Алгебра и анализ
Том 25 (2013), № 5

St. Petersburg Math. J.
Vol. 25 (2014), No. 5, Pages 755–773
S 1061-0022(2014)01315-5
Article electronically published on July 18, 2014

CHARACTERIZATION OF CYCLIC SCHUR GROUPS

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Abstract. A finite group G is called a Schur group if any Schur ring over G is associated in a natural way with a subgroup of $\mathrm{Sym}(G)$ that contains all right translations. It was proved by R. Pöschel (1974) that, given a prime $p \geq 5$, a p-group is Schur if and only if it is cyclic. We prove that a cyclic group of order n is Schur if and only if n belongs to one of the following five families of integers: p^k, pq^k, $2pq^k$, pqr, $2pqr$ where p, q, r are distinct primes, and $k \geq 0$ is an integer.

§1. Introduction

Let G be a finite group. A subring of the group ring $\mathbb{Q}G$ is called a *Schur ring* or *S-ring* over G if it is closed with respect to the componentwise multiplication and inversion. The first construction of such a ring was proposed by I. Schur [8] in connection with his famous result on permutation groups containing a regular cyclic subgroup. Namely, let Γ be a permutation group on the set G that contains the regular group G_{right} induced by right multiplications,

$$G_{\mathrm{right}} \leq \Gamma \leq \mathrm{Sym}(G).$$

Denote by Γ_1 the stabilizer of the identity of G in Γ. Then the submodule of $\mathbb{Q}G$ spanned by the Γ_1-orbits (transitivity module) is an S-ring over G. Such an S-ring was called *schurian* in [7]. The general theory of S-rings was developed by H. Wielandt in [9] where in particular he constructed an S-ring that cannot be obtained by the Schur method.

Definition (R. Pöschel)**.** A finite group G is called Schur if any S-ring over G is schurian.

The Wielandt example shows that not every finite group is Schur. More exactly, he proved that the group $\mathbb{Z}_p \times \mathbb{Z}_p$ is not Schur for any prime $p \geq 5$. This fact was used by R. Pöschel in [7] to prove the following theorem.

Theorem. *Any section of a Schur group is a Schur group. Moreover, for a prime $p \geq 5$ a p-group is Schur if and only if it is cyclic.*

Since any finite nilpotent group is a direct product of its Sylow subgroups, we immediately obtain the following result.

Corollary. *A nilpotent group of order coprime to* 6 *is Schur only if it is cyclic.*

The above results show the importance of the cyclic case for the characterization of Schur groups. It should be noted that by the Pöschel theorem any cyclic p-group is Schur for $p \geq 5$. In fact, the schurity of cyclic 3-groups was also proved in [7], whereas the same result for $p = 2$ was obtained in [5]. However, till 2001 no cyclic non-Schur group had been known, and moreover it was conjectured that all cyclic groups are Schur (the

2010 *Mathematics Subject Classification.* Primary 05E30, 20B25.

Key words and phrases. Schur ring, Schur group, permutation group, circulant cyclotomic S-ring, generalized wreath product.

This work was partially supported by the Slovenian–Russian bilateral project, grants nos. BI-RU/10-11-018 and BI-RU/12-13-035. The third author was also supported by the RFFI grant no. 11-01-00760-a.

Schur–Klin conjecture). This conjecture had also been supported by the fact that the group $\mathbb{Z}_n$, where n is a product of two distinct primes, is a Schur group [4]. The first counterexamples to the conjecture were constructed in [1]; in all those examples n was the product of at least four primes. Later in [3] the schurity of $\mathbb{Z}_n$ was proved when n is the product of at most three primes or $n = p^3q$ where p and q are distinct primes. The main result of this paper completes the characterization of cyclic Schur groups.

Theorem 1.1. *A cyclic group of order n is Schur if and only if n belongs to one of the following five* (*partially overlapped*) *families of integers*:

$$p^k,\ pq^k,\ 2pq^k,\ pqr,\ 2pqr \tag{1}$$

where p, q, r are distinct primes, and $k \geq 0$ is an integer.

Corollary 1.2. *The minimum order of a cyclic non-Schur group equals* 72*.*

Let us briefly outline the proof of Theorem 1.1. To prove the "only if" part, for each integer n satisfying the hypothesis of Theorem 2.1 we construct explicitly a nonschurian S-ring over a group $\mathbb{Z}_n$. This ring is the generalized wreath product of two smaller schurian S-rings each of which is in its turn the generalized wreath product of normal S-rings; the way is essentially the same as that used in [1]. It turns out (Lemma 2.2) that the complement to the set of all these n coincides with the set of all numbers listed in (1).

To prove the "if" part, we must verify that any S-ring over a cyclic group of order n belonging to one of the families (1) is schurian. We observe that any divisor of such n also belongs to at least one of these families.

Definition 1.3. A nonschurian S-ring $\mathcal{A}$ over a group G is said to be *minimal* if the S-ring $\mathcal{A}_S$ is schurian for any $\mathcal{A}$-section $S \neq G/1$.

It is easily seen that any nonschurian S-ring contains a section the restriction to which is minimal nonschurian. Thus, the "if" part in Theorem 1.1 immediately follows from the theorem below.

Theorem 1.4. *The order of the underlying group of a minimal nonschurian circulant S-ring cannot belong to any of the families* (1)*.*

There are two key observations to prove Theorem 1.4 that are based on the results of [3]. The first is that any nonschurian circulant S-ring is a fusion of a quasidense[1] nonschurian circulant S-ring (Theorem 3.4). The second is that such an S-ring is a proper generalized wreath product of two smaller schurian quasidense S-rings. Moreover, in the minimal case each of them is in its turn a proper generalized wreath product (Theorem 4.3). We use these observations in §4 and §5 to exclude the first, second, and fourth families, and the case of $p = 2$ in the other two families. The proof is completed in §6 by applying the criterion of schurity for S-rings of special form that is proved in §8. It should be mentioned that throughout the proof of Theorem 1.4 we use several auxiliary results on circulant S-rings that are collected in §7.

In this paper we follow the notation and terminology of [3]. When referring to that paper, we keep only the number of the statement, preceding it by the letter A (e.g., instead of [3, Theorem 4.1] we write Theorem A4.1). Some additional notation is listed below.

We write $\mathcal{A} \cong \mathcal{A}'$ when S-rings $\mathcal{A}$ and $\mathcal{A}'$ are Cayley isomorphic.

For an S-ring $\mathcal{A}$ and an $\mathcal{A}$-section S, we set $\mathrm{Hol}_{\mathcal{A}}(S) = \mathrm{Hol}(S) \cap \mathrm{Aut}(\mathcal{A}_S)$.

[1]Quasidense circulant S-rings are introduced and studied in §3.

For an S-ring $\mathcal{A}$ over a group G, we set

$$\mathcal{M}(\mathcal{A}) = \{\Gamma \leq \operatorname{Aut}(\mathcal{A}) : \Gamma \underset{2}{\approx} \operatorname{Aut}(\mathcal{A}) \text{ and } G_{\text{right}} \leq \Gamma\}.$$

For permutations $f_1 \in \operatorname{Sym}(V_1)$ and $f_2 \in \operatorname{Sym}(V_2)$ the induced permutation on $V_1 \times V_2$ is denoted by $f_1 \otimes f_2$.

For permutation groups $\Gamma_1 \leq \operatorname{Sym}(V_1)$ and $\Gamma_2 \leq \operatorname{Sym}(V_2)$ we set

$$\Gamma_1 \otimes \Gamma_2 = \{f_1 \otimes f_2 : f_1 \in \Gamma_1,\ f_2 \in \Gamma_2\}.$$

§2. The "only if" part in Theorem 1.1

Here we prove the "only if" part of Theorem 1.1. Throughout this section, $\mathbb{Z}_n$ is the additive group of integers modulo a positive integer n.

For any divisor m of n, denote by $i_{m,n} : \mathbb{Z}_m \to \mathbb{Z}_n$ and $\pi_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ the group homomorphisms taking 1 to n/m and to 1 respectively. Using them, we identify the groups $i_{m,n}(\mathbb{Z}_m)$ and $\mathbb{Z}_n/\ker(\pi_{n,m})$ with $\mathbb{Z}_m$. Thus, every section of $\mathbb{Z}_n$ of order m is identified with the group $\mathbb{Z}_m$. Moreover, the permutation $f \in \operatorname{Aut}(\mathbb{Z}_n)$ afforded by multiplication by an integer induces the permutation $f^m \in \operatorname{Aut}(\mathbb{Z}_m)$ afforded by multiplication by the same integer.

If $\mathcal{A}$ is an S-ring over $G = \mathbb{Z}_n$ and H is the $\mathcal{A}$-group of order m, then $\mathcal{A}_H$ and $\mathcal{A}_{G/H}$ are denoted respectively by $\mathcal{A}_m$ and $\mathcal{A}^{n/m}$. Let finally $\mathcal{A}_i$ be an S-ring over $\mathbb{Z}_{n_i}$ $(i = 1, 2)$, and let $(\mathcal{A}_1)^m = (\mathcal{A}_2)_m$ for some m dividing both n_1 and n_2. Then the (unique) S-ring $\mathcal{A}$ over $\mathbb{Z}_{n_1 n_2/m}$ from Theorem A3.4 is denoted by $\mathcal{A}_1 \wr_m \mathcal{A}_2$. We omit m if $m = 1$.

Below, given a positive integer m, we set

$$\Omega^*(m) = \begin{cases} \Omega(m) & \text{if } m \text{ is odd,} \\ \Omega(m/2) & \text{if } m \text{ is even,} \end{cases}$$

where $\Omega(m)$ is the total number of prime factors of m. We note that $\Omega^*(m) \leq 1$ if and only if m is a divisor of twice a prime number.

Theorem 2.1. *Let $n = n_1 n_2$, where n_1 and n_2 are coprime positive integers such that $\Omega^*(n_i) \geq 2$, $i = 1, 2$. Then the cyclic group of order n is not Schur.*

Proof. Below, for an integer $m \geq 3$ we denote by K_m the subgroup of order 2 in the group $\operatorname{Aut}(\mathbb{Z}_m)$ that is generated by multiplication by -1. Suppose first that $n_1 = ab$ and $n_2 = cd$ where $a, b, c, d \geq 3$ are integers. Set

$$\mathcal{A}_1 = \operatorname{Cyc}(K_a \times K_c, \mathbb{Z}_{ac}), \qquad \mathcal{A}_2 = \operatorname{Cyc}(K_{bc}, \mathbb{Z}_{bc}), \tag{2}$$

$$\mathcal{A}_3 = \operatorname{Cyc}(K_{ad}, \mathbb{Z}_{ad}), \qquad \mathcal{A}_4 = \operatorname{Cyc}(K_{bd}, \mathbb{Z}_{bd}). \tag{3}$$

It is easily seen that the group $\operatorname{Aut}(\mathcal{A}_i)$ is dihedral for $i = 2, 3, 4$, and is the direct product of two dihedral groups for $i = 1$. Therefore, the S-ring $\mathcal{A}_i$ is normal for all i. Furthermore, $(\mathcal{A}_1)^c = \operatorname{Cyc}(K_c, \mathbb{Z}_c) = (\mathcal{A}_2)_c$ and $(\mathcal{A}_3)^d = \operatorname{Cyc}(K_d, \mathbb{Z}_d) = (\mathcal{A}_4)_d$. Thus, we can form the S-rings

$$\mathcal{A}_{1,2} = \mathcal{A}_1 \wr_c \mathcal{A}_2 \text{ and } \mathcal{A}_{3,4} = \mathcal{A}_3 \wr_d \mathcal{A}_4.$$

It is easily seen that $(\mathcal{A}_{1,2})^{n_1} = \operatorname{Cyc}(K_a, \mathbb{Z}_a) \wr \operatorname{Cyc}(K_b, \mathbb{Z}_b) = (\mathcal{A}_{3,4})_{n_1}$. Then

$$\mathcal{A} := \mathcal{A}_{1,2} \wr_{n_1} \mathcal{A}_{3,4}$$

is an S-ring over $\mathbb{Z}_n$. Thus, it suffices to verify that $\mathcal{A}$ is not schurian.

Suppose on the contrary that $\mathcal{A}$ is schurian. Then by Theorem A1.2 the S-rings $\mathcal{A}_{1,2}$ and $\mathcal{A}_{3,4}$ are schurian, and there exist groups $\Delta_{1,2} \in \mathcal{M}(\mathcal{A}_{1,2})$ and $\Delta_{3,4} \in \mathcal{M}(\mathcal{A}_{3,4})$ such that

$$(\Delta_{1,2})^S = (\Delta_{3,4})^S,$$

where S is the section of order n_1 used in the definition of the S-ring $\mathcal{A}$. In particular, for any permutation $f_1 \in \Delta_{1,2}$ fixing 0 there exists a permutation $f_2 \in \Delta_{3,4}$ fixing 0 and such that $f_1^S = f_2^S$. We claim: *the permutation $(f_1)^H$, where H is the group of order ac, is induced by multiplication by $\varepsilon \in \{1, -1\}$.* However, if this is true, then the stabilizer of 0 in the group $(\Delta_{1,2})^H$ is contained in K_{ac}. Therefore, the basic set of the S-ring associated with the former group that contains 1 is of cardinality at most 2. On the other hand, this S-ring coincides with $\mathcal{A}_1$ by the schurity of the S-ring $\mathcal{A}_{1,2}$ and the 2-equivalence of the groups $\Delta_{1,2}$ and $\mathrm{Aut}(\mathcal{A}_{1,2})$. So the above basic set has cardinality 4, a contradiction.

To prove the claim, let $f_{1,1}$ and $f_{1,2}$ be the automorphisms of the S-rings $\mathcal{A}_1$ and $\mathcal{A}_2$ induced by f_1, and let $f_{2,3}$ and $f_{2,4}$ be the automorphisms of the S-rings $\mathcal{A}_3$ and $\mathcal{A}_4$ induced by f_2. Then the normality of these S-rings implies that $f_{1,1} \in K_a \times K_c$, $f_{1,2} \in K_{bc}$, and that $f_{2,3} \in K_{ad}$ and $f_{2,4} \in K_{bd}$. Clearly,

$$(f_{1,1})^c = (f_{1,2})^c \quad \text{and} \quad (f_{2,3})^d = (f_{2,4})^d \tag{4}$$

and, since $(f_1)^S = (f_2)^S$, also

$$(f_{1,1})^a = (f_{2,3})^a \quad \text{and} \quad (f_{1,2})^b = (f_{2,4})^b. \tag{5}$$

Next, the permutations $f_{1,2}$, $f_{2,3}$ and $f_{2,4}$ are induced respectively by multiplications by some integers $\varepsilon_{1,2}, \varepsilon_{2,3}, \varepsilon_{2,4} \in \{1, -1\}$. Therefore, by the second identities in (4) and (5), we have

$$\varepsilon_{1,2} = \varepsilon_{2,3} = \varepsilon_{2,4}.$$

Denote this number by ε. Then by the first identities in (4) and (5), the permutations $(f_{1,1})^a$ and $(f_{1,1})^c$, and hence the permutation $(f_1)^H$, are induced by multiplication by ε.

To complete the proof, we observe that the theorem is proved in all cases except for the case where one of the numbers n_1, n_2, say n_1, is equal to 8. Then obviously $n_1 = ab/2$ and $n_2 = cd$ where $a = b = 4$ and $c, d \geq 3$ are odd integers. We define S-rings $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ by formulas (2) and (3). Then again all these rings are normal,

$$(\mathcal{A}_1)^{2c} = \mathrm{Cyc}(K_{2c}, \mathbb{Z}_{2c}) = (\mathcal{A}_2)_{2c}, \quad (\mathcal{A}_3)^{2d} = \mathrm{Cyc}(K_{2d}, \mathbb{Z}_{2d}) = (\mathcal{A}_4)_{2d},$$

and we can form the S-rings

$$\mathcal{A}_{1,2} = \mathcal{A}_1 \wr_{2c} \mathcal{A}_2 \quad \text{and} \quad \mathcal{A}_{3,4} = \mathcal{A}_3 \wr_{2d} \mathcal{A}_4.$$

It should be emphasized that $\mathcal{A}_{1,2}$ and $\mathcal{A}_{3,4}$ are S-rings over the groups $\mathbb{Z}_{cn_1}$ and $\mathbb{Z}_{dn_1}$. It is also easily seen that

$$(\mathcal{A}_{1,2})^{n_1} = \mathrm{Cyc}(K_a, \mathbb{Z}_a) \wr_2 \mathrm{Cyc}(K_b, \mathbb{Z}_b) = (\mathcal{A}_{3,4})_{n_1}.$$

Then

$$\mathcal{A} := \mathcal{A}_{1,2} \wr_{n_1} \mathcal{A}_{3,4}$$

is an S-ring over $\mathbb{Z}_n$. Thus, it suffices to verify that $\mathcal{A}$ is not schurian. The rest of the proof repeats the proof of the first part literally. □

To complete the proof of the "only if" part, we note that the required statement immediately follows from Theorem 2.1 and the lemma below.

Lemma 2.2. *An integer n belongs to none of the families listed in* (1) *if and only if $n = n_1 n_2$ for some coprime positive integers n_1 and n_2 such that $\Omega^*(n_i) \geq 2$, $i = 1, 2$.*

Proof. The "if" part is straightforward by exhaustive search. To prove the "only if" part, let an integer $n = p_1^{k_1} \cdots p_s^{k_s}$ belong to none of the families (1), where $p_1, \ldots, p_s$ are pairwise distinct primes. Then without loss of generality we may assume that

$$2 \leq s \leq 4 \quad \text{and} \quad k_1 \geq k_2 \geq \cdots \geq k_s.$$

Suppose, to the contrary, that n cannot be decomposed into the product of coprime positive integers n_1 and n_2 such that $\Omega^*(n_i) \ge 2$, $i = 1, 2$. Then $k_2 = 1$, for otherwise $s = 2$ and n belongs to the third family with $p = 2$, which is impossible. Thus, $k_2 = \cdots = k_s = 1$. Therefore, $s = 3$ or $s = 4$, for otherwise $s = 2$ and n belongs to the second family. Let $s = 3$. Then $k_1 \ne 1$ because n does not belong to the fourth family. So $k_1 \ge 2$, and hence $2 \in \{p_2, p_3\}$ or $p_1^{k_1} = 4$ by the assumption. However, then n belongs to the third family, a contradiction. Finally, let $s = 4$. Then the assumption implies that $k_1 = 1$ and one of the p_i's equals 2. But then n belongs to the fifth family, a contradiction. □

§3. Quasidense S-rings

A circulant S-ring $\mathcal{A}$ is said to be *quasidense* if any primitive $\mathcal{A}$-section is of prime order. Any dense S-ring is obviously quasidense. It is also clear that the property to be quasidense is preserved by taking the restriction to any $\mathcal{A}$-section. Moreover, in the quasidense case any minimal $\mathcal{A}$-group is of prime order, any maximal $\mathcal{A}$-group is of prime index, and the S-ring $\mathcal{A}_S$ is dense for any $\mathcal{A}$-section S of prime power order.

Theorem 3.1. *Any quasidense circulant S-ring with trivial radical is cyclotomic, and hence dense.*

Proof. Let $\mathcal{A}$ be a quasidense circulant S-ring with trivial radical. Then $\mathcal{A}$ is the tensor product of a normal S-ring with trivial radical and S-rings of rank 2 by Theorem A4.1. However, any normal circulant S-ring is cyclotomic by Theorem A4.2. Furthermore, by quasidensity, the underlying group of any factor of rank 2 is of prime order. Therefore, such a factor is also cyclotomic. Thus, $\mathcal{A}$ is cyclotomic as the tensor product of cyclotomic S-rings. □

The following two statements will be used in proving the "if" part of Theorem 1.1 to find nontrivial $\mathcal{A}$-groups.

Corollary 3.2. *Let $\mathcal{A}$ be a quasidense S-ring over a cyclic group G. Then any subgroup of G that contains* $\mathrm{rad}(\mathcal{A})$ *is an $\mathcal{A}$-group. In particular, if* $\mathrm{rad}(\mathcal{A})_p = 1$ *for a prime divisor p of $|G|$, then $G_{p'}$ is an $\mathcal{A}$-group.*

Proof. The S-ring $\mathcal{A}_{G/L}$ where $L = \mathrm{rad}(\mathcal{A})$, has trivial radical. Therefore, by Theorem 3.1 it is dense. Thus, the required statement follows from the fact that a group H containing L is an $\mathcal{A}$-group if and only if the group H/L is an $\mathcal{A}_{G/L}$-group. □

Corollary 3.3. *Let $\mathcal{A}$ be a quasidense S-ring over a cyclic group G. Suppose that $\mathcal{A}$ is not the U/L-wreath product where U/L is an $\mathcal{A}$-section such that the number $p := |L|$ is prime. Then*:

1) *there exists $H \in \mathcal{G}(\mathcal{A})$ such that $H \not\le U$ and $H_{p'} \in \mathcal{G}(\mathcal{A})$;*
2) *if $q := |G/U|$ is a prime other than p, then $H_{p'} \ge G_q$ for any group H in statement* (1)*.*

Proof. To prove statement 1, we observe that by the hypothesis there exists $X \in \mathcal{S}(\mathcal{A})$ outside U such that $\mathrm{rad}(X)_p = 1$. Then $H = \langle X \rangle$ is an $\mathcal{A}$-group. Therefore, the required statement follows from Corollary 3.2 applied to the S-ring $\mathcal{A}_H$. Next, the condition of statement 2 implies that any group $H \not\le U$ contains a generator of G_q. Therefore, $H \ge G_q$, which proves this statement. □

The following theorem reduces the schurity problem for circulant S-rings to the quasidense case. The proof is based on the extension construction introduced and studied in [3].

Theorem 3.4. *Given a circulant S-ring $\mathcal{A}$, there exists a quasidense S-ring $\mathcal{A}' \geq \mathcal{A}$ such that $\mathcal{A}$ and $\mathcal{A}'$ are schurian or not simultaneously.*

Proof. We define an S-ring $\mathcal{A}'$ recursively as follows. If $\mathcal{A}$ has no singular class of composite order, then we set $\mathcal{A}' = \mathcal{A}$; otherwise we set

$$\mathcal{A}' = (\mathrm{Ext}_C(\mathcal{A}, \mathbb{Z}S))'$$

where C is a singular class of composite order and $S = S_{\min}(C)$. Then the S-ring $\mathcal{A}'$ has no singular classes of composite order. Moreover, from Theorem A6.7 it follows that $\mathcal{A}$ and $\mathcal{A}'$ are schurian or not simultaneously. To complete the proof, we verify that the S-ring $\mathcal{A}'$ is quasidense. Suppose, to the contrary, that this is not true. Then there exists a primitive $\mathcal{A}'$-section S of composite order. Then by Theorem A4.6 the class of projectively equivalent $\mathcal{A}'$-sections that contains S is singular, a contradiction. □

In general, the automorphism group of a quasidense S-ring is not solvable. However, from Theorem A8.1 it follows that in the schurian case such an S-ring can always be obtained from an appropriate solvable permutation group in a standard way. The following theorem shows that "locally" this group has a rather simple form.

Theorem 3.5. *Let $\mathcal{A}$ be a schurian quasidense circulant S-ring. Then there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that $\Gamma^S = \mathrm{Hol}_{\mathcal{A}}(S)$ for any $\mathcal{A}$-section S with* $\mathrm{rad}(\mathcal{A}_S) = 1$.

Remark 3.6. In fact, we prove that equality in the theorem statement occurs for any S such that $\mathcal{A}_S$ is the tensor product of a normal S-ring and S-rings of rank 2.

Proof. The quasidensity of $\mathcal{A}$ implies that each primitive $\mathcal{A}$-section is of prime order. Therefore by Theorems A4.6 and A8.1 there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that $\Gamma^T \leq \mathrm{Hol}(T)$ for all primitive $\mathcal{A}$-sections T. Let S be an $\mathcal{A}$-section with $\mathrm{rad}(\mathcal{A}_S) = 1$. Then by Theorem A4.1 the S-ring $\mathcal{A}_S$ is the tensor product of a normal S-ring, say $\mathcal{A}_{T_0}$, and S-rings of rank 2, say $\mathcal{A}_{T_1}, \ldots, \mathcal{A}_{T_k}$ where all T_i's are $\mathcal{A}_S$-groups. It follows that

$$\Gamma^{T_0} \leq \mathrm{Aut}(\mathcal{A}_{T_0}) \leq \mathrm{Hol}(T_0).$$

Moreover, by the above, $\Gamma^{T_i} \leq \mathrm{Hol}(T_i)$ for all $i > 0$, because the sections $T_1, \ldots, T_k$ are primitive. Thus

$$\Gamma^S \leq \prod_{i=0}^{k} \Gamma^{T_i} \leq \prod_{i=0}^{k} \mathrm{Hol}(T_i) = \mathrm{Hol}(S).$$

But then Γ^S is obviously a unique subgroup of $\mathrm{Hol}(S)$ in the set $\mathcal{M}(\mathcal{A}_S)$. Thus $\Gamma^S = \mathrm{Hol}_{\mathcal{A}}(S)$. □

§4. Excluding the families 1, 2, and 4

In the end of this section we prove the following theorem showing that any minimal nonschurian quasidense S-ring $\mathcal{A}$ over a cyclic group G contains two distinct minimal $\mathcal{A}$-groups and two distinct maximal $\mathcal{A}$-groups, the relationship between which is as in Figure 1.

Theorem 4.1. *Let $\mathcal{A}$ be a minimal nonschurian quasidense S-ring over a cyclic group G of order n belonging to one of five families* (1). *Then:*

1) *n belongs to the third or fifth families*;
2) *there exist distinct $\mathcal{A}$-groups K, L of prime orders and distinct $\mathcal{A}$-groups V, U of prime indices such that $LK \leq U \cap V$ and $\mathcal{A}$ is a proper U/L-wreath product.*

We begin with studying minimal nonschurian circulant S-rings. In the following statement we establish general properties of them.

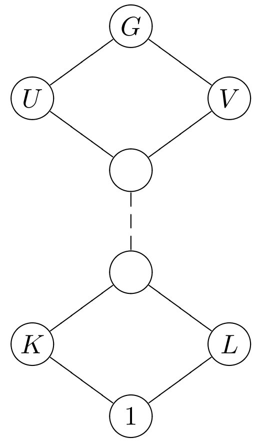

FIGURE 1

Lemma 4.2. *Let $\mathcal{A}$ be a minimal nonschurian S-ring over a cyclic group G. Then*

1) *$\mathcal{A}$ is a proper generalized wreath product*;
2) *if $\mathcal{A}$ is a proper U/L-wreath product, then* $\mathrm{rad}(\mathcal{A}_{U/L}) \neq 1$; *moreover, if $\mathcal{A}_{U/L}$ is cyclotomic, then* $|\,\mathrm{rad}(\mathcal{A}_{U/L})| > 2$.

Proof. By Corollary A4.3 we can assume that $\mathrm{rad}(\mathcal{A}) \neq 1$. So statement 1 follows from Theorem A4.1. The first part of statement 2 follows from the minimality of $\mathcal{A}$, Theorem A1.3, and Corollary A1.4. Similarly, to prove the second part of this statement it suffices to verify that the S-ring $\mathcal{A}_{U/L}$ is the tensor product of a normal S-ring and S-rings of rank 2 whenever it is cyclotomic and its radical is of order 2. However, under this condition the criterion of normality [2, Theorem 6.1] implies that $\mathcal{A}_{U/L}$ is not normal only if it is the tensor product one factor of which is an S-ring of rank 2 (over a cyclic group of prime order). Thus, the required statement follows by induction. □

By statement 1 of Lemma 4.2, any minimal nonschurian circulant S-ring is a proper generalized wreath product. The following important theorem shows that if n belongs to one of the families (1), then in the quasidense case both operands are also proper generalized wreath products.

Theorem 4.3. *Let $\mathcal{A}$ be a minimal nonschurian quasidense S-ring over a cyclic group G of order n belonging to one of families* (1). *Then* $\mathrm{rad}(\mathcal{A}_U) \neq 1$ *and* $\mathrm{rad}(\mathcal{A}_{G/L}) \neq 1$ *whenever $\mathcal{A}$ is a proper U/L-wreath product.*

Proof. Let $\mathcal{A}$ be a proper U/L-wreath product. Suppose, to the contrary, that $\mathrm{rad}(\mathcal{A}_T) = 1$ where $T \in \{U, G/L\}$. Then the quasidensity of $\mathcal{A}$ implies by Theorem 3.1 that the S-ring $\mathcal{A}_T$, and hence the S-ring $\mathcal{A}_S$ with $S = U/L$, is cyclotomic. By the minimality of $\mathcal{A}$ and Lemma 4.2 this implies that $|\,\mathrm{rad}(A_S)| > 2$. Thus, n does not belong to the fourth and the fifth families, because otherwise $|S|$ is either prime, or 4, or the product of two distinct primes. Moreover, from Theorem 7.3 for $G = T$ it follows that $S_l = 1$ for some odd prime divisor l of $|T|$. Thus n does not belong to the first family. In the remaining two cases the prime l coincides with p, because otherwise $l = q$, and hence $|S|$ divides $2p$, which is impossible by the above. This proves the following lemma.

Lemma 4.4. *Under the above assumptions we have $n = pq^k$ or $n = 2pq^k$, and $p \neq 2$. Moreover,*

1) *if* $\mathrm{rad}(\mathcal{A}_U) = 1$, *then* $L_p \neq 1$;

2) *if* $\mathrm{rad}(\mathcal{A}_{G/L}) = 1$*, then* $(G/U)_p \neq 1$.

Let $\mathrm{rad}(\mathcal{A}_U) = 1$. Assume that either $q = 2$, or $G_{2'}$ is not an $\mathcal{A}$-group. By Lemma 4.4 the number $|G/L|$ is a power of q or twice a power of q. So, by assumption, there is a unique maximal $\mathcal{A}_{G/L}$-group, say U'/L. Therefore $U' \geq U$, and hence $\mathcal{A}$ is a U'/L-wreath product. Denote by L' a maximal possible $\mathcal{A}$-group containing L for which $\mathcal{A}$ is the U'/L'-wreath product. Then the uniqueness of U' implies that $\mathrm{rad}(\mathcal{A}_{G/L'}) = 1$. (Indeed, otherwise by Theorem A4.1 the S-ring $\mathcal{A}$ is a U'/L''-wreath product for some $L'' > L'$, which contradicts the maximality of L'). Then by statement 2 of Lemma 4.4 we conclude that $(G/U')_p \neq 1$. Taking into account that $L'_p \geq L_p \neq 1$, we conclude that p^2 divides n which is impossible by Lemma 4.4. This proves the following lemma providing $\mathrm{rad}(\mathcal{A}_U) = 1$ (when $\mathrm{rad}(\mathcal{A}_{G/L}) = 1$ the proof is similar).

Lemma 4.5. *We have* $n = 2pq^k$ *and* $q \neq 2$*. Moreover,*

1) *if* $\mathrm{rad}(\mathcal{A}_U) = 1$*, then* $G_{2'}$ *is an* $\mathcal{A}$*-group*;
2) *if* $\mathrm{rad}(\mathcal{A}_{G/L}) = 1$*, then* G_2 *is an* $\mathcal{A}$*-group.*

To complete the proof of Theorem 4.3, we arrive at a contradiction under the assumption $T = U$ (the remaining case $T = G/L$ can be proved in a similar way). In this case we observe that $U' := G_{2'}$ is an $\mathcal{A}$-group by Lemma 4.5. We claim that

$$\mathcal{A} = \mathcal{A}_{U'} \wr_{U'/L} \mathcal{A}_{G/L}. \tag{6}$$

Indeed, since $\mathcal{A}$ is a U/L-wreath product, it suffices to verify that $U' \geq U$. Suppose, to the contrary, that this is not true. Then the number $|U|$ must be even. This implies that G_2 is an $\mathcal{A}$-group (we have used the fact that the S-ring $\mathcal{A}_U$ is cyclotomic, and hence dense). Since $p \neq 2$ and $q \neq 2$, this shows that G_2 is the $\mathcal{A}$-complement of U'. Therefore by Corollary 7.2 we conclude that $\mathcal{A} = \mathcal{A}_{U'} \otimes \mathcal{A}_{G_2}$. By the minimality of $\mathcal{A}$, this implies that the S-ring $\mathcal{A}$ is schurian. The obtained contradiction proves (6).

After increasing the group L in (6) (if necessary), we can assume that it is a maximal possible $\mathcal{A}$-group with that property. Then

$$(G/L)_p = 1 \quad \text{and} \quad (G/L)_2 \notin \mathcal{G}(\mathcal{A}_{G/L}). \tag{7}$$

The first relation follows from Lemma 4.4. To prove the second one, suppose, to the contrary, that the group $(G/L)_2$ is the $\mathcal{A}_{G/L}$-complement of U'/L. Therefore, by Corollary 7.2 we conclude that $\mathcal{A}_{G/L} = \mathcal{A}_{U'/L} \otimes \mathcal{A}_{(G/L)_2}$. By the minimality of $\mathcal{A}$ and Theorem 7.5, this implies that the S-ring $\mathcal{A}$ is schurian. The contradiction obtained proves (7).

Due to the quasidensity of $\mathcal{A}$, formula (7) implies that there is only one minimal $\mathcal{A}_{G/L}$-group, say L'/L, and $|L'/L| = q$. We claim that

$$\mathcal{A}_{G/L} = \mathcal{A}_{U'/L} \wr_{U'/L'} \mathcal{A}_{G/L'}. \tag{8}$$

Indeed, otherwise by Corollary 3.3 there exists an $\mathcal{A}_{G/L}$-group H/L such that $H/L \not\leq U'/L$ and $(H/L)_{q'}$ is an $\mathcal{A}_{G/L}$-group. However, it is easily seen that in our case $(H/L)_{q'} = (G/L)_2$, which contradicts the second relation in (7). The obtained contradiction proves (8).

Formulas (6) and (8) show that the S-ring $\mathcal{A}$ is a U'/L'-wreath product. However, this is impossible by the maximality of L. □

Proof of Theorem 4.1. Statement 1 immediately follows from statement 2. To prove the latter we observe that by Lemma 4.2 the S-ring $\mathcal{A}$ is a proper U/L-wreath product for some $\mathcal{A}$-groups U and L. By the quasidensity of $\mathcal{A}$, we may assume that L is of prime order and U is of prime index. Denote by $\widetilde{U}$ a minimal subgroup of U such that the S-ring $\mathcal{A}$ is a $\widetilde{U}/L$-wreath product. Then by Theorem 4.3 the S-ring $\mathcal{A}_{\widetilde{U}}$ is a proper

U'/K-wreath product for some $\mathcal{A}$-groups U' and K. Again we may assume that K is of prime order. By the minimality of $\widetilde{U}$, we conclude that $K \neq L$. Next, $KL \le U$ because $L \le U$ and $K \le U' < \widetilde{U} \le U$.

Similarly, denote by $\widetilde{L}$ a maximal subgroup of U that contains L and such that the S-ring $\mathcal{A}$ is a $U/\widetilde{L}$-wreath product. Then again by Theorem 4.3 the S-ring $\mathcal{A}_{G/\widetilde{L}}$ is a proper V/L'-wreath product for some $\mathcal{A}$-groups V and L' such that V is of prime index in G. By the maximality of $\widetilde{L}$ we conclude that $V \neq U$. Moreover, obviously $V \ge L$. To complete the proof it suffices to note that $K \le V$. Indeed, if this is not true, then any nontrivial basic set of $\mathcal{A}_{G/\widetilde{L}}$ inside $K\widetilde{L}/\widetilde{L}$, is outside $V/\widetilde{L}$ and has trivial radical (because $|K\widetilde{L}/\widetilde{L}| = |K|$ is prime), which is impossible. □

§5. Excluding the families 3 and 5 for $p \neq 2$

In the end of this section we prove the following theorem that will enable us to exclude the cases in the title.

Theorem 5.1. *Let $\mathcal{A}$ be a minimal nonschurian quasidense S-ring over a cyclic group G of order n belonging to the third or fifth of the families* (1)*. Then:*

1) $p = 2$;
2) *$\mathcal{A}$ is both a U/L- and a V/K-wreath product where K, L, U, V are $\mathcal{A}$-groups defined by*
   (a) $|K| = 2$, $|L| = q$, $|U| = 2qr$, $|V| = 4q$ *for* $n = 4qr$,
   (b) $|K| = 2$, $|L| = q$, $|U| = 2q^k$, $|V| = 4q^{k-1}$ *for* $n = 4q^k$,
   with q and r distinct odd primes and $k \ge 2$.

Throughout the rest of the section, $\mathcal{A}$ denotes an S-ring satisfying the hypothesis of Theorem 5.1. It is also assumed that we are given $\mathcal{A}$-groups K, L, U, V for which statement 2 of Theorem 4.1 holds true.

Theorem 5.2. *The number $|U/L|$ is even.*

Proof. Suppose, to the contrary, that $|U/L|$ is odd. Then either $|L| = 2$ or $|G/U| = 2$. We consider the former case, the latter one can be proved similarly. We need to find an $\mathcal{A}$-group U' such that $L \le U' \le U$ and

$$\mathcal{A} = \mathcal{A}_{U'} \wr_{U'/L} \mathcal{A}_{G/L} \quad \text{and} \quad (U')_{2'} \in \mathcal{G}(\mathcal{A}). \tag{9}$$

Indeed, if such a group does exist, then by Corollary 7.2 with $G = U'$, $H = (U')_{2'}$, $S = L/1$, and $T = U'/L$, we obtain $\mathcal{A}_U = \mathcal{A}_L \otimes \mathcal{A}_{U'/L}$. By the minimality of $\mathcal{A}$ and Theorem 7.5 with $U = U'$, this implies that the S-ring $\mathcal{A}$ is schurian, which is not true.

In the case where $n = 2pqr$, set $U' = U$. Then the left-hand side relation in (9) is obvious. To prove the other one, we observe that by Theorem A11.3 the number $|U|$ is the product of three primes, the S-ring $\mathcal{A}_U$ is not a proper wreath product and $\mathcal{A}_{U/L}$ is a proper wreath product. So, since the number $|U/L|$ is odd, the hypothesis of Lemma A11.2 is satisfied for $\mathcal{A} = \mathcal{A}_U$, $S = U/L$ and $r = 2$. By that lemma, we obtain $U_{2'} \in \mathcal{G}(\mathcal{A}_U)$, and we are done.

In the case of $n = 2pq^k$, set U' to be a minimal $\mathcal{A}$-subgroup of U for which the first relation in (9) holds true. We may assume that

$$(U')_p \neq 1. \tag{10}$$

Indeed, otherwise $|U'| = 2q^i$ for some $i > 0$ ($|U'| = 2$ cannot happen, for otherwise $\mathcal{A}$ would be a proper wreath product, contradicting that it is a minimal nonschurian S-ring). Moreover, the minimality of U' implies that the S-ring $\mathcal{A}_{U'}$ is not a U''/L-wreath product, where U'' is the subgroup of U' of index q. Then, by Corollary 3.3 with $p = 2$, there exists an $\mathcal{A}_{U'}$-group H_1 such that $(U')_q \le H_1 \le (U')_{2'}$. Hence, $H_1 = (U')_q = (U')_{2'}$, and the second relation in (9) follows.

From (10) it follows that $|G/V| = p$, and since $|U/L|$ is odd, p is also odd. Therefore $U' \cap V \ne U'$. By the minimality of U', this implies that the S-ring $\mathcal{A}_{U'}$ is not a $(U' \cap V)/L$-wreath product. So by Corollary 3.3 with $(p,q) = (2,p)$, there exists an $\mathcal{A}$-group H_1 such that

$$(U')_p \le H_1 \le (U')_{2'}. \tag{11}$$

Denote by H a maximal $\mathcal{A}$-subgroup of U' that contains H_1. Then due to (11) we have $|U'/H| \in \{2, q\}$. If $|U'/H| = 2$, then the second relation in (9) holds true and we are done. Finally, if $|U'/H| = q$, then by the minimality of U' the S-ring $\mathcal{A}_{U'}$ is not an H/L-wreath product. By Corollary 3.3 with $(p,q) = (2,q)$, there exists an $\mathcal{A}$-group H_2 such that

$$(U')_q \le H_2 \le (U')_{2'}. \tag{12}$$

Thus, by (11) and (12) we have $(U')_{2'} = H_1 H_2$, and hence $(U')_{2'} \in \mathcal{G}(\mathcal{A})$. □

Theorem 5.3. *The order of G is divisible by* 4.

Proof. Suppose first that $n = 2pqr$. Then by statement 1 of Theorem A11.4 (where the lattice of $\mathcal{A}$-groups is found) there are exactly two maximal $\mathcal{A}$-groups and exactly two minimal $\mathcal{A}$-groups; the former are of prime index whereas the latter are of prime order. From statement 2 of Theorem 4.1 it follows that these groups are U, V and K, L, respectively, and also

$$|U/L| \cdot |V/K| = |G|. \tag{13}$$

On the other hand, we claim that the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are nonnormal. Indeed, otherwise by statement 3 of Theorem A11.4 the number $|U/L|$ is a square of a prime. In our case this is possible only for $p = 2$. But in this case $|U/L| = 4$, which is impossible by the theorem quoted above. The claim is proved. So by statement 5 of the same theorem, the S-ring $\mathcal{A}$ is a V/K-wreath product. Then by Theorem 5.2 of this paper the number $|V/K|$ is even. Thus, 4 divides $|G|$.

Let $n = 2pq^k$. Suppose, to the contrary, that $p \ne 2$. Since q is odd and $|U/L|$ is even (Theorem 5.2), from statement 2 of Theorem 4.1 it follows that $|G/V| = 2$ or $|K| = 2$. We consider the former case, the latter one can be proved similarly. In this case $|V|$ is odd. Therefore, by Theorem 5.2 the S-ring $\mathcal{A}$ is neither a V/K- nor a V/L-wreath product. So by Corollary 3.3 with $(p,q) = (2,p)$ and $(p,q) = (2,q)$, there exist $\mathcal{A}$-groups H_1 and H_2 such that

$$G_2 \le H_1 \le G_{p'} \quad \text{and} \quad G_2 \le H_2 \le G_{q'}.$$

Thus $G_2 = H_1 \cap H_2$, and hence G_2 is an $\mathcal{A}$-group. By Corollary 7.2, this implies that $\mathcal{A} = \mathcal{A}_{G_2} \otimes \mathcal{A}_V$, which is impossible by the minimality of $\mathcal{A}$. □

Theorem 5.4. *Without loss of generality we may assume that* 4 *does not divide* $|U/L|$.

Proof. By Theorem 5.3 we have $p = 2$. So $n = 4qr$ or $n = 4q^k$. In the former case $|U/L|$ is divisible by 4 only if $|U/L| = 4$. However, in this case the S-ring $\mathcal{A}_{U/L}$ is cyclotomic and $|\operatorname{rad}(\mathcal{A}_{U/L})| \le 2$, which contradicts statement 2 of Lemma 4.2. Thus, we may assume that $n = 4q^k$ and 4 divides $|U/L|$. Then from statement 2 of Theorem 4.1 it follows that

$$|G/U| = |L| = q \quad \text{and} \quad |G/V| = |K| = 2.$$

Therefore, it suffices to verify that the S-ring $\mathcal{A}$ is either a U/K- or a V/L-wreath product. Suppose, to the contrary, that this is not true. Then, by Corollary 3.3 with $(p,q)=(2,q)$ and $(p,q)=(q,2)$, there exist $\mathcal{A}$-groups H_1 and H_2 such that

$$G_q \le H_1 \le G_{2'} \quad\text{and}\quad G_2 \le H_2 \le G_{q'}.$$

It follows that $H_1 = G_q$ and $H_2 = G_2$. Thus, G_q and G_2 are $\mathcal{A}$-groups. By the quasidensity of $\mathcal{A}$, this implies that $\mathcal{A}$ is dense.

Denote by $\widetilde{U}$ the minimal $\mathcal{A}$-subgroup of U for which the S-ring $\mathcal{A}$ is a $\widetilde{U}/L$-wreath product. Then by Theorem 4.3 with $U=\widetilde{U}$ the radical of the ring $\mathcal{A}_{\widetilde{U}}$ is nontrivial. Since this S-ring is dense, from [6, Theorem 3.4] (see also statement (1) of [2, Theorem 5.4]) it follows that it is a U'/L'-wreath product, where the number $|L'|=|\widetilde{U}/U'|$ is the greatest prime divisor of $|\operatorname{rad}(\mathcal{A}_{\widetilde{U}})|$. By the minimality of the group $\widetilde{U}$, we conclude that this prime divisor is equal to 2. Thus

$$L' = K = \operatorname{rad}(\mathcal{A}_{\widetilde{U}}) \quad\text{and}\quad |\widetilde{U}/U'| = 2. \tag{14}$$

By Corollary 7.2, we have $\mathcal{A}_{U'} = \mathcal{A}_K \otimes \mathcal{A}_{U'_q}$ and $\mathcal{A}_{\widetilde{U}/K} = \mathcal{A}_{\widetilde{U}_2/K} \otimes \mathcal{A}_{U'/K}$. Since $\mathcal{A}_{\widetilde{U}}$ is a U'/K-wreath product, this implies that

$$\mathcal{A}_{\widetilde{U}} = \mathcal{A}_{\widetilde{U}_2} \otimes \mathcal{A}_{\widetilde{U}_q}. \tag{15}$$

Therefore, $\mathcal{A}_{\widetilde{U}/L} \cong \mathcal{A}_{\widetilde{U}_2} \otimes \mathcal{A}_{\widetilde{U}_q/L}$. Moreover, the S-ring $\mathcal{A}_{\widetilde{U}_2}$ being a dense S-ring over a cyclic group of order 4, it is cyclotomic and $|\operatorname{rad}(\mathcal{A}_{\widetilde{U}_2})| \le 2$. Finally, by Corollary 7.4 the S-ring $\mathcal{A}_{\widetilde{U}_q/L}$ has trivial radical, because by (14) and (15) so is the S-ring $\mathcal{A}_{\widetilde{U}_q}$. Therefore, the last-mentioned S-ring is cyclotomic by Theorem 3.1. Thus, the S-ring $\mathcal{A}_{\widetilde{U}/L}$ is cyclotomic and $|\operatorname{rad}(\mathcal{A}_{\widetilde{U}/L})| \le 2$, which contradicts statement 2 of Lemma 4.2. □

Proof of Theorem 5.1. By Theorem 4.1, we may assume that the hypothesis under which Theorems 5.2, 5.3, and 5.4 were proved is true for the S-ring $\mathcal{A}$. Then statement (1) immediately follows from Theorem 5.3. Thus,

$$n = 4qr \quad\text{or}\quad n = 4q^k,$$

where q and r are distinct odd primes and $k \ge 2$. To prove statement 2, choose the groups K, L, U, V as above. Then obviously $|KL| = 2q$ when $n = 4q^k$. The same is also true for $n = 4qr$ after interchanging q and r (if necessary). By Theorems 5.2 and 5.4, we can also assume $|U/L| = 2 \pmod 4$.

Let $n = 4qr$. Then by the above assumptions the number $|U/L|$ is not a prime square. By statement 3 of Theorem A11.4, this implies that neither of the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ is normal. Therefore, by statement 5 of that theorem, the S-ring $\mathcal{A}$ is a V/K-wreath product. Moreover, $|V/K| = 2 \pmod 4$. Thus, without loss of generality we may assume that $|K| = 2$. Then $|L| = q$ and $|G/U| = 2$. It follows that $|U| = 2qr$ and $|V| = 4q$, which completes the proof in this case.

Let $n = 4q^k$. Then by the above assumptions one of the following occurs:

(a) $|K| = 2$, $|L| = q$, $|U| = 2q^k$, $|V| = 4q^{k-1}$,
(b) $|K| = q$, $|L| = 2$, $|U| = 4q^{k-1}$, $|V| = 2q^k$.

Thus, it suffices to verify that $\mathcal{A}$ is a V/K-wreath product. Suppose that this is not true.

In case (a), by Corollary 3.3 with $p = 2$, there exists an $\mathcal{A}$-group H_1 such that $G_q \le H_1 \le G_{2'}$. It follows that $H_1 = G_q$, whence G_q is an $\mathcal{A}$-group. By Corollary 7.2, this implies that

$$\mathcal{A}_U = \mathcal{A}_{G_q} \otimes \mathcal{A}_K.$$

Denote by H the maximal $\mathcal{A}$-group such that $L \le H \le G_q$ and $\mathrm{rad}(\mathcal{A}_H) = 1$. Set $U' = HK$. Then the radical of any basic set inside $U \setminus U'$ contains L. Since the same is true also for any basic set outside U, the S-ring $\mathcal{A}$ is a U'/L-wreath product. Next, $\mathrm{rad}(\mathcal{A}_{U'}) = 1$ because $\mathcal{A}_{U'} = \mathcal{A}_H \otimes \mathcal{A}_K$ and $|K| = 2$. Then $\mathrm{rad}(\mathcal{A}_{U'/L}) = 1$. This is trivially true if $U' = LK$, and if $U' > LK$, then that follows from statement 2 of Theorem 7.3 with $G = U'$. However, this contradicts statement 2 of Lemma 4.2. Thus, case (a) is impossible.

In case (b) one can similarly prove that G_2 is an $\mathcal{A}$-group, and

$$\mathcal{A}_{G/L} = \mathcal{A}_{G_2/L} \otimes \mathcal{A}_{V/L}. \tag{16}$$

Recalling that $\mathcal{A}$ is the U/L-wreath product, we see that for any basic set X outside U we have $L \le \mathrm{rad}(X)$, and hence X is highest in $\mathcal{A}$ or in $\mathcal{A}_V$. Due to (16) this implies that

$$\mathrm{rad}(X) = \mathrm{rad}(\mathcal{A}_V) =: L'.$$

Thus, the S-ring $\mathcal{A}$ is the U/L'-wreath product. Moreover, $\mathrm{rad}(\mathcal{A}_{G/L'}) = 1$ because $\mathcal{A}_{G/L'} = \mathcal{A}_{H/L'} \otimes \mathcal{A}_{V/L'}$ by (16), and $|H/L'| = 2$ where $H = G_2L'$. However, this is impossible by Theorem 4.3. □

§6. Proof of Theorem 1.4

Let $\mathcal{A}$ be a minimal nonschurian S-ring over a cyclic group G of order n. Suppose, to the contrary, that n belongs to one of the families (1). Since any divisor of n also belongs to one of these families and the quasidensity is preserved by taking restrictions to $\mathcal{A}$-sections, Lemma 4.2 shows that without loss of generality we may assume that $\mathcal{A}$ is quasidense. Then by Theorems 4.1 and 5.1 we have $n = 4qr$ or $n = 4q^k$, and $\mathcal{A}$ is both a U/L- and a V/K-wreath product, where K, L, U, and V are $\mathcal{A}$-groups defined by

(a) $|K| = 2$, $|L| = q$, $|U| = 2qr$, $|V| = 4q$ for $n = 4qr$,
(b) $|K| = 2$, $|L| = q$, $|U| = 2q^k$, $|V| = 4q^{k-1}$ for $n = 4q^k$,

with q and r distinct odd primes and $k \ge 2$. In both cases we shall verify that the hypothesis of Theorem 8.1 is satisfied for some $\mathcal{A}$-groups so that the generalized wreath product for $\mathcal{A}$ defined there is proper. Then, by that theorem, $\mathcal{A}$ is schurian because, due to the minimality of $\mathcal{A}$, so are the operands of this product, a contradiction.

Suppose that we are in case (a). Set $H_1 = H_2 = H$ where $H = KL = U \cap V$. First, we observe that relations (24) and (25) are obviously satisfied. Furthermore, $|U/L| = 2r$ is not a prime square, and hence by statement 3 of Theorem A11.4 the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are not normal. By statement 4 of that theorem, this implies that these S-rings are respectively a H/K- and a V/H-wreath product. Besides, condition 1 of Theorem 8.1 is trivially satisfied because the underlying groups of the S-rings $\mathcal{A}_{H/K}$, $\mathcal{A}_{U/H}$, $\mathcal{A}_{V/H}$ and $\mathcal{A}_{H/L}$ are of prime orders, whereas condition 2 is satisfied because $|K| = |G/U| = 2$. Thus, the hypothesis of Theorem 8.1 is satisfied.

Suppose that we are in case (b). To define the $\mathcal{A}$-groups from the hypothesis of Theorem 8.1, we need preliminary work. Set M to be the minimal $\mathcal{A}$-subgroup of G that contains G_2, and N to be the maximal $\mathcal{A}$-subgroup of G_q. We claim that

$$G_2 \ne M, \quad M_q \le N, \quad N \ne G_q. \tag{17}$$

Indeed, if $G_2 = M$, then the radical of the highest basic set in G_2 has trivial q-part. However, this is impossible because $\mathcal{A}$ is a U/L-wreath product. Similarly, if $N = G_q$, then the radical of the highest basic set in G_q has trivial 2-part. However, this is impossible because $\mathcal{A}$ is a V/K-wreath product. To prove the rest, we observe that by Theorem 5.2 the S-ring $\mathcal{A}$ is not a U/K-wreath product. Then by statement 1 of Corollary 3.3 with $p = 2$, there exists an $\mathcal{A}$-group H such that $G_2 \le H$ and H_q is an

$\mathcal{A}$-group. So $M_q \le H_q$ and $N \ge H_q$ by the choice of M and N, respectively. This proves the claim.

We verify that the hypothesis of Theorem 8.1 is satisfied for the $\mathcal{A}$-groups $K, \widetilde{L}, M, N, U, \widetilde{V}$, where

$$\widetilde{L} = M \cap N \quad \text{and} \quad \widetilde{V} = MN.$$

Then obviously $\widetilde{L} \le N$ and $M \le \widetilde{V}$. Moreover, from (17) it also follows that $H_1 \le H_2$ where $H_1 = K\widetilde{L}$ and $H_2 = \widetilde{V} \cap U$. Since also

$$H_2 = K \times N \quad \text{and} \quad G/H_1 = M/H_1 \times U/H_1, \tag{18}$$

relations (24) and (25) hold true. A part of the $\mathcal{A}$-group lattice is given in Figure 2.

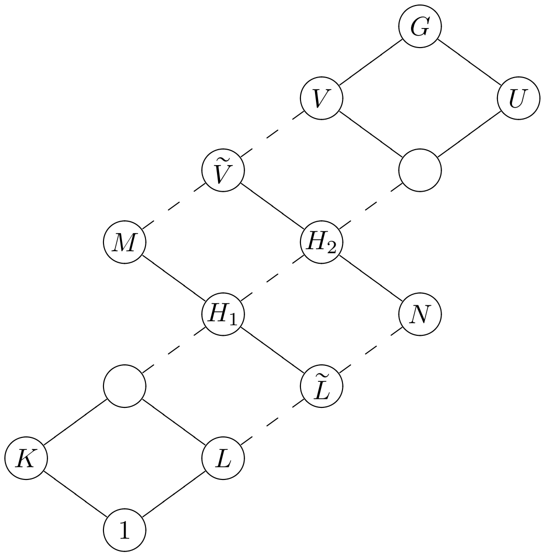

FIGURE 2

To verify the rest of the hypothesis of Theorem 8.1, we observe that condition 2 is satisfied because $|K| = |G/U| = 2$. We claim that

$$\mathcal{A} = \mathcal{A}_U \wr_{U/\widetilde{L}} \mathcal{A}_{G/\widetilde{L}}. \tag{19}$$

Suppose, to the contrary, that this is not true. Then there exists a basic set X outside U such that $\mathrm{rad}(X)_q < \widetilde{L}$. Then obviously $M' := G_2 \,\mathrm{rad}(X)$ is a proper subgroup of M that contains G_2, which is an $\mathcal{A}$-group by Corollary 3.2. However, this contradicts the minimality of M. Next, we verify that

$$\mathcal{A}_U = \mathcal{A}_{H_2} \wr_{H_2/K} \mathcal{A}_{U/K} \quad \text{and} \quad \mathcal{A}_{G/\widetilde{L}} = \mathcal{A}_{\widetilde{V}/\widetilde{L}} \wr_{\widetilde{V}/H_1} \mathcal{A}_{G/H_1}. \tag{20}$$

To prove the first identity, conversly suppose that S-ring $\mathcal{A}_U$ is not an H_2/K-wreath product. Then by statement 1 of Corollary 3.3 with $p = 2$, there exists an $\mathcal{A}_U$-group $H \not\le H_2$ such that H_q is an $\mathcal{A}$-group. However, this is impossible by the maximality of N. The second identity can be proved in a similar way. Thus, by Remark 8.2 we only need to prove that

$$\mathrm{rad}(\mathcal{A}_{H_2/K}) = 1, \quad \mathrm{rad}(\mathcal{A}_{U/H_1}) = 1, \quad \mathrm{rad}(\mathcal{A}_{H_2/H_1}) = 1. \tag{21}$$

We observe that the third relation follows from the first and Corollary 7.4 for $G = H_2/K$. To prove the first relation in (21), suppose, to the contrary, that $\mathrm{rad}(\mathcal{A}_{H_2/K}) > 1$.

To get a contradiction, we use the idea from the proof of case 1 in Theorem 5.1. First, we observe that by Corollary 7.2 we have

$$\mathcal{A}_{H_2} = \mathcal{A}_N \otimes \mathcal{A}_K.$$

Set $U' = N'K$ where N' is the maximal $\mathcal{A}$-group such that $L \le N' \le N$ and $\operatorname{rad}(\mathcal{A}_{N'}) = 1$. Then $N' < N$ by the above assumption and the fact that $\mathcal{A}_N \cong \mathcal{A}_{H_2/K}$. Next, let X be a basic set outside U'. Then $L \le \operatorname{rad}(X)$ for $X \subset G \setminus U$ because $\mathcal{A}$ is a U/L-wreath product and for $X \subset H_2 \setminus U'$ by the definition of U'. The same is also true for $X \subset U \setminus H_2$. Indeed, otherwise set $Q = \langle X \rangle / \operatorname{rad}(X)$, and let S be the image of the section H_2/K in Q. Then $\operatorname{rad}(\mathcal{A}_S) = 1$ by Theorem 7.3 applied to the S-ring $\mathcal{A}_Q$ and the section S. On the other hand, $\operatorname{rad}(\mathcal{A}_S) > 1$ because $\operatorname{rad}(\mathcal{A}_{H_2/K}) > 1$ and $\operatorname{rad}(X) \le K$, a contradiction. Thus, the S-ring $\mathcal{A}$ is a U'/L-wreath product. Moreover, $\operatorname{rad}(\mathcal{A}_{U'}) = 1$ because $\operatorname{rad}(\mathcal{A}_{N'}) = 1$ and $A_{U'} = \mathcal{A}_{N'} \otimes \mathcal{A}_K$. By statement 2 of Theorem 7.3 for $G = U'$, this implies that $\operatorname{rad}(\mathcal{A}_{U'/L}) = 1$. However, this contradicts statement 2 of Lemma 4.2. The second relation in (21) is proved similarly, following the proof of case (b) in Theorem 5.1. □

§7. Auxiliary statements on S-rings

Given an S-ring $\mathcal{A}$ over a group G, we define an *$\mathcal{A}$-complement* of an $\mathcal{A}$-group H to be an $\mathcal{A}$-group H' such that $G = H \times H'$. When the group G is cyclic, the group H' is obviously uniquely determined.

Theorem 7.1. *Let $\mathcal{A}$ be an S-ring over a cyclic group G. Suppose that an $\mathcal{A}$-group H has an $\mathcal{A}$-complement and $\mathcal{A}_S = \mathbb{Z}S$ where S is an $\mathcal{A}$-section projectively equivalent to G/H. Then, given an $\mathcal{A}$-section T projectively equivalent to H, the S-rings $\mathcal{A}$ and $\mathcal{A}_S \otimes \mathcal{A}_T$ are Cayley isomorphic.*

Proof. Denote by H' the $\mathcal{A}$-complement of H. Then obviously $H'/1$ and G/H are respectively the smallest and greatest $\mathcal{A}$-sections in the class of projectively equivalent $\mathcal{A}$-sections that contains G/H. This implies that the section S is projectively equivalent to (in fact, a multiple of) $H'/1$. By Theorem A3.2, the S-rings $\mathcal{A}_S$ and $\mathcal{A}_{H'}$ as well as $\mathcal{A}_T$ and $\mathcal{A}_H$ are Cayley isomorphic. Thus, without loss of generality we may assume that $S = H'/1$ and $T = H/1$. Then $\mathcal{A}_{H'} = \mathbb{Z}H'$, whence

$$\operatorname{rk}(\mathcal{A}) = |H'| \operatorname{rk}(\mathcal{A}_H) = \operatorname{rk}(\mathcal{A}_{H'}) \operatorname{rk}(\mathcal{A}_H).$$

Since also $\mathcal{A} \ge \mathcal{A}_H \otimes \mathcal{A}_{H'}$ by Lemma A2.1, we have $\mathcal{A} = \mathcal{A}_H \otimes \mathcal{A}_{H'}$. □

Corollary 7.2. *Theorem* 7.1 *remains true with the condition $\mathcal{A}_S = \mathbb{Z}S$ replaced by $|S| = 2$.*

Some parts of the following statement appeared in a number of papers. Here we formulate it in a more or less general form, because it is used throughout the paper several times.

Theorem 7.3. *Let $\mathcal{A}$ be a cyclotomic S-ring with trivial radical over a cyclic group G. Suppose that S is an $\mathcal{A}$-section such that $S_p \ne 1$ for any odd prime divisor p of $|G|$. Then*:

1) $|\operatorname{rad}(\mathcal{A}_S)| \le 2$,
2) $|\operatorname{rad}(\mathcal{A}_S)| = 1$ *unless* $|S_2| = 4$.

Proof. By [6, Lemma 3.5], given a set $X \in \mathcal{S}(\mathcal{A})$ with $\operatorname{rad}(X) = 1$ and a prime p such that p^2 divides $m = |\langle X \rangle|$, we have $\operatorname{rad}(X^p) = 1$ unless $p = 2$ and $m = 8m'$ with m' odd. This shows that $\operatorname{rad}(\mathcal{A}_U) = 1$ where U is the subgroup of G of index p, unless $p = 2$ and $|G| = 8m'$ with m' odd. Since $\mathcal{A}_{G/L} \cong \mathcal{A}_U$ where L is the subgroup of G of order p, we have $\mathrm{rad}(\mathcal{A}_{G/L}) = 1$ under the same conditions. Thus, applying these results recursively, we reduce the lemma to the case where

$$|S_2| \le 4, \quad |G_2| \le 8, \quad S_{2'} = G_{2'}.$$

However, from [6, Proposition 3.1] with $m = |G|$ and $l = |G_{2'}|$ we see that $\mathrm{rad}(\mathcal{A}_{G_{2'}}) = 1$. On the other hand, $\mathrm{rad}(\mathcal{A}_S) \le \mathrm{rad}(\mathcal{A}_{S_2})\,\mathrm{rad}(\mathcal{A}_{S_{2'}})$ because $\mathcal{A}_S \ge \mathcal{A}_{S_2} \otimes \mathcal{A}_{S_{2'}}$ (see Lemma A2.2). Thus, $\mathrm{rad}(\mathcal{A}_S) \le \mathrm{rad}(\mathcal{A}_{S_2})$. Since $|S_2| \le 4$, we are done. □

From Theorems A4.1, A4.2, and 7.3 we immediately obtain the following useful result.

Corollary 7.4. *Let $\mathcal{A}$ be an S-ring with trivial radical over a cyclic p-group, where p is odd. Then $\mathrm{rad}(\mathcal{A}_S) = 1$ for any $\mathcal{A}$-section S.*

The following statement gives a necessary and sufficient condition for the schurity of an U/L-wreath product when the section U/L is one of two sections forming an isolated pair of sections in the corresponding S-ring (see Definition A6.1).

Theorem 7.5. *Let $\mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}$ be an S-ring over a cyclic group G. Suppose that either $\mathcal{A}_U \cong \mathcal{A}_L \otimes \mathcal{A}_{U/L}$ or $\mathcal{A}_{G/L} \cong \mathcal{A}_{U/L} \otimes \mathcal{A}_{G/U}$. Then the S-ring $\mathcal{A}$ is schurian if and only if so are the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$.*

Proof. The "only if" part is obvious because, given an $\mathcal{A}$-section S, the S-ring $\mathcal{A}_S$ is schurian whenever so is $\mathcal{A}$. We prove the "if" part under the assumption $\mathcal{A}_U \cong \mathcal{A}_L \otimes \mathcal{A}_{U/L}$ (the rest can be proved similarly). Let $f : U \to L \times (U/L)$ be the corresponding Cayley isomorphism. Then $L^f = L$ and $H^f = U/L$ for a uniquely determined $\mathcal{A}$-group H. It follows that $\mathcal{A}_U = \mathcal{A}_L \otimes \mathcal{A}_H$. Set

$$\Delta_0 = \mathrm{Aut}(\mathcal{A}_{G/L}) \quad \text{and} \quad \Delta_1 = \mathrm{Aut}(\mathcal{A}_L) \otimes \Delta_H,$$

where Δ_H is the full $f^{U/L}$-preimage of the group $(\Delta_0)^{U/L}$ in the group $\mathrm{Aut}(\mathcal{A}_H)$. Clearly,

$$(G/L)_{\mathrm{right}} \le \Delta_0, \quad U_{\mathrm{right}} \le \Delta_1, \quad (\Delta_0)^{U/L} = (\Delta_1)^{U/L}.$$

Moreover, since the S-ring $\mathcal{A}_{G/L}$ is schurian, the latter group is 2-equivalent to the group $\mathrm{Aut}(\mathcal{A}_{U/L})$. It follows that the groups Δ_H and $\mathrm{Aut}(\mathcal{A}_H)$ are 2-equivalent. So, since the S-ring $\mathcal{A}_U$ is schurian, the groups Δ_1 and $\mathrm{Aut}(\mathcal{A}_U) = \mathrm{Aut}(\mathcal{A}_L) \otimes \mathrm{Aut}(\mathcal{A}_H)$ are also 2-equivalent. Thus, by Theorem A1.2 the S-ring $\mathcal{A}$ is schurian and we are done. □

§8. A special generalized wreath product

In this section under special assumptions we prove a necessary and sufficient condition for a U/L-wreath product to be schurian when the restriction of it to U/L is also a generalized wreath product. We start with the description of elements of the canonical generalized wreath product introduced in Definition A5.3[2].

Let G be an Abelian group, and let $L \le U \le G$. Suppose we are given groups $\Delta_0 \le \mathrm{Sym}(G/L)$ and $\Delta_1 \le \mathrm{Sym}(U)$ such that U/L is both a Δ_0- and a Δ_1-section and

$$(G/L)_{\mathrm{right}} \le \Delta_0, \quad U_{\mathrm{right}} \le \Delta_1, \quad (\Delta_0)^{U/L} = (\Delta_1)^{U/L}.$$

Then an element of the canonical generalized wreath product

$$\Gamma = \Delta_1 \wr_{U/L} \Delta_0$$

can be described explicitly as follows. Fix bijections $h_X \in (G_{\mathrm{right}})^{U,X}$ where $X \in G/U$. Suppose we are given a permutation $f_0 \in \Delta_0$ and a family $\{f_X \in \Delta_1 : X \in G/U\}$ of permutations such that

$$(f_X)^{U/L} = (h_X)^{U/L}\, f_0^{X/L} \left((h_{X'})^{U/L}\right)^{-1} \tag{22}$$

[2]The group that was denoted there by Δ_U is denoted here by Δ_1.

for all $X \in G/U$, where X' is the U-coset for which $X'/L = (X/L)^{f_0}$. Then obviously there exists a uniquely determined permutation $f \in \mathrm{Sym}(G)$ for which

$$f^{G/L} = f_0 \quad \text{and} \quad f^X = (h_X)^{-1} f_X h_{X'}$$

for all $X \in G/U$. We emphasize that this permutation depends on the choice of the permutations h_X. Denote it by $\{f_X\} \wr_{U/L} f_0$. Then the definition of the generalized wreath product of permutation groups implies immediately that

$$\Gamma = \big\{\{f_X\} \wr_{U/L} f_0 \,:\, f_0 \in \Delta_0,\ f_X \in \Delta_1 \ \text{ for all } \ X \in G/U\big\}. \tag{23}$$

We turn to the main theorem of this section. Let $\mathcal{A}$ be a quasidense S-ring over a cyclic group G. Suppose we are given $\mathcal{A}$-groups K, L, M, N, U, V such that $L \le N$, $M \le V$,

$$H_1 := KL \le U \cap V := H_2 \tag{24}$$

and also

$$H_2 = K \times N \ \text{ and } \ G/H_1 = M/H_1 \times U/H_1. \tag{25}$$

The corresponding part of the $\mathcal{A}$-group lattice is represented in Figure 3.

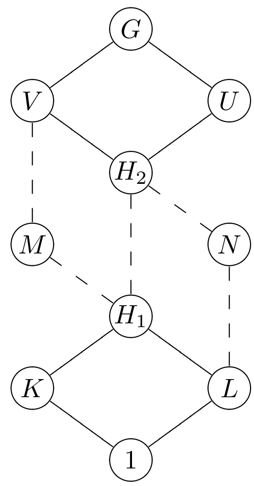

Figure 3

Theorem 8.1. *In the above notation, suppose that the S-rings $\mathcal{A}$, $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are respectively the U/L-, H_2/K- and V/H_1-wreath products such that*

1) *the S-rings $\mathcal{A}_{H_2/K}$, $\mathcal{A}_{U/H_1}$ and $\mathcal{A}_{V/H_1}$, $\mathcal{A}_{H_2/L}$ are of trivial radicals,*
2) *$\mathcal{A}_K = \mathbb{Z}K$ and $\mathcal{A}_{G/U} = \mathbb{Z}G/U$.*

Then the S-ring $\mathcal{A}$ is schurian if and only if so are the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$.

Remark 8.2. Formulas (25) together with condition 2 imply by Theorem 7.1 that $\mathcal{A}_{V/H_1} \cong \mathcal{A}_{G/U} \otimes \mathcal{A}_{H_2/H_1}$ and $\mathcal{A}_{H_2/L} \cong \mathcal{A}_K \otimes \mathcal{A}_{H_2/H_1}$. Thus, in our case the second part of condition 1 is equivalent to the identity $\mathrm{rad}(\mathcal{A}_{H_2/H_1}) = 1$.

Proof. The "only if" part is obvious. To prove the "if" part, suppose that the S-rings $\mathcal{A}_U$ and $\mathcal{A}_{G/L}$ are schurian. Set

$$\Gamma_1 = \mathrm{Hol}_{\mathcal{A}}(U/H_1) \ \text{ and } \ \Gamma_2 = \mathrm{Hol}_{\mathcal{A}}(H_2/L).$$

Then obviously $(\Gamma_1)^{H_2/H_1} = (\Gamma_2)^{H_2/H_1}$. So, one can define the generalized wreath product $\Delta = \Gamma_2 \wr_{H_2/H_1} \Gamma_1$. Thus, by Theorem A1.2, to complete the proof it suffices to find groups $\Delta_1 \in \mathcal{M}(\mathcal{A}_U)$ and $\Delta_0 \in \mathcal{M}(\mathcal{A}_{G/L})$ such that

$$(\Delta_1)^{U/L} = \Delta = (\Delta_0)^{U/L}. \tag{26}$$

To do this, we observe that, by Theorem 3.5 and condition 1, there exist groups $\Gamma_3 \in \mathcal{M}(\mathcal{A}_{U/K})$ and $\Gamma_6 \in \mathcal{M}(\mathcal{A}_{V/L})$ such that

$$(\Gamma_3)^{H_2/K} = \mathrm{Hol}_{\mathcal{A}}(H_2/K), \quad (\Gamma_3)^{U/H_1} = \mathrm{Hol}_{\mathcal{A}}(U/H_1), \tag{27}$$

$$(\Gamma_6)^{V/H_1} = \mathrm{Hol}_{\mathcal{A}}(V/H_1), \quad (\Gamma_6)^{H_2/L} = \mathrm{Hol}_{\mathcal{A}}(H_2/L). \tag{28}$$

Set

$$\Gamma_4 = \mathrm{Hol}_{\mathcal{A}}(H_2), \quad \Gamma_5 = \mathrm{Hol}_{\mathcal{A}}(G/H_1). \tag{29}$$

Then clearly $(\Gamma_4)^{H_2/K} = (\Gamma_3)^{H_2/K}$ and $(\Gamma_6)^{V/H_1} = (\Gamma_5)^{V/H_1}$. Therefore, one can define generalized wreath products

$$\Delta_1 = \Gamma_4 \wr_{H_2/K} \Gamma_3 \quad \text{and} \quad \Delta_0 = \Gamma_6 \wr_{V/H_1} \Gamma_5.$$

First, we prove that $\Delta_1 \in \mathcal{M}(\mathcal{A}_U)$ and $\Delta_0 \in \mathcal{M}(\mathcal{A}_{G/L})$. Indeed, since $U_{\mathrm{right}} \leq \Delta_1$ and $(G/L)_{\mathrm{right}} \leq \Delta_0$, it suffices to verify that

$$\Delta_1 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_U) \quad \text{and} \quad \Delta_0 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{G/L}).$$

In its turn, to prove these relations, by Corollary A5.7 applied to the S-ring $\mathcal{A}_U$ and the groups Γ_4, Γ_3, and the S-ring $\mathcal{A}_{G/L}$ and the groups Γ_6, Γ_5, it suffices to verify that

$$\Gamma_4 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{H_2}), \;\; \Gamma_3 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{U/K}), \;\; \Gamma_6 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{V/L}), \;\; \Gamma_5 \underset{2}{\approx} \mathrm{Aut}(\mathcal{A}_{G/H_1}).$$

However, the statements on Γ_3 and Γ_6 hold true by the definition of these groups. Next, the hypothesis $\mathcal{A}_K = \mathbb{Z}K$ implies by Theorem 7.1 that $\mathcal{A}_{H_2}$ is the tensor product of the cyclotomic rings $\mathcal{A}_K$ and $\mathcal{A}_N \cong \mathcal{A}_{H_2/K}$ (the latter S-ring is cyclotomic by Theorem 3.1). Therefore, the S-ring $\mathcal{A}_{H_2}$ is cyclotomic, and hence the groups $\mathrm{Aut}(\mathcal{A}_{H_2})$ and Γ_4 are 2-equivalent. Similarly, one can prove that the group Γ_5 is 2-equivalent to the group $\mathrm{Aut}(\mathcal{A}_{G/H_1})$.

To prove (26), we note that, by (27), (28) and (29), we have

$$(\Gamma_3)^{U/H_1} = \Gamma_1 = (\Gamma_5)^{U/H_1} \quad \text{and} \quad (\Gamma_4)^{H_2/L} = \Gamma_2 = (\Gamma_6)^{H_2/L}. \tag{30}$$

Therefore, both $(\Delta_1)^{U/L}$ and $(\Delta_0)^{U/L}$ are contained in the group Δ. To prove the reverse inclusion we observe that, by (24) and (25), there is an isomorphism

$$G/V \to U/H_2, \quad X \mapsto X \cap U =: Y. \tag{31}$$

In what follows, the factor sets X and Y modulo L are denoted by $\bar{X}$ and $\bar{Y}$, respectively. For each $X \in G/V$, we fix a bijection $h_X \in (G_{\mathrm{right}})^{V,X}$ that takes H_2 to Y, and set

$$h_Y = (h_X)^Y, \quad h_{\overline{X}} = (h_X)^{\overline{X}}, \quad h_{\bar{Y}} = (h_X)^{\bar{Y}}. \tag{32}$$

Then due to (23), any element $\bar{f} \in \Delta$ can be written in the form

$$\bar{f} = \{f_{\bar{Y}}\} \wr_{H_2/H_1} \bar{f}_0$$

for some permutation $\bar{f}_0 \in \Gamma_1$ and a family of permutations $f_{\bar{Y}} \in \Gamma_2$, where $Y \in U/H_2$, such that

$$(f_{\bar{Y}})^{H_2/H_1} = (h_{\bar{Y}})^{H_2/H_1} \, (\bar{f}_0)^{Y/H_1} \left((h_{\overline{Y'}})^{H_2/H_1}\right)^{-1} \tag{33}$$

for all $Y \in U/H_2$ where Y' is the H_2-coset in U for which $Y'/H_1 = (Y/H_1)^{\bar{f}_0}$. In what follows we find some elements of the groups Δ_1 and Δ_0 the restrictions of which to U/L coincide with $\bar{f}$.

To find the required permutation in Δ_0, we observe that $\mathcal{A}_{M/H_1} = \mathbb{Z}M/H_1$ because $\mathcal{A}_{M/H_1} \cong \mathcal{A}_{G/U}$, and the latter is a group ring by condition 2. So, by Theorem 7.1 we have

$$\Gamma_5 = (M/H_1)_{\mathrm{right}} \otimes \Gamma_1. \tag{34}$$

Therefore this group contains the permutation $f_0 = \mathrm{id}_{M/H_1} \otimes \bar{f}_0$. Clearly,

$$(f_0)^{U/H_1} = \bar{f}_0. \tag{35}$$

Next, let $X \in G/V$. Then by (28) there exists a permutation $f_{\overline{X}} \in \Gamma_6$ that leaves the set H_2/L fixed and satisfies

$$(f_{\overline{X}})^{H_2/L} = f_{\bar{Y}}. \tag{36}$$

Below we show that

$$(f_{\overline{X}})^{V/H_1} = (h_{\overline{X}})^{V/H_1}\,(f_0)^{X/H_1}\,\left((h_{\overline{X'}})^{V/H_1}\right)^{-1}, \tag{37}$$

where X' is the V-coset in G for which $X'/H_1 = (X/H_1)^{f_0}$. Then one can define a permutation $f = \{f_{\overline{X}}\} \wr_{V/H_1} f_0$ belonging to the group Δ_0. This is what we wanted to find, because $f^{U/L} = \bar{f}$ by (32), (35) and (36).

To prove (37), we observe that $(\Gamma_6)^{V/H_1} = (V/H_2)_{\mathrm{right}} \otimes \mathrm{Hol}_{\mathcal{A}}(V/M)$ because $\mathcal{A}_{V/H_2} = \mathbb{Z}V/H_2$ (see above). So

$$(f_{\overline{X}})^{V/H_1} = (f_{\overline{X}})^{V/H_2} \otimes (f_{\overline{X}})^{V/M} = (f_{\overline{X}})^{V/H_2} \otimes (f_{\bar{Y}})^{V/M}.$$

On the other hand, the permutation $f_{\overline{X}}$ leaves the set H_2/L fixed. So, $(f_{\overline{X}})^{V/H_2}$ lives the set H_2 fixed. Since also $(f_{\overline{X}})^{V/H_2} \in (V/H_2)_{\mathrm{right}}$, this implies that $(f_{\overline{X}})^{V/H_2} = \mathrm{id}_{V/H_2}$. Thus, (37) is true by (33) and the choice of the bijections h_X.

To find a permutation $f \in \Delta_1$ such that $f^{U/L}$ coincides with the permutation $\bar{f}$ defined in (31), we observe that, by (27), there exists a permutation $f_0 \in \Gamma_3$ such that (35) holds true. Next, for each $Y \in U/H_2$ we define a permutation of H_2/K by

$$g_Y = (h_Y)^{H_2/K}\,(f_0)^{Y/K}\,\left((h_{Y'})^{H_2/K}\right)^{-1} \tag{38}$$

where Y' is the H_2-coset in U for which $Y'/K = (Y/K)^{f_0}$. However, the bijection h_X by its choice leaves the set U fixed. Hence,

$$(h_Y)^{H_2/K}\,(f_0)^{Y/K}\,\left((h_{Y'})^{H_2/K}\right)^{-1} = \left((h_X)^{U/K}\,f_0\,((h_{X'})^{U/K})^{-1}\right)^{H_2/K}.$$

Thus, g_Y belongs to the group $(\Gamma_3)^{H_2/K} = (\Gamma_4)^{H_2/K}$. Therefore, due to condition 2 of the theorem, the permutation

$$f_Y := g_Y \otimes (f_{\bar{Y}})^{H_2/N}$$

belongs to the group Γ_4. Moreover, formulas (35), (38), and (33) imply that

$$(f_Y)^{H_2/L} = (h_Y)^{H_2/L}\,(f_0)^{Y/L}\,\left((h_{Y'})^{H_2/L}\right)^{-1}.$$

Thus, one can define a permutation $f = \{f_Y\} \wr_{H_2/L} f_0$ belonging to the group Δ_1. By the choice of f_0 and f_Y, we have $f^{U/L} = \bar{f}$, which completes the proof. $\square$

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Received 7/SEP/2012

Originally published in English