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METHOD FOR COMPUTING WAVEGUIDE SCATTERING MATRICES IN THE VICINITY OF THRESHOLDS

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Abstract. A waveguide occupies a domain G in $\mathbb{R}^{n+1}$, $n \geq 1$, having several cylindrical outlets to infinity. The waveguide is described by the Dirichlet problem for the Helmholtz equation. The scattering matrix $S(\mu)$ with spectral parameter μ changes its size when μ crosses a threshold. To calculate $S(\mu)$ in a neighborhood of a threshold, an "augmented" scattering matrix $\mathcal{S}(\mu)$ is introduced, which keeps its size near the threshold and is analytic in μ there. A minimizer of a quadratic functional $J^R(\,\cdot\,,\mu)$ serves as an approximation to a row of the matrix $\mathcal{S}(\mu)$. To construct such a functional, an auxiliary boundary-value problem is solved in the bounded domain obtained by cutting off the waveguide outlets to infinity at a distance R. As $R \to \infty$, the minimizer $a(R,\mu)$ tends exponentially to the corresponding row of $\mathcal{S}(\mu)$ uniformly with respect to μ in a neighborhood of the threshold. The neighborhood may contain some waveguide eigenvalues corresponding to eigenfunctions exponentially decaying at infinity. Finally, the elements of the "ordinary" scattering matrix $S(\mu)$ are expressed in terms of those of the augmented matrix $\mathcal{S}(\mu)$.

If an interval $[\mu_1,\mu_2]$ of the continuous spectrum contains no thresholds, the corresponding functional $J^R(\,\cdot\,,\mu)$ should be defined for the usual matrix $S(\mu)$ and, as $R \to \infty$, its minimizer $a(R,\mu)$ tends to the row of the scattering matrix at exponential rate uniformly with respect to $\mu \in [\mu_1,\mu_2]$.

§1. Introduction

The waveguide considered in this paper occupies a domain G in the space $\mathbb{R}^{n+1}$ with smooth boundary ∂G and finitely many cylindrical outlets to infinity (cylindrical ends). By this we mean that, outside a large ball centered at the origin, the domain G coincides with the union of nonoverlapping semicylinders $\Pi_+^1,\dots,\Pi_+^{\mathcal{T}}$; here

$$\Pi_+^p = \bigl\{(y^p,t^p)\,:\,y^p \in \Omega^p, t^p > 0\bigr\},$$

the (y^p,t^p) are local coordinates in Π_+^p, and the cross-section Ω^p of the cylinder Π^p is a bounded domain in $\mathbb{R}^n$. The waveguide is described by the Dirichlet problem for the operator $-\Delta-\mu$, where μ is a spectral parameter and Δ is the Laplace operator. The continuous spectrum of the problem coincides with the semiaxis $\{\mu \in \mathbb{R}\,:\,\tau_1 \leq \mu\}$, where τ_1 is a positive number. For every point $\mu \in [\tau_1,+\infty)$ there exist a finite number $\varkappa(\mu)$ of solutions, $\varkappa(\mu) < \infty$, for the homogeneous problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= 0, \qquad x \in G,\\ u(x) &= 0, \qquad x \in \partial G, \end{aligned} \tag{1.1}$$

satisfying $|u(x)| \leq \mathrm{Const}(1+|x|)$ in G and linearly independent modulo $L_2(G)$. Such solutions are called the continuous spectrum eigenfunctions and the number $\varkappa(\mu)$ is called

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the multiplicity of the continuous spectrum. The threshold values (thresholds) form a sequence $\tau_1 < \tau_2, \dots, \tau_n \to +\infty$. The multiplicity $\varkappa(\mu)$ is constant on every interval $[\mu', \mu'']$ of the continuous spectrum containing no thresholds. The function $\mu \mapsto \varkappa(\mu)$ has discontinuity at every threshold, being continuous from the right. This is a monotone increasing function, so $\varkappa(\mu) \to +\infty$ as $\mu \to +\infty$.

There may also exist eigenvalues of problem (1.1) embedded into the continuous spectrum whose eigenfunctions belong to $L_2(G)$. Such an eigenfunction exponentially decays at infinity, any eigenvalue is of finite multiplicity, and the eigenvalues can only accumulate at infinity. An eigenvalue μ_0 does not affect the multiplicity of the continuous spectrum at μ_0 because $\varkappa(\mu_0)$ takes into account only linear independence modulo $L_2(G)$. If μ is not an eigenvalue, then the linear independence modulo $L_2(G)$ in the definition of $\varkappa(\mu)$ can be changed for the usual linear independence.

It is known [1] that, for every $\mu \in [\tau_1, +\infty]$, in the space of continuous spectrum eigenfunctions there exists a basis $Y_1(\,\cdot\,,\mu), \dots, Y_M(\,\cdot\,,\mu)$ modulo $L_2(G)$ such that

$$Y_j(x,\mu) = u_j^+(x,\mu) + \sum_{k=1}^{\varkappa(\mu)} S_{jk}(\mu) u_k^-(x,\mu) + O(e^{-\varepsilon|x|}) \tag{1.2}$$

as $|x| \to \infty$ and $j = 1, \dots, \varkappa(\mu)$; here ε is a sufficiently small number, $u_j^+(\,\cdot\,,\mu)$ is an "incoming" wave, and $u_j^-(\,\cdot\,,\mu)$ is an "outgoing" wave (see Subsection 2.2 for the precise definitions see). The matrix $S(\mu) = \|S_{jk}(\mu)\|$ is unitary; it is called the scattering matrix.

In [2] and [3], a method for approximate computing the matrix $S(\mu)$ was discussed under the condition that μ varies on an interval $[\mu', \mu'']$ of the continuous spectrum containing no thresholds. Briefly, a minimizer $a(R,\mu)$ of a quadratic functional $J_l^R(\,\cdot\,,\mu)$ was chosen as an approximation to the lth row $S_l(\mu) = (S_{l,1}(\mu), \dots, S_{l,M}(\mu))$ of the scattering matrix $S(\mu)$. To construct such a functional, one needs to solve an auxiliary boundary value problem in the bounded domain G^R obtained from G by cutting off the cylindrical ends at a sufficiently large distance R from the coordinate origin. For $R \geq R_0$ and all $\mu \in [\mu', \mu'']$, the estimate

$$\|a(R,\mu) - S_l(\mu)\| \leq Ce^{-\gamma R} \tag{1.3}$$

was proved with a positive number γ and a constant C independent of R and μ. In [2], the two-dimensional waveguides and the Helmholtz operator were considered; the waveguides of arbitrary dimension and selfadjoint elliptic systems of any order were discussed in [3]. The approach suggested in [3] turns out to be new also for the Helmholtz operator and it is simpler than that in [2].

The present paper is devoted to a method for approximate computing the scattering matrices in a neighborhood of the thresholds. Let us outline the method. We assume $\tau' < \tau''$ to be thresholds of problem (1.1) such that the interval (τ', τ'') contains a single threshold τ. We also suppose that the three thresholds are related to one and the same cylindrical end. We intend to (approximately) calculate the scattering matrix $S(\mu)$ in (1.2) for $\mu \in [\mu', \mu'']$ with $\tau \in [\mu', \mu''] \subset (\tau', \tau'')$.

On the interval (τ, τ''), one can choose a basis of incoming waves $w_1^+(\,\cdot\,,\mu), \dots, w_\varkappa^+(\,\cdot\,,\mu)$ and outgoing waves $w_1^-(\,\cdot\,,\mu), \dots, w_\varkappa^-(\,\cdot\,,\mu)$ with analytic functions $(\tau, \tau'') \ni \mu \mapsto w_j^\pm(\,\cdot\,,\mu)$ that admit analytic continuation to (τ', τ''); here $\varkappa = \varkappa(\mu'')$ (recall that $\varkappa(\mu) = \text{const}$ for $\mu \in [\tau, \tau'')$). Such a basis is said to be *stable* at the threshold τ. For $\mu \in (\tau', \tau)$, some incoming waves and the same number of outgoing waves turn out to be exponentially growing as $x \to \infty$. On the interval (τ, τ''), in the space of continuous spectrum eigenfunctions there exists a basis $\mathcal{Y}_1(\,\cdot\,,\mu), \dots, \mathcal{Y}_\varkappa(\,\cdot\,,\mu)$ satisfying the

conditions

$$\mathcal{Y}_j(x,\mu) = w_j^+(x,\mu) - \sum_{k=1}^{M} \mathcal{S}_{jk}(\mu) w_k^-(x,\mu) + O(e^{-\varepsilon|x|}). \tag{1.4}$$

The functions $\mu \mapsto \mathcal{Y}_j(\,\cdot\,,\mu)$ and $\mu \mapsto \mathcal{S}_{jk}(\mu)$ are analytic and admit analytic continuation to (τ',τ''). In contrast to $S(\mu)$, the new matrix $\mathcal{S}(\mu) = \|\mathcal{S}_{jk}(\mu)\|$ keeps its size on this interval; the matrix is unitary for all $\mu \in (\tau',\tau'')$. The entries of $S(\mu)$ can be expressed in terms related to the matrix $\mathcal{S}(\mu)$ only. In particular, this enables us to prove the existence of finite limits of $S(\mu)$ as $\mu \to \tau \pm 0$, to calculate the limits, and, in essence, to reduce approximate calculation of the matrix $S(\mu)$ with $\mu \in [\mu',\mu'']$ to that of the augmented matrix $\mathcal{S}(\mu)$. As an approximation to a row of $\mathcal{S}(\mu)$, we take a minimizer for a quadratic functional $\mathcal{J}^R(\,\cdot\,,\mu)$. To construct such a functional, we use a boundary-value problem in the bounded domain G^R obtained from G by cutting off the cylindrical ends at a distance R. We set

$$\Pi_+^{r,R} = \{(y^r,t^r) \in \Pi^r \,:\, t^r > R\}, \quad G^R = G \setminus \cup_{r=1}^{\mathcal{T}} \Pi_+^{r,R},$$
$$\partial G^R \setminus \partial G = \Gamma^R = \cup_r \Gamma^{r,R}, \quad \Gamma^{r,R} = \big\{(y^r,t^r) \in \Pi^r \,:\, t^r = R\big\}$$

for large R and introduce the boundary-value problem

$$\begin{aligned} -\Delta \mathcal{X}_j^R - \mu \mathcal{X}_j^R &= 0, \quad x \in G^R;\\ \mathcal{X}_j^R &= 0, \quad x \in \partial G^R \setminus \Gamma^R;\\ (-\partial_n + i\zeta)\mathcal{X}_j^R &= (-\partial_n + i\zeta)\Big(w_j^+ + \sum_{k=1}^{\varkappa} a_k w_k^-\Big), \quad x \in \Gamma^R, \end{aligned}$$

where the $w_j^\pm$ constitute a stable basis in the space of waves, $\zeta \in \mathbb{R} \setminus \{0\}$ is an arbitrary fixed number, and the a_k are complex numbers. As an approximation to the row $(\mathcal{S}_{j1}(\mu),\dots,\mathcal{S}_{j\varkappa}(\mu))$, we take the minimizer $a^0(R,\mu) = (a_1^0(R,\mu),\dots,a_\varkappa^0(R,\mu))$ of the functional

$$\mathcal{J}_j^R(a_1,\dots,a_\varkappa) = \Big\| \mathcal{X}_j^R(\,\cdot\,,\mu) - w_j^+(\,\cdot\,,\mu) - \sum_{k=1}^{\varkappa} a_k w_k^-(\,\cdot\,,\mu); L_2(\Gamma^R) \Big\|^2,$$

where $\mathcal{X}_j^R(\,\cdot\,,\mu)$ is a solution of the above boundary value problem. If $\tau \in [\mu',\mu''] \subset (\tau',\tau'')$, then we have

$$\|a(R,\mu) - \mathcal{S}_j(\mu)\| \le C(\Lambda) e^{-\Lambda R}$$

for all $\mu \in [\mu',\mu'']$ and $R \ge R_0$, with positive constants Λ and $C(\Lambda)$ independent of μ and R.

Note that the use of stable bases is not uncommon in asymptotic studies of various "threshold" situations. In this connection we refer to [4, 5], where the asymptotics of solutions of elliptic boundary value problems were investigated near singularities of the boundary (see also the references therein). In [6], the asymptotics of the scattering matrix for a two-dimensional diffraction grating was justified, in essence, with the help of a stable basis in the space of waves.

In the present paper, §2 is devoted to constructing a stable basis of waves in a neighborhood of a threshold for the waveguide in a domain G. The continuous spectrum eigenfunctions and the scattering matrices $S(\mu)$ and $\mathcal{S}(\mu)$ are introduced in §3; here we also prove the analyticity of the matrices on the corresponding intervals of the continuous spectrum. We describe the relationship between the matrices $S(\mu)$ and $\mathcal{S}(\mu)$ and calculate the one-sided limits of $S(\mu)$ at a threshold in §4. The last §5 contains the statement and justification of the method for approximate computation of the scattering matrices.

§2. Augmented space of waves

2.1. Waves in a cylinder. In a cylinder $\Pi = \{(y,t) : y = (y_1,\dots,y_n) \in \Omega,\ t \in \mathbb{R}\}$, we consider the problem

$$\begin{aligned} (-\Delta - \mu)u(y,t) &= 0, \quad (y,t) \in \Pi, \\ u(y,t) &= 0 \quad (y,t) \in \partial\Pi, \end{aligned} \tag{2.1}$$

where

$$\Delta = \Delta_y + \partial_t^2, \quad \Delta_y = \sum_{j=1}^n \partial_j^2, \quad \partial_j = \partial/\partial y_j.$$

With problem (2.1) we associate an operator pencil $\mathbb{C} \ni \lambda \mapsto \mathfrak{A}(\lambda,\mu)$ by setting

$$\mathfrak{A}(\lambda,\mu)v(y) = (-\Delta_y + \lambda^2 - \mu)v(y), \quad y \in \Omega; \quad v|_{\partial\Omega} = 0. \tag{2.2}$$

We also consider the problem

$$\begin{aligned} (-\Delta_y - \mu)v(y) &= 0, \quad y \in \Omega, \\ v(y) &= 0, \quad y \in \partial\Omega. \end{aligned} \tag{2.3}$$

The eigenvalues of problem (2.3) are called the thresholds of problem (2.1). The thresholds form a positive sequence $\tau_1 < \tau_2 < \dots$, which is strictly monotone increasing to infinity. We introduce the monotone nondecreasing sequence $\{\mu_k\}_{k=1}^\infty$ of eigenvalues of problem (2.3), counted with their multiplicities (generally, the numbering of τ_l and that of μ_k are different; every μ_k coincides with one of the thresholds τ_l). We assume that the corresponding eigenvectors φ_k are orthogonal and normalized by the condition

$$\int_\Omega \varphi_k(y)\overline{\varphi_k(y)}\,dy = 1. \tag{2.4}$$

The spectrum of the operator pencil (for every fixed $\mu \in \mathbb{R}$) consists of isolated eigenvalues on the axes of the complex plane. For any μ, the eigenvalues $\lambda_k^\pm$ of the pencil $\lambda \mapsto \mathfrak{A}(\lambda,\mu)$ are defined by $\lambda_k^\pm = \pm(\mu - \mu_k)^{1/2}$. If $\lambda_k^\pm \neq 0$, then the eigenvalues $\lambda_k^\pm$ are associated with one and the same eigenvector φ_k, which is also an eigenvector of problem (2.3) corresponding to the eigenvalue μ_k. There is no generalized eigenvector in this case. If $\mu_{k-1} < \mu < \mu_k$, then the $\lambda_k^\pm, \lambda_{k+1}^\pm, \dots$ are imaginary and the $\lambda_1^\pm, \dots, \lambda_{k-1}^\pm$ are real. If $\mu = \mu_k$, then the eigenvalue $0 = \lambda_k^+ = \lambda_k^-$ gives rise to a Jordan chain φ_k^0, φ_k^1, where φ_k^0 is an eigenvector and φ_k^1 is a generalized eigenvector.

We fix a real $\mu \neq \mu_k$, $k = 1, 2, \dots$, that is, this μ is not a threshold, and introduce the complex linear space spanned by the functions

$$(y,t) \mapsto \exp(i\lambda_k^\pm t)\varphi_k(y) \tag{2.5}$$

with real $\lambda_k^\pm = \pm(\mu - \mu_k)^{1/2}$; these functions satisfy (2.1). We denote this space by $W(\mu)$ and call it the *space of waves*. Its dimension is equal to twice the number of the μ_k such that $\mu_k < \mu$. The functions

$$u_k^\pm(y,t;\mu) = (2|\lambda_k^\mp|)^{-1/2}\exp(i\lambda_k^\mp t)\varphi_k(y) \tag{2.6}$$

form a basis in $W(\mu)$. We call $u_k^+(\,\cdot\,,\mu)$ a *wave incoming from* $+\infty$, and $u_k^-(\,\cdot\,,\mu)$ a *wave outgoing to* $+\infty$.

Assume now that $\mu = \tau$ is a threshold and, consequently, μ is an eigenvalue of (2.3) with multiplicity $\varkappa \geq 1$. Then $\varkappa$ numbers μ_l satisfy $\mu_l = \tau$. For each l, the functions $\exp(i\lambda_l^+ t)\varphi_l(y)$ and $\exp(i\lambda_l^- t)\varphi_l(y)$ coincide. Therefore, the number $\varkappa$ of linearly independent functions of the form (2.5) for $\mu = \tau$ is less than the number of such functions for μ satisfying $\tau < \mu < \tau + \beta$ with small $\beta > 0$. However, for a more general notion of the waves, the dimension of the space $W(\mu)$ is continuous from the right at the

threshold. In this case, the definition of incoming and outgoing waves is based on energy considerations, as in the Sommerfeld and Mandelstamm principles.

For the definition, we introduce the form

$$\begin{aligned} q_N(u,v) := {} & ((-\Delta-\mu)u,v)_{\Pi(N)} + (u,-\partial_\nu v)_{\partial\Pi(N)\cap\partial\Pi} \\ & -(u,(-\Delta-\mu)v)_{\Pi(N)} - (-\partial_\nu u,v)_{\partial\Pi(N)\cap\partial\Pi}, \end{aligned} \tag{2.7}$$

where $\Pi(N) = \{(y,t) \in \Pi : t < N\}$, the number $\mu \in \mathbb{R}$ is not a threshold for the time being, $u = \chi f$ and $v = \chi g$, while f and g are some functions of the form (2.6) corresponding to real $\lambda_k^\pm(\mu)$ (possibly with distinct indices); χ stands for a smooth cut-off function, $\chi(t) = 0$ for $t < T-1$ and $\chi(t) = 1$ for $t > T$ with $T < N$. Integrating by parts, we see that

$$iq_N(\chi u_k^\pm, \chi u_l^\mp) = 0 \quad \text{for all} \quad k,l, \tag{2.8}$$

$$iq_N(\chi u_k^\pm, \chi u_l^\pm) = \mp\delta_{kl}, \tag{2.9}$$

so that the result is independent of N and χ; in what follows, we drop N but keep χ. We say that the wave $u_k^+(u_k^-)$ is *incoming* (*outgoing*) for $-(+)$ on the right in (2.9) and obtain the definition of incoming (outgoing) waves equivalent to the previous definition.

We are going to construct a basis in the (augmented) space of waves "stable at a threshold". Let $\mu \in \mathbb{R}$ be a regular value of the spectral parameter of problem (2.3) and μ_m the eigenvalue with the greatest number satisfying $\mu_m < \mu$. We also assume that $\mu_l < \mu_{l+1} = \cdots = \mu_m$. Then the numbers $\tau' := \mu_l$, $\tau := \mu_{l+1} = \cdots = \mu_m$, and $\tau'' := \mu_{m+1}$ turn out to be three successive thresholds $\tau' < \tau < \tau''$ of problem (2.1) in the cylinder Π. (We discuss the general situation; the cases where $l+1 = m$, $m = 1$, and so on, can be handled with evident simplifications.)

We set

$$w_k^\pm(y,t;\mu) = 2^{-1/2}\left(\frac{e^{it\sqrt{\mu-\mu_k}} + e^{-it\sqrt{\mu-\mu_k}}}{2} \mp \frac{e^{it\sqrt{\mu-\mu_k}} - e^{-it\sqrt{\mu-\mu_k}}}{2\sqrt{\mu-\mu_k}}\right)\varphi_k(y), \tag{2.10}$$

$$w_p^\pm(y,t;\mu) = u_p^\pm(y,t;\mu), \tag{2.11}$$

where $k = l+1,\dots,m$, $p = 1,\dots,l$, and the $u_p^\pm$ are defined as in (2.6).

Proposition 2.1. *The functions $\mu \mapsto w_k^\pm(y,t;\mu), k = l+1,\dots,m$, admit analytic continuation to the entire complex plane. These analytic functions smoothly depend on the parameters $y \in \bar\Omega$ and $t \in \mathbb{R}$ (i.e., any derivatives in y and t are also analytic functions).*

The functions $\mu \mapsto w_p^\pm(y,t;\mu)$ are analytic on the complex plane cut along the ray $\{\mu \in \mathbb{R} : -\infty < \mu \le \mu_p\}$, $p = 1,\dots,l$; they smoothly depend on y and t.

All the functions $w_k^\pm$, $k = 1,\dots,m$, are solutions of problem (2.1). *For every μ with $\tau' < \mu < +\infty$, the functions* (2.10), (2.11) *satisfy the orthogonality and normalization conditions*

$$iq\big(\chi w_r^\pm(\,\cdot\,;\mu), \chi w_s^\mp(\,\cdot\,;\mu)\big) = 0 \quad \textit{for all } r,s = 1,\dots,m, \tag{2.12}$$

$$iq\big(\chi w_r^\pm(\,\cdot\,;\mu), \chi w_s^\pm(\,\cdot\,;\mu)\big) = \mp\delta_{rs}. \tag{2.13}$$

Proof. The first and second fractions in (2.10) can be expanded in the series

$$\sum_{l\ge0} \frac{(\mu_k-\mu)^l t^{2l}}{(2l)!} \quad \text{and} \quad it\sum_{l\ge0} \frac{(\mu_k-\mu)^l t^{2l}}{(2l+1)!}, \tag{2.14}$$

which are absolutely and uniformly convergent on any compact set

$$K \subset \big\{(\mu,t) : \mu \in \mathbb{C}, t \in \mathbb{R}\big\}.$$

This implies the analyticity properties of $w_k^{\pm}(y,t;\mu)$ for $k=l+1,\dots,m$. The corresponding assertions about $w_p^{\pm}(y,t;\mu)$ with $p=1,\dots,l$ are evident.

It remains to verify the orthogonality and normalization conditions. First, we assume that $\mu>\tau$ and consider, for instance, formula (2.13). If r and s are distinct, then (2.13) follows from the orthogonality of φ_r and φ_s (like (2.8) and (2.9)). In the case where $r=s\le l$, (2.9) contains the required formula. Finally, assume that $r=s>l$ and substitute the expressions (2.10) in $q(\chi w_r^{\pm},\chi w_s^{\pm})$. Setting $\lambda:=\sqrt{\mu-\tau}$, we obtain

$$\begin{aligned} iq(\chi w_s^{\pm},\chi w_s^{\pm}) = \lambda^{-2}\big((\lambda\pm1)(\lambda\mp1)iq^{+-} + (\lambda\mp1)(\lambda\pm1)iq^{-+} \\ + (\lambda\mp1)^2iq^{++} + (\lambda\pm1)^2iq^{--}\big), \end{aligned} \tag{2.15}$$

where, for example, $q^{+-}=2^{-3}q(\chi e^{it\lambda}\varphi_s,\chi e^{-it\lambda}\varphi_s)$, and so on. Recalling (2.6), (2.8), and (2.9), we arrive at (2.13).

Now we consider the function

$$\begin{aligned} \mathbb{C}\ni\mu\mapsto q_N(u,v;\mu) := ((-\Delta-\mu)u,v)_{\Pi(N)} + (u,-\partial_\nu v)_{\partial\Pi(N)\cap\partial\Pi} \\ -(u,(-\Delta-\bar\mu)v)_{\Pi(N)} - (-\partial_\nu u,v)_{\partial\Pi(N)\cap\partial\Pi}, \end{aligned} \tag{2.16}$$

where $\Pi(N)$, N, and χ are the same as in (2.7), $u=\chi w_r^{\pm}(\,\cdot\,;\mu)$, and $v=\chi w_s^{\mp}(\,\cdot\,;\bar\mu)$). Since u and $\bar v$ are analytic, so is the function $\mu\mapsto q_N(u,v;\mu)$. Therefore, identities (2.13) (with $r=s>l$) are valid for all $\mu\in\mathbb{C}$. □

From (2.10) it follows that $w_k^{\pm}(y,t;\tau)=2^{-1/2}(1\mp it)\varphi_k(y)$, $k=l+1,\dots,m$, and, in the case where $\mu<\tau$, the amplitudes of the waves grow exponentially as $t\to\infty$. The space spanned by the waves (2.10) and (2.11) is called the *augmented space of waves* for $\tau'<\mu<\tau$ and is denoted by $W_a(\mu)$. Let $W(\mu)$ stand for the linear hull of the functions (2.10) and (2.11) for $\tau\le\mu<\tau''$ and the linear hull of the functions (2.11) for $\tau'<\mu<\tau$. The lineal $W(\mu)$ is called the *space of waves*. An element $w\in W_a(\mu)$ (or $W(\mu)$) is called a *wave incoming from* $+\infty$ (*outgoing to* $+\infty$) if $iq(\chi w,\chi w)<0$ ($iq(\chi w,\chi w)>0$).

The collection of waves $\{w_k^{\pm}\}_{k=1}^m$ defined by (2.10) and (2.11) is called a *basis of waves stable in a neighborhood of the threshold* τ. By definition, a basis of waves of the form (2.6) is *stable* on (μ',μ'') if the interval $[\mu',\mu'']$ contains no thresholds.

2.2. Waves in the domain G. Let G be a domain in $\mathbb{R}^{n+1}$ with smooth boundary ∂G coinciding, outside a large ball, with the union $\Pi_+^1\cup\dots\cup\Pi_+^{\mathcal{T}}$ of finitely many nonoverlapping semicylinders

$$\Pi_+^r=\{(y^r,t^r)\,:\,y^r\in\Omega^r,\,t^r>0\},$$

where (y^r,t^r) are local coordinates in Π_+^r and Ω^r is a bounded domain in $\mathbb{R}^n$.

We introduce the problem

$$\begin{aligned} -\Delta u(x)-\mu u(x)=0, &\quad x\in G, \\ u(x)=0, &\quad x\in\partial G. \end{aligned} \tag{2.17}$$

With every Π_+^r, we associate a problem of the form (2.1) in the cylinder $\Pi^r=\{(y^r,t^r)\,:\,y^r\in\Omega^r,t^r\in\mathbb{R}\}$. Let $\chi\in C^\infty(\mathbb{R})$ be a cut-off function, $\chi(t)=0$ for $t<0$ and $\chi(t)=1$ for $t>1$. We multiply each wave in Π^r by the function $t\mapsto\chi(t^r-t_0^r)$ with a certain $t_0^r>0$ and then extend it by zero to the domain G. All functions (for all Π^r) obtained in this way will be called *waves in* G. A number τ is called a *threshold* for problem (2.17) if τ is a threshold for at least one of problems of the form (2.1) in Π^r, $r=1,\dots,\mathcal{T}$. Let $\tau'<\tau<\tau''$ be three successive thresholds for problem (2.17); then the intervals (τ',τ) and (τ,τ'') are free from thresholds.

For $\mu\in(\tau',\tau)$, we introduce the augmented space $\mathcal{W}_a(\mu,G)$ of waves in G as the union of the waves in G corresponding to the waves in $W_a(\mu)$ for Π^r, $r=1,\dots,\mathcal{T}$; if a space $W_a(\mu)$ is not introduced on the interval $\tau'<\mu<\tau$ for a certain Π^r (which

means that this τ is not a threshold for problem (2.1) in this cylinder), then, for this cylinder, into the space $\mathcal{W}_a(\mu, G)$ we include the waves generated by the elements of the corresponding $W(\mu)$. By definition, for $\mu \in (\tau', \tau'')$ the space $\mathcal{W}(\mu, G)$ of waves in G is the union of the waves in G that correspond to the waves in $W(\mu)$ for all Π^r.

The bases $\{u_j^{\pm}(\,\cdot\,, \mu)\}$ and $\{w_j^{\pm}(\,\cdot\,, \mu)\}$ of waves in $\mathcal{W}(\mu, G)$ and $\mathcal{W}_a(\mu, G)$ are comprised by the waves obtained in G from the basis waves in Π^r, $r = 1, \dots, \mathcal{T}$. The basis waves in the spaces $\mathcal{W}(\mu, G)$ and $\mathcal{W}_a(\mu, G)$ are subject to orthogonality and normalization conditions like (2.8) and (2.9) or (2.12) and (2.13) with the form q in a cylinder replaced by the form q_G in G:

$$\begin{aligned} q_G(u, v) := {} & ((-\Delta - \mu)u, v)_G + (u, -\partial_\nu v)_{\partial G} \\ & -(u, (-\Delta - \mu)v)_G - (-\partial_\nu u, v)_{\partial G}. \end{aligned} \tag{2.18}$$

An element w in $\mathcal{W}_a(\mu, G)$ (or in $\mathcal{W}(\mu, G)$) is called a *wave incoming from* ∞ (*outgoing to* ∞) if $iq_G(w, w) < 0$ ($iq_G(w, w) > 0$).

A basis of waves in G is said to be *stable near a value* ν of the spectral parameter if the basis consists of bases in the cylinders $\Pi^1, \dots, \Pi^{\mathcal{T}}$ stable near ν.

§3. Continuous spectrum eigenfunctions. Scattering matrices

Let $\tau' < \tau < \tau''$ be three successive thresholds for problem (2.17). For simplicity, we assume that these three numbers are thresholds for a problem of the form (2.1) only in one of the cylinders $\Pi^1, \dots, \Pi^{\mathcal{T}}$, for instance in $\Pi^1 = \Omega^1 \times \mathbb{R}$. Moreover, we suppose that $\tau' = \mu_l$, $\tau = \mu_{l+1} = \dots = \mu_m$, and $\tau'' = \mu_{m+1}$, where the μ_k are eigenvalues of problem (2.3) in Ω^1. Thus, for $\Pi = \Pi^1$ we deal with the situation considered in Subsection 2.1.

3.1. Intrinsic and expanded radiation principles. We consider the boundary value problem

$$\begin{aligned} -\Delta u(x) - \mu u(x) &= f(x), \quad x \in G, \\ u(x) &= g(x), \quad x \in \partial G, \end{aligned} \tag{3.1}$$

and recall two statements of the problem with radiation conditions at infinity: the intrinsic and expanded radiation principles. In the first principle, the intrinsic radiation conditions involve only outgoing waves in the space $\mathcal{W}(\mu, G)$. The second (expanded) principle involves outgoing waves in the augmented space $\mathcal{W}_a(\mu, G)$. For the general elliptic problems selfadjoint with respect to the Green formula, the first statement was discussed in [1] and the second was considered in [7, 8] (for various geometric situations). We shall apply the intrinsic principle with spectral parameter outside a neighborhood of the thresholds. In the vicinity of a threshold, we make use of the expanded principle employing the stable basis of waves in $\mathcal{W}_a(\mu, G)$ constructed in §2.

First, we define the required function spaces. For an integer $l \geq 0$, we denote by $H^l(G)$ the Sobolev space with the norm

$$\|v; H^l(G)\| = \left(\sum_{j=0}^{l} \int_G \sum_{|\alpha|=j} |D_x^\alpha v(x)|^2 \, dx \right)^{1/2},$$

and let $H^{l-1/2}(\partial G)$ with $l \geq 1$ stand for the space of traces on ∂G of the functions in $H^l(G)$. Assume that ρ_γ is a smooth function positive on $\bar{G}$ and given on Π_+^r by $\rho_\gamma(y^r, t^r) = \exp(\gamma t^r)$ with $\gamma \in \mathbb{R}$. We also introduce the spaces $H_\gamma^l(G)$ and $H_\gamma^{l-1/2}(\partial G)$

with norms $\|u; H^l_\gamma(G)\| = \|\rho_\gamma u; H^l(G)\|$ and $\|v; H^{l-1/2}_\gamma(\partial G)\| = \|\rho_\gamma v; H^{l-1/2}(\partial G)\|$. The operator of problem (3.1) implements a continuous mapping

$$\mathcal{A}_\gamma(\mu)\colon H^2_\gamma(G) \to H^0_\gamma(G) \times H^{3/2}_\gamma(\partial G). \tag{3.2}$$

As is known, the operator (3.2) is Fredholm if and only if the line $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \gamma\}$ is free of the eigenvalues of the pencils $\lambda \mapsto \mathfrak{A}^r(\lambda, \mu)$, $r = 1, \dots, \mathcal{T}$, where $\mathfrak{A}^r$ is a pencil of the form (2.2) for the problem (2.1) in the cylinder Π^r. (An operator is said to be *Fredholm* if its range is closed and the kernel and cokernel are finite-dimensional.)

We proceed to the intrinsic radiation principle. Assume that μ does not coincide with a threshold, $\mu \in (\tau', \tau'')$, and $\mu \neq \tau$. Let δ denote a small positive number such that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \le \delta\}$ contains only real eigenvalues of the pencils $\mathfrak{A}^r(\,\cdot\,, \mu)$, $r = 1, \dots, \mathcal{T}$; the number of such eigenvalues (counted with their multiplicities) is denoted by $2M = 2M(\mu)$. There exist collections of elements $\{Y^+_1(\,\cdot\,,\mu), \dots, Y^+_M(\,\cdot\,,\mu)\}$ and $\{Y^-_1(\,\cdot\,,\mu), \dots, Y^-_M(\,\cdot\,,\mu)\}$ in the kernel $\ker \mathcal{A}_{-\delta}(\mu)$ of $\mathcal{A}_{-\delta}(\mu)$ such that

$$\left(Y^+_j(\,\cdot\,,\mu) - u^+_j(\,\cdot\,,\mu) - \sum_{k=1}^M S_{jk}(\mu) u^-_k(\,\cdot\,,\mu)\right) \in H^2_\delta(G), \tag{3.3}$$

$$\left(Y^-_j(\,\cdot\,,\mu) - u^-_j(\,\cdot\,,\mu) - \sum_{k=1}^M T_{jk}(\mu) u^+_k(\,\cdot\,,\mu)\right) \in H^2_\delta(G), \tag{3.4}$$

where $S(\mu) = \|S_{jk}(\mu)\|$ is a unitary scattering matrix and $S(\mu)^{-1} = T(\mu) = \|T_{jk}(\mu)\|$. Every collection $\{Y^+_1(\,\cdot\,,\mu), \dots, Y^+_M(\,\cdot\,,\mu)\}$ and $\{Y^-_1(\,\cdot\,,\mu), \dots, Y^-_M(\,\cdot\,,\mu)\}$ is a basis modulo $\ker \mathcal{A}_\delta(\mu)$ in $\ker \mathcal{A}_{-\delta}(\mu)$. This means that any $v \in \ker \mathcal{A}_{-\delta}(\mu)$ is a linear combination of the functions $Y^+_1(\,\cdot\,,\mu), \dots, Y^+_M(\,\cdot\,,\mu)$ up to a term in $\ker \mathcal{A}_\delta(\mu)$; the same is true also for $Y^-_1(\,\cdot\,,\mu), \dots, Y^-_M(\,\cdot\,,\mu)$. If μ is not an eigenvalue of the operator (3.2) with $\gamma = \delta$, that is, $\ker \mathcal{A}_\delta(\mu) = \{0\}$, then every collection $\{Y^+_j\}$ and $\{Y^-_j\}$ is a basis of $\ker \mathcal{A}_{-\delta}(\mu)$ in the usual sense.

The elements $Y(\,\cdot\,,\mu)$ in $\ker \mathcal{A}_{-\delta}(\mu) \setminus \ker \mathcal{A}_\delta(\mu)$ are called the continuous spectrum eigenfunctions of problem (2.17) corresponding to μ.

Denote by $\mathfrak{N}$ the linear hull $\mathfrak{L}(u^-_1, \dots, u^-_M)$. We define the norm of $u = \sum c_j u^-_j + v \in \mathfrak{N} \dotplus H^2_\delta(G)$ with $c_j \in \mathbb{C}$ and $v \in H^2_\delta(G)$ by

$$\|u\| = \sum |c_j| + \|v; H^2_\delta(G)\|.$$

Let $\mathbb{A}(\mu)$ be the restriction of the operator $\mathcal{A}_{-\delta}(\mu)$ to the space $\mathfrak{N} \dotplus H^2_\delta(G)$. The map

$$\mathbb{A}(\mu) : \mathfrak{N} \dotplus H^2_\delta(G) \to H^0_\delta(G) \times H^{3/2}_\delta(\partial G) =: \mathcal{H}_\delta(G) \tag{3.5}$$

is continuous. The next theorem provides the statement of problem (3.1) with intrinsic radiation conditions at infinity (the numbers μ and δ are assumed to satisfy the requirements given before (3.3)).

Theorem 3.1. *Suppose $z_1, \dots, z_d$ form a basis in the space $\ker \mathcal{A}_\delta(\mu)$, $\{f, g\} \in \mathcal{H}_\delta(G)$, and $(f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G} = 0$, $j = 1, \dots, d$. The following statements hold.*

1. *There exists a solution $u \in \mathfrak{N} \dotplus H^2_\delta(G)$ of the equation $\mathbb{A}(\mu)u = \{f, g\}$, determined up to an arbitrary term in $\ker \mathcal{A}_\delta(\mu)$.*
2. *We have*

$$v \equiv u - c_1 u^-_1 - \dots - c_M u^-_M \in H^2_\delta(G) \tag{3.6}$$

with $c_j = i(f, Y^-_j)_G + i(g, -\partial_\nu Y^-_j)_{\partial G}$.

3. *The inequality*

$$\|v; H^2_\delta(G)\| + |c_1| + \dots + |c_M| \le \text{const}\,(\|\{f, g\}; \mathcal{H}_\delta(G)\| + \|\rho_\delta v; L_2(G)\|) \tag{3.7}$$

holds true with v and $c_1, \dots, c_M$ as in (3.6). *A solution u_0 satisfying the additional conditions $(u_0, z_j)_G = 0$ for $j = 1, \dots, d$ is unique and obeys* (3.7) *with the right-hand side changed for* const $\|\{f, g\}; \mathcal{H}_\delta(G)\|$.

4. *If $\{f, g\} \in \mathcal{H}_\delta(G) \cap \mathcal{H}_{\delta'}(G)$ and the strip $\{\lambda \in \mathbb{C} : \min\{\delta, \delta'\} \le \operatorname{Im}\lambda \le \max\{\delta, \delta'\}\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\,\cdot\,, \mu)$, $r = 1, \dots, \mathcal{T}$, then the solutions $u \in \mathfrak{N} \dotplus H^2_\delta(G)$ and $u' \in \mathfrak{N} \dotplus H^2_{\delta'}(G)$ coincide, and the choice between δ and δ' affects in essence only the constant in* (3.7).

Remark 3.2. In Theorem 3.1, the numbers δ and "const" in (3.7) can be taken the same for all μ in $[\mu', \mu''] \subset (\tau, \tau'')$ (in $[\mu', \mu''] \subset (\tau', \tau)$). If μ'' approaches τ'' (or τ), then δ must tend to zero: an admissible interval for δ should be narrowed because the imaginary eigenvalues of the pencils move closer to the real axis; the constant in (3.7) grows. It can be seen from the proof of Theorem 3.1 in [1] that the constant also grows when μ' approaches τ (or τ').

Now we turn to the expanded radiation principle in a neighborhood of τ. To this end, for problem (2.17), we construct a basis of waves stable at the threshold τ. We build such a basis from the waves generated by the functions (2.10), (2.11) and from the waves corresponding to the real eigenvalues of the pencils $\mathfrak{A}^r(\,\cdot\,, \mu)$, $r = 2, \dots, \mathcal{T}$. In accordance with our assumption (at the beginning of §3), the interval $[\tau', \tau'']$ contains no thresholds for problems of the form (2.1) in the cylinders $\Pi^2, \dots, \Pi^{\mathcal{T}}$. Therefore, the number of real eigenvalues for every one of the pencils $\mathbb{R} \ni \lambda \to \mathfrak{A}^r(\lambda, \mu)$, $r = 2, \dots, \mathcal{T}$, remains invariant for $\mu \in [\tau', \tau'']$. Thus, when passing from the cylinder Π^1 to the domain G, the dimension of wave space increases by the same number for all $\mu \in (\tau', \tau'')$. We set $2L = \dim \mathcal{W}(\mu, G)$ for $\mu \in (\tau', \tau)$ and $2M = \dim \mathcal{W}(\mu, G)$ for $\mu \in (\tau, \tau'')$; then $M - L = m - l$, where m and l are the same as in (2.10), (2.11), and $\dim \mathcal{W}_a(\mu, G) = 2M$ for $\mu \in (\tau', \tau)$.

The number γ for the operators $\mathcal{A}_{\pm\gamma}(\mu)$ is chosen so as to fit for all μ in a neighborhood of the threshold $\tau = \mu_m$. Let us explain such a choice. We have $\lambda_k^\pm(\mu) = \pm(\mu - \mu_k)^{1/2}$, $\mu_{l+1} = \dots, \mu_m$, so that $\lambda_k^\pm(\tau) = 0$ with $k = l+1, \dots, m$. The interval of the imaginary axis with the ends $-i(\mu_{m+1} - \mu_m)^{1/2}$, $i(\mu_{m+1} - \mu_m)^{1/2}$ punctured at the coordinate origin is free from the spectra of the pencils $\mathfrak{A}^q(\,\cdot\,, \mu_m)$, $q = 1, \dots, \mathcal{T}$. If μ moves a little along $\mathbb{R}$, the eigenvalues of the pencils $\mathfrak{A}^q(\,\cdot\,, \mu)$ move slightly along the coordinate axes. Therefore, for a small $\alpha > 0$, there exists $\beta > 0$ such that for $\mu \in (\mu_m - \beta, \mu_m + \beta)$ the intervals $iI_{\pm\alpha} := \pm i(\alpha, (\mu_{m+1} - \mu_m)^{1/2} - \alpha)$ are free from the spectra of the pencils $\mathfrak{A}^q(\,\cdot\,, \mu)$. Thus, the lines $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = \pm\gamma\}$ with $\gamma \in I_\alpha$ do not intersect the spectra of the pencils $\mathfrak{A}^q(\,\cdot\,, \mu)$, while the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \le \gamma\}$ contains only real eigenvalues of the pencils and the numbers $\lambda_k^\pm(\mu) = \pm(\mu - \mu_k)^{1/2} = \pm(\mu - \mu_m)^{1/2}$ occurring in (2.10), $k = l+1, \dots, m$.

Let $\mu \in (\tau - \beta, \tau + \beta)$, let $\gamma \in I_\alpha$, and let $\{w_k^\pm(\,\cdot\,, \mu)\}$ be the stable basis of waves in G described in Subsections 2.1 and 2.2. In the kernel $\ker \mathcal{A}_{-\gamma}(\mu)$ of $\mathcal{A}_{-\gamma}(\mu)$, there exist collections of elements $\{\mathcal{Y}_1^+(\,\cdot\,, \mu), \dots, \mathcal{Y}_M^+(\,\cdot\,, \mu)\}$ and $\{\mathcal{Y}_1^-(\,\cdot\,, \mu), \dots, \mathcal{Y}_M^-(\,\cdot\,, \mu)\}$ such that

$$\left(\mathcal{Y}_j^+(\,\cdot\,, \mu) - w_j^+(\,\cdot\,, \mu) - \sum_{k=1}^M \mathcal{S}_{jk}(\mu) w_k^-(\,\cdot\,, \mu)\right) \in H^2_\gamma(G), \tag{3.8}$$

$$\left(\mathcal{Y}_j^-(\,\cdot\,, \mu) - w_j^-(\,\cdot\,, \mu) - \sum_{k=1}^M \mathcal{T}_{jk}(\mu) w_k^+(\,\cdot\,, \mu)\right) \in H^2_\gamma(G), \tag{3.9}$$

where $\mathcal{S}(\mu) = \|\mathcal{S}_{jk}(\mu)\|$ is a unitary matrix and $\mathcal{S}(\mu)^{-1} = \mathcal{T}(\mu) = \|\mathcal{T}_{jk}(\mu)\|$. The collections $\{\mathcal{Y}_1^+(\,\cdot\,, \mu), \dots, \mathcal{Y}_M^+(\,\cdot\,, \mu)\}$ and $\{\mathcal{Y}_1^-(\,\cdot\,, \mu), \dots, \mathcal{Y}_M^-(\,\cdot\,, \mu)\}$ are bases (modulo $\ker \mathcal{A}_\gamma(\mu)$) in $\ker \mathcal{A}_{-\gamma}(\mu)$.

The elements $\mathcal{Y}(\,\cdot\,,\mu)$ in $\ker\mathcal{A}_{-\gamma}(\mu)\setminus\ker\mathcal{A}_{\gamma}(\mu)$ are called the *continuous spectrum eigenfunctions* of problem (2.17) corresponding to the number μ. The matrix $\mathcal{S}(\mu)$ (with $\mu\in(\tau-\beta,\tau)$) is called the *augmented scattering matrix*.

Let $\mathfrak{K}$ denote the linear hull $\mathfrak{L}(w_1^-,\dots,w_M^-)$. We define the norm of $w=\sum c_jw_j^-+v\in\mathfrak{K}\dotplus H^2_\gamma(G)$, where $c_j\in\mathbb{C}$ and $v\in H^2_\gamma(G)$, by the formula

$$\|w\|=\sum|c_j|+\|v;H^2_\gamma(G)\|.$$

Let $\mathbb{A}(\mu)$ be the restriction of $\mathcal{A}_{-\gamma}(\mu)$ to the space $\mathfrak{K}\dotplus H^2_\gamma(G)$; then the mapping

$$\mathbb{A}(\mu)\,:\,\mathfrak{K}\dotplus H^2_\gamma(G)\to H^0_\gamma(G)\times H^{3/2}_\gamma(\partial G)=:\mathcal{H}_\gamma(G). \tag{3.10}$$

is continuous.

Theorem 3.3. *Let $\mu\in(\tau-\beta,\tau+\beta)$, let $\gamma\in I_\alpha$, and let $\{w_k^\pm(\,\cdot\,,\mu)\}$ be the stable basis of waves in G mentioned above. Suppose $z_1,\dots,z_d$ form a basis in the space $\ker\mathcal{A}_\gamma(\mu)$, $\{f,g\}\in\mathcal{H}_\gamma(G)$, and $(f,z_j)_G+(g,-\partial_\nu z_j)_{\partial G}=0$, $j=1,\dots,d$. The following statements hold.*

1. *There exists a solution $w\in\mathfrak{K}\dotplus H^2_\gamma(G)$ of the equation $\mathbb{A}(\mu)w=\{f,g\}$, determined up to an arbitrary term in the linear space $\mathcal{L}(z_1,\dots,z_d)$.*
2. *We have*

$$v\equiv w-c_1w_1^--\dots-c_Mw_M^-\in H^2_\gamma(G) \tag{3.11}$$

with $c_j=i(f,\mathcal{Y}_j^-)_G+i(g,-\partial_\nu\mathcal{Y}_j^-)_{\partial G}$.

3. *Such a solution w satisfies the inequality*

$$\|v;H^2_\gamma(G)\|+|c_1|+\dots+|c_M|\le\mathrm{const}\,(\|\{f,g\};\mathcal{H}_\gamma(G)\|+\|\rho_\gamma v;L_2(G)\|)\,. \tag{3.12}$$

A solution w_0 satisfying the conditions $(w_0,z_j)_G=0$ for $j=1,\dots,d$ is unique, and estimate (3.12) *is valid with the right-hand side changed for* $\mathrm{const}\,\|\{f,g\};\mathcal{H}_\gamma(G)\|$.

4. *If $\{f,g\}\in\mathcal{H}_\gamma(G)\cap\mathcal{H}_{\gamma'}(G)$ and the strip $\{\lambda\in\mathbb{C}\,:\,\min\{\gamma,\gamma'\}\le\operatorname{Im}\lambda\le\max\{\gamma,\gamma'\}\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\,\cdot\,,\mu)$, $r=1,\dots,\mathcal{T}$, then the solutions $w(\,\cdot\,,\mu)\in\mathfrak{K}\dotplus H^2_\gamma(G)$ and $w'(\,\cdot\,,\mu)\in\mathfrak{K}\dotplus H^2_{\gamma'}(G)$ of the equation $\mathbb{A}(\mu)w=\{f,g\}$ coincide, while the choice between γ and γ' affects, in essence, only the constant in* (3.12).

We would like to extend relations of the form (3.8) and (3.9) to the interval (τ',τ'') with analytic functions $\mu\mapsto\mathcal{Y}_j^\pm(\mu)$. Unlike the situation in Remark 3.2, generally speaking, relations (3.8) and (3.9) cannot be extended to an arbitrary interval $[\mu',\mu'']\subset(\tau',\tau'')$ with the same index γ. However, for this, one can use a finite collection of indices for various parts of $[\mu',\mu'']$. The following lemma explains how to compile such a collection.

Lemma 3.4. *For any interval $[\mu',\mu'']\subset(\tau',\tau'')$, there exists a finite covering $\{U_p\}_{p=0}^N$ of open intervals and a collection of indices $\{\gamma^p\}_{p=0}^N$ subject to the following conditions (with a certain nonnegative number N).*

1) *$\mu'\in U_0$, $\mu''\in U_N$; $U_0\cap U_p=\varnothing$, $p=2,\dots,N$; $U_N\cap U_p=\varnothing$, $p=0,\dots,N-2$; moreover, U_p overlaps only U_{p-1} and U_{p+1}, $1\le p\le N-1$.*

2) *The line $\{\lambda\in\mathbb{C}\,:\,\operatorname{Im}\lambda=\gamma^p\}$ is free from the spectra of the pencils $\mathfrak{A}^r(\,\cdot\,,\mu)$, $r=1,\dots,\mathcal{T}$, for all $\mu\in U_p\cap[\mu',\mu'']$ and $p=0,\dots,N$.*

3) *The strip $\{\lambda\in\mathbb{C}\,:\,\gamma^p\le\operatorname{Im}\lambda\le\gamma^{p+1}\}$ is free from the spectra of the pencils $\mathfrak{A}^r(\,\cdot\,,\mu)$, $r=1,\dots,\mathcal{T}$, for all $\mu\in U_p\cap U_{p+1}$ and $p=0,\dots,N-1$.*

4) *The inequality $|\operatorname{Im}(\mu-\tau)^{1/2}|<\gamma^p$ is fulfilled for $\mu\in U_p\cap[\mu',\mu'']$ (recall that the numbers $\pm(\mu-\tau)^{1/2}$ are eigenvalues of $\mathfrak{A}^1(\,\cdot\,,\mu)$, $\tau=\mu_{l+1}=\dots=\mu_m$); there are no other eigenvalues of the pencils $\mathfrak{A}^r(\,\cdot\,,\mu)$, $r=1,\dots,\mathcal{T}$, in the strip $\{\lambda\in\mathbb{C}\,:\,|\operatorname{Im}\lambda|\le\gamma^p\}$, except for the real ones, $p=0,\dots,N$.*

Proof. We outline the proof. Consider an interval $[\mu', \mu'']$ assuming $\tau \in (\mu', \mu'')$. Before formulas (3.8) and (3.9), we defined an interval $(\tau - \beta, \tau + \beta)$ that can be taken as an element of the desired covering. It has been shown that, as an index γ for such an element, one can take any number in $I_\alpha = (\alpha, (\mu_{m+1} - \mu_m)^{1/2} - \alpha)$ with small positive α; the number β depends on α.

We take some $\nu \in (\tau, \tau+\beta)$. The eigenvalue $\lambda_m(\mu) = (\mu - \mu_m)^{1/2}$ of the pencil $\mathfrak{A}^1(\,\cdot\,, \mu)$ is real for $\mu > \nu$, the eigenvalue $\lambda_{m+1}(\mu) = i(\mu_{m+1} - \mu)^{1/2}$ of this pencil tends to zero as μ increases from ν to $\tau'' = \mu_{m+1}$, and the interval $\{z \in \mathbb{C} : z = it,\ 0 < t < (\mu_{m+1} - \mu'')^{1/2}\}$ of the imaginary axis remains free from the spectra of the pencils $\mathfrak{A}^r(\,\cdot\,, \mu)$, $\mu' \le \mu \le \mu''$, $r = 1, \dots, \mathcal{T}$. Therefore, the interval $(\nu, \widetilde{\nu})$ with $\mu'' < \widetilde{\nu} < \tau''$ can serve as an element of the covering, and any number $\gamma \in (0, (\mu_{m+1} - \mu'')^{1/2})$ can serve as an index for the element. Finally, we choose elements U_p to the left of the threshold τ so that the graphs of the functions $U_p \ni \mu \mapsto \gamma^p = \text{const}$ be located between the graphs of the functions $(\tau', \tau) \ni \mu \mapsto \operatorname{Im} \lambda_k(\mu) = (\mu_k - \mu)^{1/2}$, $k = m, m+1$, and the indices form a monotone decreasing sequence $\gamma^0 > \gamma^1 > \dots$. $\square$

3.2. Analyticity of scattering matrices with respect to the spectral parameter. Consider the bases $\{\mathcal{Y}_j^+\}$ and $\{\mathcal{Y}_j^-\}$ in the spaces of continuous spectrum eigenfunctions (CSE) defined near the threshold τ (see (3.8) and (3.9)). First, we show that the functions $\mu \mapsto \mathcal{Y}_j^\pm(\,\cdot\,, \mu)$ admit analytic extension to the interval (τ', τ''). In what follows, by the analyticity of a function on an interval we mean the possibility of analytic continuation of that function to a complex neighborhood of every point in the interval. Then we prove the analyticity of the scattering matrix $\mu \mapsto \mathcal{S}(\mu)$ on (τ', τ''). This analyticity does not exclude the existence of eigenvalues of problem (2.17) embedded into the continuous spectrum; however, this eliminates the arbitrariness in the choice of CSE. Moreover, we establish the analyticity of the elements $\mu \mapsto Y_j^\pm(\,\cdot\,, \mu)$ in (3.3) and (3.4) as well as the analyticity of the corresponding scattering matrix $\mu \mapsto S(\mu)$ on (τ', τ) and (τ, τ'').

In a neighborhood of any point of the interval (τ', τ''), one can define an operator $\mathcal{A}_\gamma(\mu)$ required for relations like (3.8) and (3.9). The index γ is provided by Lemma 3.4: one and the same number γ^p can serve for all $\mu \in U_p$. Therefore, for $\mu \in U_p$ there exist families $\{\mathcal{Y}_j^\pm(\,\cdot\,, \mu)\} \subset \ker \mathcal{A}_{-\gamma^p}(\mu)$ satisfying relations like (3.8) and (3.9) with a unitary matrix $\mathcal{S}(\mu)$, so that Theorem 3.3 is valid with $\mu \in U_p$. Thus, it suffices to prove the analyticity of the "local families" $\{\mathcal{Y}_j^\pm(\,\cdot\,, \mu)\}$ and that of the matrix $\mathcal{S}(\mu)$ on U_p and to verify the compatibility of such families on the intersections of neighborhoods.

First, we obtain a representation of the operator $\mathbb{A}(\mu)^{-1}$, where $\mathbb{A}(\mu)$ is the operator (3.5) or (3.10), in a neighborhood of an eigenvalue of problem (2.17). For this, we recall some facts of the theory of holomorphic operator-valued functions (see, e.g., [9]). Let $\mathcal{D}$ be a domain in a complex plane, B_1 and B_2 Banach spaces, and $\mathbb{A}$ a holomorphic operator-valued function $\mathcal{D} \ni \mu \mapsto \mathbb{A}(\mu)\colon B_1 \to B_2$. The spectrum of the function $\mathbb{A}(\,\cdot\,)$ is the set of points $\mu \in \mathcal{D}$ such that $\mathbb{A}(\mu)$ is a noninvertible operator. A number μ_0 is called an *eigenvalue* of $\mathbb{A}$ if there exists a nonzero vector $\varphi_0 \in B_1$ such that $\mathbb{A}(\mu_0)\varphi_0 = 0$; then φ_0 is called an *eigenvector*. Let μ_0 and φ_0 be an eigenvalue and an eigenvector. Elements $\varphi_1, \dots, \varphi_{m-1}$ are called *generalized eigenvectors* if

$$\sum_{q=0}^{n} \frac{1}{q!} (\partial_\mu^q \mathbb{A})(\mu_0) \varphi_{n-q} = 0,$$

where $n = 1, \dots, m$. A holomorphic function $\mathbb{A}$ is said to be *Fredholm* if the operator $\mathbb{A}(\mu)\colon B_1 \to B_2$ is Fredholm for all $\mu \in \mathcal{D}$ and is invertible for at least one μ. The spectrum of a Fredholm function $\mathbb{A}$ consists of isolated eigenvalues of finite algebraic multiplicity. The holomorphic function $\mathbb{A}^*$ adjoint to $\mathbb{A}$ is defined on the set $\{\mu : \bar{\mu} \in \mathcal{D}\}$

by $\mathbb{A}^*(\mu) = (\mathbb{A}(\bar\mu))^*\colon B_2^* \to B_1^*$. If one of the functions $\mathbb{A}$ and $\mathbb{A}^*$ is Fredholm, then so is the other. A number μ_0 is an eigenvalue of $\mathbb{A}$ if and only if $\bar\mu_0$ is an eigenvalue of $\mathbb{A}^*$; the algebraic and geometric multiplicities of $\bar\mu_0$ coincide with those of μ_0.

Consider the operator-valued function $\mu \mapsto \mathbb{A}(\mu)$ occurring in (3.5) or (3.10) on a segment $[\mu', \mu'']$ lying in one of the intervals (τ', τ) or (τ, τ''). By Remark 3.2, we can choose one and the same index δ in (3.5) and in Theorem 3.1 for all $\mu \in [\mu', \mu'']$. When considering the function $\mu \mapsto \mathbb{A}(\mu)$ in (3.10) on an interval $[\mu', \mu''] \subset (\tau', \tau'')$, we suppose that the interval is so small that Lemma 3.4 enables us to take one and the same γ in (3.10) and in Theorem 3.3 for all $\mu \in [\mu', \mu'']$. By Proposition 2.1, the waves in the definitions of the operators (3.5) and (3.10) are holomorphic in a complex neighborhood of the corresponding interval $[\mu', \mu'']$. Therefore, the functions $\mu \mapsto \mathbb{A}(\mu)$ in Theorems 3.1 and 3.3 are holomorphic in the same neighborhood.

Proposition 3.5. i). *Let $\mu \mapsto \mathbb{A}(\mu)$ be the function in Theorem* 3.3, *μ_0 an eigenvalue of operator* (3.2), *and $(z_1, \dots, z_d)$ a basis of* $\ker \mathcal{A}_\gamma(\mu_0)$. *Then in a punctured neighborhood of μ_0 we have the representation*

$$\mathbb{A}^{-1}(\mu)\{f, g\} = (\mu - \mu_0)^{-1}\mathbb{P}\{f, g\} + \mathbb{R}(\mu)\{f, g\}, \tag{3.13}$$

where $\{f, g\} \in \mathcal{H}_\gamma(G)$,

$$\mathbb{P}\{f, g\} = -\sum_{j=1}^{d}\big((f, z_j)_G + (g, -\partial_\nu z_j)_{\partial G}\big)z_j, \tag{3.14}$$

and the function $\mathbb{R}(\mu)\colon \mathcal{H}_\gamma(G) \to \mathfrak{K} \dotplus H^2_\gamma(G)$ is holomorphic in a neighborhood of μ_0.

ii). *Let $\mu \mapsto \mathbb{A}(\mu)$ be the operator-valued function in Theorem* 3.1, *μ_0 an eigenvalue of operator* (3.2) *in (τ', τ) or (τ, τ''), and $(z_1, \dots, z_d)$ a basis of* $\ker \mathcal{A}_\delta(\mu_0)$. *Then in a punctured neighborhood of μ_0 we have the representation* (3.13), *where $\mathbb{P}\{f, g\}$ is defined by* (3.14) *and the function $\mathbb{R}(\mu)\colon \mathcal{H}_\delta(G) \to \mathfrak{N} \dotplus H^2_\delta(G)$ is holomorphic in a neighborhood of μ_0.*

Proof. i). By Theorem 3.3, item 1, the operator $\mathbb{A}(\mu)$ is Fredholm at any $\mu \in [\mu', \mu'']$. We may assume that $\mathbb{A}(\mu)$ is Fredholm in a neighborhood U (the Fredholm property is stable with respect to perturbations that are small in the operator norm). Moreover, the operator $\mathbb{A}(\mu)$ is invertible for all $\mu \in [\mu', \mu'']$ except for the eigenvalues of the operator (3.2), which are real and isolated. Hence, the function $\mu \mapsto \mathbb{A}(\mu)$ is Fredholm in a neighborhood of μ_0 in the complex plane. From Theorem 3.3, item 4, it follows that the eigenspaces of operators (3.10) and (3.2) coincide, that is, $\ker \mathbb{A}(\mu_0) = \ker \mathcal{A}_\gamma(\mu_0) \subset H^2_\gamma(G)$. It is easy to verify that the operator-valued function $\mathbb{A}$ has no generalized eigenvectors at μ_0. Then the Keldysh theorem on the resolvent of holomorphic operator-valued function (see [9]) provides the identity

$$\mathbb{A}^{-1}(\mu)\{f, g\} = (\mu - \mu_0)^{-1}\mathbb{T}\{f, g\} + \mathbb{R}(\mu)\{f, g\}; \tag{3.15}$$

here $\mathbb{T}\{f, g\} = \sum_{j=1}^{d}\langle\{f, g\}, \{\psi_j, \chi_j\}\rangle z_j$, the duality $\langle\,\cdot\,,\,\cdot\,\rangle$ on the pair $\mathcal{H}_\gamma(G)$, $\mathcal{H}_\gamma(G)^*$ is defined by $\langle\{f, g\}, \{\psi, \chi\}\rangle = (f, \psi)_G + (g, \chi)_{\partial G}$, and $(\,\cdot\,,\,\cdot\,)_G$ and $(\,\cdot\,,\,\cdot\,)_{\partial G}$ are the extensions of the inner products on $L_2(G)$ and $L_2(\partial G)$ to the pairs $H^0_\gamma(G), H^0_\gamma(G)^*$ and $H^{3/2}_\gamma(\partial G), H^{3/2}_\gamma(\partial G)^*$, respectively. The elements $\{\psi_j, \chi_j\} \in \ker \mathbb{A}(\mu_0)^* \subset W(G; \gamma)^*$ are subject to the orthogonality and normalization conditions

$$\langle(\partial_\mu \mathbb{A})(\mu_0)z_j, \{\psi_k, \chi_k\}\rangle = \delta_{jk}, \quad j, k = 1, \dots, d. \tag{3.16}$$

Furthermore, $(\partial_\mu \mathbb{A})(\mu_0)z_j = \{-z_j, 0\} \in W(G; \gamma)$. The elements $\{\psi_k, \chi_k\}$ can be interpreted in terms of the Green formula and, by (3.16), can be rewritten in the form

$\{\psi_k, \chi_k\} = \{-z_k, \partial_\nu z_k\}$ (see, e.g., [1]). Now $\mathbb{T}\{f, g\}$ coincides with $\mathbb{P}\{f, g\}$ in (3.14) and (3.15) takes the form (3.13).

ii). The argument in (i) can be repeated with evident modifications. □

Now we are ready to discuss the analyticity of bases in the space of the continuous spectrum eigenfunctions. For instance, we consider the basis $\{\mathcal{Y}_j^+\}$ in (3.8). From the definition of the wave w_j^+ in G (see 2.2), it follows that the function $G \ni x \mapsto w_j^+(x, \mu)$ is supported by one of the cylindrical ends of G,

$$-\Delta w_j^+(x, \mu) - \mu w_j^+(x, \mu) = f_j(x, \mu), \quad x \in G,$$
$$w_j^+(x, \mu) = 0, \quad x \in \partial G,$$

and the support of the function $x \mapsto f_j(x, \mu)$ is compact. Consider the equation

$$\mathbb{A}(\mu) w(\,\cdot\,, \mu) = \{f_j(\,\cdot\,, \mu), 0\} \tag{3.17}$$

on an interval $[\mu', \mu''] \subset (\tau', \tau'')$. First, we assume that $[\mu', \mu'']$ is free from the eigenvalues of the operator-valued function $\mu \mapsto \mathbb{A}(\mu)$. By Theorem 3.3, for all $\mu \in [\mu', \mu'']$ there exists a unique solution $w = v + c_1 w_1^- + \cdots + c_M w_M^-$ of equation (3.17),

$$w(\,\cdot\,, \mu) = \{c_1(\mu), \dots, c_M(\mu), v(\,\cdot\,, \mu)\} \in \mathfrak{K} \dot{+} H^2_\gamma(G). \tag{3.18}$$

Since the functions $\mu \mapsto \mathbb{A}(\mu)^{-1}$ and $\mu \mapsto f_j(\,\cdot\,, \mu)$ are holomorphic in a complex neighborhood of the interval $[\mu', \mu'']$, so are the components of the vector-valued function $\mu \mapsto w(\,\cdot\,, \mu)$. Therefore, the analyticity of the function $\mu \mapsto \mathcal{Y}_j^+(\,\cdot\,, \mu)$ in the same neighborhood follows from the relation

$$\mathcal{Y}_j^+ = w_j^+ - w. \tag{3.19}$$

Assume now that the interval $[\mu', \mu'']$ contains an eigenvalue μ_0 of the operator-valued function $\mu \mapsto \mathbb{A}(\mu)$. We find the residue $\mathbb{P}\{f, g\}$ in (3.13) for $\{f, g\} = \{f_j, 0\}$ on the right-hand side of (3.17). For $z \in \ker \mathcal{A}_\gamma(\mu_0)$, we have

$$(f, z)_G + (g, -\partial_\nu z)_{\partial G} = (f_j, z)_G = (-\Delta w_j^+ - \mu w_j^+, z)_G = (w_j^+, -\Delta z - \mu z)_G = 0.$$

Hence $\mathbb{P}\{f_j, 0\} = 0$ and, by (3.13),

$$w(\,\cdot\,, \mu) = \mathbb{A}(\mu)^{-1}\{f_j, 0\} = \mathbb{R}(\mu)\{f_j, 0\},$$

which means that the function $\mu \mapsto w(\,\cdot\,, \mu)$ is analytic in a neighborhood of μ_0. This implies the analyticity of the function $\mu \mapsto \mathcal{Y}_j^+(\,\cdot\,, \mu)$.

The analyticity of the functions $\mu \mapsto \mathcal{Y}_j^-(\,\cdot\,, \mu)$ can be proved in the same way. When verifying the analyticity of functions of the form $\mu \mapsto Y_j^+(\,\cdot\,, \mu)$ and $\mu \mapsto Y_j^-(\,\cdot\,, \mu)$ in (3.3) and (3.4) in a complex neighborhood of the interval $[\mu', \mu''] \subset (\tau', \tau)$ or $[\mu', \mu''] \subset (\tau, \tau'')$, it suffices to make only evident modifications in the above argument.

Lemma 3.4 and Theorem 3.3, item 4 enable us to extend formulas (3.8) and (3.9) to the entire interval (τ', τ'') for the analytic families $\mu \mapsto \mathcal{Y}_j^\pm(\,\cdot\,, \mu)$; however, one index γ must be replaced by a collection of indices. Nonetheless, in a neighborhood of any given point $\mu \in (\tau', \tau'')$, one can do with one index γ. Remark 3.2 and Theorem 3.1, item 4 allow us to extend (3.3) and (3.4) to the intervals (τ', τ) and (τ, τ'') for the analytic families $\mu \mapsto Y_j^\pm(\,\cdot\,, \mu)$.

Theorem 3.6. *Let τ' and τ'' be thresholds of problem* (2.17) *such that $\tau' < \tau''$ and the interval (τ', τ'') contains a single threshold τ. We also suppose that the three thresholds are related to the same cylindrical end.*

i) *On the intervals (τ', τ) and (τ, τ''), there exist analytic bases $\{\mu \mapsto Y_j^\pm(\,\cdot\,, \mu)\}$ in the spaces of continuous spectrum eigenfunctions of problem* (2.17) *satisfying* (3.3) *and* (3.4) *with the scattering matrix $\mu \mapsto S(\mu)$ analytic on the intervals mentioned.*

ii) *On the interval* (τ', τ''), *there exist analytic bases* $\{\mu \mapsto \mathcal{Y}_j^{\pm}(\,\cdot\,,\mu)\}$ *in the spaces of continuous spectrum eigenfunctions of problem* (2.17) *satisfying* (3.8) *and* (3.9) *with the scattering matrix* $\mu \mapsto \mathcal{S}(\mu)$ *analytic on* (τ', τ'').

Proof. The argument in Subsection 3.2 shows that it suffices to verify the analyticity of the scattering matrices. For example, consider the matrix $\mu \mapsto \mathcal{S}(\mu)$. Identity (3.19), the representation $w = v + c_1 w_1^- + \dots + c_M w_M^-$, and relation (3.18) yield

$$\mathcal{Y}_j^+(\,\cdot\,,\mu) = w_j^+(\,\cdot\,,\mu) - \sum_{k=1}^{M} c_k(\mu) w_k^-(\,\cdot\,,\mu) \in H^2_\gamma(G).$$

Therefore, $\mathcal{S}_{jk}(\mu) = -c_k(\mu)$, $k = 1, \dots, M$. It remains to recall that the functions $\mu \mapsto c_k(\mu)$ are analytic on (τ', τ''). □

For the basis $\{\mathcal{Y}_j^+(\,\cdot\,,\mu)\}_{j=1}^M$ (see Theorem 3.6, item ii), we introduce the columns $\mathcal{Y}_{(\mathbf{1})}^+ = (\mathcal{Y}_1^+, \dots, \mathcal{Y}_L^+)^t$ and $\mathcal{Y}_{(\mathbf{2})}^+ = (\mathcal{Y}_{L+1}^+, \dots, \mathcal{Y}_M^+)^t$ and write the scattering matrix in the form

$$\mathcal{S}(\mu) = \begin{pmatrix} \mathcal{S}_{(11)}(\mu) & \mathcal{S}_{(12)}(\mu) \\ \mathcal{S}_{(21)}(\mu) & \mathcal{S}_{(22)}(\mu) \end{pmatrix},$$

where $\mathcal{S}_{(11)}(\mu)$ is a block of size $L \times L$ and $\mathcal{S}_{(22)}(\mu)$ is a block of size $(M-L) \times (M-L)$, and $\mu \in (\tau', \tau'')$. We also set

$$D = ((\mu - \tau)^{1/2} + 1)/((\mu - \tau)^{1/2} - 1)$$

with $(\mu - \tau)^{1/2} = i(\tau - \mu)^{1/2}$ for $\mu \le \tau$ and $(\tau - \mu)^{1/2} \ge 0$. The next assertion will be used in §4.

Lemma 3.7. *Assume that* $\mu \in (\tau', \tau]$ *and* $\mathcal{S}(\mu)$ *is the scattering matrix in Theorem* 3.6, item ii. *Then*

$$\ker(D + \mathcal{S}_{(22)}(\mu)) \subset \ker \mathcal{S}_{(12)}(\mu), \tag{3.20}$$

$$\operatorname{im}(D + \mathcal{S}_{(22)}(\mu)) \supset \operatorname{im} \mathcal{S}_{(21)}(\mu). \tag{3.21}$$

Therefore, the operator $\mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}$ *is defined on* $\operatorname{im}(D + \mathcal{S}_{(22)}(\mu))$.

Proof. We consider (3.20). Assume that $h \in \ker(D + \mathcal{S}_{(22)}(\mu))$ and $(0, h)^t \in \mathbb{C}^M$. Then

$$\begin{pmatrix} \mathcal{S}_{(11)}(\mu) & \mathcal{S}_{(12)}(\mu) \\ \mathcal{S}_{(21)}(\mu) & \mathcal{S}_{(22)}(\mu) \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} \mathcal{S}_{(12)}(\mu) h \\ -Dh \end{pmatrix}.$$

Since the matrix $\mathcal{S}(\mu)$ is unitary and $|D| = 1$, we have $\|h\|^2 = \|\mathcal{S}_{(12)}(\mu)h\|^2 + \|h\|^2$, so that $\mathcal{S}_{(12)}(\mu)h = 0$ and (3.20) is valid. Inclusion (3.21) is equivalent to

$$\ker(D + \mathcal{S}_{(22)}(\mu))^* \subset \ker \mathcal{S}_{(21)}(\mu)^*. \tag{3.22}$$

Moreover, since

$$\mathcal{S}(\mu)^* = \begin{pmatrix} \mathcal{S}_{(11)}(\mu)^* & \mathcal{S}_{(21)}(\mu)^* \\ \mathcal{S}_{(12)}(\mu)^* & \mathcal{S}_{(22)}(\mu)^* \end{pmatrix}$$

and the matrix $\mathcal{S}(\mu)^*$ is unitary, relation (3.22) can be proved by using the same argument as for (3.20). □

§4. Other properties of the scattering matrices

Here we clarify the relationship between the matrices $\mathcal{S}(\mu)$ and $S(\mu)$ on the interval $\tau' < \mu < \tau$, prove the existence of the one-sided finite limits of $S(\mu)$ as $\mu \to \tau \pm 0$, and describe the transformation of the scattering matrix under the changes of bases in the space of waves $\mathcal{W}(\mu, G)$ for $\mu \in (\tau, \tau'')$. We keep the assumptions introduced at the very beginning of §3.

4.1. The relationship between $\mathcal{S}(\mu)$ and $S(\mu)$ for $\tau' < \mu < \tau$. We recall the description of the stable basis chosen for the definition of $\mathcal{S}(\mu)$. In the semicylinder Π^1_+, we introduce the functions

$$\Pi^1_+ \ni (y,t) \mapsto e^{\pm}_k(y,t;\mu) := \chi(t)\exp(\pm it\sqrt{\mu-\mu_k})\varphi_k(y), \tag{4.1}$$

where $k = l+1,\dots,m$ (the notation is the same as in (2.10); as before, $\mu_{l+1} = \dots = \mu_m = \tau$). We extend the functions by zero to the entire domain G and set

$$w^{\pm}_{L+j}(\,\cdot\,;\mu) = 2^{-1/2}\left(\frac{e^+_{l+j}(\,\cdot\,;\mu)+e^-_{l+j}(\,\cdot\,;\mu)}{2} \mp \frac{e^+_{l+j}(\,\cdot\,;\mu)-e^-_{l+j}(\,\cdot\,;\mu)}{2\sqrt{\mu-\mu_{l+j}}}\right) \tag{4.2}$$

for $j = 1,\dots,m-l = M-L$ (the fact that $m-l = M-L$ was explained after Remark 3.2). All the other waves with supports in Π^1_+ that were obtained from the functions (2.11) and the waves of the same type with supports in $\Pi^2_+,\dots,\Pi^{\mathcal{T}}_+$ will be numbered by one index $j = 1,\dots,L$ and denoted by $w^{\pm}_1(\,\cdot\,;\mu),\dots,w^{\pm}_L(\,\cdot\,;\mu)$. The resulting collection $\{w^{\pm}_1,\dots,w^{\pm}_M\}$ is a basis of waves in G stable in a neighborhood of the threshold τ. Finally, we introduce the columns $\mathbf{w}^{\pm}_{(\mathbf{1})} = (w^{\pm}_1,\dots,w^{\pm}_L)^t$, $\mathbf{w}^{\pm}_{(\mathbf{2})} = (w^{\pm}_{L+1},\dots,w^{\pm}_M)^t$, and $(\mathbf{w}^{\pm}_{(\mathbf{1})},\mathbf{w}^{\pm}_{(\mathbf{2})}) = (w^{\pm}_1,\dots,w^{\pm}_M)^t$, where t indicates the transpose. The components of the vector $\mathbf{w}^{\pm}_{(\mathbf{1})}$ are bounded, and the components of $\mathbf{w}^{\pm}_{(\mathbf{2})}$ grow exponentially at infinity in Π^1_+. Setting $\mathbf{e}^{\pm}_{(\mathbf{1})} = (e^{\pm}_1,\dots,e^{\pm}_L)^t$ and $\mathbf{e}^{\pm}_{(\mathbf{2})} = (e^{\pm}_{L+1},\dots,e^{\pm}_M)^t$, we arrive at

$$\mathbf{w}^{\pm}_{(\mathbf{2})} = D^{\mp}\mathbf{e}^{+}_{(\mathbf{2})} + D^{\pm}\mathbf{e}^{-}_{(\mathbf{2})} \tag{4.3}$$

with

$$D^{\pm} = ((\mu-\tau)^{1/2} \pm 1)/2\sqrt{2}(\mu-\tau)^{1/2}.$$

In essence, the following assertion is contained in [7].

Proposition 4.1. *Let $\mu \in (\tau',\tau)$, and let $S(\mu)$ and $\mathcal{S}(\mu)$ be the scattering matrices in Theorem* 3.6. *Then*

$$S(\mu) = \mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(D+\mathcal{S}_{(22)}(\mu))^{-1}\mathcal{S}_{(21)}(\mu), \tag{4.4}$$

where

$$D = D^+/D^- = ((\mu-\tau)^{1/2}+1)/((\mu-\tau)^{1/2}-1).$$

Proof. We verify (4.4). Rewrite (3.8) in the form

$$\begin{aligned} &\mathcal{Y}^+_{(\mathbf{1})} - \mathbf{w}^+_{(\mathbf{1})} - \mathcal{S}_{(11)}\mathbf{w}^-_{(\mathbf{1})} - \mathcal{S}_{(12)}\mathbf{w}^-_{(\mathbf{2})} \in H^2_\gamma(G),\\ &\mathcal{Y}^+_{(\mathbf{2})} - \mathbf{w}^+_{(\mathbf{2})} - \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} - \mathcal{S}_{(22)}\mathbf{w}^-_{(\mathbf{2})} \in H^2_\gamma(G). \end{aligned} \tag{4.5}$$

Recall that $\gamma > 0$ was chosen in accordance with Lemma 3.4, so that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \gamma\}$ contains the eigenvalues $\pm(\mu-\tau)^{1/2}$ of the pencil $\mathfrak{A}^1(\,\cdot\,,\mu)$. We take $\delta > 0$ such that the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| < \delta\}$ contains only the real eigenvalues of the pencils $\mathfrak{A}^r(\,\cdot\,,\mu)$, $r = 1,\dots,\mathcal{T}$; then $\delta < \gamma$ and $H^2_\gamma(G) \subset H^2_\delta(G)$. Instead of $\mathbf{w}^{\pm}_{(\mathbf{2})}$, we substitute their expressions given by (4.3) in (4.5); if δ is as above, the vector-valued function $\mathbf{e}^+_{(\mathbf{2})}$ belongs to $H^2_\delta(G)$. As a result, we obtain

$$\mathcal{Y}^+_{(\mathbf{1})} = \mathbf{w}^+_{(\mathbf{1})} + \mathcal{S}_{(11)}\mathbf{w}^-_{(\mathbf{1})} + \mathcal{S}_{(12)}D^-\mathbf{e}^-_{(\mathbf{2})} + \mathfrak{R}_{(\mathbf{1})}, \tag{4.6}$$

$$\mathcal{Y}^+_{(\mathbf{2})} = \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} + (D+\mathcal{S}_{(22)})D^-\mathbf{e}^-_{(\mathbf{2})} + \mathfrak{R}_{(\mathbf{2})}, \tag{4.7}$$

where $\mathfrak{R}_{(\mathbf{1})},\mathfrak{R}_{(\mathbf{2})} \in H^2_\delta(G)$. Introducing the orthogonal projection

$$\mathcal{P}\colon \mathbb{C}^{M-L} \to \operatorname{im}(D+\mathcal{S}_{(22)}(\mu))$$

and using (3.21) and (4.7), we see that

$$\mathcal{P}\mathcal{Y}^+_{(\mathbf{2})} = \mathcal{S}_{(21)}\mathbf{w}^-_{(\mathbf{1})} + (D + \mathcal{S}_{(22)})D^-\mathbf{e}^-_{(\mathbf{2})} + \mathcal{P}\mathfrak{R}_{(\mathbf{2})}. \tag{4.8}$$

We apply the operator $\mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}$ to the two sides of (4.8) and subtract the resulting identity from (4.6). We have

$$Z = \mathbf{w}^+_{(\mathbf{1})} + \big(\mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{S}_{(21)}(\mu)\big)\mathbf{w}^-_{(\mathbf{1})} + R, \tag{4.9}$$

where

$$Z = \mathcal{Y}^+_{(\mathbf{1})} - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{P}\mathcal{Y}^+_{(\mathbf{2})}, \tag{4.10}$$

$$R = \mathfrak{R}_{(\mathbf{1})} - \mathcal{S}_{(12)}(\mu)(D + \mathcal{S}_{(22)}(\mu))^{-1}\mathcal{P}\mathfrak{R}_{(\mathbf{2})}. \tag{4.11}$$

The components of the vectors $\mathcal{Y}^+_{(\mathbf{1})}$ and $\mathcal{Y}^+_{(\mathbf{2})}$ satisfy problem (2.17) and the same is true for the components of the vector Z by (4.10). Moreover, since $\mathfrak{R}_{(\mathbf{1})}, \mathfrak{R}_{(\mathbf{2})} \in H^2_\delta(G)$, from (4.11) it follows that $R \in H^2_\delta(G)$. Hence, formula (4.9) describes the scattering of the vector $\mathbf{w}^+_{(\mathbf{1})}$ of incoming waves in the basis $\mathbf{w}^+_{(\mathbf{1})}, \mathbf{w}^-_{(\mathbf{1})}$, like formula (3.3), and we obtain (4.4). □

4.2. The relationship between $\mathcal{S}(\mu)$ and $S(\mu)$ for $\tau < \mu < \tau''$. We consider two bases in the wave space $\mathcal{W}(\mu, G)$ for $\tau < \mu < \tau''$. One of the bases consists of the waves in G that correspond to functions of the form $u^\pm_q(\,\cdot\,,\mu)$ in (2.6), while the other comprises the waves generated by the functions $w^\pm_q(\,\cdot\,,\mu)$ (see (2.10), (2.11)). As before, the scattering matrices defined in these bases are denoted by $S(\mu)$ and $\mathcal{S}(\mu)$ (see Theorem 3.6); this time, that is, for $\mu \in (\tau, \tau'')$, these matrices are of the same size $M \times M$.

The scattering matrices are independent of the choice of a cut-off function χ in the definition of the space $\mathcal{W}(\mu, G)$. Identifying "equivalent" waves, we can eliminate such a cut-off function from consideration. For this, we introduce the quotient space

$$\dot{\mathcal{W}}(\mu, G) := (\mathcal{W}(\mu, G) \dot{+} H^2_\gamma(G))/H^2_\gamma(G).$$

Let $\dot{v}$ stand for the class in $\dot{\mathcal{W}}(\mu, G)$ with representative $v \in \mathcal{W}(\mu, G)$. In what follows, waves of the form $\chi u^\pm_q(\,\cdot\,,\mu)$ and $\chi w^\pm_q(\,\cdot\,,\mu)$ in G are denoted by $u^\pm_q(\,\cdot\,,\mu)$ and $w^\pm_q(\,\cdot\,,\mu)$. The collections $\{\dot{u}^\pm_q(\,\cdot\,,\mu)\}_{j=1}^M$ and $\{\dot{w}^\pm_k(\,\cdot\,,\mu)\}_{k=1}^M$ are bases in the space $\dot{\mathcal{W}}(\mu, G)$, so that $\dim \dot{\mathcal{W}}(\mu, G) = 2M$. The form $q_G(u, v)$ in (2.18) is independent of the choice of representatives in $\dot{u}$ and $\dot{v}$; hence, it is defined on $\dot{\mathcal{W}}(\mu, G) \times \dot{\mathcal{W}}(\mu, G)$. From (2.8) and (2.9) it follows that

$$iq_G(\dot{u}^\pm_k(\,\cdot\,;\mu), \dot{u}^\mp_l(\,\cdot\,;\mu)) = 0 \quad \text{for all } k, l = 1, \dots, M, \tag{4.12}$$

$$iq_G(\dot{u}^\pm_k(\,\cdot\,;\mu), \dot{u}^\pm_l(\,\cdot\,;\mu)) = \mp\delta_{kl}, \tag{4.13}$$

and (2.12) and (2.13) lead to

$$iq_G(\dot{w}^\pm_r(\,\cdot\,;\mu), \dot{w}^\mp_s(\,\cdot\,;\mu)) = 0 \quad \text{for all } r, s = 1, \dots, M, \tag{4.14}$$

$$iq_G(\dot{w}^\pm_r(\,\cdot\,;\mu), \dot{w}^\pm_s(\,\cdot\,;\mu)) = \mp\delta_{rs}. \tag{4.15}$$

Thus, $\dot{\mathcal{W}}(\mu, G)$ turns out to be a $2M$-dimensional complex space with the indefinite inner product $< \dot{u}, \dot{v} >:= -iq_G(\dot{u}, \dot{v})$. The projection

$$\pi : \mathcal{W}(\mu, G) \dot{+} H^2_\gamma(G) \to \dot{\mathcal{W}}(\mu, G) \tag{4.16}$$

maps the space of continuous spectrum eigenfunctions onto a subspace in $\dot{\mathcal{W}}(\mu, G)$ of dimension M; we denote this subspace by $\mathcal{E}(\mu)$.

Let $V_1, \dots, V_{2M}$ be a basis in $\dot{\mathcal{W}}(\mu, G)$ satisfying to the orthogonality and normalization conditions

$$\langle V_j, V_l \rangle = -\delta_{j\,l},\ \langle V_{j+M}, V_{l+M} \rangle = \delta_{j\,l} \quad \text{for} \quad j, l = 1, \dots, M. \tag{4.17}$$

The elements $V_1, \dots, V_M$ are called *incoming waves*, while the elements $V_{M+1}, \dots, V_{2M}$ are called *outgoing waves*. Assume that $X_1, \dots, X_M$ is a basis of $\mathcal{E}(\mu)$ that, in the basis of waves $V_1, \dots, V_{2M}$, determines the scattering matrix $\mathfrak{S}(\mu)$ of size $M \times M$ (compare with (3.3)). We represent the vectors X_j as coordinate rows and form the $(M \times 2M)$-matrix $X = (X_1, \dots, X_M)^t$ (which is a column of the letters $X_1, \dots, X_M$). Finally, let I stand for the unit matrix of size $M \times M$. Then a relation of the form (3.3) shows that

$$X = (I, \mathfrak{S}(\mu))V, \tag{4.18}$$

where V is the $(2M \times 2M)$-matrix $(V_1, \dots, V_{2M})^t$ consisting of the coordinate rows of the vectors V_j, and $(I, \mathfrak{S}(\mu))$ is a matrix of size $M \times 2M$.

Assume that $\widetilde{V}_1, \dots, \widetilde{V}_{2M}$ is another basis of waves subject to conditions of the form (4.17), $\widetilde{X}_1, \dots, \widetilde{X}_M$ is a basis of $\mathcal{E}(\mu)$, and $\widetilde{\mathfrak{S}}(\mu)$ is the corresponding scattering matrix such that

$$\widetilde{X} = (I, \widetilde{\mathfrak{S}}(\mu))\widetilde{V}. \tag{4.19}$$

We suppose that $\widetilde{V} = \mathfrak{T}V$ and write the $(2M \times 2M)$-matrix $\mathfrak{T}$ as $\mathfrak{T} = (\mathfrak{T}_{(k\,l)})_{k,\,l=1}^2$ with blocks $\mathfrak{T}_{(k\,l)}$ of size $M \times M$.

Lemma 4.2. *The matrices* $\mathfrak{T}_{(1\,1)} + \widetilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)}$ *and* $\mathfrak{T}_{(1\,2)} + \widetilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}$ *are invertible, and*

$$\mathfrak{S}(\mu) = (\mathfrak{T}_{(1\,1)} + \widetilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)})^{-1}(\mathfrak{T}_{(1\,2)} + \widetilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}). \tag{4.20}$$

Proof. For the bases $X_1, \dots, X_M$ and $\widetilde{X}_1, \dots, \widetilde{X}_M$, there exists a nonsingular $(M \times M)$-matrix B such that $\widetilde{X} = BX$. Therefore, by (4.19), we have

$$BX = (I, \widetilde{\mathfrak{S}}(\mu))\mathfrak{T}V.$$

Using (4.18), we obtain $B(I, \mathfrak{S}(\mu))V = (I, \widetilde{\mathfrak{S}}(\mu))\mathfrak{T}V$, whence

$$B(I, \mathfrak{S}(\mu)) = (I, \widetilde{\mathfrak{S}}(\mu))\mathfrak{T}.$$

We write this in the form

$$(B, B\mathfrak{S}(\mu)) = \big(\mathfrak{T}_{(1\,1)} + \widetilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,1)}, \mathfrak{T}_{(1\,2)} + \widetilde{\mathfrak{S}}(\mu)\mathfrak{T}_{(2\,2)}\big).$$

Now the assertions of the lemma are evident. □

We intend to make use of (4.20), taking as $\widetilde{V}$ the image, under the canonical projection (4.16), of the stable basis of $\mathcal{W}(\mu, G)$ occurring in (3.8) and as V the image of the wave basis occurring in (3.3). As $\widetilde{\mathfrak{S}}(\mu)$ and $\mathfrak{S}(\mu)$, we choose $\mathcal{S}(\mu)$ and $S(\mu)$, respectively. We proceed to computing the matrix $\mathfrak{T}$ in the identity $\widetilde{V} = \mathfrak{T}V$. In doing so, instead of $\widetilde{V}$ and V we can consider their preimages in $\mathcal{W}(\mu, G)$ mentioned above. We set

$$u_j := u_j^+, \qquad u_{j+M} := u_j^-, \qquad j = 1, \dots, M, \tag{4.21}$$

where the $u_j^\pm$ are the waves in $\mathcal{W}(\mu, G)$ generated by functions of the form (2.6). We also introduce

$$\begin{aligned} &w_j := w_j^+ = u_j^+, \qquad w_{j+M} := w_j^- = u_j^-, \qquad j = 1, \dots, L, \\ &w_p := w_p^+, \qquad w_{p+M} := w_p^-, \qquad p = L+1, \dots, M, \end{aligned} \tag{4.22}$$

where the $w_p^\pm$ are the waves in $\mathcal{W}(\mu, G)$ generated by functions (2.10). For the matrix $\mathfrak{T}$, we have $w = \mathfrak{T}u$ with the columns $w = (w_1, \dots, w_{2M})^t$ and $u = (u_1, \dots, u_{2M})^t$. For

convenience, here we denote the functions (2.10) in the same way as the waves $w_p^\pm$, writing these functions in the form

$$w_p^\pm(\mu) = 2^{-1/2}\big((e^{it\lambda} + e^{-it\lambda})/2 \mp (e^{it\lambda} - e^{-it\lambda})/2\lambda\big)\varphi_p,$$

where $\lambda = \sqrt{\mu - \tau}$ and τ is a threshold; we also write the functions (2.6) in the form

$$u_p^\pm(\mu) = (2\lambda)^{-1/2} e^{\mp it\lambda}\varphi_p.$$

Then we have

$$w_p^\pm = (1/2)\big(u_p^+(\lambda^{1/2} \pm \lambda^{-1/2}) + u_p^-(\lambda^{1/2} \mp \lambda^{-1/2})\big), \qquad p = L+1, \dots, M;$$

here by $w_p^\pm$ and $u_p^\pm$ one can mean the functions in the cylinder and the corresponding waves in the domain G. Together with (4.21) and (4.22), this leads to the following description of the blocks $\mathfrak{T}_{(ij)}$ of the matrix $\mathfrak{T}$.

Lemma 4.3. *Each of the matrices $\mathfrak{T}_{(ij)}$ consists of four blocks and is block-diagonal. We have*

$$\mathfrak{T}_{(11)}(\mu) = \mathfrak{T}_{(22)}(\mu) = \operatorname{diag}\{I_L,\ 2^{-1}(\lambda^{1/2} + \lambda^{-1/2})I_{M-L}\}, \tag{4.23}$$

$$\mathfrak{T}_{(21)}(\mu) = \mathfrak{T}_{(12)}(\mu) = \operatorname{diag}\{O_L,\ 2^{-1}(\lambda^{1/2} - \lambda^{-1/2})I_{M-L}\}, \tag{4.24}$$

where I_K is the unit matrix of size $K \times K$, O_L is the zero matrix of size $L \times L$, and $\lambda = \sqrt{\mu - \tau}$ with $\mu \in (\tau, \tau'')$.

We return to (4.20) with $\mathcal{S}$ and S in place of $\widetilde{\mathfrak{S}}$ and $\mathfrak{S}$. We split the matrix $\mathcal{S}$ into four blocks with $\mathcal{S}_{(11)}$ of size $L \times L$ and $\mathcal{S}_{(22)}$ of size $(M-L) \times (M-L)$. We also set $d^\pm = 2^{-1}(\lambda^{1/2} \pm \lambda^{-1/2})$. Then

$$\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)} = \begin{pmatrix} I_L & \mathcal{S}_{(12)}d^- \\ O & \mathcal{S}_{(22)}d^- + I_{M-L}d^+ \end{pmatrix}. \tag{4.25}$$

By Lemma 4.2, the matrix $\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)}$ is invertible, so that the matrix $\mathcal{S}_{(22)}d^- + I_{M-L}d^+$ is also invertible; therefore,

$$(\mathfrak{T}_{(11)} + \mathcal{S}\mathfrak{T}_{(21)})^{-1} = \begin{pmatrix} I_L & -\mathcal{S}_{(12)}d^-(\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1} \\ O & (\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1} \end{pmatrix}. \tag{4.26}$$

Now, (4.20) leads to the following statement.

Proposition 4.4. *For $\mu \in (\tau, \tau'')$, the blocks $S_{(ij)}$ of the scattering matrix*

$$S(\mu) = (\mathfrak{T}_{(11)} + \mathcal{S}(\mu)\mathfrak{T}_{(21)})^{-1}(\mathfrak{T}_{(12)} + \mathcal{S}(\mu)\mathfrak{T}_{(22)})$$

admit the representations

$$S_{(11)} = \mathcal{S}_{(11)} - \mathcal{S}_{(12)}d^-(\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}\mathcal{S}_{(21)}, \tag{4.27}$$

$$S_{(12)} = \mathcal{S}_{(12)}d^+ - \mathcal{S}_{(12)}d^-(\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}(\mathcal{S}_{(22)}d^+ + I_{M-L}d^-), \tag{4.28}$$

$$S_{(21)} = (\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}\mathcal{S}_{(21)}, \tag{4.29}$$

$$S_{(22)} = (\mathcal{S}_{(22)}d^- + I_{M-L}d^+)^{-1}(\mathcal{S}_{(22)}d^+ + I_{M-L}d^-). \tag{4.30}$$

4.3. The limits of $S(\mu)$ as $\mu \to \tau \pm 0$. To calculate the one-sided limits of $S(\mu)$, we use formula (4.4) as $\mu \to \tau - 0$ and apply (4.27)–(4.30) as $\mu \to \tau + 0$. The computation procedure depends on whether or not the number 1 is an eigenvalue of the matrix $\mathcal{S}_{(22)}(\tau)$.

4.3.1. *The limits of $S(\mu)$ as $\mu \to \tau \pm 0$ provided* 1 *is not an eigenvalue of $\mathcal{S}_{(22)}(\tau)$.* Recall that the functions $\mu \mapsto \mathcal{S}_{(kl)}(\mu)$ are analytic in a neighborhood of $\mu = \tau$. Therefore, from (4.4) it immediately follows that

$$\lim_{\mu\to\tau-0} S(\mu) = \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau). \tag{4.31}$$

We proceed to the computation of $\lim S(\mu)$ as $\mu \to \tau + 0$. By (4.27) and (4.31),

$$\begin{aligned}\lim_{\mu\to\tau+0} S_{(11)}(\mu) &= \lim_{\mu\to\tau+0}\big(\mathcal{S}_{(11)}(\mu) - \mathcal{S}_{(12)}(\mu)(\mathcal{S}_{(22)}(\mu) + d^+(\mu)/d^-(\mu))^{-1}\mathcal{S}_{(21)}(\mu)\big)\\ &= \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau) = \lim_{\mu\to\tau-0} S(\mu).\end{aligned} \tag{4.32}$$

In accordance with (4.30),

$$\begin{aligned}\lim_{\mu\to\tau+0} S_{(22)}(\mu) &= \lim_{\mu\to\tau+0}(\mathcal{S}_{(22)} + d^+/d^-)^{-1}(\mathcal{S}_{(22)}d^+/d^- + 1)\\ &= (\mathcal{S}_{(22)}(\tau) - 1)^{-1}(-\mathcal{S}_{(22)}(\tau) + 1) = -I_{M-L}.\end{aligned} \tag{4.33}$$

Formula (4.29) implies that

$$S_{(21)}(\mu) = (\mathcal{S}_{(22)} + d^+/d^-)^{-1}\mathcal{S}_{(21)}/d^-.$$

Since $d^-(\mu) = 2^{-1}((\mu-\tau)^{1/2} - 1)/(\mu-\tau)^{1/4}$, we see that

$$S_{(21)}(\mu) = O((\mu-\tau)^{1/4}) \to 0 \quad \text{as} \quad \mu \to \tau + 0. \tag{4.34}$$

Finally, we consider $S_{(12)}(\mu)$. We rewrite (4.28) in the form

$$\begin{aligned}S_{(12)} &= \mathcal{S}_{(12)}d^+\big(1 - (\mathcal{S}_{(22)} + d^+/d^-)^{-1}(\mathcal{S}_{(22)} + d^-/d^+)\big)\\ &= \mathcal{S}_{(12)}d^+(\mathcal{S}_{(22)} + d^+/d^-)^{-1}(d^+/d^- - d^-/d^+).\end{aligned}$$

Since

$$d^+(\mu)(d^+/d^- - d^-/d^+) = 2(\mu-\tau)^{1/4}/((\mu-\tau)^{1/2} - 1),$$

we obtain

$$S_{(12)}(\mu) = O((\mu-\tau)^{1/4}) \to 0 \quad \text{as} \quad \mu \to \tau + 0. \tag{4.35}$$

4.3.2. *The limits of $S(\mu)$ as $\mu \to \tau \pm 0$ provided* 1 *is an eigenvalue of $\mathcal{S}_{(22)}(\tau)$.* We set $\lambda = \sqrt{\mu - \tau}$, $\mu = \tau + \lambda^2$, and consider the function $\lambda \mapsto \Phi(\lambda)\colon \mathbb{C}^{M-L} \to \mathbb{C}^{M-L}$ given by

$$\Phi(\lambda) := \mathcal{S}_{(22)}(\mu) + d^+(\mu)/d^-(\mu) = \mathcal{S}_{(22)}(\tau + \lambda^2) + (\lambda+1)/(\lambda-1). \tag{4.36}$$

The number $\lambda = 0$ is an eigenvalue of the function $\lambda \mapsto \Phi(\lambda)$ if and only if 1 is an eigenvalue of the matrix $\mathcal{S}_{(22)}(\tau)$; in this case $\ker(\mathcal{S}_{(22)}(\tau) - 1) = \ker \Phi(0)$. To calculate the limits of $S(\mu)$ as $\mu \to \tau \pm 0$, we need some information about the resolvent $\lambda \mapsto \Phi(\lambda)^{-1}$ in a neighborhood of $\lambda = 0$. Propositions 4.5 and 4.6 provide the required facts.

Proposition 4.5. *We have*

$$\ker(\mathcal{S}_{(22)}(\tau) - 1) = \ker(\mathcal{S}_{(22)}(\tau)^* - 1). \tag{4.37}$$

Proof. Assume that $h \in \ker(\mathcal{S}_{(22)}(\tau) - 1)$. Then, as was shown in the proof of Lemma 3.7, the vector $(0, h)^t \in \mathbb{C}^M$ belongs to $\ker(\mathcal{S}(\tau) - 1)$ and $\mathcal{S}_{(12)}(\tau)h = 0$. The same argument with $\mathcal{S}(\tau)^*$ in place of $\mathcal{S}(\tau)$ shows that if $g \in \ker(\mathcal{S}_{(22)}(\tau)^* - 1)$, then

$$(0, g)^t \in \ker(\mathcal{S}(\tau)^* - 1) \quad \text{and} \quad \mathcal{S}_{(21)}(\tau)^* g = 0.$$

Since $\mathcal{S}(\tau)^* = \mathcal{S}(\tau)^{-1}$, we have

$$\ker(\mathcal{S}(\tau) - 1) = \ker(\mathcal{S}(\tau)^* - 1). \tag{4.38}$$

Let $h_1, \dots, h_\varkappa$ be a basis of $\ker(\mathcal{S}_{(22)}(\tau) - 1)$ and $g_1, \dots, g_\varkappa$ a basis of $\ker(\mathcal{S}_{(22)}(\tau)^* - 1)$. We set $\widetilde{h}_j = (0, h_j)^t$ and $\widetilde{g}_j = (0, g_j)^t$. From (4.38) it follows that

$$\widetilde{h}_j, \widetilde{g}_j \in \ker(\mathcal{S}(\tau) - 1) = \ker(\mathcal{S}(\tau)^* - 1), \qquad j = 1, \dots, \varkappa.$$

Therefore, any vector of the collection $h_1, \dots, h_\varkappa$ is a linear combination of the vectors $g_1, \dots, g_\varkappa$, and *vice versa*. □

Proposition 4.6. *Let Φ be the matrix-valued function as in* (4.36)*, and let*

$$\dim\ker\Phi(0) = \varkappa > 0.$$

Then, in a punctured neighborhood of $\lambda = 0$, the resolvent $\lambda \mapsto \Phi(\lambda)^{-1}$ admits the representation

$$\Phi(\lambda)^{-1} = -(2\lambda)^{-1} \sum_{j=1}^{\varkappa} (\,\cdot\,, h_j) h_j + \Gamma(\lambda); \tag{4.39}$$

here the collection $h_1, \dots, h_\varkappa$ is an orthonormal basis of $\ker(\mathcal{S}_{(22)}(\tau) - 1)$*, (u, v) is the inner product on the space $\mathbb{C}^{M-L}$, and $\lambda \mapsto \Gamma(\lambda)\colon \mathbb{C}^{M-L} \to \mathbb{C}^{M-L}$ is a matrix function holomorphic in a neighborhood of $\lambda = 0$.*

Proof. It is known (see, e.g., [9, 10]) that, under certain conditions, the resolvent $\mathfrak{A}(\lambda)^{-1}$ of a holomorphic operator-valued function $\lambda \mapsto \mathfrak{A}(\lambda)$ in a punctured neighborhood of an isolated eigenvalue λ_0 admits the representation

$$\mathfrak{A}(\lambda)^{-1} = (\lambda - \lambda_0)^{-1} \sum_{j=1}^{\varkappa} (\,\cdot\,, \psi_j) \phi_j + \Gamma(\lambda), \tag{4.40}$$

where $\phi_1, \dots, \phi_\varkappa$ and $\psi_1, \dots, \psi_\varkappa$ are bases of the spaces $\ker\mathfrak{A}(\lambda_0)$ and $\ker\mathfrak{A}(\lambda_0)^*$ satisfying the orthogonality and normalization conditions

$$(\partial_\lambda \mathfrak{A}(\lambda_0)\phi_j, \psi_k) = \delta_{jk}, \qquad j, k = 1, \dots, \varkappa, \tag{4.41}$$

and Γ is an operator-valued function holomorphic in a neighborhood of λ_0. Formula (4.40) is related to the case where the operator-valued function $\lambda \mapsto \mathfrak{A}(\lambda)$ has no generalized eigenvectors at the point λ_0. To justify (4.39), we need to show that there are no generalized eigenvectors of the function $\lambda \mapsto \Phi(\lambda)$ at the point $\lambda = 0$ and to verify that formulas (4.39) and (4.40) agree.

First, we treat the generalized eigenvectors. Assume that $0 \neq h^0 \in \ker\Phi(0)$. The equation $\Phi(0)h^1 + (\partial_\lambda\Phi)(0)h^0 = 0$ for a generalized eigenvector h^1 is of the form

$$(\mathcal{S}_{(22)}(\tau) - 1)h^1 = 2h^0.$$

The orthogonality of h^0 to $\ker(\mathcal{S}_{(22)}(\tau)^* - 1) = \ker(\mathcal{S}_{(22)}(\tau) - 1)$ is necessary for the solvability of this equation (see (4.37)). Since $0 \neq h^0 \in \ker\Phi(0) = \ker(\mathcal{S}_{(22)}(\tau) - 1)$, the solvability condition is not fulfilled, showing that no generalized eigenvectors can exist.

Let us compare (4.39) and (4.40). We have $(\partial_\lambda\Phi)(0) = -2I_{M-L}$. Moreover, by (4.37), the bases $\phi_1, \dots, \phi_\varkappa$ and $\psi_1, \dots, \psi_\varkappa$ in (4.40) can be chosen so as to satisfy $\phi_j = -\psi_j = h_j/\sqrt{2}$, and for the role of $h_1, \dots, h_\varkappa$ we can take an orthonormal basis of $\ker(\mathcal{S}_{22}(\tau) - 1)$. Then

$$\big((\partial_\lambda\Phi)(0)\phi_j, \psi_k\big) = \delta_{jk}, \quad j, k = 1, \dots, \varkappa,$$

and the representation (4.40) takes the form of (4.39). □

We calculate $\lim S(\mu)$ as $\mu \to \tau - 0$. By Lemma 3.7,

$$\operatorname{im}(\mathcal{S}_{(22)}(\tau) - 1) \supset \operatorname{im} \mathcal{S}_{(21)}(\tau).$$

Therefore, Proposition 4.5 shows that $(\mathcal{S}_{(21)}(\tau)f, h_j) = 0$ for any $f \in \mathbb{C}^L$ and $h_1, \dots, h_\varkappa$ as in (4.39). Since the function $\mu \to \mathcal{S}_{(21)}(\mu)$ is analytic, we have $\mathcal{S}_{(21)}(\mu) = \mathcal{S}_{(21)}(\tau) + O(|\mu - \tau|)$; recall that $|\mu - \tau| = |\lambda|^2$. Applying (4.39), we obtain

$$(\mathcal{S}_{(22)}(\mu) + D(\mu))^{-1}\mathcal{S}_{(21)}(\mu) = \Gamma(\lambda)\mathcal{S}_{(21)}(\mu) + O(|\lambda|). \tag{4.42}$$

Now from (4.4) it follows that

$$\lim_{\mu \to \tau - 0} S(\mu) = \mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)\Gamma(0)\mathcal{S}_{(21)}(\tau), \tag{4.43}$$

and Lemma 3.7 allows us to treat the right-hand side as the operator

$$\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)(\mathcal{S}_{(22)}(\tau) - 1)^{-1}\mathcal{S}_{(21)}(\tau)$$

(see (4.31)). As $\mu \to \tau - 0$, we have the estimate

$$S(\mu) - (\mathcal{S}_{(11)}(\tau) - \mathcal{S}_{(12)}(\tau)\Gamma(0)\mathcal{S}_{(21)}(\tau)) = O(|\mu - \tau|^{(1/2)}). \tag{4.44}$$

We proceed to calculating the limits as $\mu \to \tau + 0$. We compute $\lim_{\mu \to \tau + 0} S_{(11)}(\mu)$ in the same way as $\lim_{\mu \to \tau - 0} S(\mu)$, obtaining

$$\lim_{\mu \to \tau + 0} S_{(11)}(\mu) = \lim_{\mu \to \tau - 0} S(\mu). \tag{4.45}$$

By (4.30),

$$\begin{aligned} S_{(22)}(\mu) &= \left(\mathcal{S}_{(22)}(\mu) + d^+/d^-\right)^{-1} \left(\mathcal{S}_{(22)}(\mu) + d^-/d^+\right) d^+/d^- \\ &= d^+/d^- + \left(\mathcal{S}_{(22)}(\mu) + d^+/d^-\right)^{-1} \left(d^-/d^+ - d^+/d^-\right) d^+/d^-. \end{aligned}$$

Applying the resolvent representation (4.39), we write the last identity in the form

$$S_{(22)}(\mu) = \frac{\lambda + 1}{\lambda - 1}\left(I + \frac{2}{\lambda^2 - 1}\sum_{j=1}^{\varkappa}(\,\cdot\,, h_j)h_j - \frac{4\lambda}{\lambda^2 - 1}\Gamma(\lambda)\right). \tag{4.46}$$

Hence,

$$\lim_{\mu \to \tau + 0} S_{(22)}(\mu) = 2\sum_{j=1}^{\varkappa}(\,\cdot\,, h_j)h_j - I = P - Q, \tag{4.47}$$

where $P = \sum_{j=1}^{\varkappa}(\,\cdot\,, h_j)h_j$ is the orthogonal projection of $\mathbb{C}^{M-L}$ onto $\ker(\mathcal{S}_{(22)}(\tau) - 1)$ and $Q = I - P$. Moreover, as $\mu \to \tau + 0$, from (4.46) it follows that

$$S_{(22)}(\mu) - P + Q = O(|\mu - \tau|^{1/2}). \tag{4.48}$$

In accordance with (4.29),

$$S_{(21)}(\mu) = (\mathcal{S}_{22}(\mu) + I_{M-L}d^+/d^-)^{-1}\mathcal{S}_{(21)}/d^-.$$

Using (4.42) and the fact that $d^- = (\lambda - 1)/2\sqrt{\lambda}$, we obtain

$$S_{(21)}(\mu) = \left(\Gamma(\lambda)\mathcal{S}_{(21)}(\mu) + O(|\lambda|)\right) 2\sqrt{\lambda}/(\lambda - 1).$$

Consequently,

$$S_{(21)}(\mu) = O(|\mu - \tau|^{1/4}) \to 0 \qquad \text{as} \qquad \mu \to \tau + 0. \tag{4.49}$$

It remains to find the limit of $S_{(12)}(\mu)$. By (4.28),

$$S_{(12)}(\mu) = \mathcal{S}_{(12)}(\mu)d^+ \left(I - (\mathcal{S}_{(22)}(\mu) + d^+/d^-)^{-1}(\mathcal{S}_{(22)}(\mu) + d^-/d^+)\right).$$

Since

$$(\mathcal{S}_{(22)}(\mu)+d^+/d^-)^{-1}(\mathcal{S}_{(22)}(\mu)+d^-/d^+)=I-\frac{4\lambda}{\lambda^2-1}(\mathcal{S}_{(22)}(\mu)+d^+/d^-)^{-1},$$

we get

$$S_{(12)}(\mu)=\frac{2\sqrt{\lambda}}{\lambda-1}\mathcal{S}_{(12)}(\mu)\Big(-\frac{1}{2\lambda}\sum(\,\cdot\,,h_j)h_j+\Gamma(\lambda)\Big).$$

Recall that $h_j\in\ker(\mathcal{S}_{(22)}(\tau)-1)\subset\ker\mathcal{S}_{(12)}(\tau)$ (see (3.20)), $\mathcal{S}_{(12)}(\mu)=\mathcal{S}_{(12)}(\tau)+O(|\mu-\tau|)$, and $\mu-\tau=\lambda^2$. Therefore, as $\mu\to\tau+0$, we have

$$S_{(12)}(\mu)=O(|\mu-\tau|^{1/4})\to 0. \tag{4.50}$$

§5. Method for computing the scattering matrix

First, we recall the method for calculating the scattering matrix $S(\mu)$ in Theorem 3.6, item i) with $\mu'\le\mu\le\mu''$, where $[\mu',\mu'']\subset(\tau',\tau)$ or $[\mu',\mu'']\subset(\tau,\tau'')$. The interval $[\mu',\mu'']$ may contain eigenvalues of the operator (3.5). This method was justified for the Laplace operator in [2] and generalized to elliptic systems in [3]. We set

$$\Pi_+^{r,R}=\{(y^r,t^r)\in\Pi^r\,:\,t^r>R\},\quad G^R=G\setminus\cup_{r=1}^{\mathcal{T}}\Pi_+^{r,R},$$
$$\partial G^R\setminus\partial G=\Gamma^R=\cup_r\Gamma^{r,R},\quad \Gamma^{r,R}=\{(y^r,t^r)\in\Pi^r\,:\,t^r=R\}$$

for large R and introduce the boundary-value problem

$$\begin{gathered}-\Delta X_j^R(x,\mu)-\mu X_j^R(x,\mu)=0,\quad x\in G^R;\\ X_j^R(x,\mu)=0\quad x\in\partial G^R\setminus\Gamma^R;\\ (-\partial_n+i\zeta)X_j^R(x,\mu)=(-\partial_n+i\zeta)\Big(u_j^+(x,\mu)+\sum_{k=1}^M a_k u_k^-(x,\mu)\Big),\ x\in\Gamma^R,\end{gathered}\tag{5.1}$$

where $\zeta\in\mathbb{R}\setminus\{0\}$ is an arbitrary fixed number, the a_k are complex numbers, and the $u_j^\pm$ are the waves as in (2.6). As an approximation to the row $(S_{j1},\dots,S_{jM})$, we take a minimizer $a^0(R,\mu)=(a_1^0(R,\mu),\dots,a_M^0(R,\mu))$ of the functional

$$J_j^R(a_1,\dots,a_M;\mu)=\Big\|X_j^R(\,\cdot\,,\mu)-u_j^+(\,\cdot\,,\mu)-\sum_{k=1}^M a_k u_k^-(\,\cdot\,,\mu);L_2(\Gamma^R)\Big\|^2,\tag{5.2}$$

where X_j^R is a solution of problem (5.1). To clarify the dependence of X_j^R on the parameters $a_1,\dots,a_M$, we consider the problems

$$\begin{gathered}-\Delta v_j^\pm-\mu v_j^\pm=0,\quad x\in G^R,\\ v_j^\pm=0,\quad x\in\partial G^R\setminus\Gamma^R;\\ (-\partial_n+i\zeta)v_j^\pm=(-\partial_n+i\zeta)u_j^\pm,\ x\in\Gamma^R,\quad j=1,\dots,M.\end{gathered}\tag{5.3}$$

We have $X_j^R=v_{j,R}^++\sum_k a_k v_{k,R}^-$. Introducing the $(M\times M)$-matrices with the entries

$$\mathbf{E}_{jk}^R=\big(v_j^--u_j^-,v_k^--u_k^-\big)_{\Gamma^R},$$
$$\mathbf{F}_{jk}^R=\big(v_j^+-u_j^+,v_k^--u_k^-\big)_{\Gamma^R}$$

and setting

$$\mathbf{G}_j^R=\big(v_j^+-u_j^+,v_j^+-u_j^+\big)_{\Gamma^R},$$

we can write the functional (5.2) in the form

$$J_j^R(a)=\langle a\mathbf{E}^R,a\rangle+2\,\mathrm{Re}\langle\mathbf{F}_j^R,a\rangle+\mathbf{G}_j^R,$$

where $\mathbf{F}_j^R$ is the jth row of the matrix $\mathbf{F}^R$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product on $\mathbb{C}^M$. The minimizer $a^0(R,\mu)$ satisfies $a^0(R,\mu)\mathbf{E}^R + \mathbf{F}_j^R = 0$; the matrix $\mathbf{E}^R$ is nonsingular.

In [2] it was shown that the minimizer $a^0(R,\mu)$ tends exponentially to the row $(S_{j1},\dots,S_{jM})$ as $R\to+\infty$. More precisely, we have

$$\sum_{k=1}^{M} |S_{jk}(\mu) - a_k^0(R,\mu)| \le Ce^{-\delta R}$$

for any $R \ge R_0$, where δ is the number occurring in (3.3), R_0 is a sufficiently large positive number, and the constant C is independent of R and $\mu\in[\mu',\mu'']$.

Now we proceed to calculating the matrix $\mathcal{S}(\mu)$ in Theorem 3.6, item ii) with $\mu\in[\mu',\mu'']\subset(\tau',\tau'')$. The interval $[\mu',\mu'']$ may contain the threshold τ and also some eigenvalues of the operator (3.10). Introduce the boundary-value problem

$$\begin{aligned} -\Delta\mathcal{X}_j^R - \mu\mathcal{X}_j^R &= 0, \quad x\in G^R;\\ \mathcal{X}_j^R &= 0, \quad x\in\partial G^R\setminus\Gamma^R;\\ (-\partial_n + i\zeta)\mathcal{X}_j^R &= (-\partial_n+i\zeta)\Big(w_j^+ + \sum_{k=1}^{M} a_k w_k^-\Big), \quad x\in\Gamma^R, \end{aligned} \tag{5.4}$$

where $w_j^\pm$ is the stable basis (2.10), (2.11) in the space of waves, $\zeta\in\mathbb{R}\setminus\{0\}$, and $a_k\in\mathbb{C}$. As an approximation to the row $(\mathcal{S}_{j1},\dots,\mathcal{S}_{jM})$, we suggest a minimizer $a^0(R) = (a_1^0(R),\dots,a_M^0(R))$ of the functional

$$\mathcal{J}_j^R(a_1,\dots,a_M) = \Big\|\mathcal{X}_j^R - w_j^+ - \sum_{k=1}^{M} a_k w_k^-; L_2(\Gamma^R)\Big\|^2, \tag{5.5}$$

where $\mathcal{X}_j^R$ is a solution of problem (5.4). We consider the problems

$$\begin{aligned} -\Delta z_j^\pm - \mu z_j^\pm &= 0, \quad x\in G^R;\\ z_j^\pm &= 0, \quad x\in\partial G^R\setminus\Gamma^R;\\ (-\partial_n+i\zeta)z_j^\pm &= (-\partial_n+i\zeta)w_j^\pm,\ x\in\Gamma^R; \quad j=1,\dots,M, \end{aligned}$$

set

$$\begin{aligned} \mathcal{E}_{jk}^R &= \big(z_j^- - w_j^-, z_k^- - w_k^-\big)_{\Gamma^R},\\ \mathcal{F}_{jk}^R &= \big(z_j^+ - w_j^+, z_k^- - w_k^-\big)_{\Gamma^R},\\ \mathcal{G}_j^R &= \big(z_j^+ - w_j^+, z_j^+ - w_j^+\big)_{\Gamma^R}, \end{aligned} \tag{5.6}$$

and rewrite the functional (5.5) in the form

$$\mathcal{J}_j^R(a) = \langle a\mathcal{E}^R, a\rangle + 2\,\mathrm{Re}\langle\mathcal{F}_j^R, a\rangle + \mathcal{G}_j^R,$$

where $\mathcal{F}_j^R$ is the jth row of the matrix $\mathcal{F}^R$. Thus, the minimizer $a^0(R)$ is a solution of the system $a^0(R)\mathcal{E}^R + \mathcal{F}_j^R = 0$.

The justification of this method is similar to that in [3]. The following Propositions 5.1 and 5.2 can be verified in the same way as their counterparts in [3].

Proposition 5.1. *The matrix $\mathcal{E}^R(\mu)$ with the entries* (5.6) *is nonsingular for all $\mu\in[\mu',\mu'']$ and $R\ge R_0$, where R_0 is a sufficiently large number.*

Proposition 5.2. *Let u be a solution of the problem*

$$\begin{aligned} -\Delta u - \mu u &= 0, \quad x \in G^R, \\ u &= 0, \quad x \in \partial G^R \setminus \Gamma^R, \\ (-\partial_n + i\zeta)u &= h, \quad x \in \Gamma^R, \end{aligned}$$

with $h \in L_2(\Gamma^R)$. Then

$$\|u; L_2(\Gamma^R)\| \le \frac{1}{|\zeta|}\|h; L_2(\Gamma^R)\|. \tag{5.7}$$

Proposition 5.3. *Let $a^0(R,\mu) = (a_1^0(R,\mu), \dots, a_M^0(R,\mu))$ be a minimizer of the functional $\mathcal{J}_l^R$ given by* (5.5). *Then*

$$\mathcal{J}_l^R\left(a^0(R,\mu)\right) \le Ce^{-2\gamma R} \quad \text{as } R \to \infty, \tag{5.8}$$

where the constant C is independent of $R \ge R_0$, $\mu \in [\mu', \mu'']$, and $\gamma = \gamma(\mu)$ is the piecewise constant index described in Lemma 3.4. *For all $R \ge R_0$ and all $\mu \in [\mu', \mu'']$, the components of the vector $a^0(R,\mu)$ are uniformly bounded,*

$$|a_j^0(R,\mu)| \le \text{const} < \infty, \quad j = 1, \dots, M.$$

Proof. Relation (5.8) is obtained in the same way as in [3]. We verify the uniform boundedness of the minimizer $a^0(R,\mu)$. By Lemma 3.4, the interval $[\mu', \mu'']$ admits a finite covering by intervals I_p such that for each I_p one can choose a number $\gamma(\mu)$ in (3.8) (and consequently in (5.8)) independent of μ. Moreover, $\max_{\mu \in I_p} \operatorname{Re} \sqrt{\tau - \mu} < \gamma < \min_{\mu \in I_p} \operatorname{Re} \sqrt{\tau'' - \mu}$. We assume that μ runs through one of the covering intervals. Denote by Z_l^R the solution of problem (5.4) corresponding to $a^0(R,\mu) = (a_1^0(R,\mu), \dots, a_M^0(R,\mu))$. Setting $u = v = Z_l^R$ in the Green formula, we obtain

$$(-\partial_\nu Z_l^R, Z_l^R)_{\Gamma^R} - (Z_l^R, -\partial_\nu Z_l^R)_{\Gamma^R} = 0. \tag{5.9}$$

By (5.8),

$$\left\| Z_l^R - \left(w_l^+ + \sum_{j=1}^M a_j(R,\mu) w_j^- \right); L_2(\Gamma^R) \right\| = O(e^{-\gamma R}), \quad R \to \infty, \tag{5.10}$$

uniformly with respect to μ. Since

$$(-\partial_\nu + i\zeta) Z_l^R|_{\Gamma^R} = (-\partial_\nu + i\zeta)\left(w_l^+ + \sum_{j=1}^M a_j^0(R) w_j^- \right)|_{\Gamma^R},$$

from (5.9) it follows that

$$\left\| -\partial_\nu \left(Z_l^R - \left(w_l^+ + \sum_{j=1}^M a_j^0(R) w_j^- \right) \right); L_2(\Gamma^R) \right\| = O(e^{-\gamma R}), \quad R \to \infty. \tag{5.11}$$

Recall that, for $\mu > \tau$, the waves $w_l^\pm$ are bounded functions; the waves $w_l^\pm$ with $L < l \le M$ defined by (4.2) grow at infinity as $O(e^{\sqrt{\tau-\mu}\,|x|})$ for $\mu < \tau$ and as $O(|x|)$ for $\mu = \tau$. We use (5.10) and (5.11) to reduce (5.9) to the form

$$(-\partial_\nu \varphi_l, \varphi_l)_{\Gamma^R} - (\varphi_l, -\partial_\nu \varphi_l)_{\Gamma^R} = |a^0(R)|\, O(e^{-(\gamma - \sqrt{\tau-\mu} - \varepsilon)R}),$$

where $\varphi_l = w_l^+ + \sum a_j^0(R) w_j^-$, $\sqrt{\tau - \mu} = i\sqrt{\mu - \tau}$ for $\mu > \tau$, and ε is an arbitrarily small positive number. By (2.12) and (2.13), the left-hand side is equal to $-i(1 - \sum |a_j^0(R)|^2)$. Therefore,

$$|a^0(R)|^2 = 1 + o(|a^0(R)|),$$

which leads to $|a^0(R)| = 1 + o(1)$. Looking over all elements of the covering, we obtain the desired estimate everywhere on $[\mu', \mu'']$. □

Theorem 5.4. *For all $R \geq R_0$, where R_0 is a sufficiently large number, and for all $\mu \in [\mu', \mu''] \subset (\tau', \tau'')$, there exists a unique minimizer $a^0(R,\mu) = (a_1^0(R,\mu), \dots, a_M^0(R,\mu))$ of the functional $\mathcal{J}_l^R$ in* (5.2). *The estimates*

$$\sum_{k=1}^{M} |\mathcal{S}_{jk}(\mu) - a_k^0(R,\mu)| \leq Ce^{-\Lambda R} \tag{5.12}$$

are valid for all $R \geq R_0$, $\mu \in [\mu', \mu'']$, and $0 < \Lambda < \min_{\mu\in[\mu',\mu'']} \operatorname{Re}(\sqrt{\tau''-\mu} - \sqrt{\tau-\mu})$, where $\sqrt{\tau-\mu} = i\sqrt{\mu-\tau}$ for $\mu > \tau$ and the constant $C = C(\Lambda)$ is independent of R and μ.

Proof. As in the proof of the preceding assertion, we assume that μ runs through an interval I_p of the covering of $[\mu', \mu'']$ described in Lemma 3.4, so that the number γ in (3.8), (5.10), and (5.11) is independent of μ, and $\max_{\mu\in I_p} \operatorname{Re} \sqrt{\tau-\mu} < \gamma < \min_{\mu\in I_p} \operatorname{Re} \sqrt{\tau''-\mu}$.

Let Y_l^R be a solution of problem (5.4) where the a_j, $j = 1, \dots, M$, are taken to be the entries $\mathcal{S}_{lj}$ of the scattering matrix $\mathcal{S}$, and let Z_l^R and $(a_1^0(R,\mu), \dots, a_M^0(R,\mu))$ be the same as in Proposition 5.3. We substitute $u = v = U_l := Y_l - Z_l^R$ in the Green formula. Since U_l satisfies the first two equations in (5.4), we have

$$(-\partial_\nu U_l, U_l)_{\Gamma^R} - (U_l, -\partial_\nu U_l)_{\Gamma^R} = 0. \tag{5.13}$$

Setting

$$\varphi_l = w_l^+ + \sum_{j=1}^{M} a_j^0(R,\mu) w_j^-, \quad \psi_l = w_l^+ + \sum_{j=1}^{M} S_{lj}(\mu) w_j^-, \tag{5.14}$$

we write U_l in the form

$$U_l = Y_l - Z_l^R = (Y_l - \psi_l) + (\psi_l - \varphi_l) + (\varphi_l - Z_l^R).$$

Note that $(Y_l - \psi_l)|_{\Gamma^R} = O(e^{-\gamma R})$ by (3.8). Moreover, by Proposition 5.3, the components of the minimizer $a_j(R,\mu)$ are uniformly bounded. In view of (5.10) and (5.11), this allows us to pass from (5.13) to the relation

$$\big(-\partial_\nu(\psi_l - \varphi_l), (\psi_l - \varphi_l)\big)_{\Gamma^R} - \big((\psi_l - \varphi_l), -\partial_\nu(\psi_l - \varphi_l)\big)_{\Gamma^R} = O(e^{-(\gamma-\sqrt{\tau-\mu}-\varepsilon)R}), \tag{5.15}$$

where ε is an arbitrarily small positive number. A straightforward calculation shows that the left-hand side in (5.15) is equal to $i\sum_{j=1}^{M} |a_j^0(R,\mu) - \mathcal{S}_{lj}(\mu)|^2$ (it suffices to use (5.13), (2.12), and (2.13)). Hence,

$$\sum_{j=1}^{M} |a_j^0(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O\big(e^{-(\gamma-\sqrt{\tau-\mu}-\varepsilon)R}\big)$$

and we arrive at (5.12) for $\mu \in I_p$ and $\Lambda \leq \min_{\mu\in I_p}(\gamma - \operatorname{Re} \sqrt{\tau-\mu} - \varepsilon)/2$.

Now we prove the inequality

$$\sum_{j=1}^{M} |a_j(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O\big(e^{-2(\gamma-\sqrt{\tau-\mu}-\varepsilon)(1-2^{-N})R}\big) \tag{5.16}$$

for any positive integer N. Since for $N = 1$ this inequality has already been obtained, it suffices to derive from (5.16) the same estimate with $N + 1$ in place of N. We have

$$\psi_l - \varphi_l = \sum_{j=1}^{M} \big(\mathcal{S}_{lj}(\mu) - a_j(R,\mu)\big) w_j^- = O\big(e^{-[(\gamma-\sqrt{\tau-\mu})(1-2^{-N})-\sqrt{\tau-\mu}-\varepsilon]R}\big).$$

Therefore, we can pass from (5.13) to (5.15) with the right-hand side replaced by $O(e^z)$ with

$$z = -[(\gamma-\sqrt{\tau-\mu}-\varepsilon)(1-2^{-N}) - \sqrt{\tau-\mu}-\varepsilon+\gamma]R = -2(\gamma-\sqrt{\tau-\mu}-\varepsilon)(1-2^{-N-1})R.$$

Calculating the left hand-side of (5.15) once again, we obtain

$$\sum_{j=1}^{M} |a_j(R,\mu) - \mathcal{S}_{lj}(\mu)|^2 = O\big(e^{-2(\gamma-\sqrt{\tau-\mu}-\varepsilon)(1-2^{-N-1})R}\big).$$

Thus, (5.16) is proved for any positive integer N, with $\mu \in I_p$ and $\Lambda \le \min_{\mu\in I_p}(\gamma - \operatorname{Re}\sqrt{\tau-\mu}-\varepsilon)(1-2^{-N})$. Increasing N and reducing ε, we can make Λ to be arbitrarily close to $\gamma-\operatorname{Re}\sqrt{\tau-\mu}$. Looking over all intervals I_p of the covering, we obtain the required estimate everywhere on $[\mu',\mu'']$, with $\Lambda < \min_{\mu\in[\mu',\mu'']}(\gamma(\mu) - \operatorname{Re}\sqrt{\tau-\mu})$. Finally, for the difference between $\gamma(\mu)$ and $\operatorname{Re}\sqrt{\tau''-\mu}$ to be as small as we need, it suffices to refine the covering of $[\mu',\mu'']$. □

In a neighborhood of the threshold τ, the matrix $\mathcal{S}(\mu)$ can be calculated by the method presented in this paper. Since the limits of $S(\mu)$ as $\mu \to \tau \pm 0$ are finite, the relationship between $S(\mu)$ and $\mathcal{S}(\mu)$ allows us to calculate $S(\mu)$ for μ in the vicinity of τ.

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