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ATOMIC OPERATORS, RANDOM DYNAMICAL SYSTEMS AND INVARIANT MEASURES

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Dedicated to the memory of Professor M. E. Drakhlin

ABSTRACT. It is proved that the existence of invariant measures for families of the so-called atomic operators (nonlinear generalized weighted shifts) defined over spaces of measurable functions follows from the existence of appropriate invariant bounded sets. Typically, such operators come from infinite-dimensional stochastic differential equations generating not necessarily regular solution flows, for instance, from stochastic differential equations with time delay in the diffusion term (regular solution flows called also Carathéodory flows are those almost surely continuous with respect to the initial data). Thus, it is proved that to ensure the existence of an invariant measure for a stochastic solution flow it suffices to find a bounded invariant subset, and no regularity requirement for the flow is necessary. This result is based on the possibility to extend atomic operators by continuity to a suitable set of Young measures, which is proved in the paper. A motivating example giving a new result on the existence of an invariant measure for a possibly nonregular solution flow of some model stochastic differential equation is also provided.

§1. INTRODUCTION

In this paper we deal with the existence of invariant measures for stochastic solution flows, i.e., flows generated by stochastic differential equations. The invariant measures for such flows are usually random (see, e.g., [2]), i.e., are defined on the product of the conventional phase space with the underlying probability space. It is known that the principal difficulty in studying the existence of such invariant measures is the possibility of extending the flow by continuity to the space of random measures endowed with a suitable weak topology. Once this is done, the existence of an invariant bounded subset for the flow (which is usually obtained by some *a priori* estimates on solutions of the underlying equations) gives the existence of an invariant measure through the standard Krylov–Bogolyubov procedure [2]. Thus, the existence of a continuous extension of stochastic solution flows is the main subject of the present paper. It is important to remark that, presently, we only know that it is valid in the case of the so-called *regular* random dynamical systems, i.e., those generating Carathéodory solution flows, that is, solution flows that almost surely consist of continuous paths with respect to the initial data. In fact, only such flows were studied in [2]. Regularity in the above sense is a rather strong requirement for a random dynamical system which, first and foremost,

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might be difficult to verify (and in fact, is known to be fulfilled only in quite a limited number of situations, e.g., for stochastic ODE's with "nice" right-hand side involving the standard Brownian motion), and, what is more important, may happen to be false in general. In fact, there are some natural examples of stochastic differential equations, especially in infinite-dimensional spaces, for which regularity fails, i.e., which produce non-Carathéodory flows. The most prominent example of this kind is a stochastic delay equation, where delay is incorporated in the diffusion term (see, e.g., [14], or the recent paper [19]). Another example can be found in the present paper (Example 6.2).

The crucial difference between Carathéodory (i.e., regular) and non-Carathéodory (i.e., nonregular) solution flows is their behavior with respect to the natural topology on the set of measures. Any Carathéodory flow can easily be extended to a continuous solution flow defined on the set of measures equipped with the suitable weak (narrow) topology [2]. This is due to the fact that the measures of interest are linear functionals on the linear space of Carathéodory functions, so that $f_\sharp\mu$ (defined by $f_\sharp\mu(A) := \mu(f^{-1}(A))$) is again a measure for any Carathéodory function f, and the desired extension is simply $\mu \mapsto f_\sharp\mu$. This argument breaks completely down if the flow is non-Carathéodory. This is not surprising because nonregular random dynamical systems usually provide an erratic behavior [14].

In the present paper we solve this problem by proving the existence of a continuous extension of a general non-Carathéodory flow to an appropriate set of random measures. In particular, this gives an opportunity to define the very notion of an invariant measure for non-Carathéodory flows coming from nonregular random dynamical systems and to prove results on the existence of invariant measures for general random dynamical systems, including those that do not generate Carathéodory solution flows and, therefore, cannot be covered by the existing theory presented in [2]. Summarizing, the main result of the paper can be stated as follows: to ensure the existence of an invariant measure of a stochastic solution flow it is unnecessary to check the regularity of the flow and, thus, it suffices to find a bounded invariant subset.

1.1. Structure of the paper and principal results. The paper is organized as follows. In §2 we recall the definition of the so-called "*atomic*" operator introduced and studied in [10] and provide some examples of these operators.

In §3, which is central to this work, we explain how atomic operators can be extended by continuity to the set of measures we are interested in (which, roughly speaking, is the closure in the narrow topology of the set of random Dirac measures, i.e., measures concentrated over graphs of functions). Namely, we show that

- every continuous atomic operator between Lebesgue–Bochner spaces can be extended by continuity to an operator between the spaces of measurable functions, and the extension is still a continuous atomic operator (Proposition 3.1);
- every continuous atomic operator between spaces of measurable functions can be extended by continuity to a continuous operator defined on Young measures, namely, over the closure in the narrow topology of the set of random Dirac measures (in fact, even to a linear continuous operator defined on a much wider space dual to a special space of Carathéodory functions), and such an extension is unique (Theorem 3.3).

It is the latter result that is most important for applications to random dynamical systems. The idea behind it is to observe that an extension to the linear space dual to that of Carathéodory functions (containing the space of Young measures) of any atomic operator should be linear (as is, for instance, the respective extension of a Nemytskiĭ operator generated by a Carathéodory function, which is a particular case of an atomic operator). However, in the non-Carathéodory case it seems to be problematic to arrange an explicit formula for the extension. That is why we first look at the predual space consisting of Carathéodory functions and construct there the adjoint operator, which again should be linear, and then obtain the desired continuous extension of the given atomic operator by the standard duality argument. This fact is crucial for §4, where we prove the results on the existence of invariant measures for (families of) atomic operators.

We also show by means of a series of examples in §3, sometimes even elementary, that the results on extensions of atomic operators are rather sharp in the sense that the property to be atomic is quite essential for them. Namely, there exist continuous operators that can be extended neither from Lebesgue–Bochner spaces to spaces of measurable functions, nor from spaces of measurable functions to Young measures, and there also exist operators extendible to measures but with extensions not coming from linear operators.

In §5 we show that normally stochastic differential equations give rise to solution flows (also called *evolution families*) consisting of *local* operators: for this property to hold for a stochastic differential equation one only needs the well-posedness of the respective initial value problem. This fact is in sharp contrast with the problem of existence for a Carathéodory flow, which often requires much more sophisticated analysis of solutions.

Of course, any Carathéodory flow consists of local operators, but the converse is false in general. This is shown via examples in §6. The main goal of that section is introducing the notion of a generalized cocycle property. The difference between the classical cocycle property (see, e.g., [2, p. 5]) and the generalized cocycle property is exactly the difference between Carathéodory and non-Carathéodory flows.

In §7 we show that an invariant measure for a stochastic solution flow is a common fixed point for a family of *atomic* operators constructed from the generalized cocycle property and extended to Young measures by continuity in the narrow topology. That section contains also the existence result for a model stochastic differential equation (Theorem 7.2). This result is only intended as an illustration of the applicability of the abstract theory developed in the paper. In fact, it refers to the situation where the solution flow may be not regular. Thus, it may be viewed as a motivating example for the technique developed in this paper.

Since the results we provide often require quite a lot of technicalities that are not always easy to follow, it is our explicit intention to put all the necessary technical statements in the appendices. In particular, Appendix A contains some auxiliary results on local functionals and local operators. In particular, under certain assumptions, Corollary A.6 gives the representation of local operators as Nemytskiĭ operators generated by Carathéodory functions. Though being far less general than the representation theorem for local operators in [16] (presented there without detailed proof), it suffices for our purposes, and its proof is independent of that outlined in [16] and is shorter. In Appendix B we provide the lengthy and technical proof of Theorem 7.2, which is a motivating example for this paper. Finally, Appendix C contains some nondifficult auxiliary results on the tightness of sets of functions as well as of local operators that we need in the paper.

§2. Atomic operators

2.1. Notation and preliminaries. The triple $(\Omega, \Sigma, \mathbb{P})$, where $\mathbb{P}$ is a finite positive (resp. probability) measure defined on a σ-algebra Σ of $\mathbb{P}$-measurable subsets of a set Ω, is as usual called measure (resp. probability) space. By default, in the sequel we shall assume all measure spaces we shall be dealing with to be complete (i.e., the respective σ-algebra Σ is complete with respect to $\mathbb{P}$). Also, throughout the paper, unless explicitly stated otherwise, we assume all the finite measures to be probability measures just as a matter of technical assumption simplifying the notation (in fact, all the results of this paper remain true if probability measures are replaced by finite measures). We recall also that $(\Omega, \Sigma, \mathbb{P})$ is called a *standard measure space* if Ω is a Polish space (i.e., separable metrizable with complete distance), $\mathbb{P}$ is a Borel measure on Ω, Σ is either the σ-algebra of Borel subsets of the space Ω or its $\mathbb{P}$-completion. All the metric spaces are also tacitly assumed to be complete, unless explicitly stated otherwise. The norm in a normed space X is denoted by $\|\cdot\|_X$.

By $L^p(\Omega, \Sigma, \mathbb{P}; X)$ we denote the classical Lebesgue–Bochner space of (classes of $\mathbb{P}$-equivalent) functions integrable with exponent $p > 0$ over Ω with respect to the measure $\mathbb{P}$ and taking values in some normed space X; the respective norm is denoted by $\|\cdot\|_p$. By $L^0(\Omega, \Sigma, \mathbb{P}; X)$, where X is a metric space with distance d, we denote the space of (classes of) X-valued measurable functions equipped with the distance

$$d^0(u, v) := \int_\Omega \big(d(u(\omega), v(\omega)) \wedge 1\big)\, d\mathbb{P}(\omega),$$

inducing the topology of convergence in measure.

The characteristic function of a subset $e \subset \Omega$ will in the sequel be denoted by $\mathbf{1}_e(\omega)$. We also find it convenient to write $x|_e = y|_e$ for $\{x, y\} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ if $x(\omega) = y(\omega)$ for $\mathbb{P}$-a.e. $\omega \in e \subset \Omega$ (for too rigorous readers: this may be interpreted as $(x-y)\mathbf{1}_e = 0$).

If $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ and $(\Omega_2, \Sigma_2, \mathbb{P}_2)$ are two measure spaces, a map $F\colon \Sigma_1 \to \Sigma_2$ is called a *σ-homomorphism* if $F(\Omega_1) = \Omega_2$, $F(\Omega_1 \setminus e) = \Omega_2 \setminus F(e)$ whenever $e \in \Sigma_1$, and

$$F\left(\bigsqcup_{i=1}^\infty e_i\right) = \bigsqcup_{i=1}^\infty F(e_i)$$

for any pairwise disjoint collection of $\mathbb{P}_1$-measurable sets $\{e_i\}_{i=1}^\infty$, where $\sqcup$ stands for the disjoint union. It is further called *nullset preserving* if

$$\mathbb{P}_2(F(e_1)) = 0 \quad \text{when} \quad \mathbb{P}_1(e_1) = 0.$$

2.2. Local and Nemytskiĭ operators. Let $\mathcal{X}_i := L^0(\Omega, \Sigma, \mathbb{P}; X_i)$, $i = 1, 2$.

Definition 2.1. An operator $T\colon \mathcal{X}_1 \to \mathcal{X}_2$ is said to be *local* if $x|_e = y|_e$ for $\{x, y\} \subset \mathcal{X}_1$ implies $T(x)|_e = T(y)|_e$.

The above general definition is due to Shragin [20]. The following example is also classical, and in fact it motivated the study of local operators.

Example 2.2. Let X_1 and X_2 be separable metric spaces, and let $f\colon \Omega \times X_1 \to X_2$ be a sup-measurable function (i.e., $f(\cdot, x(\cdot))$ is $\mathbb{P}$-measurable whenever $x(\cdot)$ is $\mathbb{P}$-measurable). Then the Nemytskiĭ operator $N_f\colon L^0(\Omega, \Sigma, \mathbb{P}; X_1) \to L^0(\Omega, \Sigma, \mathbb{P}; X_2)$ (commonly known also under the name of the *superposition operator* [1]) defined by

$$(N_f x)(\omega) := f(\omega, x(\omega))$$

is local. If $f\colon \Omega \times X_1 \to X_2$ is a Carathéodory function (i.e., $f(\omega, \cdot)$ is continuous for $\mathbb{P}$-almost every $\omega \in \Omega$ and $f(\cdot, x)$ is $\mathbb{P}$-measurable for all $x \in X_1$), then the Nemytskiĭ operator N_f becomes continuous in measure (i.e., as an operator in L^0).

2.3. Atomic operators. Now we introduce another definition generalizing the notion of a local operator. Here $\mathcal{X}_i := L^{p_i}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, and $0 \le p_i \le +\infty$.

Definition 2.3. An operator $T\colon \mathcal{X}_1 \to \mathcal{X}_2$ is said to be *atomic* if there is a nullset-preserving σ-homomorphism $F\colon \Sigma_1 \to \Sigma_2$ such that $x|_{e_1} = y|_{e_1}$ for $\{x, y\} \subset \mathcal{X}_1$ implies $T(x)|_{F(e_1)} = T(y)|_{F(e_1)}$.

In the rare case when the reference to a particular σ-homomorphism F in the above definition should be made, we shall call the operator T atomic with respect to F, so that a local operator is atomic with respect to the identity σ-homomorphism.

It should be emphasized that in [10], first the notions of the so-called measure-theoretic memory and comemory of an operator were introduced and then the definition of an atomic operator was given based on such notions. Though being more abstract, this opens the way to an intrinsic definition of the concept of an atomic operator. However, we do not follow this way in the present paper in order not to overburden it with too many abstract notions.

Obviously, every local operator is atomic. However, the class of atomic operators is richer, as one can conclude from the following example.

Example 2.4. Consider the *generalized shift* operator

$$T_F : \; L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X),$$

associated with a given nullset-preserving σ-homomorphism F: $\Sigma_1 \to \Sigma_2$, where X is a separable metric space. We define T_F by setting

$$T_F(\mathbf{1}_{e_1} z) := \mathbf{1}_{F(e_1)} z \tag{2.1}$$

for all $e_1 \in \Sigma_1$ and $z \in X$, extending it by linearity to all simple (i.e., finite-valued) functions, and then by continuity to the entire space $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X)$. For this operator, one has $\operatorname{Im} T_F = L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X)$ (see Lemma 3.1 in [10] for details, which are provided there for the case where X is a Banach space, but are also valid without any change for the general case of a metric space X).

Clearly, if for $\{x, y\} \subset L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X)$ one has $x|_{e_1} = y|_{e_1}$, then $T(x)|_{F(e_1)} = T(y)|_{F(e_1)}$, that is, the generalized shift operator T_F is atomic. In particular, we see that any *shift* (sometimes also called *inner superposition*) operator

$$T_g : \; L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X),$$

defined by

$$(T_g x)(\omega_2) := x(g(\omega_2)) \tag{2.2}$$

where g: $\Omega_2 \to \Omega_1$ is a (Σ_2, Σ_1)-measurable function, is atomic. For this operator to be well defined on the classes of measurable functions, we require

$$\mathbb{P}_2(g^{-1}(e_1)) = 0 \text{ for } e_1 \in \Sigma_1, \, \mathbb{P}_1(e_1) = 0. \tag{2.3}$$

The class of atomic operators is obviously closed under compositions.

2.4. Representation. The following important result is an easy extension of a similar one proved in [10].

Theorem 2.5. *Let X_i, $i = 1, 2$, be Polish spaces. Then for every operator*

$$T\colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$$

atomic with respect to a nullset-preserving σ-homomorphism F: $\Sigma_1 \to \Sigma_2$, there is a local operator N: $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ such that

$$T = N \circ T_F. \tag{2.4}$$

The operator T is continuous if and only if so is the restriction of the operator N to the subspace $L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X_1) \subset L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$. Moreover, if $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ is a standard measure space, then there is a (Σ_1, Σ_2)-measurable function g: $\Omega_2 \to \Omega_1$ satisfying (2.3) *and such that $T = N \circ T_g$.*

Proof. We rely completely on the proof of Theorem 3.1 in [10] which is a similar result for the case where X_i are Banach spaces. Namely, define on $\operatorname{Im} T_F$ an operator N: $\operatorname{Im} T_F \to L^0(\Omega_2, \Sigma_2, \mu_2; X_2)$ by setting $N(y) := T(x)$ if $y = T_F(x)$. To show that the above definition of N is consistent, assume that $y = T_F(x) = T_F(x')$ with $x \neq x'$. Let

$$\begin{aligned} e_1' &:= \big\{\omega_1 \in \Omega_1 \,:\, x(\omega_1) = x'(\omega_1)\big\}, \\ e_1 &:= \Omega_1 \setminus e_1'. \end{aligned}$$

We shall prove that $\mu_F(e_1) := \mathbb{P}_2(F(e_1)) = 0$, hence $F(e_1') = \Omega_2$ modulo a $\mathbb{P}_2$-nullset, and $T(x) = T(x')$, thus showing the consistency of the definition of N. In fact, if $\mu_F(e_1) > 0$, then $\mu_F(e_1 \cap E_1^\alpha) > 0$ for some $\alpha > 0$, where $E_1^\alpha \subset \Omega_1$ is defined by

$$E_1^\alpha := \Big\{\omega_1 \in \Omega_1 \,:\, \frac{d\mu_F}{d\mathbb{P}_1} \geq \alpha\Big\}.$$

Consider two sequences of simple functions $\{x_\nu\}$ and $\{x_\nu'\}$ in $L^0(\Omega_1, \Sigma_1, \mu_1; X_1)$ converging to x and x' in measure relative to $\mathbb{P}_1$ on Ω_1, respectively, and such that $x_\nu\big|_{e_1'} = x_\nu'\big|_{e_1'}$. We have

$$\begin{aligned} y_\nu &:= x_\nu|_{e_1 \cap E_1^\alpha} = \sum_{i=1}^{N_\nu} \mathbf{1}_{e_1^{i,\nu}} z_{i,\nu}, \\ y_\nu' &:= x_\nu'|_{e_1 \cap E_1^\alpha} = \sum_{i=1}^{N_\nu} \mathbf{1}_{e_1^{i,\nu}} z_{i,\nu}', \end{aligned}$$

for some $\{z_{i,\nu}, z_{i,\nu}\} \subset X_1$ and disjoint sets $e_1^{i,\nu} \in \Sigma_1$. Denoting by d_i^0 the distance in $L^0(\Omega_i, \Sigma_i, \mu_i; X_i)$ and by d_i the distance in X_i, $i = 1, 2$, we get

$$\begin{aligned} d_2^0(T_F(y_\nu), T_F(y_\nu')) &= \sum_{i=1}^{N_\nu} \mu_F(e_1^{i,\nu})\delta^{i,\nu} \geq \alpha \sum_{i=1}^{N_\nu} \mathbb{P}_1(e_1^{i,\nu})\delta^{i,\nu}, \\ d_1^0(y_\nu, y_\nu') &= \sum_{i=1}^{N_\nu} \mathbb{P}_1(e_1^{i,\nu})\delta^{i,\nu}, \quad \text{where} \quad \delta^{i,\nu} := d_1(z_{i,\nu}, z_{i,\nu}') \wedge 1. \end{aligned}$$

Minding that $d_2^0(T_F(y_\nu), T_F(y_\nu')) \to 0$ as $\nu \to \infty$, we have $d_1^0(y_\nu, y_\nu') \to 0$, and hence

$$x\big|_{e_1 \cap E_1^\alpha} = x'\big|_{e_1 \cap E_1^\alpha},$$

which is the desired contradiction.

The rest of the proof is simply a word-for-word repetition of the proof of Theorem 3.1 in [10]. □

We note that even if T: $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ is continuous, the operator N defined in Theorem 2.5 may fail to be continuous (only its restriction to a certain subspace is). Therefore, the function f: $\Omega_2 \times X_1 \to X_2$, generating together with g: $\Omega_2 \to \Omega_1$ the operator T, may fail to be a Carathéodory function. An example from the theory of stochastic processes that we are going to discuss now shows that there indeed exist atomic operators T not representable by a composition of a Nemytskiĭ operator generated by a Carathéodory function with a shift operator.

Example 2.6. Consider a probability space $(\Omega, \Sigma, \mathbb{P})$, the standard Wiener process W_t, and the Wiener shift $g := \theta_{-1}$: $\Omega \to \Omega$ inducing an isomorphism of the σ-subalgebra Σ_0, and let $\Sigma_1 := g^{-1}(\Sigma_0)$. Let $X := L^2((0,1), \widehat{\Sigma}, \mathcal{L}^1)$, where $\widehat{\Sigma}$ stands for the Lebesgue σ-algebra of $(0,1)$, $\mathcal{L}^1$ stands for the Lebesgue measure. Define an operator

$$T \,:\, L^0(\Omega, \Sigma_1, \mathbb{P}; X) \to L^0(\Omega, \Sigma_1, \mathbb{P}; X)$$

as stochastic integration with respect to the Wiener process,

$$(Tx)(\omega) := \int_0^{(\,\cdot\,)} x(s, g(\omega))\, dW_s(\omega).$$

Note that we have shifted the Σ_1-measurable integrand $x(t,\omega)$ with the help of g. In this way, the stochastic process $x(s, g(\omega))$ becomes Σ_0-measurable, so that the stochastic integral is well defined. The operator T is atomic because it is a composition of the stochastic integral (which is local) and the shift T_g. However, the stochastic integral cannot be represented by a Nemytskiĭ operator generated by a Carathéodory function. Otherwise, the stochastic integral could have been, by the Riesz representation theorem, reduced to the usual Lebesgue–Stieltjes integral, which is impossible (see, e.g., [16] or [17]).

§3. Young measures and atomic operators

Throughout this section, X will by default stand for a Polish space, and $\mathscr{B}(X)$ for its Borel σ-algebra.

We use the following notation.

- $C_b(X)$ stands for the space of real-valued continuous bounded functions on X equipped with the supremum norm $\|\cdot\|_\infty$;
- $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$ stands for the set of real-valued $\Sigma\otimes\mathscr{B}(X)$-measurable functions $f\colon \Omega\times X \to \mathbb{R}$ such that for $\mathbb{P}$-a.e. $\omega\in\Omega$ we have $f(\omega,\,\cdot\,)\in C_b(X)$ and

$$\|f\|_{\mathrm{Car}_b} := \int_\Omega \|f(\omega,\,\cdot\,)\|_\infty \, d\mathbb{P}(\omega) < +\infty. \tag{3.1}$$

  Note that all the elements of $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$ are Carathéodory functions. Further, we observe that (3.1) over $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$ defines a norm making this space naturally isomorphic to $L^1(\Omega, \Sigma, \mathbb{P}; C_b(X))$.
- $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ stands for the set of positive measures ν over $\Omega\times X$ whose projections to Ω (i.e., the image measures $\pi_{\Omega\sharp}\nu$ under the projection map $\pi_\Omega\colon \Omega\times X\to\Omega$ defined by $\pi_\Omega(\omega,x) := \omega$) equal $\mathbb{P}$, i.e., $\nu(A\times X) = \mathbb{P}(A)$ for each $A\in\Sigma$. The elements of $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ are called Young measures with marginal $\mathbb{P}$.

A lot of basic facts about $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ for the case of a Polish space X can be found in the classical works [8, 21] (a more recent monograph [6] treating the more general case of a generic topological space X should also be mentioned).

We equip the set of Young measures $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ with the narrow topology [21], i.e., the weakest topology that makes all the maps

$$\nu \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X) \mapsto \int_{\Omega\times X} f\, d\nu$$

continuous, where $f\in\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$. This topology is Hausdorff, see [8, 21]. It is also important to mention that it is, generally speaking, not metrizable, unless Σ is countably generated. This is in sharp contrast with the space of finite Borel measures over X equipped with the weak topology generated by the duality with $C_b(X)$ (the latter topology is also frequently referred to as narrow).

Note that the space $L^0(\Omega, \Sigma, \mathbb{P}; X)$ can be assumed to be imbedded in $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ through a natural identification of every $u\in L^0(\Omega, \Sigma, \mathbb{P}; X)$ with the measure $\delta_u \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ (usually not quite appropriately called a *Dirac random measure*) defined by

$$\delta_u(E) := \mathbb{P}\big(\{\omega\in\Omega : u(\omega)\in E\}\big)$$

for every $E \in \Sigma \times \mathscr{B}(X)$. Then, of course,

$$\int_{\Omega \times X} f\, d\delta_u = \int_\Omega f(\omega, u(\omega))\, d\mathbb{P}(\omega)$$

for $f \in \mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$. Clearly, with the above identification, $L^0(\Omega, \Sigma, \mathbb{P}; X)$ is not closed (in the narrow topology) in $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$.

3.1. Extension of atomic operators. First, we prove that, roughly speaking, every continuous atomic operator

$$T \colon L^p(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^q(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2),$$

where $\{p, q\} \subset [1, +\infty]$ and X_1 and X_2 are Banach spaces, may be extended by continuity in a unique way to an operator

$$\bar{T} \colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2),$$

the extension still being atomic.

Proposition 3.1. *Let X_1 and X_2 be Banach spaces (not necessarily separable), and let $\{p, q\} \subset [1, +\infty]$. Then every atomic operator*

$$T \colon L^p(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^q(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$$

sending the norm convergent sequences to those convergent in measure convergent, admits a unique extension to a continuous (in measure) operator

$$\bar{T} \colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2).$$

The extended operator $\bar{T}$ is still atomic.

Proof. For every $u \in L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ and every $c \in \mathbb{R}^+$, we define

$$u^c(\omega_1) := \begin{cases} u(\omega_1) & \text{if } \omega_1 \notin E^c(u), \\ 0 & \text{otherwise}, \end{cases} \tag{3.2}$$

where $E^c(u) := \{\omega_1 \in \Omega_1 : \|u\|_{X_1} \geq c\}$. Set now

$$\bar{T}(u) := \lim_{\nu \to \infty} T(u^\nu), \tag{3.3}$$

where the limit is intended in measure, relative to $\mathbb{P}_2$. In fact, the limit exists because T is atomic, so for $m \geq n$ we have

$$T(u^m)(\omega_2) = T(u^m)(\omega_2)$$

for all $\omega_2 \in (\Omega_2 \setminus F(E^n(u))) \cup F(E^m(u))$, whence

$$\begin{aligned} \mathbb{P}_2\big(\{\omega_2 \in \Omega_2 : \|T(u^m)(\omega_2) - T(u^n)(\omega_2)\|_{X_2} \neq 0\}\big) \\ = \mathbb{P}_2(\Omega_2) - \mathbb{P}_2(F(E^m(u))) + \mathbb{P}_2(F(E^n(u))), \end{aligned}$$

which means that the sequence $\{T(u^\nu)\}$ is fundamental in $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$. Now we show that

(i) $\bar{T} \colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ is atomic;
(ii) $\bar{T}$ is continuous (in measure);
(iii) for every continuous (in measure) extension

$$\widetilde{T} \colon L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$$

of T we have $\widetilde{T} = \bar{T}$.

Clearly, (iii) follows from the density of $L^p(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ is dense in $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$. To show (i), we observe that, for $\{u, v\} \subset L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$,

$$u|_e = v|_e \text{ implies } u^\nu|_{e \setminus E^\nu(u)} = v^\nu|_{e \setminus E^\nu(v)},$$

and hence,

$$T(u^\nu)|_{F(e \setminus E^\nu(u))} = T(v^\nu)|_{F(e \setminus E^\nu(v))}.$$

Thus, passing to a limit in measure (we mean $\mathbb{P}_2$) as $\nu \to \infty$ in the above relationship, and minding that

$$\mathbb{P}_1(E^\nu(u)) \to 0, \quad \mathbb{P}_1(E^\nu(v)) \to 0,$$

which implies

$$\mathbb{P}_2(F(E^\nu(u))) \to 0, \quad \mathbb{P}_2(F(E^\nu(v))) \to 0,$$

we get

$$T(u)|_{F(e)} = T(v)|_{F(e)}.$$

Finally, to show (ii), assume that $\{u_\nu\} \subset L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$, $u_\nu \to u$ in measure (we mean $\mathbb{P}_1$ this time). Then $\{u_\nu^c\} \subset L^p(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ and $u_\nu^c \to u^c$ in norm, whence $T(u_\nu^c) \to T(u^c)$ in measure. For every $k \in \mathbb{N}$ we choose $c = c(k)$ such that $d^0(Tu^c, \bar{T}u) \leq 1/k$, where d^0 stands for the distance in $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$. Next, let $\nu = \nu(k)$ be such that

$$d^0(T(u_\nu^{c(k)}), T(u_\nu^{c(k)})) \leq 1/k \text{ and } \mathbb{P}_2(F(E^{c(k)}(u_\nu))) \leq 1/k.$$

Since this relation implies $d^0(\bar{T}(u_\nu^{c(k)}), \bar{T}(u_\nu)) \leq 1/k$ in view of the atomicity of $\bar{T}$, the triangle inequality yields $d^0(\bar{T}(u_{\nu(k)}), \bar{T}(u)) \leq 3/k \to 0$ as $k \to \infty$, and the claim follows because the above argument can be applied to an arbitrary subsequence of the original sequence $\{u_\nu\}$. □

The following elementary example shows that the property for T to be atomic in Proposition 3.1 is essential, that is, even for very simple nonatomic operators it may happen that no continuous extension from the space L^p to the space L^0 exists.

Example 3.2. Suppose $p \geq 1$, $\Omega := [0, 1]$, $\mathbb{P} := \mathcal{L}^1 \llcorner \Omega$ is the Lebesgue measure over Ω, and consider the continuous (in norm) operator $T\colon L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^1(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined by the formula

$$(Tu)(\omega) := \mathbf{1}_\Omega(\omega) \int_\Omega u(z)\, d\mathbb{P}(z).$$

Clearly this operator cannot be extended by continuity over the whole $L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$. In fact, for the sequence of functions $\{u_\nu\} \subset L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined by

$$u_\nu(\omega) := \begin{cases} 2\nu^3\omega, & \omega \in [0, 1/\nu], \\ 0, & \omega \in (1/\nu, 1], \end{cases}$$

we have $u_\nu \to 0$ in measure, though, clearly, the sequence Tu_ν does not converge in measure because $(Tu_\nu)(\omega) := \nu \mathbf{1}_\Omega(\omega)$.

Now we are able to prove the main theorem of this section.

Theorem 3.3. *Let $\mathcal{K}_i := L^0(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, where the X_i are Polish spaces, $i = 1, 2$. Then every nonlinear continuous (in measure) atomic operator $T\colon \mathcal{K}_1 \to \mathcal{K}_2$ admits a linear continuous extension*

$$\bar{T}\colon \operatorname{Car}_b'(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to \operatorname{Car}_b'(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2),$$

where $\operatorname{Car}_b'(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ stands for the dual of $\operatorname{Car}_b(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$.

If, moreover, $\bar{\mathcal{K}}_i \subset \mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, *are such that*

$$\begin{aligned} \{\delta_u\}_{u \in \mathcal{K}_1} &\subset \bar{\mathcal{K}}_1, \\ \bar{T}\bar{\mathcal{K}}_1 &\subset \bar{\mathcal{K}}_2, \end{aligned}$$

then $\bar{T} \colon \bar{\mathcal{K}}_1 \to \bar{\mathcal{K}}_2$ *is continuous in the narrow topology of Young measures. In particular, for* $\bar{\mathcal{K}}_i$ *one can take the sequential closures of the sets* $\{\delta_u\}_{u \in \mathcal{K}_i}$ *in the* $*$*-weak topology of* $\mathrm{Car}'_b(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ *(or, equivalently, in the narrow topology of* $\mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$*),* $i = 1, 2$.

Remark 3.4. The heart of the proof is the construction of the extension of the original atomic operator as dual to some continuous linear operator between the spaces of Carathéodory functions. The extended operator obtained in this way is *a priori* defined, continuous, and even linear between the respective dual spaces, which clearly include the sets of Young measures $\mathcal{Y}_i := \mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, but are not reduced to them. In the sequel, however, we shall be interested in extending this operator to smaller sets that consist only of Young measures, which explains the second part of the above statement.

Proof. First, we construct the desired extension of T. This will require several technical steps and will be essentially based on the representation Theorem 2.5.

By Theorem 2.5, T is the composition $T = N \circ T_F$ of a continuous local operator N: $L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ and a generalized shift operator T_F: $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, F\Sigma_1, \mathbb{P}_2; X_1)$. We shall treat local and shift operators separately.

Step 1. We define a special continuous linear operator

$$\widetilde{N} \,:\, \mathrm{Car}_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2) \to \mathrm{Car}_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1).$$

Consider an arbitrary $f \in \mathrm{Car}_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$. Define an operator

$$h_f \,:\, L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1) \to L^1(\Omega_2, \Sigma_F, \mathbb{P}_2; \mathbb{R}),$$

where Σ_F stands for the completion of $F\Sigma_1$ with respect to $\mathbb{P}_2$, by setting

$$(h_f(u))(\,\cdot\,) := \mathbb{E}(f(\,\cdot\,, (N(u))(\,\cdot\,); \Sigma_F), \tag{3.4}$$

where $\mathbb{E}(\,\cdot\,; \Sigma_F)$ denotes the conditional expectation with respect to Σ_F. We emphasize that, by the definition of the class Car_b (namely, because of (3.1)), the operator h_f acts into $L^1(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$, and, moreover, is continuous (i.e., sends measure convergent sequences to L^1 convergent ones). In fact, if $u_\nu \in L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1)$ and $u_\nu \to u$ in measure, then $N(u_\nu) \to N(u)$ in $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1)$ in view of the continuity of N. Therefore, $(N_f \circ N)(u_\nu) \to (N_f \circ N)(u)$ in $L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; \mathbb{R})$, where N_f stands for the Nemytskiĭ operator generated by f. By (3.1) and the Lebesgue dominated convergence theorem, we have $(N_f \circ N)(u_\nu) \to (N_f \circ N)(u)$ also in $L^1(\Omega_2, \Sigma_2, \mathbb{P}_2; \mathbb{R})$, which proves the desired continuity of h_f, by the continuity of the conditional expectation.

Now we show that h_f is local. For this, we pick an arbitrary pair of functions $\{u, v\} \subset L^0(\Omega_2, \Sigma_F, \mathbb{P}_2; X_1)$ such that $u(\omega_2) = v(\omega_2)$ for $\mathbb{P}_2$-a.e. $\omega_2 \in A$, where $A \in \Sigma_F$. Since both N and N_f are local, we have $f(\omega_2, (N(u))(\omega_2)) = f(\omega_2, (N(v))(\omega_2))$ for $\mathbb{P}_2$-a.e. $\omega_2 \in A$, or in other words,

$$\mathbf{1}_A (N_f \circ N)(u) = \mathbf{1}_A (N_f \circ N)(v)$$

$\mathbb{P}_2$-a.e. on Ω_2. Hence,

$$\begin{aligned} \mathbf{1}_A \mathbb{E}((N_f \circ N)(u); \Sigma_F) &= \mathbb{E}(\mathbf{1}_A (N_f \circ N)(u); \Sigma_F) \\ &= \mathbb{E}(\mathbf{1}_A (N_f \circ N)(v); \Sigma_F) = \mathbf{1}_A \mathbb{E}((N_f \circ N)(v); \Sigma_F) \end{aligned}$$

$\mathbb{P}_2$-a.e. on Ω_2, which proves the locality of h_f.

Since $h_f\colon L^0(\Omega_2,\Sigma_F,\mathbb{P}_2;X_1)\to L^1(\Omega_2,\Sigma_F,\mathbb{P};\mathbb{R})$ is local and continuous, and

$$|(h_f(u))(\omega_2)|\le \|f(\omega_2,\,\cdot\,)\|_\infty\in L^1(\Omega_2,\Sigma_2,\mathbb{P}_2;\mathbb{R}),$$

Corollary A.6 yields the existence of a Carathéodory function $\gamma_f\colon \Omega_2\times X_1\to\mathbb{R}$ generating the operator h_f, such that $\gamma_f(\,\cdot\,,x)$ is Σ_F-measurable in for all $x\in X_1$. Namely,

$$(h_f(u))(\omega_2)=\gamma_f(\omega_2,u(\omega_2))$$

for all $u\in L^0(\Omega_2,\Sigma_F,\mathbb{P}_2;X_1)$. Moreover, such a function is unique among all the Σ_F-measurable functions representing the operator h_f in the above sense.

We claim that

$$\gamma_f\in \mathrm{Car}_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_1). \tag{3.5}$$

To prove this, let

$$\widehat{\gamma}(\omega_2):=\sup_{x_1\in X_1}|\gamma_f(\omega_2,x_1)|.$$

This function is Σ_F-measurable. In fact, since X_1 is separable, while $\gamma_f(\omega_2,\,\cdot\,)$ is continuous for $\mathbb{P}_2$-a.e. $\omega_2\in\Omega_2$, we have $\widehat{\gamma}(\omega_2)=\sup_{i\in\mathbb{N}}|\gamma_f(\omega_2,x_1^i)|$, where $\{x_1^i\}_{i=1}^\infty\subset X_1$ is a countable dense subset of X_1. The Σ_F-measurability of each $\gamma_f(\,\cdot\,,x_1^i)$ shows then the desired Σ_F-measurability of $\widehat{\gamma}$.

Now, let $v\in L^1(\Omega_2,\Sigma_F,\mathbb{P}_2;\mathbb{R})$ be a positive integrable (with respect to $\mathbb{P}_2$) and Σ_F-measurable function. Consider the set

$$\{(\omega_2,x_1)\,:\,|\gamma_f(\omega_2,x_1)|\ge\widehat{\gamma}(\omega_2)-v(\omega_2)\}.$$

It is nonempty and $\Sigma_F\otimes\mathscr{B}(X)$-measurable. Hence, by the Aumann measurable selection theorem [7, theorem III.22], there exists a Σ_F-measurable function $u\colon\Omega_2\to X_1$ the graph of which belongs to this set, namely,

$$|(h_f(u))(\omega_2)|=|\gamma_f(\omega_2,u(\omega_2))|\ge\widehat{\gamma}(\omega_2)-v(\omega_2)$$

for $\mathbb{P}_2$-a.e. $\omega_2\in\Omega_2$. Minding that both $h_f(u)$ and v belong to $L^1(\Omega_2,\Sigma_F,\mathbb{P}_2;\mathbb{R})$, it follows from the above inequality that also $\widehat{\gamma}\in L^1(\Omega_2,\Sigma_F,\mathbb{P}_2;\mathbb{R})$ hence proving (3.1) and therefore also the above claim.

We are able now to define the map

$$\widetilde{N}\,:\,\mathrm{Car}_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_2)\to\mathrm{Car}_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_1)$$

by setting

$$\widetilde{N}f:=\gamma_f.$$

The linearity of γ_f with respect to f, and hence that of $\widetilde{N}$, is immediate from the construction, while the boundedness of the latter is practically contained in the proof of the fact that $\gamma_f\in\mathrm{Car}_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_1)$. Indeed, the careful inspection of that proof gives

$$\|\gamma_f\|_{\mathrm{Car}_b}=\|\widehat{\gamma}\|_1\le\|v\|_1+\|h_f(u)\|_1\le\|v\|_1+\|f\|_{\mathrm{Car}_b},$$

as desired.

Step 2. Now we define another auxiliary linear and continuous map

$$\widetilde{T}_F\,:\,\mathrm{Car}_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_1)\to\mathrm{Car}_b(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1).$$

Consider an arbitrary function $f\in\mathrm{Car}_b(\Omega_2,\Sigma_2,\mathbb{P}_2;X_1)$. We shall use the following lemma.

Lemma 3.5. *With the above notation, there is a function* $\widetilde{f}\in\mathrm{Car}_b(\Omega_1,\Sigma_1,\mathbb{P}_1;X_1)$ *satisfying*

$$\int_{F(e_1)} f(\omega_2,(T_Fu)(\omega_2))\,d\mathbb{P}_2(\omega_2)=\int_{e_1}\widetilde{f}(\omega_1,u(\omega_1))\,d\mathbb{P}_1(\omega_2) \tag{3.6}$$

for all $u \in L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ *and all* $e_1 \in \Sigma_1$. *Moreover, the function* $\widetilde{f}$ *satisfying* (3.6) *is unique in the following sense. Assume that an integrand* g: $\Omega_2 \times X_1 \to \mathbb{R}$ *satisfies*

$$\int_{F(e_1)} f(\omega_2, (T_F u)(\omega_2))\, d\mathbb{P}_2(\omega_2) = \int_{e_1} g(\omega_1, u(\omega_1))\, d\mathbb{P}_1(\omega_2) \tag{3.7}$$

for all $u \in L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ *and all* $e_1 \in \Sigma_1$. *Then there is a set* $N_1 \subset \Omega_1$, *such that* $\mathbb{P}_1(N_1) = 0$ *and*

$$g(\omega_1, x_1) = \widetilde{f}(\omega_1, x_1) \tag{3.8}$$

for all $(\omega_1, x_1) \in (\Omega_1 \setminus N_1) \times X_1$. *Finally,* $\|\widetilde{f}\|_{\mathrm{Car}_b} \le \|f\|_{\mathrm{Car}_b}$.

We set

$$\widetilde{T}_F f := \widetilde{f}$$

and observe that, with this notation, relation (3.6) implies

$$\int_{\Omega_2} f(\omega_2, (T_F u)(\omega_2))\, d\mathbb{P}_2(\omega_2) = \int_{\Omega_1} (\widetilde{T}_F f)(\omega_1, u(\omega_1))\, d\mathbb{P}_1(\omega_1).$$

The linearity of $\widetilde{T}_F$ is also immediate from (3.6).

Step 3. We are finally able to define the desired extension of the operator T. For every $\nu_1 \in \mathrm{Car}_b{}'(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$ we set $\bar{T}\nu_1$ to be such that

$$\langle f_2, \bar{T}\nu_1 \rangle_2 := \langle (\widetilde{T}_F \circ \widetilde{N}) f_2, \nu_1 \rangle_1$$

for every $f_2 \in \mathrm{Car}_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$, where $\langle \cdot\,, \cdot \rangle_i$ stands for the duality pairings between $\mathrm{Car}_b(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ and the respective dual $\mathrm{Car}_b'(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$. Clearly, $\bar{T}$: $\mathrm{Car}_b'(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to \mathrm{Car}_b'(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ is linear and continuous, while for $u_1 \in \mathcal{K}_1$ one has

$$\begin{aligned}
\langle f_2, \bar{T}\delta_{u_1} \rangle_2 = \langle \widetilde{T}_F \circ \widetilde{N} f_2, \delta_{u_1} \rangle_1 &= \int_{\Omega_1 \times X_1} (\widetilde{T}_F \circ \widetilde{N}) f_2 \, d\delta_{u_1} \\
&= \int_{\Omega_1} ((\widetilde{T}_F \circ \widetilde{N}) f_2)(\omega_1, u_1(\omega_1))\, d\mathbb{P}_1(\omega_1) \\
&= \int_{\Omega_2} (\widetilde{N} f_2)(\omega_2, T_F u_1(\omega_2))\, d\mathbb{P}_2(\omega_2) \\
&= \int_{\Omega_2} (h_{f_2}(T_F u_1))(\omega_2)\, d\mathbb{P}_2(\omega_2) \\
&= \int_{\Omega_2} f_2(\omega_2, (N(T_F u_1))(\omega_2))\, d\mathbb{P}_2(\omega_2) \\
&= \int_{\Omega_2 \times X_2} f_2 \, d\delta_{N \circ T_F(u_1)} = \langle f_2, \delta_{N \circ T_F(u_1)} \rangle_2
\end{aligned}$$

for every $f_2 \in \mathrm{Car}_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$, which means that $\bar{T}\delta_{u_1} = \delta_{Tu_1}$, i.e., $\bar{T}$ is an extension of T. This concludes the proof of the first part of the statement.

Step 4. The second part of the statement follows immediately from the fact that the narrow topology of Young measures is generated by the duality with bounded Carathéodory functions. Finally, if for $\bar{\mathcal{K}}_i$ one takes the sequential closures of the sets $\{\delta_u\}_{u \in \mathcal{K}_i}$ in the $*$-weak topology of $\mathrm{Car}_b'(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, then $\bar{\mathcal{K}}_i \subset \mathcal{Y}(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$ by Lemma C.1. Moreover, if $\nu_1 \in \bar{\mathcal{K}}_1$, then there is a sequence $\{\delta_{u_1^n}\} \subset \bar{\mathcal{K}}_1$ such that $\delta_{u_1^n} \to \nu_1$, whence

$$\bar{T}\delta_{u_1^n} = \delta_{Tu_1^n} \to \bar{T}\nu_1$$

as $n \to \infty$ (the convergence in each of these cases is meant in the $*$-weak topology of $\mathrm{Car}'_b(\Omega_i, \Sigma_i, \mathbb{P}_i; X_i)$, $i = 1, 2$, which, by Lemma C.1, is equivalent to narrow convergence of measures). This implies $\bar{T}\nu_1 \in \bar{\mathcal{K}}_2$, concluding the proof of the last claim. □

Proof of Lemma 3.5. Let a functional I: $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \times \Sigma_1 \to \mathbb{R}$ be defined by the relation

$$I(u, e_1) := \int_{F(e_1)} f(\omega_2, (T_F u)(\omega_2))\, d\mathbb{P}_2(\omega_2).$$

Clearly, I is local and additive. Moreover, since

$$f \in \mathrm{Car}_b(\Omega_2, \Sigma_2, \mathbb{P}_2; X_1),$$

the Lebesgue theorem shows that for every $e_1 \in \Sigma_1$ the functional

$$I(\,\cdot\,, e_1) \,:\, L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to \mathbb{R}$$

is continuous and bounded both from above and from below. By Corollary A.5, there exists a Carathéodory function $\widetilde{f}$: $\Omega_1 \times X_1 \to \mathbb{R}$ satisfying (3.6), and, moreover, this function is unique in the sense indicated in the statement of Lemma 3.5. It remains therefore to show that $\widetilde{f} \in \mathrm{Car}_b(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1)$. Clearly, (3.6) implies that $\widetilde{f}(\,\cdot\,, x_1)$ is the Radon–Nikodym derivative of the signed measure $I(\,\cdot\,, x_1\mathbf{1}_{\Omega_1})$ with respect to $\mathbb{P}_1$, namely,

$$\widetilde{f}(\,\cdot\,, x_1) = \frac{dI(x_1\mathbf{1}_{\Omega_1}, \,\cdot\,)}{d\mathbb{P}_1}.$$

Define a new measure J over Σ_1 by the formula

$$J(e_1) = \int_{F(e_1)} \sup_{x_1 \in X_1} |f(\omega_2, x_1)|\, d\mathbb{P}_2(\omega_2),$$

and let $j \in L^1(\Omega_1, \Sigma_1, \mathbb{P}_1; \mathbb{R})$ stand for the Radon–Nikodym derivative of J with respect to $\mathbb{P}_1$. Let $\{x_1^i\}_{i=1}^{\infty} \subset X_1$ be a countable dense subset of X_1. Since

$$|I(x_1^i \mathbf{1}_{\Omega_1}, e_1)| \le J(e_1)$$

for all $e_1 \in \Sigma_1$ and all $i \in \mathbb{N}$, we have

$$|\widetilde{f}(\omega_1, x_1^i)| \le j(\omega_1)$$

for all $i \in \mathbb{N}$ and all $\omega_1 \in \Omega_1 \setminus N_1^i$, where $N_1^i \subset \Omega_1$ satisfies $\mathbb{P}_1(N_1^i) = 0$. Minding that $\widetilde{f}$ is a Carathéodory function, for $\mathbb{P}_1$-a.e. $\omega_1 \in \Omega_1 \setminus \cup_i N_1^i$ and hence also for $\mathbb{P}_1$-a.e. $\omega_1 \in \Omega_1$ one has the estimate

$$\sup_{x_1 \in X_1} |\widetilde{f}(\omega_1, x_1))| = \sup_{i \in \mathbb{N}} |\widetilde{f}(\omega_1, x_1^i))| \le j(\omega_1),$$

hence also

$$\|\widetilde{f}\|_{\mathrm{Car}_b} \le \int_{\Omega_1} j(\omega_1)\, d\mathbb{P}_1(\omega_1) = J(\Omega_1) = \|f\|_{\mathrm{Car}_b},$$

which shows the announced claim. □

From the above construction it is clear that the continuous extensions of atomic operators to duals of Carathéodory functions are linear. However, in the example below we show that this is not necessarily the case for all the operators admitting extension by continuity. Namely, there are operators between spaces of measurable functions that possess continuous extensions to the respective spaces of measures but do not come from any linear operator on a larger space.

Example 3.6. Consider the operator $T : L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined by the formula

$$T(u)(\omega) := \mathbf{1}_\Omega(\omega) \int_\Omega (0 \vee u(z) \wedge 1)\, d\mathbb{P}(z),$$

where $\Omega := [0,1]$, $\mathbb{P} := \mathcal{L}^1$ is the Lebesgue measure. Clearly, T can be represented as a composition $T = T_0 \circ N_f$ of the operator

$$T_0 \colon L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$$

defined by the formula

$$(T_0 u)(\omega) := \mathbf{1}_\Omega(\omega) \int_\Omega u(z)\, d\mathbb{P}(z)$$

(this operator was considered in Example 3.2) with the Nemytskiĭ operator

$$N_f \colon L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^p(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$$

generated by the function $f(\omega, u) := 0 \vee u \wedge 1$. The latter extends to the continuous linear operator $\bar{N}_f \colon \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) \to \mathcal{Y}(\Omega \times [0,1], \Sigma, \mathbb{P})$ defined on Young measures by the formula

$$\bar{N}_f \mu := f_\sharp \mu \quad \text{for all} \quad \mu \in \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}).$$

The operator T_0 can be extended to the operator $\bar{T}_0$ defined on the set $\mathcal{Y}_1(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) \subset \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P})$ of Young measures, the second marginal of which has finite first order moment, i.e.,

$$\mathcal{Y}_1(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) := \left\{ \mu \in \mathcal{Y}(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}) \ :\ \int_\mathbb{R} |x|\, d\pi_{\mathbb{R}\sharp}\mu(x) < +\infty \right\},$$

where $\pi_{\mathbb{R}\sharp}\mu$ stands for the second marginal of the measure μ. Minding that if $\mu = \delta_u$, then $\pi_{\mathbb{R}\sharp}\mu$ is the distribution law of u, i.e., $\pi_{\mathbb{R}\sharp}\mu(e) = \mathbb{P}(\{\omega \in \Omega \ :\ u(\omega) \in e\})$ for every Borel set $e \subset \mathbb{R}$, so that

$$\int_\Omega u(\omega)\, d\mathbb{P}(\omega) = \int_\mathbb{R} x\, d\pi_{\mathbb{R}\sharp}\mu(x),$$

for the operator $\bar{T}_0$ we get the formula

$$\bar{T}_0(\mu) := \delta_{\mathbf{1}_\Omega(\,\cdot\,) \int_\mathbb{R} x\, d\pi_{\mathbb{R}\sharp}\mu(x)} = \delta_{\int_\mathbb{R} x\, d\pi_{\mathbb{R}\sharp}\mu(x)} \otimes \mathbb{P}.$$

Observe that, clearly,

$$\mathcal{Y}(\Omega \times [0,1], \Sigma, \mathbb{P}) \subset \mathcal{Y}_1(\Omega \times \mathbb{R}, \Sigma, \mathbb{P}),$$

and that the restriction of $\bar{T}_0$ to this set is continuous in the narrow topology of Young measures. Hence, the operator $\bar{T} := \bar{T}_0 \circ \bar{N}_f$ extends the operator T by continuity to the space of Young measures, though this extension does not come from any linear operator in the space dual to that of Carathéodory functions. Indeed, this operator is given by the formula

$$\bar{T}(\mu) = \delta_{\mathbf{1}_\Omega(\,\cdot\,) \int_\mathbb{R} (0 \vee x \wedge 1)\, d\pi_{\mathbb{R}\sharp}\mu(x)} = \delta_{\int_\mathbb{R} (0 \vee x \wedge 1)\, d\pi_{\mathbb{R}\sharp}\mu(x)} \otimes \mathbb{P},$$

and hence, in general, $\bar{T}(\mu_1/2 + \mu_2/2) \neq \bar{T}(\mu_1)/2 + \bar{T}(\mu_2)/2$.

Further, observe that the continuous extension is uniquely determined over the narrow closure of random Dirac measures (i.e., on the entire set of Young measures, because $\mathbb{P}$ is nonatomic [7]).

The last example in this section describes operators that are continuous in measure yet not continuous in the narrow topology, and thus cannot be extended continuously to the space of Young measures.

Example 3.7. Let $\Omega := (0, 2\pi)$ be equipped with the usual Lebesgue measure $\mathbb{P} := d\omega$ and the usual Lebesgue σ-algebra Σ. Chosen a number $\lambda \in \mathbb{R}$, $\lambda \neq 1$, consider the operator T: $L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^0(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ defined as follows:

$$T(u) := \widetilde{T}(-1 \wedge u \vee 1), \text{ where}$$

the operator $\widetilde{T}$: $L^2(\Omega, \Sigma, \mathbb{P}; \mathbb{R}) \to L^2(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$ sends each function

$$u(\omega) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \sin k\omega + b_k \cos k\omega\right)$$

to the function

$$(\widetilde{T}u)(\omega) := \sum_{k=1}^{\infty} \widetilde{a}_k \sin k\omega,$$

with $\widetilde{a}_k = a_k$ for k even and $\widetilde{a}_k = \lambda a_k$ for k odd. Clearly, the operator T is continuous in measure, since the operator $\widetilde{T}$ is linear and bounded (in L^2). However, T cannot be extended by continuity to Young measures. Indeed, if we consider, for instance, the sequence $u_k(\omega) := \sin k\omega$, then $\delta_{u_k} \rightharpoonup \phi \otimes d\omega$ in the narrow sense of Young measures as $k \to \infty$, where ϕ is the measure on $\mathbb{R}$ concentrated on $[-1, 1]$ and defined by

$$\phi = \frac{1}{\pi\sqrt{1 - x^2}}\, dx.$$

On the other hand, setting $v_k := T(u_k) = \widetilde{T}u_k$, we have $v_{2k} = u_{2k}$ and $v_{2k+1} = \lambda u_{2k}$, whence $\delta_{u_{2k}} \rightharpoonup \phi \otimes d\omega$, but $\delta_{u_{2k+1}} \rightharpoonup \psi \otimes d\omega$ in the narrow sense of Young measures as $k \to \infty$, where ψ is the measure on $\mathbb{R}$ concentrated on $[-\lambda, \lambda]$ and defined by

$$\psi = \begin{cases} \frac{1}{\pi\sqrt{\lambda^2 - x^2}}\, dx, & \lambda \neq 0, \\ \delta_0, & \lambda = 0, \end{cases}$$

so that $\psi \neq \phi$ as $\lambda \neq 1$.

§4. Invariant measures for atomic operators

Throughout this section, again by default, X will stand for a Polish space. Recall the following notion [8, 21].

Definition 4.1. A set of Young measures $\mathcal{H} \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ is said to be *tight* if for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset X$ such that

$$\sup_{\nu \in \mathcal{H}} \nu(\Omega \times (X \setminus K_\varepsilon)) \leq \varepsilon.$$

Next, a set of functions $\mathcal{K} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ is *tight* if it is tight as a set of Young measures, i.e., the set $\{\delta_u\}_{u \in \mathcal{K}}$ is tight. In other words, $\mathcal{K} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ is tight if for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset X$ such that

$$\sup_{u \in \mathcal{K}} \mathbb{P}(\{\omega \in \Omega : u(\omega) \notin K_\varepsilon\}) \leq \varepsilon.$$

Let $\mathbb{T}$ be an additive subset of the set $\mathbb{R}$. Typical examples are $\mathbb{R}^+$, $\gamma\mathbb{Z}^+$ $(\gamma > 0)$ etc.

As an immediate consequence of the extension Theorem 3.3 we obtain the following result.

Theorem 4.2. *Let $\mathcal{K} \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ be a tight set. Let T_τ: $L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X)$, where $\tau \in \mathbb{T}$ is a one-parameter family of commuting continuous* (*in measure*) *atomic operators sending $\mathcal{K}$ into itself. Then this family admits a common invariant measure $\nu \in \bar{\mathcal{K}}$, where $\bar{\mathcal{K}}$ stands for the narrow closure of the set $\mathcal{K}$ in the space of Young measures. In particular, every continuous* (*in measure*) *atomic operator T: $L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X)$ sending $\mathcal{K}$ into itself admits an invariant measure.*

Proof. Denote by $\bar{\mathcal{K}}$ the closure of the set $\mathcal{K}$ in the $*$-weak topology of $\mathrm{Car}'_b(\Omega, \Sigma, \mathbb{P}; X)$. By Lemma C.2, we have $\bar{\mathcal{K}} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, and hence, $\bar{\mathcal{K}}$ coincides with the narrow closure of the set $\mathcal{K}$ in the space of Young measures. The set $\bar{\mathcal{K}}$ is tight by [21, Lemma 9]; consequently, it is narrow compact in $\mathcal{Y}(\Omega; \Sigma; \mathbb{P}; X)$ by [21, Theorem 11] (or [8, Theorem 4.4]).

In the notation of the extension Theorem 3.3, there is a continuous extension $\bar{T}_\tau$: $\mathrm{Car}'_b(\Omega, \Sigma, \mathbb{P}; X) \to \mathrm{Car}'_b(\Omega, \Sigma, \mathbb{P}; X)$ sending $\bar{\mathcal{K}}$ into itself. Note that the operators $\bar{T}_\tau$ are continuous linear and still form a commuting family. Minding that $\mathcal{K}$ is also convex, the reference to the Markov–Kakutani fixed point theorem concludes the proof. □

Now we introduce the notion of tightness for operators between spaces of measurable functions, which, as we shall see, often is a good substitute for the compactness property.

Definition 4.3. An operator T: $L^0(\Omega_1, \Sigma_1, \mathbb{P}_1; X_1) \to L^0(\Omega_2, \Sigma_2, \mathbb{P}_2; X_2)$ is said to be *tight* if it sends bounded sets into tight ones.

Now we may claim the following result.

Corollary 4.4. *Assume that a family of commuting continuous* (*in measure*) *atomic operators*

$$T_\tau : L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X), \quad \tau \in \mathbb{T},$$

maps a set $B \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ into its tight subset. Then all T_τ, $\tau \in \mathbb{T}$, admit a common invariant measure in $\bar{B} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, where $\bar{B}$ stands for the narrow closure of the set $\{\delta_u\}_{u\in B}$. In particular, every continuous (*in measure*) *atomic tight operator T: $L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X)$ that maps a bounded set $B \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ into itself has an invariant measure in $\bar{B}$.*

Proof. Since the set $D := T(B)$ is tight and T maps D into itself, we may apply Theorem 4.2 to obtain the desired result. □

Corollary 4.5. *Assume that a family of commuting continuous* (*in measure*) *atomic operators*

$$T_\tau : L^0(\Omega, \Sigma, \mathbb{P}; X) \to L^0(\Omega, \Sigma, \mathbb{P}; X), \quad \tau \in \mathbb{T},$$

maps some bounded set $B \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ into itself. Next, suppose that there exists $s \in \mathbb{T}$ such that T_s is tight. Then all T_τ, $\tau \in \mathbb{T}$, admit a common invariant measure in $\bar{B}$, where $\bar{B}$ stands for the narrow closure of the set $\{\delta_u\}_{u\in B}$.

Proof. Consider the following commuting family of operators:

$$\mathcal{A} := \{T_{t_1} \circ T_{t_2} \circ \dots T_{t_k} \circ T_s\}_{\{t_j\}\subset\mathbb{T}, k\in\mathbb{N}}.$$

For every $\{t_j\}_{j=1}^k \subset \mathbb{T}$, $k \in \mathbb{N}$, we have

$$T_{t_1} \circ T_{t_2} \circ \dots T_{t_k} \circ T_s \circ T_s(B) = T_s \circ T_{t_1} \circ T_{t_2} \circ \dots T_{t_k}(B) \subset T_s(B),$$

the last subset being tight, and hence, by Corollary 4.4, the family $\mathcal{A}$ has a common invariant measure μ, i.e., $\bar{A}\mu = \mu$ for every $A \in \mathcal{A}$. We show that this measure is in fact invariant for the original family $\{T_t\}_{t\in\mathbb{T}}$. For this, we fix $\tau \in \mathbb{T}$ and observe that $T_t \circ T_\tau \circ T_s \in \mathcal{A}$ and $T_\tau \circ T_s \in \mathcal{A}$, so that

$$\bar{T}_t \circ \bar{T}_\tau \circ \bar{T}_s\mu = \mu,$$
$$\bar{T}_\tau \circ \bar{T}_s\mu = \mu,$$

which implies $\bar{T}_t\mu = \mu$, concluding the proof. □

§5. Stochastic evolution equations

Let $V \subset H \subset V'$ be a Gelfand triple, consisting of a separable Hilbert space H, a reflexive Banach space V, and its conjugate V', each embedding being continuous and dense in the respective topologies. The pairing $a \cdot b$ between V and V' coincides with the inner product in H if $b \in H$.

The following equation is considered:

$$dx = f(t, x)\, dZ(t), \quad t \in [0, T], \tag{5.1}$$

where Z is an m-dimensional semimartingale ($m \in \mathbb{N}$), and $F : \Omega \times \mathbb{R} \times V \to (V')^m$, $(V')^m := V' \times \cdots \times V'$ (m times), and $T > 0$ is fixed. Assume that for any $s \in [0, T)$ equation (5.1) has a unique (mild) solution $x(\,\cdot\,) \colon \Omega \times [s, T) \to H$ for any $x(s) \in L^p(\Omega, \Sigma_s, \mathbb{P}; H)$. It is also assumed that this solution belongs to $L^p(\Omega, \Sigma_t, \mathbb{P}; H)$ for each $t \in [0, T)$ (in particular, this implies that the solution flow is *adapted*). Finally, for each $t \in [s, T)$, the value $x(t)$ of the solution continuously depends on the initial values $x(s)$ in the sense of the natural topologies on $L^p(\Omega, \Sigma_s, \mathbb{P}; H)$ and $L^p(\Omega, \Sigma_t, \mathbb{P}; H)$. These assumptions give rise to a continuous evolution operator $U_t^s \colon L^p(\Omega, \Sigma_s, \mathbb{P}; H) \to L^p(\Omega, \Sigma_t, \mathbb{P}; H)$ determined by the solution flow.

Proposition 5.1. *The operator U_t^s is local in ω and satisfies the evolution property*

$$U_t^\sigma \circ U_\sigma^s = U_t^s \ \text{ for all } \ 0 \le s \le \sigma \le t. \tag{5.2}$$

Proof. First, we observe that, by the properties of the stochastic integral,

$$\int_s^t \vartheta(u) \mathbf{1}_e \, dZ(u) = \int_s^t \vartheta(u)\, dZ(u) \mathbf{1}_e$$

for all $e \in \Sigma_s$, provided that the stochastic integral exists. This immediately implies that the stochastic process $\varkappa(t) = x(t)\mathbf{1}_e + y(t)\mathbf{1}_{\Omega\setminus e}$ $(t \ge s)$ is (a unique) solution of (5.1) satisfying $\varkappa(s) = x(s)\mathbf{1}_e + y(s)\mathbf{1}_{\Omega\setminus e}$. Assume now that $x(s)|_e = y(s)|_e$ a.s. for some $e \in \Sigma_s$. Then $\varkappa(s) = y(s)$ a.s., implying, due to the uniqueness property, that $\varkappa(t) = y(t)$ a.s. for any $t \ge s$. In particular, $x(t)|_e = y(t)|_e$ a.s. This yields locality of the evolution operator U_t^s for $0 \le s \le t \le T$.

The evolution property follows directly from uniqueness. □

Remark 5.2. Assume in addition that all solutions of (5.1) belong to $L^p(\Omega, \Sigma_T, \mathbb{P}; \widetilde{H}_s)$ for some space $\widetilde{H}_s$ of functions $x \colon [s, T] \to H$ (i.e., of continuous or of cadlag functions). Then, by the same argument, the uniqueness of solutions of (5.1) gives the locality of the operator

$$U_{(\,\cdot\,)}^\sigma \colon L^p(\Omega, \Sigma_s, \mathbb{P}; H) \to L^p(\Omega, \Sigma_T, \mathbb{P}; \widetilde{H}_s)$$

defined by

$$(U_{(\,\cdot\,)}^\sigma x)(t) := U_t^\sigma x(s).$$

Using Proposition 3.1, we obtain the following result.

Corollary 5.3. *For any $s \in [0, T)$, equation* (5.1) *has a unique* (*mild*) *solution $x(\,\cdot\,) \colon \Omega \times [s, T) \to H$ for any $x(s) \in L^0(\Omega, \Sigma_s, \mathbb{P}; H)$. For any $t \in [s, T)$, the value $x(t)$ of this solution continuously depends on the initial values $x(s)$ in the sense of the natural topologies on $L^0(\Omega, \Sigma_s, \mathbb{P}; H)$ and $L^0(\Omega, \Sigma_t, \mathbb{P}; H)$. The evolution operator $U_t^s \colon L^0(\Omega, \Sigma_s, \mathbb{P}; H) \to L^0(\Omega, \Sigma_t, \mathbb{P}; H)$ is local and continuous.*

The locality property refers to the pathwise nature of stochastic differential equations, where the evolution of a bunch of trajectories for $\omega \in e$, where $e \subset \Omega$ is an arbitrary measurable subset of a positive measure, does not depend (up to a $\mathbb{P}$-null set) on the evolution of the trajectories for ω outside e.

From Theorem 3.3 we immediately get the following result.

Corollary 5.4. *The solution flow U_t^s for $0 \le s \le t \le T$ of equation* (5.1) *extends continuously to the solution flow $\bar{U}_t^s \colon \mathcal{Y}(\Omega, \Sigma_s, \mathbb{P}; H) \to \mathcal{Y}(\Omega, \Sigma_t, \mathbb{P}; H)$.*

Now we outline another example of stochastic evolution coming from stochastic hereditary equations. For the solution's trajectories on an interval $[a, b]$, one can use either the space $D([a, b]; \mathbb{R}^n)$ of cadlag functions (if Z is discontinuous), or its subspace $C([a, b]; \mathbb{R}^n)$ containing continuous functions (if Z is so). In both cases one can use more general trajectory spaces like the so-called the Delfour–Mitter space $L^2([a, b], \mathcal{B}, \mathcal{L}^1; \mathbb{R}^n) \times \mathbb{R}^n$ [14], where $\mathcal{L}^1$ is the linear Lebesgue measure, $\mathcal{B}$ is the Lebesgue σ-algebra of $[a, b]$. To abbreviate the notation in the latter we will further always omit the reference to $\mathcal{B}$ and $\mathcal{L}^1$.

For simplicity, we denote the space of trajectories by $\mathcal{S}([a, b]; \mathbb{R}^n)$. We use the following notation: $\mathcal{S} := \mathcal{S}([-h + s, s], \mathbb{R}^n)$ and $x_t(\sigma) := x(t + \sigma)$ for $\sigma \in [-h + s, s]$ and $t \ge s$.

We study the stochastic functional differential equation (see [14] for the detailed definitions)

$$dx(t) = F(t, x_t) dZ(t), \tag{5.3}$$

where $t \in (s, T)$, $T > s$ being fixed, with the initial condition

$$x(\sigma) = \varphi(\sigma), \quad \sigma \le s. \tag{5.4}$$

Here $s \in [0, +\infty)$, $Z(t)$ for $t \ge s$ is an m-dimensional semimartingale, and $F \colon \Omega \times [s, T) \times S \to \mathbb{R}^{n \times m}$ is a continuous vector-functional. We assume also that the initial function φ belongs to the space $L^p(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S})$ for some $p \ge 0$.

The solutions of (5.3) should be adapted with respect to the filtration $\{\Sigma_t\}_{t \in [0,T)}$ associated with the semimartingale Z. We denote the set of all n-dimensional $\{\Sigma_t\}$-adapted stochastic processes by $\mathcal{A}$.

Assume that for any $\varphi \in L^p(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S})$ there exists a unique solution $x(\,\cdot\,)$ of equation (5.3) satisfying (5.4) and belonging to the space $\mathcal{A} \cap L^p(\Omega, \Sigma_T, \mathbb{P}; \mathcal{S}([s - h, T); \mathbb{R}^n))$ (equipped with the topology of the second space). As in the previous example, this solution should depend continuously on the initial data in the respective topologies.

Now we introduce the evolution operator associated with the hereditary equation (5.3)

$$U_t^s : \; L^2(\Omega, \Sigma_s; \mathcal{S}) \to L^2(\Omega, \Sigma_t; \mathcal{S}), \quad t \ge s,$$

defined by

$$U_t^s(\varphi) := {}^{\varphi}x_t^s, \quad \varphi \in L^2(\Omega, \Sigma_s; \mathcal{S}), \tag{5.5}$$

where ${}^{\varphi}x^s(t)$ satisfies

$${}^{\varphi}x^s(t) = \begin{cases} \varphi(0) + \int_s^t F(u, {}^{\varphi}x_u^s)\, dZ(u), & t > s, \\ \varphi(t - s), & -h + s \le t \le s. \end{cases} \tag{5.6}$$

In quite a similar way as for (5.1), we arrive at the following results.

Proposition 5.5. *The above operator U_t^s is local in ω and satisfies the evolution property*

$$U_t^\sigma \circ U_\sigma^s = U_t^s \text{ for all } s \le \sigma \le t \le T. \tag{5.7}$$

Corollary 5.6. *For any $s \in [0, T)$, equation* (5.3) *has a unique solution $x(\,\cdot\,) \colon \Omega \times [s, T) \to \mathcal{S}$ for any $\varphi \in L^0(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S})$. For any $t \in [s, T)$, the value $x(t)$ of this solution depends continuously on the initial values $x(s)$ in the sense of the topologies on $L^0(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S})$ and $L^0(\Omega, \Sigma_t, \mathbb{P}; \mathcal{S})$. The evolution operator*

$$U_t^s \colon L^0(\Omega, \Sigma_s, \mathbb{P}; \mathcal{S}) \to L^0(\Omega, \Sigma_t, \mathbb{P}; \mathcal{S})$$

is local and continuous. Moreover, the operator

$$U^{\sigma}_{(\,\cdot\,)}\colon L^0(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})\to L^p(\Omega,\Sigma_T,\mathbb{P};\mathcal{S}[-h+s,T])$$

defined by

$$(U^{\sigma}_{(\,\cdot\,)}x)(t):=U^{\sigma}_t x(s)$$

is also local and continuous.

Corollary 5.7. *The solution flow U^s_t for $0\le s\le t\le T$ of equation* (5.3) *extends continuously to the solution flow $\bar U^s_t\colon \mathcal{Y}(\Omega,\Sigma_s,\mathbb{P};\mathcal{S})\to\mathcal{Y}(\Omega,\Sigma_t,\mathbb{P};\mathcal{S})$.*

Remark 5.8. In applications, extensions of evolution operators to the space of Young measures are of great interest. It is for instance known that in the stochastic Hopf bifurcation, even in the plane, the zero solution that passes through a critical point may produce a solution measure, so that the effect of bifurcation is only visible if such generalized solutions are taken into consideration [2]. Thus, the notion of a solution measure is important for understanding the dynamics of the solution of stochastic equations. In the case of the Carathéodory flows, the problem of extension is trivial (see [2, p. 28]). In the general case the problem is solved by Corollaries 5.4–5.7.

§6. Generalized cocycles

We keep fixed a filtered probability space

$$(\Omega,\Sigma,(\Sigma_t)_{t\in\mathbb{R}^+},\mathbb{P}) \tag{6.1}$$

satisfying the usual conditions (see, e.g., [2]). In addition, we assume that $\mathbb{T}$ is a subsemigroup of the additive group $\mathbb{R}$ with the Borel s-algebra on it. In what follows we use a measurable and measure-preserving dynamical system $(\Omega,(\theta(\tau)_{\tau\in\mathbb{T}},\mathbb{P})$ consistent with the filtration $(\Sigma_t)_{t\in\mathbb{R}^+}$, i.e., a family $\theta(\tau)\colon\Omega\to\Omega$ [2] satisfying

(i) $(\omega,\tau)\mapsto\theta(\omega,\tau)$ is measurable,
(ii) $\theta(\,\cdot\,,0)=id_\Omega$,
(iii) $\theta(\,\cdot\,,\tau+\sigma)=\theta(\,\cdot\,,\tau)\theta(\,\cdot\,,\sigma)$ for all $\tau,\sigma\in\mathbb{T}$,
(iv) $\theta(\tau)_\sharp\mathbb{P}=\mathbb{P}$ for all $\tau\in\mathbb{T}$,
(v) $\theta(\tau)(\Sigma_t)=\Sigma_{\tau+t}$ for all $\tau\in\mathbb{T},\ t\in\mathbb{R}^+$.

Now, for a family of Carathéodory mappings $V_t\colon\Omega\times X\to X$ $(t\in\mathbb{R}^+)$ defined on a Polish space X, the cocycle property [2] with respect to the semigroup $\mathbb{T}$ means that

$$V_{\tau+t}(\omega,x)=V_t(\theta(\tau,\omega),x)\circ V_\tau(\omega,x) \tag{6.2}$$

a.s. for each $t\in\mathbb{R}^+$, $\tau\in\mathbb{T}$ $(t\ge\tau)$, $x\in X$.

Example 6.1. Let $\mathbb{T}=\mathbb{R}^+$, and let $\theta(\,\cdot\,,\tau)$ be the Wiener shift satisfying

$$W(\omega,t+\tau)-W(\omega,s+\tau)=W(\theta(\tau,\omega),t)-W(\theta(\tau,\omega),s)\ \text{a.s.}$$

for every τ, t, s in $\mathbb{R}^+$ (see, e.g., [3]). Here and below $W(\omega,t)$ stands for the scalar Wiener process. Consider an Itô equation in $\mathbb{R}^n$

$$dx(t)=a(x(t))\,dt+b(x(t))\,dW(t) \tag{6.3}$$

under the usual assumptions (e.g., a and b are nonrandom and uniformly Lipschitz) implying that there is a unique (up to a natural equivalence) solution $x_\alpha(t)$ for any (random) initial value $x(0)=\alpha\in L^0(\Omega,\Sigma_0,\mathbb{P};B)$. Then the solution flow defined by $V_t(\omega,\alpha):=x_\alpha(\omega,t)$ is well known to have the cocycle property [2]. Assuming instead in (6.3) that a and b depend on time and are γ-periodic yields a cocycle over the subsemigroup $\gamma\mathbb{N}$.

On the other hand, it was shown by S.-E. A. Mohammed (see [14]) that the evolution operators constructed for stochastic delay equations (see §5) can be *nonregular* in the sense that they do not give rise to a Carathéodory solution flow (which he calls *regular*). The difference between regular and nonregular cases is crucial: no cocycle property for nonregular equations. Thus, the Lyapunov exponents can only be constructed for regular flows. A typical example of a nonregular equation can be as simple as $dx(t) = x(t-h)\,dW(t)$.

The following example shows that stochastic evolution equations may also give rise to nonregular solution flows.

Example 6.2. Consider an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$. Let $B_k(t)$ ($t\geq 0$, $k\in\mathbb{N}$) be independent standard Brownian motions.

We define an (unbounded) linear operator $\mathcal{A}$ by $\mathcal{A}\big(\sum_{k\in\mathbb{N}} x_k e_k\big) := \sum_{k\in\mathbb{N}} n x_k e_k$ and an infinite-dimensional Wiener process $W(t)$ in $\mathcal{H}$ by $W(t) := \sum_{k\in\mathbb{N}} \frac{1}{k} B_k(t) e_k$. Clearly, the covariance operator Q for this process is given by $Q = \operatorname{diag}[\frac{1}{k^2}]_{k\in\mathbb{N}}$, which is a trace-class operator, so that $W(t)$ is a Q-Wiener process (see, e.g., [9, p. 52–53]). Below we assume that the σ-algebra Σ_t is generated by W_s, $0\leq s\leq t$.

The stochastic differential equation

$$dx(t) = \mathcal{A}x(t)dW(t) \tag{6.4}$$

is diagonal with the evolution operator given by

$$U(t,s)(x) := \operatorname{diag}[g_k(t,s)]_{k\in\mathbb{N}}(x),$$

where $g_k(t,s) := \exp(B_k(t) - B_k(s) - (t-s)/2)$, $t\geq s\geq 0$.

For all $t\geq s\geq 0$, the operator

$$U(t,s)\colon L^2(\Omega,\Sigma_s,\mathbb{P};\mathcal{H}) \to L^2(\Omega,\Sigma_t,\mathbb{P};\mathcal{H})$$

is bounded. To see this, we observe that the g_k satisfy $\mathbb{E}g_k(t,s) = 1$ for all $t\geq s\geq 0$ and every $k\in\mathbb{N}$.

Now, take an arbitrary $x := \sum_{k\in\mathbb{N}} x_k e_k \in L^2(\Omega,\Sigma_s,\mathbb{P};\mathcal{H})$, the norm in the latter space denoted by $\|\cdot\|_{L^2(\mathcal{H})}$. Since $g_k(t,s)$ is independent of x_k, we have

$$\begin{aligned}\mathbb{E}\Big\|\sum g_k(t,s)x_k\Big\|^2_{\mathcal{H}} &= \mathbb{E}\sum g_k^2(t,s)x_k^2\\ &= \sum \mathbb{E}g_k^2(t,s)\mathbb{E}x_k^2 \quad \text{(by independence)}\\ &= \mathbb{E}\sum x_k^2 = \|x\|_{L^2(\mathcal{H})},\end{aligned}$$

so that for the operator norm we have $\|U(t,s)\| = 1$ for all $t\geq s\geq 0$. Moreover, $U(t,s)$ is local and thus extends to a continuous operator from $L^0(\Omega,\Sigma_s,\mathbb{P};\mathcal{H})$ to $L^0(\Omega,\Sigma_t,\mathbb{P};\mathcal{H})$.

On the other hand, the random variables $h_k(t,s) := g_k(t,s) - 1$ are independent and normally distributed with the law $\mathcal{N}(0, e^{(t-s)}-1)$, where $t\geq s\geq 0$.

For any $R>0$ and $t\geq s>0$, we have

$$\begin{aligned}\mathbb{P}\Big\{\sup_{k\in\mathbb{N}} |h_k(t,s)| < R\Big\} &= \mathbb{P}\bigg\{\bigg(\bigcap_{k\in\mathbb{N}}\{\omega\colon |h_k(t,s)|<R\}\bigg)\bigg\}\\ &= \prod_{k\in\mathbb{N}} \mathbb{P}\{\omega\colon |h_k(t,s)|<R\} = \exp\bigg\{-\frac{\sqrt{2}}{\pi}\sum_{k\in\mathbb{N}} m_R\bigg\} = 0,\end{aligned}$$

where

$$m_R := \int_{\frac{R}{\sqrt{\exp(t-s)-1}}}^{\infty} \exp\Big\{-\frac{1}{2}x^2\Big\}\,dx > 0.$$

Thus, $\sup_{k\in\mathbb{N}} \|U(t,s)(e_k)\|_{\mathcal{H}} = \infty$ a.s. This means that the evolution operator $U(t,s)$ is non-Carathéodory for any $t \geq s > 0$.

In §5 we proved that, under the existence and uniqueness assumptions, the evolution operator is always local in ω (even if it is not Carathéodory). But in the non-Carathéodory case we do not have the cocycle property. Thus, we need a generalization of this concept based on evolution operators, rather than on solution flows. That is why we must be more specific about the domains and the range of the operators involved.

Assume we are given an evolution family U_t^s of local operators (e.g., a solution flow of some stochastic differential equation) that act continuously from $L^p(\Omega,\Sigma_s,\mathbb{P};X)$ to $L^p(\Omega,\Sigma_t,\mathbb{P};X)$ for some $p \geq 0$, and in addition, we have the isometries

$$T_{\theta(\tau,\,\cdot\,)}\colon L^p(\Omega,\Sigma_t,\mathbb{P};X) \to L^p(\Omega,\Sigma_{t+\tau},\mathbb{P};X).$$

Definition 6.3. The generalized cocycle property with respect to the semigroup $\mathbb{T}$ is given by

$$U_{t+\tau} = T_{\theta(\tau,\,\cdot\,)} \circ U_t \circ T^{-1}_{\theta(\tau,\,\cdot\,)} \circ U_\tau \quad (t\in\mathbb{R}^+,\ \ \tau\in\mathbb{T};\ \ t\geq\tau\geq 0). \tag{6.5}$$

In the case where the evolution operators U are given by Carathéodory solution flows V_t, i.e., $U_t = N_{V_t}$, it is easy to check (e.g., by using an arbitrary $x \in L^p(\Omega,\Sigma_s,\mathbb{P};X)$) that (6.5) gives (6.2).

The following theorem, which deals with equations (5.1) and (5.3), justifies the above definition.

Theorem 6.4. *Assume that a semimartingale $Z(t)$ on the filtered probability space* (6.1) *is a helix with respect to the dynamical system $(\theta(\tau,\,\cdot\,))_{\tau\in\mathbb{T}}$, i.e.,*

$$Z(\omega,t+\tau) - Z(\omega,s+\tau) = Z(\theta(\tau,\omega),t) - Z(\theta(\tau,\omega),s) \ \text{a.s.} \tag{6.6}$$

for every $\tau\in\mathbb{T}$, $t,s\in\mathbb{R}^+$. Then:

(i) *if $s=0$ and $f(\theta(\tau,\omega),t,x) = f(\omega,t+\tau,x)$ a.s. for all $\tau\in\mathbb{T}$, $t\in\mathbb{R}^+$, $x\in H$, then the evolution operator U_t^0 for equation* (5.1) *satisfies the generalized cocycle property* (6.5) *in the space $X = H$*;

(ii) *if $s=0$ and $F(\theta(\tau,\omega),t,\varphi) = F(\omega,t+\tau,\varphi)$ a.s. for all $\tau\in\mathbb{T}$, $t\in\mathbb{R}^+$, $\varphi\in S$, then the evolution operator U_t^0 for equation* (5.3) *satisfies the generalized cocycle property* (6.5) *in the space $X = S$.*

Proof. We only verify statement (i), because (ii) can be proved similarly.

Due to the evolution property (5.2), it suffices to check that

$$U_t = T^{-1}_{\theta(\tau,\,\cdot\,)} \circ U^{\tau}_{t+\tau} \circ T_{\theta(\tau,\,\cdot\,)}, \tag{6.7}$$

or in other words, that, for any $\varphi \in L^p(\Omega,\Sigma_\tau,\mathbb{P};H)$, the stochastic process $y(t) := T^{-1}_{\theta(\tau,\,\cdot\,)}(U^\tau_{t+\tau}\varphi)$ will be a solution of (5.1) for $t>\tau$ satisfying $y(\tau) = T^{-1}_{\theta(\tau,\,\cdot\,)}\varphi$. The last identity is obvious, so we shall concentrate on $y(t)$ for $t>0$.

We shall use the following property of helices:

$$T^{-1}_{\theta(\tau,\,\cdot\,)}\Big(\int_s^t \vartheta(u)\,dZ(u)\Big) = \int_s^t (T^{-1}_{\theta(\tau,\,\cdot\,)}\vartheta)(u)\,dZ(u-\tau) \tag{6.8}$$

for any $t,s\in\mathbb{R}^+$, $\tau\in\mathbb{T}$, which is valid for all predictable stochastic processes ϑ that are integrable with respect to the semimartingale Z (see, e.g., [18] for the case $\mathbb{T}=\mathbb{R}^+$, the proof for $\mathbb{T}\neq\mathbb{R}^+$ is similar).

Since $Y(t+\tau) = Y(\tau) + \int_\tau^{t+\tau} f(u, Y(u))Z(u)$, where $Y(t) = U_t^\tau \varphi$, by (6.8) we obtain

$$\begin{aligned} y(t) &= T^{-1}_{\theta(\tau,\,\cdot\,)}(Y(t+\tau)) = T^{-1}_{\theta(\tau,\,\cdot\,)}\left(Y(\tau) + \int_\tau^{t+\tau} f(u, Y(u))\, dZ(u)\right) \\ &= T^{-1}_{\theta(\tau,\,\cdot\,)}(Y(\tau)) + \int_\tau^{t+\tau} T^{-1}_{\theta(\tau,\,\cdot\,)}\big(f(u, Y(u))\big)\, dZ(u-\tau) \\ &= T^{-1}_{\theta(\tau,\,\cdot\,)}(Y(\tau)) + \int_0^{t} T^{-1}_{\theta(\tau,\,\cdot\,)}\big(f(v+\tau, Y(v+\tau))\big)\, dZ(v) \\ &= T^{-1}_{\theta(\tau,\,\cdot\,)}(Y(\tau)) + \int_0^{t} \big(T^{-1}_{\theta(\tau,\,\cdot\,)} \circ T_{\theta(\tau,\,\cdot\,)}\big)\big(f(v, y(v))\big)\, dZ(v) \\ &= T^{-1}_{\theta(\tau,\,\cdot\,)}(Y(\tau)) + \int_0^{t} \big(f(v, y(v))\big)\, dZ(v), \end{aligned}$$

because $T_{\theta(\tau,\,\cdot\,)}(f(v,x) = f(v+\tau, x))$ for all $x \in V$ and $y(v) = T^{-1}_{\theta(\tau,\,\cdot\,)} Y(v+\tau)$ by assumption. This means that $y(t)$ satisfies (5.1), and the result follows. □

§7. Invariant measures for stochastic dynamical systems with the generalized cocycle property

In this section we apply the general results of § 4 to stochastic equations. First, we observe that even in the case where the solution flow is regular, a natural invariant measure will be a Young measure on $\Omega \times X$, where X is the phase space (see, e.g., [2, 14]). In the regular case, however, we can naturally extend this solution flow to measures on $\Omega \times X$ by setting $\mu \mapsto (V_t)_\sharp \mu$, which is well defined and continuous in the narrow topology. As we saw in the previous sections, the problem becomes much more involved in the nonregular case, i.e., when the evolution operators do not come from the Carathéodory solution flows.

In what follows, as before, we use the isometries

$$T_{\theta(\tau,\,\cdot\,)}\colon L^0(\Omega, \Sigma_t, \mathbb{P}; X) \to L^0(\Omega, \Sigma_{t+\tau}, \mathbb{P}; X).$$

We also recall that $\bar{T}$ is the continuous extension of an operator T to the set of Young measures μ satisfying $\mu(\Omega \times X) = 1$.

Theorem 7.1. *Assume that for some $p \in [1, +\infty)$ an evolution family U_t^s consisting of local operators that act continuously from $L^p(\Omega, \Sigma_s, \mathbb{P}; X)$ to $L^p(\Omega, \Sigma_t, \mathbb{P}; X)$, possesses the generalized cocycle property (6.5) with respect to a given additive semigroup $\mathbb{T}$. Assume further that there exist $R > 0$, $h > 0$ such that*

$$\mathbb{E}\|u\|^p < R \ \text{ implies } \ \mathbb{E}\|U_\tau^0 u\|^p < R \tag{7.1}$$

for any $u \in L^p(\Omega, \Sigma_0, \mathbb{P}; X)$ and for all $\tau \in \mathbb{T}, \tau > h$. Finally, assume that for some $s \in \mathbb{T}$ the set $U_s^0(B)$ is tight in $L^0(\Omega, \Sigma, \mathbb{P}; X)$, where $B := \{u \in L^p(\Omega, \Sigma_0, \mathbb{P}; X) : \mathbb{E}\|u\|^p < R\}$. Then there exists at least one measure in $\bar{B}$, where $\bar{B}$ stands for the narrow closure of the set $\{\delta_u : u \in B\}$, for which

$$\bar{U}^0_{t+\tau}\mu = \big(\bar{T}_{\theta(\tau,\,\cdot\,)} \circ \bar{U}^0_t\big)\mu \tag{7.2}$$

for any $\tau \in \mathbb{T}$, $t \in R^+$.

Proof. Using Proposition 3.1, we may assume that the family U_t^s consists of local and continuous operators from $L^0(\Omega, \Sigma_s, \mathbb{P}; X)$ to $L^0(\Omega, \Sigma_t, \mathbb{P}; X)$.

We wish to apply Corollary 4.5. For this, we introduce the following family of continuous atomic operators

$$T_\tau := T^{-1}_{\theta(\tau,\,\cdot\,)} \circ U_\tau^0 \colon L^0(\Omega, \Sigma_0, \mathbb{P}; X) \to L^0(\Omega, \Sigma_0, \mathbb{P}; X).$$

By the cocycle property (6.5), we have

$$\begin{aligned} T_\tau \circ T_\sigma &= T^{-1}_{\theta(\tau,\,\cdot\,)} \circ U^0_\tau \circ T^{-1}_{\theta(\sigma,\,\cdot\,)} \circ U^0_\sigma \\ &= T^{-1}_{\theta(\tau+\sigma,\,\cdot\,)} \circ T_{\theta(\sigma,\,\cdot\,)} \circ U^0_\tau \circ T^{-1}_{\theta(\sigma,\,\cdot\,)} \circ U^0_\sigma \\ &= T^{-1}_{\theta(\tau+\sigma,\,\cdot\,)} \circ U^0_{\tau+\sigma} \\ &= T_{\tau+\sigma} \end{aligned}$$

for any $\tau, \sigma \in \mathbb{T}$, which means that this family is commutative. Consider the subfamily T_τ $(\tau \in \mathbb{T}, \tau > h)$. For a sufficiently large $\tau \in \mathbb{T}$, $\tau > h$, we definitely have $\tau + s \in T$, $\tau + s > h$, and the operator $T_{\tau+s} = T_s \circ T_\tau$ is tight. Corollary 4.5 gives then a common invariant measure $\mu \in \bar{B}$ for the above subfamily. However, if we take an arbitrary $\eta \in \mathbb{T}$ and sufficiently large $\tau \in \mathbb{T}$ such that $\eta + \tau > h$, then

$$T_\eta \mu = \big(T_\eta \circ T_\tau\big)\mu = T_{\eta+\tau}\mu = \mu.$$

This proves (7.2) for $t = 0$.

Finally, taking advantage of the generalized cocycle property once again, yields

$$\bar{U}^0_{t+\tau}\mu = \big(\bar{T}_{\theta(\tau,\,\cdot\,)} \circ \bar{U}_t \circ \bar{T}^{-1}_{\theta(\tau,\,\cdot\,)} \circ \bar{U}^0_\tau\big)\mu = \big(\bar{T}_{\theta(\tau,\,\cdot\,)} \circ \bar{U}^0_t\big)\bar{T}_\tau \mu = \big(\bar{T}_{\theta(\tau,\,\cdot\,)} \circ \bar{U}^0_t\big)\mu,$$

and the result follows. □

Identity (7.2) says that if $\mathbb{T} = \mathbb{R}^+$, then the solution measure starting at μ will be stationary (in distribution), while in the case of $\mathbb{T} = \gamma\mathbb{N}$ the solution measure will be γ-periodic (again in the sense of distributions).

Finally, we consider a model example where the assumptions of Theorem 7.1 can easily be verified. The result we provide is not meant to be of the most general character, and is intended merely to illustrate the application of the abstract theory developed in the paper. However, we emphasize that it is new and covers many interesting cases, including those where very little or nothing is known about invariant measures. In the example we use the Delfour–Mitter space $S := L^2([-h, 0)]; \mathbb{R}^n) \times \mathbb{R}^n$ with the norm

$$\|\varphi\|^2_S := |\varphi(0)|^2 + \int_{-h}^0 |\varphi(\sigma)|^2\, d\sigma = \int_{-h}^0 |\varphi(\sigma)|^2\, d\lambda(\sigma), \tag{7.3}$$

where $\varphi := (\varphi(\,\cdot\,), \varphi(0))$, $\varphi(\,\cdot\,) \in S$, $\varphi(0) \in \mathbb{R}^n$, and λ is the sum of the Lebesgue measure on $[-h, 0]$ and the Dirac measure at $\sigma = 0$.

Theorem 7.2. *Assume that equation* (5.3) *satisfies the following conditions*:

(i) $Z(t) = (t, W^1(t), \dots, W^k(t))^T$, *where* $t \geq 0$ *and the* $W^i(t)$, $i = 1, \dots, k$, *are independent Wiener processes*;

(ii) $F(\omega, t, \varphi) = A\varphi + F_0(\omega, t, \varphi)$, *where* $t \geq 0$, $\varphi \in S$, $\omega \in \Omega$, A *is a stable* (*Hurwitz*) $n \times n$ *matrix and* $F_0\colon \Omega \times \mathbb{R}^+ \times S \to \mathbb{R}^{n\times m}$, $m = k + 1$, *is a* (*nonlinear*) *operator continuous in the third variable* (*i.e., in* φ) *and measurable in the first two variables, being in addition adapted in* ω *and satisfying* $F_0(\theta(\tau, \omega), t, \varphi) = F_0(\omega, t + \tau, \varphi)$ *a.s. for all* $\tau \in \mathbb{T}$, $t \in \mathbb{R}^+$, $\varphi \in C$, *where either* $\mathbb{T} = \gamma\mathbb{N}$ *for some* $\gamma > 0$, *or* $\mathbb{T} = \mathbb{R}^+$, *and* $\theta(\tau, \,\cdot\,)$ *is the standard Wiener shift*;

(iii) *for some* $\tau_0 \in \mathbb{T}$, $\tau_0 > 0$, *we have* $\mathbb{E}|F_0(t, \varphi)|^2 \leq K$ $(t \geq 0,\ \varphi \in S)$ *for some* $K > 0$;

(iv) *for every* $T > 0$ *and* $\varphi \in S$, *there exists a unique solution* $x(\,\cdot\,)$ *of equation* (5.3) *satisfying* (5.4) *and belonging to*

$$\mathcal{A} \cap L^2(\Omega, \Sigma_T, \mathbb{P}; L^2([0, T]; \mathbb{R}^n))$$

($\mathcal{A}$ *is the set of all* n*-dimensional* Σ_t*-adapted stochastic processes*) *and depending continuously* (*in the respective topologies*) *on* φ (*in particular, this is true if* F_0 *is locally Lipschitz in the third variable*).

Then there exists a generalized invariant measure μ *for the solution flow of equation* (5.3), *i.e., a Young measure satisfying*

$$\bar{U}^0_{t+\tau}\mu = \big(\bar{T}_{\theta(\tau,\,\cdot\,)} \circ \bar{U}^0_t\big)\mu \tag{7.4}$$

for any $\tau \in \mathbb{T}$, $t \in \mathbb{R}^+$, *where* $U^0_{t+\tau}$ *stands for the family of evolution operators corresponding to the solution flow of* (5.3).

The proof of this theorem will be given in Appendix B.

Appendix A. Representation of local functionals and operators

Here and below we assume that $(\Omega, \Sigma, \mathbb{P})$ is a probability space with complete σ-algebra Σ, and X is a Polish space. The space $L^1(\Omega, \Sigma, \mathbb{P}; X)$ will be then abbreviated to $L^1(\Omega; X)$. We recall the following definitions from [4].

Definition A.1. A functional I: $L^1(\Omega; X) \times \Sigma \to \bar{\mathbb{R}} := \mathbb{R} \times \{\pm\infty\}$ is said to be

(i) *local* if for every $\{u, v\} \subset L^0(\Omega; X)$ and every $A \in \Sigma$ we have

$$I(u, A) = I(v, A) \text{ whenever } u(\omega) = v(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in A;$$

(ii) *additive* if whenever $A \in \Sigma$ and $B \in \Sigma$ are disjoint, i.e., $A \cap B = \varnothing$, then

$$I(u, A \cup B) = I(u, A) + I(u, B)$$

for every $u \in L^1(\Omega; X)$.

Definition A.2. A function f: $\Omega \times X \to \bar{\mathbb{R}}$ is called

(i) *an integrand* if it is $\Sigma \otimes \mathscr{B}(X)$-measurable;
(ii) a normal integrand if it is an integrand and $f(\omega,\,\cdot\,)$ is l.s.c. for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We also need the following lemma, which is a slightly adapted version of Proposition 2.1.3 in [4].

Lemma A.3. *Let* f, g: $\Omega \times X \to \bar{\mathbb{R}}$ *be two nonnegative integrands such that*

$$\int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega) \le \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega) \tag{A.1}$$

for every $(u, e) \in L^0(\Omega; X) \times \Sigma$. *Then there is a set* $N \subset \Omega$ *with* $\mathbb{P}(N) = 0$ *such that*

$$f(\omega, x) \le g(\omega, x)$$

for all $(\omega, x) \in (\Omega \setminus N) \times X$. *In particular, if*

$$\int_e f(\omega, u(\omega))\, d\mathbb{P}(\omega) = \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega),$$

then

$$f(\omega, x) = g(\omega, x)$$

for all $(\omega, x) \in (\Omega \setminus N) \times X$.

Proof. Suppose (A.1) is true. To prove the claim, it suffices to show for that, every $k \in \mathbb{N}$,

$$f_k(\omega, x) \le g_k(\omega, x)$$

for $(\omega, x) \in (\Omega \setminus N_k) \times X$, where $f_k := f \wedge k$, $g_k := g \wedge k$, and $N_k \subset \Omega$ is a $\mathbb{P}$-nullset. For this, for every $l \in \mathbb{N}$ we define

$$S_{l,k} := \{(\omega, x) \in \Omega \times X \, : \, f_k(\omega, x) \geq g_k(\omega, x) + 1/l\},$$
$$S_{l,k}(\omega) := \{x \in X \, : \, (\omega, x) \in S_{l,k}\}.$$

Since Σ is assumed to be complete, the projection theorem [7, Theorem III.23] shows that

$$\Omega_{l,k} := \{\omega \in \Omega \, : \, S_{l,k}(\omega) \neq \varnothing\} \in \Sigma,$$

while by the Aumann measurable selection theorem [7, Theorem III.22] there is a Σ-measurable function $s_{l,k}\colon \Omega_{l,k} \to X$ such that $s_{l,k}(\omega) \in S_{l,k}(\omega)$ for all $\omega \in \Omega_{l,k}$. We extend this map to Ω by setting $s_{l,k}(\omega) := x_0$ for all $\omega \notin \Omega_{l,k}$, where $x_0 \in X$ is an arbitrarily chosen element of X.

By the definition of $s_{l,k}$,

$$f_k(\omega, s_{l,k}(\omega)) \geq g_k(\omega, s_{l,k}(\omega)) + 1/l$$

for all $\omega \in \Omega_{l,k}$. But since $f_k \leq k$, we have $g_k(\omega, s_{l,k}(\omega)) < k$, and hence, in accordance with the definition of g_k,

$$g_k(\omega, s_{l,k}(\omega)) = g(\omega, s_{l,k}(\omega))$$

for $\omega \in \Omega_{l,k}$. This implies

$$g(\omega, s_{l,k}(\omega)) + 1/l = g_k(\omega, s_{l,k}(\omega)) + 1/l \leq f_k(\omega, s_{l,k}(\omega)) \leq f(\omega, s_{l,k}(\omega)).$$

Integrating the above inequality over $\Omega_{l,k}$ provides

$$\int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega) + \mathbb{P}(\Omega_{l,k})/l \leq \int_{\Omega_{l,k}} f(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega).$$

Taking into account that

$$\int_{\Omega_{l,k}} f(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega) \leq \int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega)$$

in view of (A.1), we get

$$\int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega) + \mathbb{P}(\Omega_{l,k})/l \leq \int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega).$$

Minding now that

$$0 \leq \int_{\Omega_{l,k}} g(\omega, s_{l,k}(\omega)) \, d\mathbb{P}(\omega) \leq k$$

because $0 \leq g = g_k \leq k$ over $\Omega_{l,k}$, we arrive finally at the conclusion that $\mathbb{P}(\Omega_{l,k}) = 0$. We may set now

$$N_k := \bigcup_{l \in \mathbb{N}} \Omega_{l,k},$$

yielding $\mathbb{P}(N_k) = 0$ and

$$f_k(\omega, x) \leq g_k(\omega, x)$$

for $(\omega, x) \in (\Omega \setminus N_k) \times X$, which concludes the proof. □

Proposition A.4. *Suppose that a functional $I\colon L^0(\Omega; X) \times \Sigma \to \mathbb{R}$ is local, additive, and bounded from below, i.e., $I(u) \geq c$ for some $c \in \mathbb{R}$ and for all $u \in L^1(\Omega; X)$. Assume, moreover, that*

(i) *there exists $u_0 \in L^1(\Omega; X)$ such that $I(u_0, \cdot\,)$ is a signed measure absolutely continuous with respect to $\mathbb{P}$;*
(ii) *the functional $I(\,\cdot\,, \Omega)$ is l.s.c.*

Then there is a normal integrand $f\colon \Omega\times X\to\bar{\mathbb{R}}$ *such that*

$$I(u,A)=\int_A f(\omega,u(\omega))\,d\mathbb{P}(\omega) \tag{A.2}$$

for all $(u,A)\in L^1(\Omega,X)\times\Sigma$. *Moreover, such an integrand is unique in the sense that if for some integrand* $g\colon \Omega\times X\to\bar{R}$ *we have*

$$I(u,A)=\int_A g(\omega,u(\omega))\,d\mathbb{P}(\omega)$$

for all $(u,A)\in L^1(\Omega,X)\times\Sigma$, *then* $g(\omega,x)=f(\omega,x)$ *for all* $(\omega,x)\in(\Omega\setminus N)\times X$, *where* $N\subset\Omega$ *satisfies* $\mathbb{P}(N)=0$.

Proof. The proof follows the lines of that of the similar Theorem 2.4.2 in [4], which is formulated for functionals defined over a Lebesgue space (instead of L^0) of functions with values in a finite-dimensional space $\mathbb{R}^n$ (instead of a generic Polish space X). The generalization to our case is quite straightforward though technical, and therefore here we provide the proof merely for the reader's convenience.

Step 1. Without loss of generality we may assume that $c=0$, so that $I(u,e)\geq 0$ for all $(u,e)\in L^0(\Omega;X)\times\Sigma$. For every $k\in\mathbb{N}$, define the Moreau–Yosida transform $I_k(\,\cdot\,,e)$ of the functional $I(\,\cdot\,,e)$ by the formula

$$I_k(u,e):=\inf\left\{I(v,e)+k\int_e d(u(\omega),v(\omega))\wedge 1\,d\mathbb{P}(\omega)\ :\ v\in L^0(\Omega;X)\right\},$$

where d stands for the distance in X. Then

$$0\leq I_k(u,e)\leq k\int_e d(u(\omega),u_0(\omega))\wedge 1\,d\mathbb{P}(\omega). \tag{A.3}$$

By Proposition 1.3.7 in [4], each $I_k(\,\cdot\,,e)$ is Lipschitz continuous over $L^0(e,\Sigma\cap e,\mathbb{P};X)$ with Lipschitz constant k, namely,

$$|I_k(u,e)-I_k(v,e)|\leq k\int_e d(u(\omega),v(\omega))\wedge 1\,d\mathbb{P}(\omega). \tag{A.4}$$

For every $x\in X$, the set function $I_k(x\mathbf{1}_\Omega,\,\cdot\,)$ is additive on disjoint sets and, by (A.3) and (i), is bounded from above by a finite measure absolutely continuous with respect to $\mathbb{P}$. Therefore, $I_k(x\mathbf{1}_\Omega,\,\cdot\,)$ is a finite measure that is also absolutely continuous with respect to $\mathbb{P}$. Denote by $f_k(\omega,x)$ the Radon–Nikodym derivative of $I_k(x\mathbf{1}_\Omega,\,\cdot\,)$ with respect to $\mathbb{P}$, i.e.,

$$I_k(x\mathbf{1}_\Omega,e)=\int_e f_k(\omega,x)\,d\mathbb{P}(\omega).$$

Let $D:=\{x_i\}_{i=1}^\infty\subset X$ be a countable dense subset of X. From (A.3) and (A.4) it follows that

$$\begin{aligned}&0\leq f_k(\omega,x_i)\leq a(\omega)+k(d(x_i,u_0(\omega))\wedge 1),\\ &|f_k(\omega,x_i)-f_k(\omega,x_j)|\leq k(d(x_i,x_j)\wedge 1)\end{aligned} \tag{A.5}$$

for some $a\in L^1(\Omega;\mathbb{R})$ and for all $\{i,j\}\in\mathbb{N}$ and $\omega\in\Omega\setminus N_{ij}$, where $N_k\subset\Omega$ is some set satisfying $\mathbb{P}(N_{ij})=0$. Set

$$N:=\bigcup_{\{i,j\}\subset\mathbb{N}} N_{ij}$$

and fix $\omega\in\Omega\setminus N$. Since the function $f_k(\omega,\,\cdot\,)\colon\ D\to\mathbb{R}$ is Lipschitz continuous over D by (A.5), it admits a Lipschitz continuous extension to the entire X. Namely, there is a function $g_k(\omega,\,\cdot\,)\colon X \to \mathbb{R}$ that is still Lipschitz continuous and satisfies $f_k(\omega,x) = g_k(\omega,x)$ for all $x \in D$. Putting

$$\widetilde{f}_k(\omega,x) := \begin{cases} g_k(\omega,x), & \omega \in \Omega \setminus N, \\ 0 & \text{otherwise}, \end{cases}$$

it is easy to verify that $\widetilde{f}_k$ is a Carathéodory function (moreover, $\widetilde{f}_k(\omega,\,\cdot\,)$ is Lipschitz continuous), and

$$0 \le \widetilde{f}_k(\omega,x) \le a(\omega) + k(d(x,u_0(\omega)) \wedge 1) \tag{A.6}$$

for all $\omega \in \Omega$ and $x \in X$.

Finally, we observe that

$$I_k(u,e) = \int_e \widetilde{f}_k(\omega,u(\omega))\,d\mathbb{P}(\omega) \tag{A.7}$$

for every $u \in L^0(\Omega;X)$ and $e \in \Sigma$. Indeed, (A.7) is clearly valid for any simple function (i.e., a function with finitely many values) with values in D. But such functions form a dense subset of $L^0(\Omega;X)$. Thus, approximating an arbitrary $u \in L^0(\Omega;X)$ by a sequence $\{u_i\}_{i=1}^{\infty} \subset L^0(\Omega;X)$ of simple functions with values in D, i.e., $u_i \to u$ in $L^0(\Omega;X)$ as $i \to \infty$, we get

$$I_k(u_i,e) = \int_e \widetilde{f}_k(\omega,u_i(\omega))\,d\mathbb{P}(\omega) \to \int_e \widetilde{f}_k(\omega,u(\omega))\,d\mathbb{P}(\omega)$$

as $i \to \infty$, in view of estimate (A.6) and the Lebesgue theorem. On the other hand, since $I_k(\,\cdot\,,e)$ is Lipschitz continuous over $L^0(\Omega;X)$, which follows from (A.4), we have $I_k(u_i,e) \to I_k(u,e)$ as $i \to \infty$, implying (A.7).

Step 2. For $l \ge k$ we have $0 \le I_k(u,e) \le I_l(u,e)$ for all $(u,e) \in L^0(\Omega;X) \times \Sigma$. By Lemma A.3, this implies the existence of a set $\widetilde{N} \subset \Omega$ with $\mathbb{P}(\widetilde{N}) = 0$ such that

$$0 \le \widetilde{f}_k(\omega,x) \le \widetilde{f}_l(\omega,x)$$

for all $(\omega,x) \in (\Omega \setminus \widetilde{N}) \times X$. We set

$$f(\omega,x) := \begin{cases} \sup_{k\in\mathbb{N}} \widetilde{f}_k(\omega,x), & \omega \in \Omega \setminus \widetilde{N}, \\ 0 & \text{otherwise}. \end{cases} \tag{A.8}$$

Obviously, f is a normal integrand, being the supremum of an increasing sequence of Carathéodory functions. But the additivity and locality of $I(u,\,\cdot\,)$, (ii) implies that $I(\,\cdot\,,e)$ is l.s.c. for every $e \in \Sigma$, and hence, by Proposition 1.3.7 in [4], we have

$$I(u,e) = \sup_{k\in\mathbb{N}} I_k(u,e)$$

for every $(u,e) \in L^0(\Omega;X) \times \Sigma$. Recalling (A.7) and (A.8) and using the Beppo Levi theorem, we get

$$I(u,e) = \int_e f(\omega,u(\omega))\,d\mathbb{P}(\omega),$$

which concludes the proof of existence.

The uniqueness property follows immediately from the second claim of Lemma A.3. □

Corollary A.5. *Suppose that a functional $I\colon L^0(\Omega;X) \times \Sigma \to \mathbb{R}$ is local, additive, and bounded both from above and from below, i.e. $c \le I(u) \le C$ for some $c, C \in \mathbb{R}$ and for all $u \in L^1(\Omega;X)$. Assume, moreover, that*

(i) *there exists $u_0 \in L^1(\Omega;X)$ such that $I(u,\,\cdot\,)$ is a signed measure absolutely continuous with respect to $\mathbb{P}$;*

(ii) *the functional* $I(\,\cdot\,,\Omega)$ *is continuous.*

Then there is a Carathéodory function $f\colon \Omega\times X\to\bar{\mathbb{R}}$ *such that*

$$I(u,e)=\int_e f(\omega,u(\omega))\,d\mathbb{P}(\omega) \tag{A.9}$$

for all $(u,e)\in L^0(\Omega,X)\times\Sigma$. *Moreover, such a function is unique in the sense announced in Proposition* A.4.

Proof. By Proposition A.4, there is a unique normal integrand f: $\Omega\times X\to\bar{\mathbb{R}}$ such that (A.9) is valid. Similarly, since $-I$ also satisfies the conditions of Proposition A.4, we have

$$-I(u,e)=\int_e g(\omega,u(\omega))\,d\mathbb{P}(\omega)$$

for a unique normal integrand g: $\Omega\times X\to\bar{\mathbb{R}}$. On the other hand, from (A.9) it follows that

$$-I(u,e)=\int_e(-f(\omega,u(\omega)))\,d\mathbb{P}(\omega),$$

and the uniqueness of the integrand g representing the functional $-I$ provides

$$g(\omega,x)=-f(\omega,x)$$

for all $x\in X$ and for $\mathbb{P}$-a.e. $\omega\in\Omega$. Since $g(\omega,\,\cdot\,)$ is l.s.c. for such ω, the latter relation implies that $f(\omega,\,\cdot\,)$ is u.s.c. Combined with the lower semicontinuity of $f(\omega,\,\cdot\,)$ for $\mathbb{P}$-a.e. $\omega\in\Omega$, this proves that f is a Carathéodory function. $\square$

The following corollary is the main representation result of this section; though being less general than Theorem 1 in [16], it has a much shorter and easier proof and quite suffices for our purposes.

Corollary A.6. *Suppose that an operator* $N\colon L^0(\Omega;X)\to L^1(\Omega;\mathbb{R})$ *is local, continuous* (*in measure*), *and such that*

$$\int_\Omega |(N(u))(\omega)|\,d\mathbb{P}(\omega)\le C$$

for some $C\ge 0$ *and for all* $u\in L^0(\Omega;X)$. *Then there exists a Carathéodory function* f: $\Omega\times X\to\bar{\mathbb{R}}$ *such that*

$$(N(u))(\omega)=f(\omega,u(\omega)) \tag{A.10}$$

for all $u\in L^0(\Omega,X)$ *and* $\mathbb{P}$-*a.e.* $\omega\in\Omega$. *Moreover, such a function is unique in the sense announced in Proposition* A.4.

Proof. Define a functional I: $L^0(\Omega;X)\times\Sigma\to\mathbb{R}$ by the formula

$$I(u,e):=\int_e(N(u))(\omega)\,d\mathbb{P}(\omega).$$

Clearly, I satisfies the conditions of Corollary A.5; and hence, there is a Carathéodory function f: $\Omega\times X\to\bar{\mathbb{R}}$ such that

$$I(u,e)=\int_e f(\omega,u(\omega))\,d\mathbb{P}(\omega)$$

for all $(u,e)\in L^0(\Omega,X)\times\Sigma$. Therefore, both $f(\,\cdot\,,u(\,\cdot\,))$ and $(N(u))(\,\cdot\,)$ are Radon–Nikodym derivatives of $I(u,\,\cdot\,)$ with respect to $\mathbb{P}$, whence we see that (A.10) is valid. If there is an integrand $g\colon\Omega\times X\to\mathbb{R}$ such that

$$(N(u))(\omega)=g(\omega,u(\omega))$$

for all $u \in L^0(\Omega, X)$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, then

$$I(u, e) = \int_e g(\omega, u(\omega))\, d\mathbb{P}(\omega),$$

so that $g(\omega, x) = f(\omega, x)$ for all $x \in X$ and all $\omega \in \Omega \setminus N$, where $N \subset \Omega$ is $\mathbb{P}$-negligible, in accordance with Proposition A.4. $\square$

Appendix B. Proof of Theorem 7.2

Theorem 6.4(ii) and Theorem 7.1 applied with $p := 2$ show that the existence of an invariant measure is ensured whenever

(A) for some $C > 0$ and some norm $\||\cdot\||$ in the space S we have

$$\mathbb{E}\||\phi\||^2 \le C \ \text{ implies } \ \mathbb{E}\||U^0_\tau \phi\||^2 \le C$$

for all $\phi \in L^2(\Omega, \Sigma_0, \mathbb{P}; S)$, $\tau \in \mathbb{T}$, $\tau > h$, where U^0_τ is defined by (5.5);

(B) U^0_T is tight for some $T > h$.

We divide therefore the proof of Theorem 7.2 into two steps starting with part (A).

Step 1. Since the matrix A is stable, there exists a symmetric and positive matrix P such that $A^*P + PA = -I$, where I is the $n \times n$ identity matrix (for instance, one can put $P := \int_0^\infty \exp(A^*s)\exp(As)\, ds$, see [13]). We introduce the quadratic form $v(x) = x^*Px$ (the dot product of x and Px) on $\mathbb{R}^n$ and the Lyapunov functional V on S by the formula

$$V(\phi) := v(\phi(0)) + \int_{-h}^0 v(\phi(\sigma))\, d\sigma = \int_{-h}^0 v(\sigma)\, d\lambda(\sigma), \tag{B.1}$$

where λ is the sum of the Lebesgue measure on $[-h, 0]$ and the Dirac measure at $\sigma = 0$ (i.e., λ is the same measure as in (7.3)). Clearly, $\varphi \mapsto \sqrt{V(\varphi)}$ is a norm on S.

Below we shall always assume that $t \ge h$, so that, by (5.6), we have the following representation:

$$(U^0_t)(\phi)(\sigma) = x_t(\sigma) = x(t+\sigma) = \int_0^{t+\sigma} F(s, x_s(\,\cdot\,))\, dZ_s,$$

where

$$F(s, x_s(\,\cdot\,))\, dZ(s) = Ax(s)ds + F_{01}(s, x_s(\,\cdot\,))\, ds + F_{02}(s, x_s(\,\cdot\,))\, dW(s). \tag{B.2}$$

Applying the stochastic integration by parts formula [15] (which is a particular case of the Itô formula)

$$d\,(u^*w) = u^*dw + (du)^*w + (du)^*dw$$

to $v(x(t+\sigma))$ (with $u := x$ and $w := Px$), for an arbitrary $\sigma \in [-h, 0]$ and any $t \ge h$ we get

$$dv(x(t+\sigma)) = x^*(t+\sigma)\, d(Px(t+\sigma)) + dx^*(t+\sigma)Px(t+\sigma) + dx^*(t+\sigma)\, d(Px(t+\sigma)).$$

Using (B.2) and minding that, the formal calculation rules with the stochastic differential yield $dt\, dW = 0$ and $(dW)^*dW = 0$, see [15], we get

$$\begin{aligned} dv(x(t+\sigma)) &= x^*(t+\sigma)\, PAx(t+\sigma)\, dt + x^*(t+\sigma)\, PF_{01}(t+\sigma, x_{t+\sigma})\, dt \\ &\quad + x^*(t+\sigma)\, PF_{02}(t+\sigma, x_{t+\sigma})\, dW(t+\sigma) + \big[Ax(t+\sigma)\big]^* Px(t+\sigma)\, dt \\ &\quad + F^*_{01}(t+\sigma, x_{t+\sigma})\, Px(t+\sigma)\, dt + \big[F_{02}(t+\sigma, x_{t+\sigma})\, dW(t+\sigma)\big]^* Px(t+\sigma) \\ &\quad + \big[F_{02}(t+\sigma, x_{t+\sigma})\, dW(t+\sigma)\big]^* PF_{02}(t+\sigma, x_{t+\sigma})\, dW(t+\sigma). \end{aligned}$$

Since $a^*b = b^*a$ when a and b are vectors, while $P^* = P$ and $dW^*QdW = \operatorname{tr} Q$, we obtain

$$\begin{aligned} dv(x(t+\sigma)) &= x^*(t+\sigma)(PA+A^*P)x(t+\sigma)\,dt + 2x^*(t+\sigma)PF_{01}(t+\sigma,x_{t+\sigma})\,dt \\ &\quad + \operatorname{tr}\big(F_{02}^*(t+\sigma,x_{t+\sigma})\,PF_{02}(t+\sigma,x_{t+\sigma})\big)\,dt + x^*(t+\sigma)\,PF_{02}(t+\sigma,x_{t+\sigma})dW(t+\sigma) \\ &\quad + \big[Px(t+\sigma)\big]^* F_{02}(t+\sigma,x_{t+\sigma})\,dW(t+\sigma), \end{aligned}$$

where $-h \le \sigma \le 0$ and $t \ge h$. Thus, integrating the above relation and recalling that $\mathbb{E}W_t = 0$, we get

$$\begin{aligned} \mathbb{E}v(x(t+\sigma)) = \mathbb{E}v(x(h+\sigma)) - \int_h^t \mathbb{E}|x(s+\sigma)|^2\,ds + 2\int_h^t \mathbb{E}x^*(s+\sigma)PF_{01}(s+\sigma,x_{s+\sigma})\,ds \\ + \int_h^t \mathbb{E}\operatorname{tr}\big(F_{02}^*(s+\sigma,x_{s+\sigma})\,PF_{02}(s+\sigma,x_{s+\sigma})\big)\,ds, \end{aligned}$$

where $-h \le \sigma \le 0$ and $t \ge h$. Now we integrate the last identity with respect to the measure λ, obtaining

$$\begin{aligned} \mathbb{E}V(x_t) = \mathbb{E}V(x_h) - \int_h^t \mathbb{E}\|x_s\|_S^2\,ds + 2\int_h^t \mathbb{E}\int_{-h}^0 x^*(s+\sigma)PF_{01}(s+\sigma,x_{s+\sigma})\,ds\,d\lambda(\sigma) \\ + \int_h^t \mathbb{E}\int_{-h}^0 \operatorname{tr}\big(F_{02}^*(s+\sigma,x_{s+\sigma})\,PF_{02}(s+\sigma,x_{s+\sigma})\big)\,ds\,d\lambda(\sigma) \quad (t \ge h). \end{aligned}$$

In particular, this shows that the function $\gamma(t) := \mathbb{E}V(x_t)$ is differentiable for $t > h$ and

$$\begin{aligned} \gamma'(t) = -\mathbb{E}\|x_t\|_S^2 + 2\mathbb{E}\int_{-h}^0 x^*(t+\sigma)PF_{01}(t+\sigma,x_{t+\sigma})\,d\lambda(\sigma) \\ + \mathbb{E}\int_{-h}^0 \operatorname{tr}\big(F_{02}^*(t+\sigma,x_{t+\sigma})\,PF_{02}(t+\sigma,x_{t+\sigma})\big)\,d\lambda(\sigma) \quad (t \ge h). \end{aligned}$$

Assumption (iii) of Theorem 7.2 together with the Hölder inequality imply that

$$\gamma'(t) \le -\mathbb{E}\|x_t\|_S^2 + C_1\mathbb{E}\|x_t\|_S + C_2 \quad (t > h),$$

where C_1 and C_2 are some positive constants. Therefore, $\gamma'(t) < 0$ as soon as

$$\mathbb{E}\|x_t\|_S^2 \ge R > R_0 := \frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2} \quad \text{and} \quad t > h.$$

On the other hand, $v(x) \le \|P\|\cdot|x|^2$, where $\|P\|$ stands for the matrix norm of P, so that $V(\varphi) \le \|P\|\cdot\|\varphi\|_S^2$. Hence, for any $R > R_0$ and any $t > h$, the inequality $\mathbb{E}V(x_t) \ge R\|P\|$ always implies $\frac{d}{dt}\mathbb{E}V(x_t) < 0$. In other words,

$$\mathbb{E}V(\phi) < R\|P\| \text{ implies } \mathbb{E}V(x_t) < R\|P\| \quad (t > h).$$

Since $x_t = U_t^0\phi$, this completes the proof of part (A) (with $C := R\|P\|$ and $\|\!|\cdot|\!\|^2 := V(\cdot)$).

Step 2. Now we prove the tightness condition (B). Below we assume that $T > h$ is kept fixed.

Assumption (iii) of Theorem 7.2 gives immediately the following estimate:

$$\mathbb{E}\|F(t,\phi)\|_S^2 \le C(1 + \mathbb{E}\|\phi\|_S^2) \quad (t \ge 0,\ \ \phi \in L^0(\Omega,\Sigma_t,\mathbb{P};S)). \tag{B.3}$$

We use again the representation (5.6) to obtain

$$(U_t^0\phi)(\sigma) = \phi(t+\sigma)\mathbf{1}_{\{t+\sigma<0\}} + \left(\phi(0) + \int_0^{t+\sigma} F(s,(U_s^0\phi)(\sigma))\,dZ(s)\right)\mathbf{1}_{\{t+\sigma\ge 0\}}.$$

Squaring the latter identity, minding that $(a+b)^2 \le 2a^2 + 2b^2$, and finally taking the expectation, we get

$$\begin{aligned}\mathbb{E}|U_t^0\phi|^2 &\le \mathbb{E}|\phi(t+\sigma)|^2\mathbf{1}_{\{t+\sigma<0\}} + \left(2\mathbb{E}|\phi(0)|^2 + 2\mathbb{E}\left|\int_0^{t+\sigma} F(s,(U_s^0\phi)(\sigma))\,dZ(s)\right|^2\right)\mathbf{1}_{\{t+\sigma\ge 0\}} \\ &\le \mathbb{E}|\phi(t+\sigma)|^2\mathbf{1}_{\{t+\sigma<0\}} + \left(2\mathbb{E}|\phi(0)|^2 + C_1\int_0^{t+\sigma} \mathbb{E}\big|F(s,(U_s^0\phi)(\sigma))\big|^2\,ds\right)\mathbf{1}_{\{t+\sigma\ge 0\}} \\ &\le 2\mathbb{E}|\phi(t+\sigma)|^2\mathbf{1}_{\{t+\sigma<0\}} + 2\left(\mathbb{E}|\phi(0)|^2 + C_2\int_0^{t+\sigma} \mathbb{E}(1+\mathbb{E}\|U_s^0\phi\|_S^2)\,ds\right)\mathbf{1}_{\{t+\sigma\ge 0\}},\end{aligned}$$

where the latter estimate follows from (B.3). Integrating over the interval $[-h,0]$ with respect to the measure λ and using (7.3) gives

$$\mathbb{E}\|U_t^0\phi\|_S^2 \le 2\mathbb{E}\|\phi\|_S^2 + C_3\int_0^t (1+\mathbb{E}\|U_s^0\phi\|_S^2)\,ds.$$

Now we apply the Gronwall inequality to obtain the estimate

$$\mathbb{E}\|U_t^0\phi\|_S^2 \le C_4\big(1+\mathbb{E}\|\phi\|_S^2\big),$$

so that

$$\mathbb{E}\|F(t,U_t^0\phi)\| \le C_5\big(1+\mathbb{E}\|\phi\|_S^2\big)$$

for any $t\in[0,T]$ and any $\phi\in L^0(\Omega,\Sigma_0,\mathbb{P};S)$, by (B.3). Finally,

$$\int_0^T \mathbb{E}|F(t,U_t^0\phi)|^2\,dt \le C_6\big(1+\mathbb{E}\|\phi\|_S^2\big). \tag{B.4}$$

Let $B\subset L^0(\Omega,\Sigma_0,\mathbb{P};S)$ be an arbitrary bounded set. Estimate (B.4) says that $h\colon \phi\mapsto F(t,U_t^0\phi)$ is a bounded operator from $L^2(\Omega,\Sigma_0,\mathbb{P};S)$ to $L^2(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))$ (as before, for the sake of brevity we always omit the reference to the Lebesgue σ-algebra and the Lebesgue measure of $[0,T]$). On the other hand, the operator h is a composition of the Nemytskiĭ operator F and the local operator U_t^0 (the latter is local by Corollary 5.6). Using the locality property and boundedness in L^2 we can actually prove that h is bounded as an operator between $L^0(\Omega,\Sigma_0,\mathbb{P};S)$ and $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))$. To see this, we pick any $\phi\in B$ and any $\varepsilon>0$. Then there exists a $\phi_\varepsilon\in L^2(\Omega,\Sigma_0,\mathbb{P};S)$ such that $\mathbb{P}(\{\phi\ne\phi_\varepsilon\})<\varepsilon$. The locality of h shows that

$$\{h(\phi)\ne h(\phi_\varepsilon)\}\subset\{\phi\ne\phi_\varepsilon\},$$

whence $\mathbb{P}(\{h(\phi)\ne h(\phi_\varepsilon)\})<\varepsilon$. It remains to refer to Lemma C.3, which provides the boundedness of h in L^0.

Continuing the proof of the tightness claim (B), we recall that

$$(U_t^0\phi)(\sigma) = \phi(0) + \int_0^{t+\sigma} H_\phi(s)\,dZ(s),$$

where $H_\phi(s) := F(s,U_s^0\phi)$, $\phi\in B$, $\sigma\in[-h,0]$. We know already that the set

$$\mathcal{H} := \{H_\phi \,:\, \phi\in B\}$$

is bounded in $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))$. Thus, it remains to prove the tightness of the stochastic integral operator

$$\mathcal{T}(H,\phi)(t) := \phi(0) + \int_0^t H(s)\,dZ(s)$$

as a mapping from $L^0(\Omega,\Sigma_T,\mathbb{P};L^2([0,T];\mathbb{R}^{n\times m}))\cap\mathcal{A}$ ($\mathcal{A}$ is the set of adapted processes on $[0,T]$). This follows from Lemmas C.4 and C.5.

Remark B.1. It is well known that the evolution operator U_t^0 for deterministic delay equations is compact for $t > h$ for reasonable right hand side nonlinearities, if the delay does not exceed h. Part (B) above is the stochastic counterpart of this general statement. The suggested proof of part (B) is only based on assumptions (i) and (iv) of Theorem 7.2 and on the linear growth assumption. This means that, in the case of stochastic delay equations with reasonable nonlinearities, the evolution operator should always be expected to be tight for $t > h$, provided the delay does not exceed h.

Appendix C. Some properties of local operators and tight sets

Here we collect some auxiliary technical statements on tightness of sets and of local operators, which are used in the paper and are also of some independent interest.

Throughout this section $(\Omega, \Sigma, \mathbb{P})$ is a measure space and X is a Polish space. For every ψ in the dual of $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$, denote by $\pi_X \psi$ the functional over $C_b(X)$ defined by

$$\langle \pi_X \psi, f \rangle := \langle \psi, f \circ \pi_X \rangle$$

(we use the same notation $\langle \cdot, \cdot \rangle$ for pairings between $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$ and its dual and between $C_b(X)$ and its dual, since the context in each case is quite clear). Clearly, π_X is a continuous operator between the respective duals (equipped with their $*$-weak topologies). Note that if ψ is a measure, i.e., $\psi \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, then π_X is the usual push-forward operator with respect to the projection map $\pi_X : \Omega \times X \to X$ defined by $\pi_X(\omega, x) := x$ (again, we slightly abuse the notation by using same symbols for formally different objects), in other words, $\pi_X \psi = \pi_{X\sharp} \psi$ in this case.

The first two statements are rather general. Although being of somewhat folkloric character, they cannot be easily found in the literature (at least in the explicit form as presented below). The first is a direct generalization of the corollary to Theorem 1 in [12, Volume I, Chapter VI, §1].

Lemma C.1. *Let $\{\mu_n\} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ be such that for every $f \in \mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$ there is a limit*

$$L(f) := \lim_{n\to\infty} \int_{\Omega\times X} f(\omega, x)\, d\mu_n(\omega, x).$$

Then the sequence $\{\mu_n\}$ is a tight set, and hence, in particular, $\mu_n \rightharpoonup \mu$ in the narrow sense of measures for some measure $\mu \in \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$, as $n \to \infty$. In other words, the space of Young measures $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ is sequentially closed in the $*$-weak topology of the space $\mathrm{Car}_b'(\Omega, \Sigma, \mathbb{P}; X)$ dual to $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$, and the narrow topology of $\mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ is inherited from the $*$-weak topology of $\mathrm{Car}_b'(\Omega, \Sigma, \mathbb{P}; X)$.

Proof. Clearly, it suffices to prove the tightness of $\{\mu_n\}$ and then refer to Lemma C.2. The tightness of $\{\mu_n\}$ follows from the tightness of the set of measures $\{\pi_{X\sharp}\mu_n\}$ over X. Since for every $f \in C_b(X)$ we have

$$\lim_{n\to\infty} \int_X f(x)\, d\pi_{X\sharp}\mu_n(x) = \lim_{n\to\infty} \int_{\Omega\times X} f(\pi_X(\omega, x))\, d\mu_n(\omega, x),$$

this limit exists and, therefore, by the corollary to Theorem 1 in [12, Volume I, Chapter VI, §1], the sequence $\{\pi_{X\sharp}\mu_n\}$ is tight as requested. □

Lemma C.2. *Let a set $K \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$ be tight. Then its closure $\bar{K}$ in the $*$-weak topology of the space dual to $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$ consists of Young measures, i.e., $\bar{K} \subset \mathcal{Y}(\Omega, \Sigma, \mathbb{P}; X)$.*

Proof. Let a generalized (Moore–Smith) sequence $\{\mu_\alpha\}_{\alpha\in A} \subset K$, where A is some directed set, be such that $\mu_\alpha \to \mu$ in the $*$-weak topology of the dual space Car_b' to $\mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$, for some $\mu \in \mathrm{Car}_b'$. Since the set of Borel measures $\{\pi_{X\sharp}\mu_\alpha\}_{\alpha\in A}$ over

X is tight, for every $\varepsilon > 0$ there is a compact set $C_\varepsilon \subset X$ such that for every $u \in C_b(X)$ with $0 \le u \le 1$ and $u = 0$ over C_ε we have

$$\langle u, \pi_{X\sharp}\mu_\alpha\rangle \le \varepsilon \text{ for all } \alpha \in A.$$

Passing to the limit in α, we get $\langle u, \pi_X\mu\rangle \le \varepsilon$, which means that $\pi_X\mu$ is a Borel measure (say, by Proposition B.7 in [5]). Hence, μ is a Young measure by Proposition 4.12 in [8] (the Stone–Daniell characterization of random measures). □

Now we present some results on tight sets and local operators.

Lemma C.3. *A set $B \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ is bounded (respectively, tight) if and only if for all $\varepsilon > 0$ there is a bounded (respectively, tight) set $B_\varepsilon \subset L^0(\Omega, \Sigma, \mathbb{P}; X)$ such that for every $f \in B$ there exists $f_\varepsilon \in B_\varepsilon$ for which $\mathbb{P}(\{f \neq f_\varepsilon\}) < \varepsilon$.*

Proof. The "if" part is trivial, because we can always choose $B_\varepsilon := B$. To prove the "only if" part, choose an arbitrary $\varepsilon > 0$ and an arbitrary $f \in B$. This gives a function $f_\varepsilon \in B_\varepsilon$ such that $\mathbb{P}(\{f \neq f_\varepsilon\}) < \varepsilon$. Let a ball (respectively, compact set) $K_\varepsilon \subset X$ be chosen such that $\mathbb{P}(\{f_\varepsilon \notin K_\varepsilon\}) < \varepsilon$ (such a set exists in view of the assumption on B_ε). Since $f(\omega) \in K_\varepsilon$ when $f(\omega) = f_\varepsilon(\omega)$ and $f_\varepsilon(\omega) \in K_\varepsilon$, we get

$$\mathbb{P}(\{f \notin K_\varepsilon\}) \le \mathbb{P}(\{f \neq f_\varepsilon\}) + \mathbb{P}(\{f_\varepsilon \notin K_\varepsilon\}) < 2\varepsilon,$$

which means that B is bounded (respectively, tight). □

Lemma C.4. *The Nemytskiĭ operator h defined by the formula*

$$h(x)(\omega) := f(\omega, x(\omega)),$$

where $f \in \mathrm{Car}_b(\Omega, \Sigma, \mathbb{P}; X)$, is tight in $L^0(\Omega, \Sigma, \mathbb{P}; X)$ if $f(\omega, \cdot\,)$ is compact for almost all $\omega \in \Omega$.

Proof. Clearly, for any bounded $S \in X$ the image of the set $B = L^0(\Omega, \Sigma, \mathbb{P}; S)$ is contained in the relatively compact, random set $A(\omega, S)$. From Proposition 2.15 in [8] it follows that for any $\varepsilon > 0$ there is a compact subset $Q \subset X$ for which $\mathbb{P}\big(\{\omega : A(\omega, S) \not\subset Q\}\big) < \varepsilon$, so that $\mathbb{P}\big(\{h(x) \notin Q\}\big) < \varepsilon$ for all $x \in B$. Applying Lemma C.3, we complete the proof. □

In particular, the above result implies the tightness of the operator

$$\mathcal{J}(H)(t) := \int_0^t H(s)\, ds$$

in the space $L^0(\Omega, \Sigma_T, \mathbb{P}; L^2([0,T]; \mathbb{R})) \cap \mathcal{A}$, where $\mathcal{A}$ is the set of all adapted processes on $[0, T]$.

The tightness of the corresponding Itô integral follows from the next statement.

Lemma C.5. *The stochastic integral operator $\mathcal{I}$,*

$$\mathcal{I}(H)(t) := \int_0^t H(s)\, dW(s),$$

where $W(t)$ stands for the scalar Brownian motion, is tight as a mapping from the space $L^0(\Omega, \Sigma_T, \mathbb{P}; L^2([0,T]; \mathbb{R})) \cap \mathcal{A}$ into itself, where $\mathcal{A}$ is the set of all adapted processes on $[0, T]$.

Proof. Below $L^2 := L^2([0,T]; \mathbb{R})$. We set

$$g_t^\nu := \sum_{k=0}^{\nu-1} \frac{kT}{\nu} \mathbf{1}_{[\frac{kT}{\nu}, \frac{(k+1)T}{\nu})}(t).$$

Clearly, $g_t^\nu \le t$ and

$$\mathbb{E}\Big|\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H(s)\,dW(s)\Big|^2 = \mathbb{E}\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H^2(s)\,ds.$$

Therefore,

$$\begin{aligned}\mathbb{E}\int_0^T \Big|\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H(s)\,dW(s)\Big|^2 dt &= \int_0^T dt\,\mathbb{E}\Big|\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H(s)\,dW(s)\Big|^2\\ &= \int_0^T dt\,\mathbb{E}\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H^2(s)\,ds = \mathbb{E}\int_0^T dt\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)H^2(s)\,ds\\ &= \mathbb{E}\int_0^T H^2(s)ds\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)\,dt\\ &\le \Big(\mathbb{E}\int_0^T H^2(s)\,ds\Big)\sup_{0\le s\le T}\int_0^T \mathbf{1}_{[g_t^\nu,t]}(s)\,dt \le \frac{CT}{\nu}\mathbb{E}\|H\|_2^2,\end{aligned}$$

and finally,

$$\mathbb{E}\,\|\mathcal{I}(H)-\mathcal{I}_\nu(H)\|_2^2 \;\le\; \frac{CT}{\nu}\mathbb{E}\|H\|_2^2,$$

where

$$\mathcal{I}_\nu(H)(t) := \int_0^{g_\nu(t)} H(s)\,dW(s).$$

The operator $\mathcal{I}_\nu$ is a finite-dimensional linear Nemytskiĭ operator. By Lemma C.4, it is tight.

To prove the tightness of the operator $\mathcal{I}$, first we observe that, by its linearity and Lemma C.3, we may assume that $\mathbb{E}\|H\|_2 \le 1$. We denote the set of all such H by B.

Let $\epsilon > 0$ be given. For any $\nu \in \mathbb{N}$ we choose $k(\nu) \in \mathbb{N}$ such that

$$\mathbb{E}\,\big\|\mathcal{I}(H)-\mathcal{I}_{k(\nu)}(H)\big\|_2^2 \;\le\; \frac{\varepsilon 2^{-\nu-1}}{\nu} \tag{C.1}$$

for any $H \in B$. Using the tightness of $\mathcal{I}_\nu$, we choose a compact set $Q_\nu \in L^2$ for which

$$\mathbb{P}\big(\{\mathcal{I}_{k(\nu)}(H) \notin Q_\nu\}\big) \;\le\; \varepsilon 2^{-\nu-1} \tag{C.2}$$

for any $H \in B$ and $\nu \in \mathbb{N}$.

Letting Q_ν^δ stand for the closed neighborhood of Q_ν, we put

$$Q := \bigcap_{\nu=1}^{\infty} Q_\nu^{1/\nu}$$

and observe that Q is compact in L^2, because any $1/\nu$-net for Q_ν will be a $2/\nu$-net for Q.

Now for any $H \in B$ we use estimates (C.1), (C.2) and the Chebyshev inequality to obtain the inequality

$$\begin{aligned}\mathbb{P}\big(\{\mathcal{I}(H)\notin Q\}\big) &\le \sum_{\nu=1}^\infty \mathbb{P}\big(\{\mathcal{I}(H)\notin Q_\nu^{1/\nu}\}\big)\\ &\le \sum_{\nu=1}^\infty \Big(\mathbb{P}\Big(\Big\{\,\big\|\mathcal{I}(H)-\mathcal{I}_{k(\nu)}(H)\big\|_2^2 \ge \frac{1}{\nu}\Big\}\Big) + \mathbb{P}\big(\{\mathcal{I}_{k(\nu)}(H)\notin Q_\nu\}\big)\Big)\\ &\le \sum_{\nu=1}^\infty \Big(\nu\mathbb{E}\,\big\|\mathcal{I}(H)-\mathcal{I}_{k(\nu)}(H)\big\|_2^2 + \mathbb{P}\left(\{\mathcal{I}_{k(\nu)}(H)\notin Q_\nu\}\right)\Big)\\ &\le \sum_{\nu=1}^\infty \Big(\nu\frac{\varepsilon}{\nu}2^{-\nu-1} + \varepsilon 2^{-\nu-1}\Big) \le \varepsilon.\end{aligned}$$

Therefore, the set $\mathcal{I}(B)$ is tight, and the lemma is proved. $\square$

References

[1] J. Appell and P. P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Math., vol. 95, Cambridge Univ. Press, Cambridge, 1990. MR1066204 (91k:47168)

[2] L. Arnold, *Random dynamical systems*, Springer Monogr. in Math., Springer-Verlag, Berlin, 1998. MR1723992 (2000m:37087)

[3] L. Arnold and V. Wihtutz, *Stationary solutions of linear systems with additive and multiplicative noise*, Stochastics **7** (1982), no. 1–2, 133–156. MR665411 (83j:60058)

[4] G. Buttazzo, *Semicontinuity, relaxation and integral representation in the calculus of variations*, Pitman Research Notes in Math., vol. 207, Longman Sci. and Tech., Harlow, 1989. MR1020296 (91c:49002)

[5] G. Buttazzo, A. Pratelli, S. Solimini, and E. Stepanov, *Optimal urban networks via mass transportation*, Lecture Notes in Math., vol. 1961, Springer-Verlag, Berlin, 2009. MR2469110 (2010c:49002)

[6] C. Castaing, P. Raynaud de Fitte, and M. Valadier, *Young measures on topological spaces: with applications in control theory and probability theory*, Math. Appl., vol. 571, Kluwer Acad. Publ., Dordrecht, 2004. MR2102261 (2005k:28001)

[7] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Math., vol. 580, Springer-Verlag, Berlin, 1977. MR0467310 (57:7169)

[8] H. Crauel, *Random probability measures on Polish spaces*, Stochastics Monogr., vol. 11, Taylor & Francis, London, 2002. MR1993844 (2004e:60005)

[9] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia Math. Appl., vol. 44, Cambridge Univ. Press, Cambridge, 1992. MR1207136 (95g:60073)

[10] M. E. Drakhlin, A. Ponosov, and E. Stepanov, *On some classes of operators determined by the structure of their memory*, Proc. Edinb. Math. Soc. (2) **45** (2002), no. 2, 467–490. MR1912652 (2003g:47114)

[11] M. E. Drakhlin, E. Litsyn, A. Ponosov, and E. Stepanov, *Generalizing the property of locality: atomic/coatomic operators and applications*, J. Nonlinear Convex Anal. **7** (2006), no. 2, 139–162. MR2254115 (2008b:47058)

[12] I. I. Gikhman and A. V. Skorokhod, *Theory of random processes*. Vol. III, Nauka, Moscow, 1975; English transl.; Grundlehren Math. Wiss., vol. 232, Springer-Verlag, Berlin, 1979. MR0651014 (58:31323a)

[13] D. Liberzon, *Switching in systems and control*, Birkhäuser, Boston, MA, 2003. MR1987806 (2004i:93003)

[14] S. E. A. Mohammed, *Stochastic functional differential equations*, Research Notes in Math., vol. 99, Pitman, Boston, MA, 1984. MR0754561 (86j:60151)

[15] B. Øksendal, *Stochastic differential equations. An introduction with applications*, 6th ed., Universitext, Springer-Verlag, Berlin, 2003. MR2001996 (2004e:60102)

[16] A. Ponosov, *The Nemytskiĭ conjecture*, Dokl. Akad. Nauk SSSR **289** (1986), no. 6, 1308–1311; English transl., Soviet Math. Dokl. **34** (1987), no. 1, 231–233. MR856838 (88b:47092)

[17] ———, *A fixed-point method in the theory of stochastic differential equations*, Dokl. Akad. Nauk SSSR **299** (1988), no. 3, 562–565; English transl., Soviet Math. Dokl. **37** (1988), no. 2, 426–429. MR936492 (89g:60189)

[18] P. Protter, *Semimartingales and measure preserving flows*, Ann. Inst. H. Poincare Probab. Statist. **22** (1986), no. 2, 127–147. MR850752 (88m:60138)

[19] M. Scheutzow, *Exponential growth rate for a singular linear stochastic delay differential equation*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), no. 6, 1683–1696. MR3038775

[20] I. V. Shragin, *Abstract Nemetskiĭ operators are locally defined operators*, Dokl. Akad. Nauk SSSR **227** (1976), no. 1, 47–49; English transl., Soviet Math. Dokl. **17** (1976), no. 2, 354–357. MR0412918 (54:1039)

[21] M. Valadier, *Young measures*, Methods of Nonconvex Analysis (Varenna, 1989), Lecture Notes in Math., vol. 1446, Springer-Verlag, Berlin, 1990, pp. 152–188. MR1079763 (91j:28006)

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