb{R}), \quad \mathcal{D}(\mathcal{H}_\ell) = W_2^2(\mathbb{R}),$$

where  $V_+$  and  $V_-$  are bounded measurable compactly supported potentials, and  $\ell$  is a large positive parameter. The operator  $\mathcal{H}_\ell$  results from  $\mathcal{H}_X$  if we put  $d = n = m = 1, k = 2, A_{\beta\gamma} = 1, A_\beta = 0, X_1 = \ell, X_2 = -\ell, \mathcal{L}_1^0 = V_-, \mathcal{L}_2^0 = V_+$ , and the  $\varsigma_i, \eta_i$  are compactly supported functions with

$$\varsigma_i(|x|) \equiv 1, \quad \eta_i(|x|) \equiv 1 \quad \text{for} \quad x \in \text{supp } V_- \cup \text{supp } V_+.$$

Consider the following operators in  $L_2(\mathbb{R})$ :

$$(2.9) \quad \mathcal{H}_1 = -\frac{d^2}{dx^2} + V_-, \quad \mathcal{H}_2 = -\frac{d^2}{dx^2} + V_+, \quad \mathcal{D}(\mathcal{H}_i) = W_2^2(\mathbb{R}), \quad i = 1, 2.$$

Let  $\lambda_0$  be a simple isolated eigenvalue of  $\mathcal{H}_1$  not belonging to the spectrum of  $\mathcal{H}_2$ , and let  $\psi_0$  be the corresponding eigenfunction. Since the potential  $V_-$  is compactly supported, we have

$$(2.10) \quad \psi_0(x) = C_0 e^{-\sqrt{-\lambda_0}x} \quad \text{as } x \rightarrow +\infty,$$

where  $C_0$  is a certain constant. In our case, the series (2.2) and (2.3) take the form

$$(2.11) \quad \begin{aligned} \lambda_\ell &= \lambda_0 + \sum_{j=1}^{\infty} e^{-4j\sqrt{-\lambda_0}\ell} \tilde{\Lambda}_j(\ell), \\ \psi_\ell(x) &= \psi_0(x + \ell) + \sum_{j=1}^{\infty} e^{-4j\sqrt{-\lambda_0}\ell} \tilde{\phi}_{1,j}(x + \ell, \ell) + \sum_{j=1}^{\infty} e^{-(4j-2)\sqrt{-\lambda_0}\ell} \tilde{\phi}_{2,j}(x - \ell, \ell). \end{aligned}$$

Formulas for the coefficients of these series look like this:

$$\begin{aligned} \tilde{\Lambda}_j(\ell) &= (V_- Q_{2,j}(\cdot - 2\ell, \ell) e^{\sqrt{-\lambda_0}\cdot}, \psi_0)_{L_2(\mathbb{R})}, \quad j \geq 1, \\ \tilde{\phi}_{1,1} &= 0, \quad \tilde{\phi}_{2,1} = C_0 (\mathcal{H}_2 - \lambda_0)^{-1} V_+ e^{-\sqrt{-\lambda_0}\cdot}, \\ \tilde{\phi}_{1,j} &= \mathcal{R}_1(\lambda_0) \left( \sum_{p=1}^j \tilde{\Lambda}_p(\ell) \tilde{\phi}_{1,j-p} - V_- Q_{2,j-1}(\cdot - 2\ell, \ell) e^{-\sqrt{-\lambda_0}\cdot} \right), \quad j \geq 2, \\ \tilde{\phi}_{2,j} &= (\mathcal{H}_2 - \lambda_0)^{-1} \left( \sum_{p=1}^{j-1} \tilde{\Lambda}_p(\ell) \tilde{\phi}_{2,j-p} - V_+ Q_{1,j-1}(\cdot + 2\ell, \ell) e^{\sqrt{-\lambda_0}\cdot} \right), \quad j \geq 2, \\ \tilde{\phi}_{1,j}(x, \ell) &= Q_{1,j}(x, \ell) e^{-\sqrt{-\lambda_0}x} \quad \text{as } x \rightarrow +\infty, \quad j \geq 2, \\ \tilde{\phi}_{2,j}(x, \ell) &= Q_{2,j}(x, \ell) e^{\sqrt{-\lambda_0}x} \quad \text{as } x \rightarrow -\infty, \quad j \geq 1. \end{aligned}$$

Here the  $Q_{p,j}(x, \ell)$  are certain polynomials in the variables  $x$  and  $\ell$ , and  $Q_{1,0}(x, \ell) \equiv C_0$ . We recall that  $\mathcal{R}_1(\lambda_0)$  is the reduced resolvent of  $\mathcal{H}_1$ . The behavior described above of the functions  $\tilde{\phi}_{p,j}$  at infinity is a direct consequence of the definition of these functions and the fact that the potentials  $V_\pm$  are compactly supported (it only suffices to write the equations for  $\tilde{\phi}_{p,j}$ ). When deriving the formulas for  $\tilde{\Lambda}_j$  and  $\tilde{\phi}_{p,j}$ , we have used the identities

$$\begin{aligned} (V_+ S(-2\ell) \tilde{\phi}_{1,j})(x, \ell) &= e^{-2\ell\sqrt{-\lambda_0}} V_+(x) Q_{1,j}(x + 2\ell, \ell) e^{-\sqrt{-\lambda_0}x}, \\ (V_- S(2\ell) \tilde{\phi}_{2,j})(x, \ell) &= e^{-2\ell\sqrt{-\lambda_0}} V_-(x) Q_{2,j}(x - 2\ell, \ell) e^{\sqrt{-\lambda_0}x} \end{aligned}$$

valid for  $\ell$  sufficiently large because the  $V_\pm$  are compactly supported potentials. It can easily be seen that the functions  $\tilde{\Lambda}_j(\ell)$  and  $\tilde{\phi}_{p,j}(\cdot, \ell)$  are polynomials of  $\ell$  whose degree may grow as  $j$  increases. Observe also that the coefficients (2.11) and the terms of the series (2.2), (2.3) are related by the formulas

$$\begin{aligned} \Lambda_{2j} &= e^{-4j\sqrt{-\lambda_0}\ell} \tilde{\Lambda}_j, & \Lambda_{2j-1} &= 0, \\ \phi_{1,2j} &= e^{-4j\sqrt{-\lambda_0}\ell} \tilde{\phi}_{1,j}, & \phi_{1,2j-1} &= 0, \\ \phi_{2,2j-1} &= e^{-(4j-2)\sqrt{-\lambda_0}\ell} \tilde{\phi}_{2,j}, & \phi_{2,2j} &= 0. \end{aligned}$$

In many dimensions, an analog of the operator (2.8) looks like this:

$$\mathcal{H}_\ell = -\Delta + V_-(x + \ell) + V_+(x - \ell) \quad \text{in } L_2(\mathbb{R}^d), \quad \mathcal{D}(\mathcal{H}_\ell) = W_2^2(\mathbb{R}^d),$$

where  $V_-$  and  $V_+$  are bounded measurable compactly supported potentials,  $d \geq 2$ , and  $\ell = (\ell_1, \dots, \ell_d)$ . The operators  $\mathcal{H}_j$  are introduced by analogy with (2.9):

$$\mathcal{H}_1 = -\Delta + V_-, \quad \mathcal{H}_2 = -\Delta + V_+, \quad \mathcal{D}(\mathcal{H}_i) = W_2^2(\mathbb{R}^d), \quad i = 1, 2.$$

The behavior of the eigenfunction  $\psi_0$  of  $\mathcal{H}_1$  at infinity is determined by the behavior at infinity of the Green function:

$$(2.12) \quad \psi_0(x) = O(|x|^{-(d-1)/2} e^{-\sqrt{-\lambda_0}|x|}), \quad x \rightarrow \infty.$$

To prove this, it suffices to write the equation for the eigenfunction  $\psi_0$  in the form  $-\Delta\psi_0 = -V_- \psi_0$ . Then, treating  $-V_- \psi_0$  as a right-hand side, we can express  $\psi_0$  in terms of the Green function.

Similarly, it can easily be verified that for  $f \in L_2(\mathbb{R}^d; \mathbb{C}^n)$  and compactly supported  $\zeta_j$ , we have

$$(2.13) \quad (\mathcal{R}_i(\lambda)\zeta_i(|\cdot|)f)(x) = O(|x|^{-(d-1)/2} e^{-\operatorname{Re} \sqrt{-\lambda}|x|}), \quad x \rightarrow \infty, \quad i = 1, 2.$$

Substituting this in (2.12) and (2.1), we deduce that

$$\varepsilon(X) = O(|\ell|^{-(d-1)/2} e^{-\sqrt{-\lambda_0 - \beta}|\ell|}), \quad \ell \rightarrow \infty,$$

where  $\beta$  is a fixed number. But the structure of the series (2.2) and (2.3) in the multidimensional case is much more complicated than in dimension 1, because now the behavior of  $\psi_0$  is much more involved than in (2.10).

### §3. ESSENTIAL SPECTRUM AND THE CONVERGENCE OF THE DISCRETE SPECTRUM

In this section we prove Theorem 2.1 and the existence of a unique eigenvalue of the perturbed operator convergent to an eigenvalue  $\lambda_0$  of the limit operator.

*Proof of Theorem 2.1.* We introduce the following notation. Let  $\zeta = \zeta(r) \in C^{2m}(\mathbb{R}_+)$  be one of the functions  $\zeta_i$ , and  $\eta = \eta(r) \in C^{2m}(\mathbb{R}_+)$  one of the functions  $\eta_i$  satisfying conditions (A1) and (A2). We denote by  $\mathcal{L}^0$  either an operator among the  $\mathcal{L}_i^0$ , or one among the  $-\mathcal{L}_i^0$ , or we put  $\mathcal{L}^0 = 0$ . In  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ , we define an operator  $\mathcal{L}$  by the rule

$$(3.1) \quad (\mathcal{L}u)(x) := \zeta(|x|)(\mathcal{L}^0\eta(|\cdot|)u)(x),$$

with domain  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . Consider the operator  $\mathcal{H} = \mathcal{H}_0 + \mathcal{L}$  in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  with domain  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . Assume that  $\mathcal{H}$  is selfadjoint. If  $\lambda \in \sigma_{\text{ess}}(\mathcal{H}) \subset \mathbb{R}$ , then, by the Weyl criterion, there exists a characteristic sequence, i.e., a sequence  $\{u_p\} \subset W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  bounded and noncompact in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  and such that  $f_p = (\mathcal{H} - \lambda)u_p \rightarrow 0$  in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  as  $p \rightarrow \infty$ . Since  $(\lambda - i)$  is in the resolvent set of  $\mathcal{H}$ , the operator  $(\mathcal{H} - \lambda + i)^{-1}$  is well defined and bounded on  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . By the Banach inverse operator theorem, it is bounded as an operator from  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  to  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . Consequently, the sequence  $u_p = (\mathcal{H} - \lambda + i)^{-1}(f_p + iu_p)$  is bounded in  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  and, after passage to a subsequence, converges weakly in that space to some element  $w$ .

We put  $w_p = u_p - w$ . Since  $u_p \rightarrow w$  weakly in  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ , we have

$$(3.2) \quad w_p \rightarrow 0 \text{ weakly in } W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n) \text{ as } p \rightarrow \infty.$$

Hence,  $(\mathcal{H} - \lambda)u_p$  converges weakly to  $(\mathcal{H} - \lambda)w$  in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  and, since  $(\mathcal{H} - \lambda)u_p$  converges weakly to zero, we see that

$$(3.3) \quad (\mathcal{H} - \lambda)w_p \rightarrow 0 \text{ strongly in } L_2(\mathbb{R}^d; \mathbb{C}^n) \text{ as } p \rightarrow \infty.$$

Clearly, the sequence  $w_p$  is bounded and noncompact in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . Since the embedding of  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  in  $W_2^{2m-1}(B_r; \mathbb{C}^n)$  is compact for every  $r$ , we can use the standard

diagonal procedure to show that

$$(3.4) \quad \|w_p\|_{W_2^{2m-1}(B_{p+1};\mathbb{C}^n)} \rightarrow 0 \quad \text{as } p \rightarrow \infty$$

(after passage to a subsequence).

Let  $\chi_p \in C^{2m}(\mathbb{R}^d)$  be cut-off functions such that  $\chi_p = 0$  in  $B_p$ ,  $\chi_p = 1$  in  $\mathbb{R}^d \setminus B_{p+1}$ , and all their derivatives up to the order  $2m$  are uniformly bounded in  $p$  and  $x$  for all  $x \in \overline{B_{p+1}} \setminus B_p$ . We show that the sequence  $\chi_p w_p$  is characteristic for the operator  $\mathcal{H} + \widehat{\mathcal{L}}$  at the point  $\lambda$ , where  $\widehat{\mathcal{L}}$  is defined by analogy with  $\mathcal{L}$ , i.e., by (3.1) with  $\varsigma, \eta, \mathcal{L}^0$  replaced by  $\widehat{\varsigma}, \widehat{\eta}, \widehat{\mathcal{L}}^0$ . Here  $\widehat{\mathcal{L}}^0$  is an operator among the  $\mathcal{L}_i^0$ , or one among the  $-\mathcal{L}_i^0$ , or  $\widehat{\mathcal{L}}^0 = 0$ , and  $\widehat{\varsigma}$  and  $\widehat{\eta}$  are some functions among the  $\varsigma_i, \eta_i$ . We again take  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  for the domain of  $\mathcal{H} + \widehat{\mathcal{L}}$ , and again assume that this operator is selfadjoint.

By (3.4), we have

$$\|(1 - \chi_p)w_p\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = \|(1 - \chi_p)w_p\|_{L_2(B_{p+1}; \mathbb{C}^n)} \leq \|w_p\|_{L_2(B_{p+1}; \mathbb{C}^n)} \rightarrow 0$$

as  $p \rightarrow \infty$ . Since  $\chi_p w_p = w_p - (1 - \chi_p)w_p$  and the sequence  $w_p$  is noncompact, it follows that the sequence  $\chi_p w_p$  is noncompact. Clearly,

$$(3.5) \quad \begin{aligned} (\mathcal{H} + \widehat{\mathcal{L}} - \lambda)\chi_p w_p &= (\mathcal{H}_0 - \lambda)\chi_p w_p + (\mathcal{L} + \widehat{\mathcal{L}})\chi_p w_p \\ &= \chi_p(\mathcal{H} - \lambda)w_p + (\mathcal{L} + \widehat{\mathcal{L}})\chi_p w_p - \chi_p \mathcal{L} w_p + g_p, \end{aligned}$$

where the function  $g_p$  is expressed linearly in terms of derivatives of  $\chi_p$  and  $w_p$  and satisfies the estimates

$$(3.6) \quad \|g_p\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|w_p\|_{W_2^{2m-1}(B_{p+1}; \mathbb{C}^n)} \rightarrow 0,$$

where  $C$  is a constant independent of  $p$ . Since the functions  $\varsigma$  and  $\eta$  decay at infinity, the definition of the cut-off functions  $\chi_p$  shows that

$$\max_{\mathbb{R}^d} |\chi_p \varsigma| \rightarrow 0, \quad \max_{|\beta| \leq 2m} \max_{\mathbb{R}^d} \left| \frac{\partial^\beta (\eta \chi_p)}{\partial x^\beta} \right| \rightarrow 0, \quad \max_{|\beta| \leq 2m} \max_{\mathbb{R}^d} \left| \frac{\partial^\beta (\widehat{\eta} \chi_p)}{\partial x^\beta} \right| \rightarrow 0$$

as  $p \rightarrow +\infty$ . Therefore, we obtain the following convergence relations:

$$\begin{aligned} \|\chi_p \mathcal{L} w_p\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} &= \|\chi_p \varsigma \mathcal{L}^0 \eta w_p\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq \max_{\mathbb{R}^d} |\chi_p \varsigma| \|\mathcal{L}^0 \eta w_p\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \\ &\leq C \max_{\mathbb{R}^d} |\chi_p \varsigma| \|w_p\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} \rightarrow 0 \quad \text{as } p \rightarrow \infty, \\ \|(\mathcal{L} + \widehat{\mathcal{L}})\chi_p w_p\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} &\leq C (\|\eta \chi_p w_p\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} + \|\widehat{\eta} \chi_p w_p\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)}) \\ &\leq C \left( \max_{|\beta| \leq 2m} \max_{\mathbb{R}^d} \left| \frac{\partial^\beta (\eta \chi_p)}{\partial x^\beta} \right| + \max_{|\beta| \leq 2m} \max_{\mathbb{R}^d} \left| \frac{\partial^\beta (\widehat{\eta} \chi_p)}{\partial x^\beta} \right| \right) \|w_p\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} \rightarrow 0, \end{aligned}$$

where  $C$  stands throughout for certain constants independent of  $p$ . Since the sequence  $w_p$  is bounded, combining the above relations with (3.2), (3.3), (3.5), and (3.6) yields  $(\mathcal{H} + \widehat{\mathcal{L}} - \lambda)\chi_p w_p \rightarrow 0$  as  $p \rightarrow \infty$ . Thus, we have proved the inclusion  $\sigma_{\text{ess}}(\mathcal{H}) \subseteq \sigma_{\text{ess}}(\mathcal{H} + \widehat{\mathcal{L}})$ .

Now, putting  $\mathcal{H} = \mathcal{H}_0$ ,  $\mathcal{L} = 0$ ,  $\widehat{\mathcal{L}} = \mathcal{L}_i$ ,  $\varsigma = \widehat{\varsigma} = \varsigma_i$ , and  $\eta = \widehat{\eta} = \eta_i$ , we see that  $\sigma_{\text{ess}}(\mathcal{H}_0) \subseteq \sigma_{\text{ess}}(\mathcal{H}_i)$ . The reverse inclusion  $\sigma_{\text{ess}}(\mathcal{H}_i) \subseteq \sigma_{\text{ess}}(\mathcal{H}_0)$  is proved similarly, by putting  $\mathcal{H} = \mathcal{H}_0 + \mathcal{L}_i$ ,  $\mathcal{L} = \mathcal{L}_i$ ,  $\widehat{\mathcal{L}} = -\mathcal{L}_i$ ,  $\varsigma = \widehat{\varsigma} = \varsigma_i$ , and  $\eta = \widehat{\eta} = \eta_i$ . The identity  $\sigma_{\text{ess}}(\mathcal{H}_0) = \sigma_{\text{ess}}(\mathcal{H}_X)$  is established much as above. First, we take  $\mathcal{L} = 0$  and  $\widehat{\mathcal{L}} = \sum_{i=1}^k \mathcal{S}(-X_i) \mathcal{L}_i \mathcal{S}(X_i)$ , and then we take  $\mathcal{L} = \sum_{i=1}^k \mathcal{S}(-X_i) \mathcal{L}_i \mathcal{S}(X_i)$  and  $\widehat{\mathcal{L}} = -\sum_{i=1}^k \mathcal{S}(-X_i) \mathcal{L}_i \mathcal{S}(X_i)$ . All estimates and convergence relations presented above remain true in this setting.  $\square$

**Lemma 3.1.** *For  $\tau(X)$  sufficiently large, the neighborhood  $\bar{U}$  contains a unique eigenvalue  $\lambda_X$  of  $\mathcal{H}_X$  that converges to  $\lambda_0$  as  $\tau(X) \rightarrow \infty$ . The eigenvalues  $\lambda_X$  are simple and isolated.*

*Proof.* In the proof, the symbol  $\|\cdot\|$  stands for the operator norm  $\|\cdot\|_{L_2(\mathbb{R}^d; \mathbb{C}^n) \rightarrow L_2(\mathbb{R}^d; \mathbb{C}^n)}$ , and the symbol  $(\cdot, \cdot)$ , stands for the inner product  $(\cdot, \cdot)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}$ .

Let  $K_r$  be the circle centered at  $\lambda_0$  and with sufficiently small radius  $r$ , in order to ensure the inclusion  $K_r \subset \bar{U}$ . By repetition of the arguments in the proof of Theorem 1 in [19], it can easily be shown that we have uniform convergence

$$\left\| (\mathcal{H}_X - \lambda)^{-1} - \sum_{i=1}^k \mathcal{S}(-X_i)(\mathcal{H}_i - \lambda)^{-1} \mathcal{S}(X_i) - (k-1)(\mathcal{H}_0 - \lambda)^{-1} \right\| \rightarrow 0$$

as  $\tau(X) \rightarrow \infty$ . Integrating the last relation over the closed contour  $K_r$ , we see that

$$(3.7) \quad \left\| \int_{K_r} (\mathcal{H}_X - \lambda)^{-1} d\lambda - \int_{K_r} \sum_{i=1}^k \mathcal{S}(-X_i)(\mathcal{H}_i - \lambda)^{-1} \mathcal{S}(X_i) d\lambda - (k-1) \int_{K_r} (\mathcal{H}_0 - \lambda)^{-1} d\lambda \right\| \rightarrow 0.$$

Since the operators  $\mathcal{H}_i$ ,  $i = 0, \dots, k$ , are selfadjoint and the circle  $K_r$  does not intersect their spectra, we have

$$\int_{K_r} \sum_{i=2}^k \mathcal{S}(-X_i)(\mathcal{H}_i - \lambda)^{-1} \mathcal{S}(X_i) d\lambda = 0, \quad \int_{K_r} (\mathcal{H}_0 - \lambda)^{-1} d\lambda = 0.$$

Taking these two identities into account, we rewrite (3.7) in the form

$$(3.8) \quad \left\| \int_{K_r} (\mathcal{H}_X - \lambda)^{-1} d\lambda - \mathcal{S}(-X_1) \left( \int_{K_r} (\mathcal{H}_1 - \lambda)^{-1} d\lambda \right) \mathcal{S}(X_1) \right\| \rightarrow 0.$$

Since  $\lambda_0$  is a simple isolated eigenvalue of  $\mathcal{H}_1$  and  $\psi_0$  is the corresponding eigenfunction, by [24, Chapter V, §3, Subsection 5] we have

$$\frac{1}{2\pi i} \int_{K_r} (\mathcal{H}_1 - \lambda)^{-1} d\lambda = \psi_0(\cdot, \psi_0).$$

It follows that (3.8) reduces to

$$(3.9) \quad \left\| \frac{1}{2\pi i} \int_{K_r} (\mathcal{H}_X - \lambda)^{-1} d\lambda - (\cdot, \mathcal{S}(-X_1)\psi_0) \mathcal{S}(-X_1)\psi_0 \right\| \rightarrow 0.$$

The second summand in the above formula differs from zero. Consequently, if  $r$  is sufficiently small, there exists at least one eigenvalue of  $\mathcal{H}_X$  inside  $K_r$ , i.e., there exists at least one eigenvalue of the perturbed operator that converges to  $\lambda_0$  as  $\tau(X) \rightarrow \infty$ . We show that such an eigenvalue is unique and simple.

The integral over  $K_r$  on the left in (3.9) is a projection onto the eigenfunctions of  $\mathcal{H}_X$  corresponding to eigenvalues that tend to  $\lambda_0$  as  $\tau(X) \rightarrow \infty$ . The second summand in (3.9) is a projection to the function  $\mathcal{S}(-X_1)\psi_0$ , i.e., a rank one operator. The convergence in (3.9), formulas (4.33) and (4.43) in [24, Chapter I, §4, Subsection 6], and formulas (1.19) in [24, Chapter II, §1, Subsection 4] show that, for  $\tau(X)$  sufficiently large, the projection in (3.9) represented by the integral is also of rank 1. This means that  $\mathcal{H}_X$  has a unique eigenvalue  $\lambda_X$  convergent to  $\lambda_0$  as  $\tau(X) \rightarrow \infty$ . □

§4. REDUCTION OF THE EQUATION FOR EIGENVALUES TO AN OPERATOR EQUATION

In this section, we use the version of the Birman–Schwinger method suggested in [21, 22] to reduce the equation for eigenvalues of the perturbed operator  $\mathcal{H}_X$  to the problem of finding the zeros of a certain function.

Consider the equation

$$(4.1) \quad \mathcal{H}_X \psi_X = \lambda_X \psi_X$$

for eigenvalues. We rewrite it in the form

$$(\mathcal{H}_0 - \lambda_X) \psi_X = - \sum_{j=1}^k \mathcal{S}(-X_j) \mathcal{L}_j \mathcal{S}(X_j) \psi_X.$$

The right-hand side of this equation is a sum, so we shall seek its solution  $\psi_X(x)$  also in the form of a sum, specifically, in the form

$$(4.2) \quad \psi_X(x) = \sum_{j=1}^k \mathcal{S}(-X_j) \psi_j(x),$$

where the functions  $\psi_j$  are the unknowns. Substitution of (4.2) in (4.1) yields

$$\begin{aligned} 0 &= \left( \mathcal{H}_0 + \sum_{i=1}^k \mathcal{S}(-X_i) \mathcal{L}_i \mathcal{S}(X_i) - \lambda_X \right) \left( \sum_{j=1}^k \mathcal{S}(-X_j) \psi_j(x) \right) \\ &= \sum_{j=1}^k \mathcal{S}(-X_j) \left[ (\mathcal{H}_j - \lambda_X) \psi_j + \sum_{\substack{i=1 \\ i \neq j}}^k \mathcal{L}_j \mathcal{S}(X_j - X_i) \psi_i \right]. \end{aligned}$$

To ensure this relation, it suffices to satisfy  $k$  equations

$$(4.3) \quad (\mathcal{H}_j - \lambda_X) \psi_j + \sum_{\substack{i=1 \\ i \neq j}}^k \mathcal{L}_j \mathcal{S}(X_j - X_i) \psi_i = 0, \quad j = 1, \dots, k.$$

In what follows, we consider certain more general equations rather than (4.3), namely, the equations

$$(4.4) \quad (\mathcal{H}_j - \lambda_X) \psi_j + \delta \sum_{\substack{i=1 \\ i \neq j}}^k \frac{1}{\varepsilon(X)} \mathcal{L}_j \mathcal{S}(X_j - X_i) \psi_i = 0,$$

where  $\delta$  is a small positive parameter and  $\varepsilon(X)$  is defined by (2.1). Taking  $\delta = \varepsilon(X)$ , we return to (4.3). Our purpose will be to find a number  $\lambda_X$  close to  $\lambda_0$  for which equations (4.4) have a nontrivial solution (that is, a collection  $\psi_j$  of functions at least one of which does not vanish identically).

Equation (4.4) implies the relations

$$(4.5) \quad \begin{aligned} &(\mathcal{H}_j - \lambda_X) \psi_j = \varsigma_j(| \cdot |) g_j, \\ g_j &:= -\delta \sum_{\substack{i=1 \\ i \neq j}}^k \frac{1}{\varepsilon(X)} \mathcal{L}_j^0 \eta_j(| \cdot |) \mathcal{S}(X_j - X_i) \psi_i \in L_2(\mathbb{R}^d; \mathbb{C}^n). \end{aligned}$$

We introduce the Hilbert space

$$\mathfrak{L} := \left\{ h = \begin{pmatrix} h_1 \\ \vdots \\ h_k \end{pmatrix}, h_i \in L_2(\mathbb{R}^d; \mathbb{C}^n), i = 1, \dots, k \right\}$$

with the scalar product

$$(u, v)_{\mathfrak{L}} = \sum_{i=1}^k (u_i, v_i)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

Put

$$(4.6) \quad \Psi_X := \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_k \end{pmatrix} \in \mathfrak{L}, \quad g := \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} \in \mathfrak{L}.$$

In the space  $\mathfrak{L}$ , we introduce the operators

$$\mathcal{H} := \begin{pmatrix} \mathcal{H}_1 & 0 & \dots & 0 \\ 0 & \mathcal{H}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{H}_k \end{pmatrix}, \quad \mathcal{P} := \begin{pmatrix} \varsigma_1 & 0 & \dots & 0 \\ 0 & \varsigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varsigma_k \end{pmatrix},$$

$$\mathcal{T}_X := -\frac{1}{\varepsilon(X)} \begin{pmatrix} 0 & \mathcal{L}_1^0 \eta_1 \mathcal{S}(X_1 - X_2) & \dots & \mathcal{L}_1^0 \eta_1 \mathcal{S}(X_1 - X_k) \\ \mathcal{L}_2^0 \eta_2 \mathcal{S}(X_2 - X_1) & 0 & \dots & \mathcal{L}_2^0 \eta_2 \mathcal{S}(X_2 - X_k) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_k^0 \eta_k \mathcal{S}(X_k - X_1) & \mathcal{L}_k^0 \eta_k \mathcal{S}(X_k - X_2) & \dots & 0 \end{pmatrix}.$$

With this notation, we can rewrite (4.4) and (4.5) as equations in the space  $\mathfrak{L}$ :

$$(4.7) \quad (\mathcal{H} - \lambda_X) \Psi_X - \delta \mathcal{P} \mathcal{T}_X \Psi_X = 0,$$

$$(4.8) \quad (\mathcal{H} - \lambda_X) \Psi_X = \mathcal{P}g, \quad \Psi_X = (\mathcal{H} - \lambda_X)^{-1} \mathcal{P}g.$$

Since  $\lambda_0$  does not belong to the spectrum of  $\mathcal{H}_i$  for any  $i \geq 2$ , the operators  $(\mathcal{H}_i - \lambda)^{-1}$ ,  $i \geq 2$ , are bounded and holomorphic in  $\lambda \in \bar{U}$  as operators from  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  to  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . We recall that  $U$  is a small fixed neighborhood of  $\lambda_0$  whose closure does not contain other points of the spectra of  $\mathcal{H}_0$  or  $\mathcal{H}_i$ . Since  $\lambda_0$  is a simple eigenvalue of  $\mathcal{H}_1$ , formula (3.21) in [24, Chapter V, §3, Subsection 5] shows that the resolvent of  $\mathcal{H}_i$  is representable in the form

$$(4.9) \quad (\mathcal{H}_1 - \lambda_X)^{-1} f = \frac{(f, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}}{\lambda_0 - \lambda_X} \psi_0 + \mathcal{R}_1(\lambda_X) f,$$

where  $f \in L_2(\mathbb{R}^d; \mathbb{C}^n)$  (recall that  $\mathcal{R}_1(\lambda)$  is the reduced resolvent holomorphic in  $\lambda$  on  $\bar{U}$  and acting in the orthogonal complement of the function  $\psi_0$ ). Taking (4.9) into account, we see that the action of  $(\mathcal{H} - \lambda_X)^{-1}$  is described like this:

$$(4.10) \quad (\mathcal{H} - \lambda_X)^{-1} h = \begin{pmatrix} (\mathcal{H}_1 - \lambda_X)^{-1} h_1 \\ \vdots \\ (\mathcal{H}_k - \lambda_X)^{-1} h_k \end{pmatrix}$$

$$= \frac{(h_1, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}}{\lambda_0 - \lambda_X} \begin{pmatrix} \psi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{R}_1(\lambda_X) h_1 \\ \vdots \\ \mathcal{R}_k(\lambda_X) h_k \end{pmatrix}$$

$$= \frac{1}{\lambda_0 - \lambda_X} (h, \Psi_0)_{\mathfrak{L}} \Psi_0 + \mathcal{R}(\lambda_X) h,$$

where

$$(4.11) \quad \mathcal{R}(\lambda) := \begin{pmatrix} \mathcal{R}_1(\lambda) & 0 & \dots & 0 \\ 0 & \mathcal{R}_2(\lambda) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{R}_k(\lambda) \end{pmatrix}, \quad \Psi_0 := \begin{pmatrix} \psi_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and  $h \in \mathfrak{L}$ . Recall that  $\mathcal{R}_i(\lambda) = (\mathcal{H}_i - \lambda)^{-1}$ ,  $i \geq 2$ . We use (4.8) and replace  $(\mathcal{H} - \lambda_X)\Psi_X$  by  $\mathcal{P}g$  and  $\Psi_X$  by  $(\mathcal{H} - \lambda_X)^{-1}\mathcal{P}g$  in (4.7). Then we obtain

$$\mathcal{P}(g - \delta\mathcal{T}_X(\mathcal{H} - \lambda_X)^{-1}\mathcal{P}g) = 0.$$

To ensure the last relation, it suffices to satisfy the equation

$$g - \delta\mathcal{T}_X(\mathcal{H} - \lambda_X)^{-1}\mathcal{P}g = 0.$$

Transferring the second summand on the left to the right-hand side and using (4.10), we arrive at

$$(4.12) \quad \begin{aligned} g &= \frac{\delta}{\lambda_0 - \lambda_X}(\mathcal{P}g, \Psi_0)_{\mathfrak{L}}\mathcal{T}_X\Psi_0 + \delta\mathcal{T}_X\mathcal{R}(\lambda_X)\mathcal{P}g, \\ (\lambda - \delta\mathcal{T}_X\mathcal{R}(\lambda_X)\mathcal{P})g &= \frac{\delta}{\lambda_0 - \lambda_X}(\mathcal{P}g, \Psi_0)_{\mathfrak{L}}\mathcal{T}_X\Psi_0. \end{aligned}$$

The further study of the last equation requires some auxiliary lemmas.

**Lemma 4.1.** *Let  $\varsigma(r) \in C^{2m}(\mathbb{R}_+)$  be a nonnegative function satisfying the condition*

$$\varsigma(r) \leq Ce^{-\kappa \int_0^r a(t) dt},$$

where  $a$  is the function occurring in (A1), and  $C > 0$ ,  $\kappa > 0$  are some constants. Then for every  $h \in L_2(\mathbb{R}^d; \mathbb{C}^n)$  and all  $\lambda \in \bar{U}$ , the equation

$$(\mathcal{H}_0 - \lambda)u = \varsigma(|\cdot|)h$$

has a unique solution in the space  $W_2^{2m}(\mathbb{R}^2; \mathbb{C}^n)$ . This solution is representable in the form

$$u(x) = e^{-\rho \int_0^{|\cdot|} a(t) dt} \widehat{u}(x),$$

where  $\rho > 0$  is a fixed number independent on  $\lambda$ , and  $\widehat{u} \in W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  is a function satisfying

$$\|\widehat{u}\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} \leq c\|h\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

Here the constant  $c$  does not depend on  $\lambda$  and  $h$ .

Every nonnegative function  $\eta_j \in C^{2m}(\mathbb{R}_+)$ ,  $j = 1, \dots, k$ , satisfying (A2) obeys the inequality

$$\|\eta_j(|\cdot|)\mathcal{S}(Y)u\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} \leq C(Y)\|h\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)},$$

where  $Y \in \Gamma$ ,  $C(Y)$  is a function independent of  $j$ ,  $z$ , and  $h$ , and  $C(Y) \rightarrow 0$  as  $Y \rightarrow \infty$ .

This lemma was established in [19] (see Lemma 4 and its proof therein).

**Lemma 4.2.** *The eigenfunction  $\psi_0$  of  $\mathcal{H}_1$  is representable in the form*

$$(4.13) \quad \psi_0(x) = e^{-\rho \int_0^{|\cdot|} a(t) dt} \widehat{\psi}_0(x),$$

where  $\rho > 0$  is a fixed number,  $\widehat{\psi}_0 \in W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  is a certain function, and

$$\|\mathcal{L}_j^0 \eta_j(|\cdot|)\mathcal{S}(X_j - X_1)\psi_0\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \rightarrow 0 \quad \text{à} \quad \tau(X) \rightarrow \infty, \quad i = 2, \dots, k.$$

*Proof.* We write the equation for  $\psi_0$  in the form

$$(\mathcal{H}_0 - \lambda_0)\psi_0 = \varsigma_1(|\cdot|)h_1, \quad \text{where} \quad h_1 = -\mathcal{L}_1^0 \eta_1(|\cdot|)\psi_0 \in L_2(\mathbb{R}^d; \mathbb{C}^n).$$

Now the claim readily follows from Lemma 4.1. □

**Lemma 4.3.** *For every  $h \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ , we have the following estimates, which are uniform in  $\lambda \in \bar{U}$ :*

$$(4.14) \quad \|\mathcal{R}_j(\lambda)h\|_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)} \leq c\|h\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)},$$

$$(4.15) \quad \|\mathcal{L}_p^0 \eta_p(|\cdot|) \mathcal{S}(X_p - X_j) \mathcal{R}_j(\lambda) \varsigma_j(|\cdot|) h\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C(X) \|h\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

*Proof.* For  $h \in L_2(\mathbb{R}^d; \mathbb{C}^n)$  and  $\lambda \in \bar{U}$ , put  $v_j := \mathcal{R}_j(\lambda)h$ ,  $j = 1, \dots, k$ . By the definition of  $\mathcal{R}_j(\lambda)$ , we have

$$(4.16) \quad \|v_j\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\|h\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}$$

uniformly in  $\lambda$  and  $h$ , where  $C$  is a constant independent of  $h$ ,  $\lambda$ , and  $j$ . Also, it is easy to check that the functions  $v_j$  are representable in the form

$$(4.17) \quad \begin{aligned} v_j &= (\mathcal{H}_j - i)^{-1}(h + (\lambda - i)v_j), \quad j = 2, \dots, k, \\ v_1 &= (\mathcal{H}_1 - i)^{-1}(h - (h, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \psi_0 + (\lambda - i)v_1). \end{aligned}$$

Since the  $\mathcal{H}_j$  are selfadjoint, the operators  $(\mathcal{H}_j - i)^{-1}$ ,  $j = 1, \dots, k$ , act boundedly from  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  to  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . Next, the operators  $\mathcal{H}_j - i$  act boundedly from  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$  to  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . By the Banach inverse operator theorem, the operators  $(\mathcal{H}_j - i)^{-1}$ ,  $j = 1, \dots, k$ , act boundedly from  $L_2(\mathbb{R}^d; \mathbb{C}^n)$  to  $W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ . Together with (4.16) and (4.17), this implies (4.14).

For  $\lambda \in \bar{U}$ , we put  $u_j = \mathcal{R}_j \varsigma_j(|\cdot|) h \in W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)$ ,  $j = 1, \dots, k$ . The functions  $u_j$  satisfy the equations

$$\begin{aligned} (\mathcal{H}_j - \lambda)u_j &= \varsigma_j(|\cdot|)h, \quad j = 2, \dots, k, \\ (\mathcal{H}_1 - \lambda)u_1 &= \varsigma_1(|\cdot|)h - (\varsigma_1(|\cdot|)h, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \psi_0. \end{aligned}$$

These equations can be rewritten in the following way:

$$\begin{aligned} (\mathcal{H}_0 - \lambda)u_j &= \varsigma_j(|\cdot|)g_j, \quad j = 2, \dots, k, \\ (\mathcal{H}_0 - \lambda)u_1 &= \varsigma_1(|\cdot|)f - e^{-\rho} \int_0^{|\cdot|} a(t) dt (\varsigma_1(|\cdot|)h, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \widehat{\psi}_0, \end{aligned}$$

where  $g_j = h - \mathcal{L}_j^0 \eta_j(|\cdot|)u_j \in L_2(\mathbb{R}^d; \mathbb{C}^n)$  and  $f = h - \mathcal{L}_1^0 \eta_1(|\cdot|)u_1 \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ . Applying Lemma 4.1 to the last identities and taking (4.14) and (4.13) into account, we obtain (4.15).  $\square$

Lemmas 4.2 and 4.3 and formula (2.1) imply that  $\varepsilon(X)$  tends to zero as  $\tau(X) \rightarrow \infty$ .

We return to equation (4.12). The components of the operator  $\mathcal{T}_X \mathcal{R}(\lambda) \mathcal{P}$  have the form

$$-\varepsilon^{-1}(X) \mathcal{L}_p^0 \eta_p(|\cdot|) \sum_{\substack{j=1 \\ p \neq j}}^k \mathcal{S}(X_p - X_j) \mathcal{R}_j(\lambda) \varsigma_j(|\cdot|).$$

The choice of  $\varepsilon(X)$  shows that for  $\lambda \in \bar{U}$ , every component of  $\mathcal{T}_X \mathcal{R}(\lambda) \mathcal{P}$  is an operator bounded uniformly with respect to  $X$  and acting on  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . We have proved the following statement.

**Lemma 4.4.** *The operator  $\mathcal{T}_X \mathcal{R}(\lambda) \mathcal{P}$  is bounded uniformly with respect to  $X$  and  $\lambda \in \bar{U}$ .*

This lemma shows that, for  $\delta$  sufficiently small, the operator  $\delta \mathcal{T}_X \mathcal{R}(\lambda) \mathcal{P}$  is a contraction for all  $X$  and all  $\lambda \in \bar{U}$ . Then the operator  $(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda) \mathcal{P})^{-1}$  exists. We apply the operator  $(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1}$  to the two sides of (4.12):

$$(4.18) \quad g = \frac{\delta}{\lambda_0 - \lambda_X} (\mathcal{P}g, \Psi_0)_{\mathfrak{L}} (\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0.$$

Applying  $\mathcal{P}$ , we arrive at

$$\mathcal{P}g = \frac{\delta}{\lambda_0 - \lambda_X} (\mathcal{P}g, \Psi_0)_{\mathfrak{L}} \mathcal{P}(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0.$$

Taking inner products (in  $\mathfrak{L}$ ) with  $\Psi_0$ , we find

$$(4.19) \quad (\mathcal{P}g, \Psi_0)_{\mathfrak{L}} = \frac{\delta}{\lambda_0 - \lambda_X} (\mathcal{P}g, \Psi_0)_{\mathfrak{L}} (\mathcal{P}(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0, \Psi_0)_{\mathfrak{L}}.$$

The inner product  $(\mathcal{P}g, \Psi_0)_{\mathfrak{L}}$  is nonzero because otherwise  $g = 0$  by (4.18), and the second formula in (4.7) yields  $\Psi_X = 0$ , whereas we look for a nontrivial solution of (4.4). Dividing the two sides of (4.19) by  $(\mathcal{P}g, \Psi_0)_{\mathfrak{L}}$ , we deduce the formula

$$(4.20) \quad \lambda_X = \lambda_0 - \delta (\mathcal{P}(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0, \Psi_0)_{\mathfrak{L}}.$$

This identity is a condition on  $\lambda_X$  ensuring the existence of a nontrivial solution of equations (4.4). It is possible to find this nontrivial solution explicitly; we shall write a formula for it. Since a solution for (4.7) is determined up to a constant factor, by (4.18) the function  $g$  is representable in the form

$$g = C(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0,$$

where  $C$  is a positive constant. Taking this identity into account, we transform equation (4.20) to

$$\lambda_X = \lambda_0 - \delta C^{-1} (\mathcal{P}g, \Psi_0)_{\mathfrak{L}}.$$

Now, combining the previous identities with (4.7), (4.10), and the equation

$$\Psi_X = (\mathcal{H} - \lambda_X)^{-1} \mathcal{P}g = \frac{\Psi_0}{\lambda_0 - \lambda_X} (\mathcal{P}g, \Psi_0)_{\mathfrak{L}} + \mathcal{R}(\lambda_X) \mathcal{P}g,$$

we deduce that

$$\Psi_X = \delta^{-1} C \Psi_0 + \mathcal{R}(\lambda_X) \mathcal{P}g.$$

Taking  $C = \delta$ , we find

$$(4.21) \quad \Psi_X = \Psi_0 + \delta \mathcal{R}(\lambda_X) \mathcal{P}(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0.$$

This vector-valued function is a nontrivial solution of equations (4.4); it corresponds to the number  $\lambda_X$  determined by equation (4.20). Substitution of (4.6), (4.20), and (4.21) in equations (4.4) turn indeed them into identities, as can be checked by direct inspection. The vector-valued function  $\Psi_X$  (see (4.21)) is not equal to zero for  $\delta$  sufficiently small and  $\tau(X)$  sufficiently large, because

$$(4.22) \quad \|\Psi_X - \Psi_0\|_{\mathfrak{L}} = \mathcal{O}(\delta), \quad \Psi_0 \neq 0,$$

by (4.20) and (4.11). Since equations (4.4) turn into (4.3) for  $\delta = \varepsilon(X)$ , it follows that the solution of (4.20) for  $\delta = \varepsilon(X)$  is an eigenvalue of  $\mathcal{H}_X$  tending to  $\lambda_0$  as  $\tau(X) \rightarrow 0$ . The corresponding eigenfunction arises after substitution of (4.21) and of the identity  $\delta = \varepsilon(X)$  in (4.2). Again, this eigenfunction is nonzero by (4.22).

### §5. ASYMPTOTICS FOR THE PERTURBED EIGENVALUE AND EIGENFUNCTION

In this section, we finish the proof of Theorem 2.2. For this, we prove the solvability of equation (4.20) and clarify the dependence of the solution on  $X$  and  $\delta$ .

After the change of variables  $z = \lambda_X - \lambda_0$ , equation (4.20) takes the form

$$F(\delta, z, X) := z + \delta G(\delta, z, X) = 0,$$

where

$$G(\delta, z, X) := (\mathcal{P}(\lambda - \delta \mathcal{T}_X \mathcal{R}(z + \lambda_0) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0, \Psi_0)_{\mathfrak{L}}.$$

The function  $G(\delta, z, X)$  is analytic with respect to  $(\delta, z)$  in the domain  $|\delta| \leq \delta_0, |z| \leq z_0$ , where  $\delta_0$  and  $z_0$  are certain sufficiently small positive numbers.

The function  $z \mapsto z$  has a unique simple zero at the point  $z = 0$ . It is easily seen that for all  $X$ , the function  $G(\delta, z, X)$  satisfies the estimate

$$|G(\delta, z, X)| \leq C \quad \text{for all } |z| \leq z_0, \quad |\delta| \leq \delta_0,$$

where  $C$  is a constant independent of  $\delta, z$ , and  $X$ . Then we have

$$\delta|G(\delta, z, X)| \leq \delta C < z_0 \quad \text{for } |z| = z_0 \text{ and sufficiently small } \delta.$$

It follows that the Rouché theorem (see [25, Chapter IV, §3]) can be applied to the function  $F(\delta, z, X)$  to ensure the existence of a unique simple zero of this function in the domain  $|z| \leq z_0$ . We denote this zero by  $z_1(\delta, X)$ . Applying the Cauchy residue theorem (see [25, Chapter III, §2]), it is easy to deduce the following representation for  $z_1(\delta, X)$ :

$$(5.1) \quad z_1(\delta, X) = \frac{1}{2\pi i} \int_{|z|=z_0} z \frac{\frac{\partial F}{\partial z}(\delta, z, X)}{F(\delta, z, X)} dz = \frac{1}{2\pi i} \int_{|z|=z_0} z \frac{1 + \delta \frac{\partial G}{\partial z}(\delta, z, X)}{z + \delta G(\delta, z, X)} dz.$$

Since  $G(\delta, z, X)$  is analytic in the domain  $|\delta| \leq \delta_0, |z| \leq z_0$  and  $F(\delta, z, X)$  does not vanish for  $|\delta| \leq \delta'_0, |z| = z_0$ , where  $\delta'_0 < \delta_0$  is a sufficiently small positive number, it follows that the function

$$\Phi(\delta, z, X) := z \frac{1 + \delta \frac{\partial G}{\partial z}(\delta, z, X)}{z + \delta G(\delta, z, X)}$$

is also analytic as a function of  $\delta$  in the disk  $|\delta| \leq \delta'_0$  for  $|z| = z_0$ . Consequently, it admits a series expansion

$$(5.2) \quad \Phi(\delta, z, X) = \sum_{j=0}^{\infty} \delta^j K_j(z, X),$$

where the  $K_j(z, X)$  are certain functions of  $z$  and  $X$ . The series converges uniformly with respect to  $\delta$  for  $|\delta| \leq \delta'_0, |z| = z_0$ , and with respect to  $X$ .

**Lemma 5.1.** *For  $|z| = z_0$ , the coefficients  $K_j(z, X)$  of the series (5.2) admit the estimate*

$$|K_j(z, X)| \leq C^j,$$

where  $C$  is a constant independent of  $j, X$ , and  $z$ .

*Proof.* The definitions of  $G(\delta, z, X)$  and  $\Phi(\delta, z, X)$  imply the estimate

$$|\Phi(\delta, z, X)| \leq C$$

for all  $|\delta| \leq \delta'_0$  and  $|z| = z_0$  if  $\delta'_0$  is sufficiently small. Here  $C$  is a constant independent of  $j, \delta, z$ , and  $X$ . The coefficients  $K_j(z, X)$  are given by the formula

$$K_j(z, X) = \frac{1}{2\pi i} \int_{|\delta|=\delta'_0} \frac{\Phi(\delta, z, X)}{\delta^{j+1}} d\delta.$$

Therefore,

$$|K_j(z, X)| \leq \frac{1}{2\pi} \int_{|\delta|=\delta'_0} \frac{|\Phi(\delta, z, X)|}{|\delta|^{j+1}} d\delta \leq \frac{C}{2\pi} \int_{|\delta|=\delta'_0} \frac{d\delta}{|\delta|^{j+1}} = C(\delta'_0)^{-j}. \quad \square$$

We return to (5.1) and show that the zero  $z_1(\delta, X)$  can be expanded in a series that converges uniformly with respect to small  $\delta$  and  $X$ . A series expansion is a direct consequence of the identity

$$\begin{aligned} z_1(\delta, X) &= \frac{1}{2\pi i} \int_{|z|=z_0} \Phi(\delta, z, X) dz = \frac{1}{2\pi i} \int_{|z|=z_0} \sum_{j=0}^{\infty} \delta^j K_j(z, X) dz \\ &= \frac{1}{2\pi i} \sum_{j=0}^{\infty} \delta^j \int_{|z|=z_0} K_j(z, X) dz = \sum_{j=0}^{\infty} \delta^j \lambda_j(X), \end{aligned}$$

where

$$\lambda_j(X) := \frac{1}{2\pi i} \int_{|z|=z_0} K_j(z, X) dz.$$

Lemma 5.1 shows that

$$(5.3) \quad |\lambda_j(X)| \leq C^j,$$

where  $C$  is a constant independent of  $j$  and  $X$  for  $\tau(X)$  sufficiently large. The last inequality yields

$$\sum_{j=0}^{\infty} |\delta^j \lambda_j(X)| \leq \sum_{j=0}^{\infty} C^j \delta^j,$$

where the series on the right converges for all sufficiently small  $\delta$  and  $X$ . Consequently, the same is true for the series  $\sum_{j=0}^{\infty} \delta^j \lambda_j(X)$ .

Thus, there exists a unique solution of equation (4.20), which is representable in the form

$$(5.4) \quad \lambda_X(\delta) = \lambda_0 + z_1(\delta, X) = \lambda_0 + \sum_{j=1}^{\infty} \delta^j \lambda_j(X),$$

and the series converges uniformly with respect to small  $\delta$  and  $X$ . We put  $\delta = \varepsilon(X)$ ; then this solution is an eigenvalue of  $\mathcal{H}_X$  that converges to  $\lambda_0$  as  $\tau(X) \rightarrow \infty$ . Since such an eigenvalue is unique by Lemma 3.1, we see that the series (5.4) with  $\delta = \varepsilon(X)$  represents this eigenvalue:

$$(5.5) \quad \lambda_X(\varepsilon(X)) = \lambda_0 + \sum_{j=1}^{\infty} \varepsilon^j(X) \lambda_j(X).$$

The last series converges uniformly in  $X$  for  $\tau(X)$  sufficiently large.

Now, we consider identity (4.21). Since the operator  $\mathcal{R}(\lambda)$  is analytic in  $\lambda$  and  $\lambda_X(\delta)$  is analytic in  $\delta$ , the operator  $\mathcal{R}(\lambda_X(\delta))$  is analytic in  $\delta$ . Consequently,

$$(5.6) \quad \Psi_X(\delta) = \Psi_0 + \sum_{j=1}^{\infty} \delta^j \Psi_j(X),$$

where the  $\Psi_j(X)$  are given by the formula

$$(5.7) \quad \Psi_j(X) = \frac{1}{2\pi i} \int_{|\delta|=\delta'_0} \delta^{-j} \mathcal{R}(\lambda_X(\delta)) \mathcal{P}(\lambda - \delta \mathcal{T}_X \mathcal{R}(\lambda_X(\delta)) \mathcal{P})^{-1} \mathcal{T}_X \Psi_0 d\delta.$$

As in the proof of Lemma 5.1, the last formula implies the estimates

$$(5.8) \quad \|\Psi_j(X)\|_{\mathfrak{L}} \leq C^j,$$

where  $C$  is a constant independent of  $j$ ,  $\delta$ , and  $X$ . Consequently, the nontrivial solution  $\Psi_X$  of equation (4.7) corresponding to  $\lambda_X$  is represented by the series (5.6), convergent in  $\mathfrak{L}$  uniformly with respect to  $X$  and sufficiently small  $\delta$ .

We introduce the Hilbert space

$$\tilde{\mathfrak{L}} := \left\{ h = \begin{pmatrix} h_1 \\ \vdots \\ h_k \end{pmatrix}, h_i \in W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n), i = 1, \dots, k \right\}$$

with the inner product

$$(u, v)_{\tilde{\mathfrak{L}}} = \sum_{i=1}^k (u_i, v_i)_{W_2^{2m}(\mathbb{R}^d; \mathbb{C}^n)}.$$

Much as we did when proving (5.8), we use (4.14) and (5.7) to deduce that

$$(5.9) \quad \|\Psi_j(X)\|_{\tilde{\mathfrak{L}}} \leq C^j,$$

where  $C$  is a constant independent of  $j$ ,  $\delta$ , and  $X$ . Thus, the series (5.6) converges in the norm of  $\tilde{\mathfrak{L}}$  uniformly with respect to  $X$  and sufficiently small  $\delta$ . Now, taking  $\delta = \varepsilon(X)$ , we obtain

$$(5.10) \quad \Psi_X(\varepsilon(X)) = \Psi_0 + \sum_{j=1}^{\infty} \varepsilon^j(X) \Psi_j(X).$$

Combined with (4.2), this implies a series expansion (2.3) for the eigenfunction  $\psi_X$  corresponding to the eigenvalue  $\lambda_X(\varepsilon(X))$ ; the series converges uniformly in  $X$  for  $\tau(X)$  sufficiently large. Estimate (2.7) for the terms of this series follows from (5.9) and the identity  $\delta = \varepsilon(X)$ .

We determine the terms  $\lambda_j$  and  $\Psi_j$  of the expansions (5.5) and (5.10), respectively. We rewrite (5.6) componentwise:

$$\psi_p = \sum_{j=0}^{\infty} \delta^j \tilde{\phi}_{p,j}, \quad p = 1, \dots, k,$$

where the  $\tilde{\phi}_{p,j}(x, X)$  are the components of the vector  $\Psi_j$ . Denote

$$\phi_{p,j}(x, X) := \varepsilon^j(X) \tilde{\phi}_{p,j}(x, X).$$

By (5.6) and the definition of  $\Psi_0$ , we have

$$(5.11) \quad \phi_{1,0} = \psi_0, \quad \phi_{p,0} = 0, \quad p = 2, \dots, k,$$

and by (5.9) the functions  $\phi_{p,j}(x, X)$  enjoy inequalities (2.7). We shall find the coefficients  $\lambda_j$  in the form

$$\lambda_j(X) = \Lambda_j(X) \varepsilon^{-j}(X),$$

where the  $\Lambda_j$  are certain functions satisfying (5.3) by (2.7). Now, we substitute (5.11) and the series

$$\lambda_X(\delta) = \lambda_0 + \sum_{j=1}^{\infty} \delta^j \varepsilon^{-j}(X) \Lambda_j(X), \quad \psi_p = \sum_{j=0}^{\infty} \delta^j \varepsilon^{-j}(X) \phi_{p,j}$$

in (4.4):

$$\begin{aligned} & \left( \mathcal{H}_p - \lambda_0 - \sum_{j=1}^{\infty} \delta^j \varepsilon^{-j}(X) \Lambda_j(X) \right) \sum_{s=0}^{\infty} \delta^s \varepsilon^{-s}(X) \phi_{p,s} \\ & + \delta \varepsilon^{-1}(X) \sum_{\substack{t=1 \\ t \neq p}}^k \mathcal{L}_p \mathcal{S}(X_p - X_t) \sum_{s=0}^{\infty} \delta^s \varepsilon^{-s}(X) \phi_{t,s} = 0, \quad p = 2, \dots, k. \end{aligned}$$

Expanding and then equating the coefficients of equal powers of  $\delta$ , we obtain

$$\begin{aligned}
 (\mathcal{H}_1 - \lambda_0)\phi_{1,1} &= \Lambda_1\psi_0, \\
 (\mathcal{H}_1 - \lambda_0)\phi_{1,j} &= \Lambda_j\psi_0 + \sum_{t=1}^{j-1} \Lambda_t\phi_{1,j-t} - \sum_{t=2}^k \mathcal{L}_1\mathcal{S}(X_1 - X_t)\phi_{t,j-1}, \\
 (\mathcal{H}_p - \lambda_0)\phi_{p,1} &= -\mathcal{L}_p\mathcal{S}(X_p - X_1)\psi_0, \\
 (\mathcal{H}_p - \lambda_0)\phi_{p,j} &= \sum_{t=1}^{j-1} \Lambda_t\phi_{p,j-t} - \sum_{\substack{t=1 \\ t \neq p}}^k \mathcal{L}_p\mathcal{S}(X_p - X_t)\phi_{t,j-1},
 \end{aligned}
 \tag{5.12}$$

where  $p = 2, \dots, k, j \geq 2$ .

**Lemma 5.2.** *We have*

$$(\phi_{1,j}, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = 0, \quad j \geq 1.$$

*Proof.* We calculate the inner product

$$(\Psi_X, \Psi_0)_{\mathfrak{L}} = (\psi_1, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = \sum_{j=0}^{\infty} \delta^j \varepsilon^{-j}(X) (\phi_{1,j}, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

On the other hand, by (4.11), (4.21), and the normalization for  $\psi_0$ , the inner product  $(\Psi_X, \Psi_0)_{\mathfrak{L}}$  has the form

$$\begin{aligned}
 (\Psi_X, \Psi_0)_{\mathfrak{L}} &= (\Psi_0, \Psi_0)_{\mathfrak{L}} - \delta(\mathcal{R}(\lambda_X)\mathcal{P}(\lambda - \delta\mathcal{T}_X\mathcal{R}(\lambda_X)\mathcal{P})^{-1}\mathcal{T}_X\Psi_0, \Psi_0)_{\mathfrak{L}} \\
 &= 1 - \delta(\mathcal{R}_1(\lambda_X)h, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)},
 \end{aligned}$$

where  $h \in L_2(\mathbb{R}^d; \mathbb{C}^n)$  is the first component of the vector

$$\mathcal{P}(\lambda - \delta\mathcal{T}_X\mathcal{R}(\lambda_X)\mathcal{P})^{-1}\mathcal{T}_X\Psi_0.$$

The operator  $\mathcal{R}_1(\lambda_0)$  acts in the orthogonal complement of  $\psi_0$ , whence

$$(\mathcal{R}_1(\lambda_X)h, \psi_0)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = 0.$$

Consequently,

$$(\Psi_X, \Psi_0)_{\mathfrak{L}} = 1.$$

Together with (5.13), this implies the claim. □

Since  $\mathcal{H}_1$  is selfadjoint, we see that the solvability condition for equations (5.12) is the orthogonality of the right-hand sides to  $\psi_0$  in  $L_2(\mathbb{R}^d; \mathbb{C}^n)$ . Applying this solvability conditions and taking Lemma 5.2 into account, we see that  $\Lambda_1 = 0$  and deduce (2.4). Again by Lemma 5.2 and the definition of  $\mathcal{R}_1$ , we obtain (2.5) and (2.6). The proof of Theorem 2.2 is complete.

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DEPARTMENT OF PHYSICS AND MATHEMATICS, M. AKMULLA BASHKIR STATE PEDAGOGICAL UNIVERSITY, UL. OKTYABR'SKOI REVOLYUTSII 3A, UFA 450055, RUSSIA  
*E-mail address:* `nastya_gm@mail.ru`

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