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SIDEL′NIKOV INEQUALITY

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Abstract. The Sidel′nikov integral inequality was established in 1974 with a very difficult proof. A discrete version of this inequality was studied by Goethals and Seidel (1979) and by B. B. Venkov (2001). They found certain conditions under which equality occurs in the discrete inequality. In the paper, a simple proof of the Sidel′nikov inequality is suggested, based on an idea of Venkov.

§1. Sidel′nikov inequality

Let $\langle x, y\rangle$ be the usual scalar product in $\mathbb{R}^n$, $n \geq 2$, $\|x\|^2 = \langle x, x\rangle$. Let t be a positive integer, let $U \subset \mathbb{R}^n$ and let μ be a measure on U such that $\int_U \|x\|^t \, d\mu_x < \infty$.

In [1] Sidel′nikov proved two inequalities. The first of them is very simple, while the second (for t even) is difficult and substantial.

The following inequality is valid for an arbitrary positive integer t:

$$\int_U \int_U \langle x, y\rangle^t \, d\mu_x \, d\mu_y \geq 0. \tag{1}$$

This can be proved easily with the help of the multiindex techniques. Suppose $x = (x_1, x_2, \dots, x_n)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}$, $\alpha_i \geq 0$. Then $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$ and (1) is proved as follows:

$$\int_U \int_U \langle x, y\rangle^t \, d\mu_x \, d\mu_y = \int_U \int_U \Bigl(\sum_{|\alpha|=t} \frac{t!}{\alpha!} x^\alpha y^\alpha \Bigr) \, d\mu_x \, d\mu_y = \sum_{|\alpha|=t} \frac{t!}{\alpha!} \Bigl(\int_U x^\alpha \, d\mu_x \Bigr)^2 \geq 0.$$

Theorem 1 (V. M. Sidel′nikov [1]). *Suppose $U \subset \mathbb{R}^n$, μ is a measure on U, and t is an even number. Then*

$$\int_U \int_U \langle x, y\rangle^t \, d\mu_x \, d\mu_y \geq c_t \Bigl(\int_U \|x\|^t \, d\mu_x \Bigr)^2, \tag{2}$$

where

$$c_t = \frac{1}{\sigma_n} \int_{S^{n-1}} \xi_1^t \, dS_\xi \tag{3}$$

is the mean value of the function $(\xi_1, \dots, \xi_n) \mapsto \xi_1^t$ on the sphere

$$S^{n-1} = \bigl\{ \xi \in \mathbb{R}^n \colon \|\xi\| = 1 \bigr\},$$

and σ_n is the area of the sphere S^{n-1}, $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$.

This is a strong result, and the proof in [1] is fairly hard. We present a simpler proof, but one that requires some auxiliary statements and constructions.

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Lemma 1. *For any $x \in \mathbb{R}^n$ we have*

$$\int_{S^{n-1}} \langle x, \xi\rangle^t \, dS_\xi = \sigma_n c_t \|x\|^t. \tag{4}$$

Proof. The constant (3) can be written in the form

$$c_t = \frac{1}{\sigma_n} \int_{S^{n-1}} \langle \eta, \xi\rangle^t \, dS_\xi,$$

where η is arbitrary vector in S^{n-1}. Taking $\eta = x/\|x\|$, we get identity (4). □

Let $\mathrm{Hom}(t)$ denote the space of homogeneous polynomials of degree t over $\mathbb{R}$ in n variables. Following Venkov [6], we introduce a scalar product in $\mathrm{Hom}(t)$. For two polynomials

$$f(x) = \sum_{|\alpha|=t} a_\alpha x^\alpha, \quad g(x) = \sum_{|\alpha|=t} b_\alpha x^\alpha$$

in $\mathrm{Hom}(t)$, we put

$$[f, g] = \sum_{|\alpha|=t} \frac{\alpha!}{t!} a_\alpha b_\alpha.$$

For any $y \in \mathbb{R}^n$ we introduce the polynomial $\rho_y(x) = \langle y, x\rangle^t$. It is easy to check that

$$[\rho_y, f] = f(y), \quad f \in \mathrm{Hom}(t). \tag{5}$$

Indeed,

$$[\rho_y, f] = \sum_{|\alpha|=t} \frac{\alpha!}{t!} \frac{t!}{\alpha!} y^\alpha a_\alpha = f(y).$$

Lemma 2. *For the polynomial $\omega_t(x) = \|x\|^t$ we have*

$$[\omega_t, \omega_t] = \frac{1}{c_t}. \tag{6}$$

Proof. We take the scalar product of each side in (4) by the polynomial ω_t, obtaining

$$\int_{S^{n-1}} [\rho_\xi, \omega_t] \, dS_\xi = \sigma_n c_t [\omega_t, \omega_t].$$

By the reproducing property (5), we have $\big[\rho_\xi, \omega_t\big] = \omega_t(\xi) = 1$ because $\|\xi\| = 1$. Therefore, the above integral is equal to σ_n. Thus, $1 = c_t[\omega_t, \omega_t]$. □

Proof of Theorem 1. It turns out that inequality (2) is none other than the Cauchy–Schwartz inequality

$$[A, A][\omega_t, \omega_t] \geq [A, \omega_t]^2 \tag{7}$$

for the polynomials

$$A(x) = \int_U \langle y, x\rangle^t \, d\mu_y, \quad \omega_t(x) = \|x\|^t.$$

Represent $A(x)$ in the form

$$A(x) = \sum_{|\alpha|=t} \frac{t!}{\alpha!} M_\alpha x^\alpha,$$

where $M_\alpha = \int_U y^\alpha \, d\mu_y$. To calculate the scalar products $[A, A]$ and $[A, \omega_t]$, we observe that for any polynomial $f(x) = \sum_{|\alpha|=t} a_\alpha x^\alpha$ we have

$$[A, f] = \sum_{|\alpha|=t} \frac{\alpha!}{t!} \frac{t!}{\alpha!} M_\alpha a_\alpha = \int_U \sum_{|\alpha|=t} y^\alpha a_\alpha \, d\mu_y = \int_U f(y) \, d\mu_y.$$

It follows that

$$[A,\, A] = \int_U A(z)\, d\mu_z = \int_U \int_U \langle y,\, z\rangle^t\, d\mu_y\, d\mu_z,$$
$$[A,\, \omega_t] = \int_U \omega_t(y)\, d\mu_y = \int_U \|y\|^t\, d\mu_y.$$

Also, we have $[\omega_t,\, \omega_t] = 1/c_t$ by Lemma 2. To get (2), it remains to plug these scalar products in (7) and to multiply the resulting inequality by c_t. □

The above proof implies a condition under which inequality (2) turns into equality.

Proposition. *Let $I_t = \int_U \|x\|^t\, d\mu_x$. Inequality* (2) *turns into equality if and only if the following identity is fulfilled*:

$$\int_U \langle y,\, x\rangle^t\, d\mu_y \equiv c_t I_t \|x\|^t, \quad x \in \mathbb{R}^n. \tag{8}$$

Proof. Suppose equality occurs in (2). Then the same is true for the Cauchy–Schwartz inequality (7). Therefore, there is a constant λ such that $A = \lambda\omega_t$. Multiplying this scalarly by ω_t, we get $[A,\, \omega_t] = \lambda[\omega_t,\, \omega_t]$. In the proof of Theorem 1 we saw that $[A,\, \omega_t] = I_t$ and $[\omega_t,\, \omega_t] = 1/c_t$. It follows that $\lambda = c_t I_t$ and $A = c_t I_t \omega_t$, and the latter identity coincides with (8).

Conversely, if (8) is true, then $A = c_t I_t \omega_t$, implying that equality occurs in (7) and (2). □

The particular case where $U = S^{n-1}$ and μ is the usual Lebesgue measure on the sphere, $\mu(S^{n-1}) = \sigma_n$, deserves a separate mention. Then

$$I_t = \int_{S^{n-1}} \|x\|^t\, d\mu_x = \int_{S^{n-1}} d\mu = \sigma_n.$$

Now, (8) is fulfilled by Lemma 1. Hence, for $U = S^{n-1}$ we have equality in (2). This fact was observed in [1]. Another particular case – a discrete set U – will be treated in §2.

Computation of the constant c_t. We must evaluate the integral (3), but we shall deal with a more general integral, which will be needed in what follows.

Lemma 3. *Suppose that all indices k_j in a multiindex $k = (k_1, \dots, k_n)$ are even, and that $|k| = k_1 + \dots + k_n = t$. Then*

$$I(k) := \frac{1}{\sigma_n} \int_{S^{n-1}} x_1^{k_1} \dots x_n^{k_n}\, dS = \frac{(k_1-1)!! \dots (k_n-1)!!}{n(n+2)\dots(n+t-2)}. \tag{9}$$

Here $(-1)!! = 1$, and for $t = 2$ the factor $n+2$ is absent.

Proof. Consider the integral $V = \int_{\mathbb{R}^n} x^k e^{-\|x\|^2}\, dx$. It splits into a product of one-dimensional integrals: $V = \Gamma\big(\frac{k_1+1}{2}\big) \dots \Gamma\big(\frac{k_n+1}{2}\big)$. Passing to the spherical coordinates, we get

$$V = \frac{1}{2}\Gamma\Big(\frac{t+n}{2}\Big)\sigma_n I(k).$$

This yields an expression for $I(k)$, which is easily reshaped to (9). □

A formula for c_t is obtained from (9) as a special case:

$$c_t = \frac{1}{\sigma_n} \int_{S^{n-1}} x_1^t\, dS = \frac{(t-1)!!}{n(n+2)\dots(n+t-2)}. \tag{10}$$

§2. A discrete version of Sidel′nikov's inequality

An interesting version of Sidel′nikov's inequality is obtained when U is a finite set, $U = \{x_1, \dots, x_m\}$, $\mu\big(\{x_i\}\big) = 1$, $i = 1, \dots, m$. Theorem 1 says that

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\langle x_i, x_j\rangle^t \geq c_t\left(\sum_{i=1}^{m}\|x_i\|^t\right)^2. \tag{11}$$

Recall that t is even. In (11), the vectors $x_1, \dots, x_m$ need not be distinct, so that instead of a set U we can talk of a finite sequence.

Some authors, being unaware of the paper [1], obtained inequality (11) independently. For $t = 2$, inequality (11) was proved in [2], and for $t \geq 4$ in [3]. The question about the equality case in (11) was also studied in [2, 3].

This question is of special interest. The results turn out to be more impressive if we assume that the sequence $U = \{x_i\}_{i=1}^m$ lies on the sphere S^{n-1}. Theorem 1 and Proposition, see §1, yield the following.

Theorem 2. *For any sequence $U = \{x_i\}_{i=1}^m$ of points of the sphere S^{n-1} we have*

$$\frac{1}{m^2}\sum_{i,j=1}^{m}\langle x_i,\, x_j\rangle^t \geq c_t. \tag{12}$$

Equality in (12) *is attained if and only if*

$$\frac{1}{m}\sum_{i=1}^{m}\langle x_i,\, x\rangle^t \equiv c_t\|x\|^t, \quad x \in \mathbb{R}^n. \tag{13}$$

We adopt the following definition.

Definition 1. A sequence $U = \{x_i\}_{i=1}^m$ on the sphere S^{n-1} is called a *spherical semidesign of order* t if identity (13) is fulfilled.

Theorem 2 says that inequality (12) turns into an identity on the spherical semidesigns and only on them.

Identity (13) will be called the Waring identity, after the name of an English mathematician E. Waring (1734–1798) who was interested in the representation of the form $\|x\|^t$ as a sum of linear forms taken in the power t. Many mathematician of the 19th century obtained Waring identities for various n, t, m. A collection of such identities is presented in [5, p. 103].

Example (E. Lucas, 1876)**.** The following identity is verified easily:

$$6\|\xi\|^4 = \sum_{i<j}(\xi_i \pm \xi_j)^4$$

for any $\xi = (\xi_1, \dots, \xi_4) \in \mathbb{R}^4$. Here we have $2\binom{4}{2} = 12$ summands. It is not hard to write out vectors $x_1, \dots, x_{12} \in S^3$ such that (13) is fulfilled for $m = 12$, $n = 4$, $t = 4$. These 12 vectors form a spherical semidesign of order 4.

Criterion for being a semidesign in terms of integrals over the sphere. Standardly, a spherical design of order t is defined as a system of points on the sphere S^{n-1} such that for any polynomial of order at most t the mean value over the sphere is equal to that over the design (see [7]).

A similar criterion is valid for the spherical semidesigns.

Theorem 3. *Suppose t is even, $t \geq 2$. A sequence $U = \{x_i\}_{i=1}^m$ of points of the sphere S^{n-1} is a spherical semidesign of order t if and only if the identity*

$$\frac{1}{\sigma_n}\int_{S^{n-1}} Q(x)\,dS = \frac{1}{m}\sum_{i=1}^m Q(x_i) \tag{14}$$

is fulfilled for any homogeneous polynomial $Q(x)$ of degree t.

A homogeneous polynomial of degree t can be written as $Q(x) = \sum_{|k|=t} a_k x^k$, where $k = (k_1, \dots, k_n)$ is a multiindex, $|k| = k_1 + \dots + k_n$, $x^k = x_1^{k_1} \dots x_n^{k_n}$.

Proof. The "only if" part. Let $\{x_1, \dots, x_m\} \subset S^{n-1}$ be a semidesign of order t. Then (13) is true. We have

$$S(x) := \frac{1}{m}\sum_{i=1}^m \langle x_i,\, x\rangle^t = \sum_{|k|=t} \frac{t!}{k!}\left(\frac{1}{m}\sum_{i=1}^m x_i^k\right)x^k.$$

Put $s = t/2$. Then the right-hand side of (13) can be written in the form

$$R(x) := c_t\|x\|^t = c_t \sum_{|l|=s} \frac{s!}{l!}x^{2l}.$$

Since $S(x) \equiv R(x)$, the coefficients of the same powers of x are equal. Let $k = (k_1, \dots, k_n)$, and let at least one k_j be odd. Then

$$\frac{1}{m}\sum_{i=1}^m x_i^k = 0. \tag{15}$$

Now suppose that all k_j are even, $k_j = 2l_j$. Then $k = 2l$ and

$$\frac{t!}{k!}\left(\frac{1}{m}\sum_{i=1}^m x_i^k\right) = c_t\frac{s!}{l!}. \tag{16}$$

It suffices to check (14) for $Q(x) = x^k$, $|k| = t$. If at least one k_j is odd, then, by (15),

$$\frac{1}{\sigma_n}\int_{S^{n-1}} x^k\,dS = 0 = \frac{1}{m}\sum_{i=1}^m x_i^k.$$

Now, let $k = 2l$, where $|l| = s$. Then, by Lemma 3,

$$I(k) := \frac{1}{\sigma_n}\int_{S^{n-1}} x^k\,dS = \frac{(k_1-1)!!\dots(k_n-1)!!}{n(n+2)\dots(n+t-2)}.$$

By (16), on the right in (14) we get the sum

$$T(k) := \frac{1}{m}\sum_{i=1}^m x_i^k = \frac{k!}{t!}c_t\frac{s!}{l!} = \frac{k!}{t!}\frac{s!}{l!}\frac{(t-1)!!}{n(n+2)\dots(n+t-2)},$$

and elementary calculations show that $I(k) = T(k)$. Thus, (14) is true for $Q(x) = x^k$. *The "if" part* is proved by reversing the above arguments. □

Theorems 2 and 3 imply the following statement, which was proved for the first time by Goethals and Seidel in [4]. This is a special case of the Sidel′nikov inequality, but equality conditions were found for the first time.

Theorem 4. *Inequality* (12) *holds true for any sequence $U = \{x_i\}_{i=1}^m$ of points on the sphere. Equality is attained if and only if condition* (14) *is fulfilled for any homogeneous polynomial $Q(x)$ of degree t.*

Recalling Theorem 3, once again we see that inequality (12) turns into an identity on the spherical semidesigns and only on them.

More information on semidesigns can be found in [3].

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