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ON SOLUTIONS AND WARING'S FORMULAS FOR SYSTEMS OF n ALGEBRAIC EQUATIONS FOR n UNKNOWNS

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Abstract. A system of n algebraic equations for n unknowns is considered, in which the collection of exponents is fixed, and the coefficients are variable. Since the solutions of such systems are $2n$-homogeneous, two coefficients in each equation can be fixed, which makes it possible to pass to the corresponding reduced systems. For the reduced systems, a formula for the solution (and also for any monomial of the solution) is obtained in the form of a hypergeometric type series in the coefficients. Such series are represented as a finite sum of Horn's hypergeometric series: the ratios of the neighboring coefficients of the latter series are rational functions of summation variables. The study is based on the linearization procedure and on the theory of multidimensional residues. As an application of the main formula, a multidimensional analog is presented of the Waring formula for powers of the roots of the system.

§1. Introduction

In 1921, Mellin [1] obtained a formula for the solution of a general reduced algebraic equation

$$y^m + x_1 y^{m_1} + \cdots + x_p y^{m_p} - 1 = 0. \tag{1.1}$$

He represented the solution $y(x) = y(x_1, \dots, x_p)$ as a multiple integral (of Mellin–Barnes's type, see [2]) and also as a power series of hypergeometric type. Such series are finite sums of Horn's hypergeometric series [3]: the ratios of the neighboring coefficients of the latter series are rational functions of the summation variables.

The reduced algebraic equation (1.1) is obtained by fixing two coefficients in the general algebraic equation of degree m. Since the solution of the latter depends bihomogeneously on the coefficients, the fixing as above can be done for any pair of monomials, and without any loss of information about solutions, see [4].

In the past years, the active development of the theory of multidimensional hypergeometric series (see, e.g., [5, 6, 7]) and the adaptation of the multidimensional residue methods to the case of the Mellin–Barnes integrals have led to new interest and new possibilities for the study of algebraic equations. In particular, in 2000, in the papers [9] and [10], the investigations of Mellin [1] and Birkeland [11] were continued, as applied to equation (1.1). The tools like hypergeometric series and multidimensional residues made it possible to obtain a new method for the description of the monodromy of the general algebraic function $y(x)$, based of the analytic extensions to each other of hypergeometric series and Mellin–Barnes integrals.

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The passage from the scalar equation (1.1) to a system was started in Antipova's paper [13]; following Mellin's approach, she got a solution for a lower-triangular system of algebraic equation, i.e., for the case where the first equation depends only on the first unknown y_1, the second equation depends only on the first two unknowns y_1, y_2, an so on, the last nth equation depending on all the unknowns $y_1, \dots, y_n$.

Mellin's approach looks like this. First, we calculate the Mellin transform of the solution, and then use the inversion formula for this transformation to get an integral representation of the solution (in the form of a Mellin–Barnes integral). In its turn, the theory of residues allows us to reduce the integral representation to a hypergeometric type series. However, application of Mellin's approach to a class of systems wider than the lower-triangular ones meets difficulties. Namely, as a rule, the corresponding formal Mellin–Barnes integral for more general systems converges nowhere.

Our purpose in the present paper is to obtain a hypergeometric series type formula for solution of a system of general algebraic equations and, as an application, to get a multidimensional analog of Waring's formula for the power sums of roots. Namely, we consider a reduced system of n equations

$$y_j^{m_j} + \sum_{\lambda \in \Lambda^{(j)}} x_\lambda^{(j)} y^\lambda - 1 = 0, \quad j = 1, \dots, n, \tag{1.2}$$

for n unknowns $y = (y_1, \dots, y_n)$, where the collection $\Lambda^{(j)} \subset \mathbb{Z}_{\geq 0}^n$ of exponents is fixed, and all the coefficients $x_\lambda^{(j)}$ are variable. Of course, it is assumed that the set $\Lambda^{(j)}$ in the jth equation does not include the points $\lambda = (0, \dots, m_j, \dots, 0)$ or $\lambda = 0$ that occur as exponents for the distinguished monomials $y_j^{m_j}$ and y^0 with the fixed coefficients 1 and -1. By using rather simple algebraic manipulations, we can reduce as arbitrary system of n polynomial equations with n unknowns to the form (1.2), see [14].

We denote by Λ the disjoint sum $\bigsqcup \Lambda^{(j)}$ and by N the number of coefficients in (1.2) (i.e., the cardinality of Λ). The exponents of the λ-monomials $y^\lambda = y_1^{\lambda_1} \dots y_n^{\lambda_n}$ in (1.2) can be thought of as an $(n \times N)$-matrix

$$\Phi = (\lambda^1, \dots, \lambda^N),$$

where λ^k is a column-vector in Λ. It is assumed that, within each equation, the order of the columns λ is fixed arbitrarily. The coordinates of the vectors $x = (x_\lambda)$ of coefficients are indexed by the elements $\lambda \in \Lambda$. The entire space of coefficients will be denoted by $\mathbb{C}^N$.

Let $\widehat{y}(x)$ denote the branch of the solution $y(x) = (y_1(x), \dots, y_n(x))$ of system (1.2) that satisfies the condition $y(0) = (1, \dots, 1)$; we name this branch the *principal solution*.

To state our main results we need the following notation. For each row φ_j of the matrix of exponents Φ and every $\mu_j \in \mathbb{Z}_{\geq 0}$, we introduce the affine function

$$l_j(\alpha) = \frac{\mu_j}{m_j} + \frac{1}{m_j} \langle \varphi_j, \alpha \rangle, \quad j = 1, \dots, n.$$

In terms of these function, we define the following set of indices:

$$P = P(\alpha) = \big\{ j \in \{1, \dots, n\} \,:\, l_j(\alpha) \neq 0 \big\}.$$

Observe that Φ is naturally split into blocks that correspond to the $\Lambda^{(j)}$; therefore each row φ_i of Φ has the form of a sequence of vectors $\varphi_i^{(1)}, \dots, \varphi_i^{(n)}$.

Now we can formulate our main claim about formulas not only for the solution itself of system (1.2), but also for any monomial function of the solution.

Theorem 1. *A monomial $\widehat{y}^{\mu} = \widehat{y}^{\mu}(x)$ of the principal solution of system* (1.2) *can be represented as a hypergeometric type series*

$$\widehat{y}^{\mu} = \sum_{\alpha \in \mathbb{Z}^N_{\geq 0}} c_\alpha x^\alpha$$

with the coefficients

$$c_\alpha = \frac{(-1)^\alpha}{\alpha!} \frac{\prod_{j=1}^n \Gamma\big(l_j(\alpha)+1\big)}{\prod_{j=1}^n \Gamma\big(l_j(\alpha) - |\alpha^{(j)}| + 1\big)} \det \left\| \delta_i^j - \frac{\langle \varphi_i^{(j)}, \alpha^{(j)} \rangle}{m_j l_j(\alpha)} \right\|_{(i,j)\in P\times P}. \tag{1.3}$$

Note that, for some subclass of systems (1.2), an "incomplete analog" of formula (1.3) was obtained in [15] on the basis of Mellin's formal approach, i.e., via ignoring the divergence of the Mellin transform for the function $y^\mu(x)$ and the divergence of the Mellin–Barnes integral.

To obtain multidimensional Waring formulas, there is no need in fixing the constant terms in equations (1.2); therefore, we consider the system

$$y_j^{m_j} + \sum_{\lambda \in \Lambda^{(j)} \cup \{0\}} x_\lambda^{(j)} y^\lambda = 0, \quad j = 1, \dots, n. \tag{1.4}$$

However, now we impose the following condition on the set $\Lambda^{(j)}$: for any $\lambda \in \Lambda^{(j)}$ we have

$$|\lambda| = \lambda_1 + \dots + \lambda_n < m_j, \ j = 1, \dots, n. \tag{1.5}$$

Then, by the Bezout theorem, system (1.4) has $M = m_1 \cdot \dots \cdot m_n$ solution $y^{(\nu)}(x)$. Observe that, in (1.4), the vector x is an element of $\mathbb{C}^{N+n}$.

A *power sum* of degree $\mu \in \mathbb{Z}^n_{\geq 0}$ is defined as the expression

$$S_\mu = \sum_{\nu=1}^M \big(y^{(\nu)}(x)\big)^\mu.$$

Consider the $(n \times N)$-matrix χ the ith row of which represents the indicator function of the subset $\Lambda^{(i)}$ of Λ, i.e., the entries in this row are equal to 1 at the places $\lambda \in \Lambda^{(i)}$ and to 0 at all other places $\lambda \in \Lambda$.

Theorem 2 claims that the power sums S_μ of roots of system (1.4) can be expressed through the coefficients $x = (x_\lambda^{(j)})$ via the (multidimensional Waring) formula:

$$\sum_{\substack{\alpha \in \mathbb{Z}^{N+n}_{\geq 0} \\ (D_m \chi - \Phi)\alpha = \mu}} \frac{(-1)^{|\alpha|}}{\alpha!} \prod_{j=1}^n |\alpha^{(j)}|! \det \left\| m_j \delta_j^i - \frac{\langle \varphi_j^{(i)}, \alpha^{(i)} \rangle}{l_j(\alpha)} \right\|_{(i,j)\in P\times P} x^\alpha, \tag{1.6}$$

where D_m is the diagonal $(n \times n)$-matrix with diagonal entries m_j.

Earlier, V. A. Bolotov (see [16, p. 227]) obtained a multidimensional version of Waring's formulas only for the power sums of the form

$$\sum_{\nu=1}^N \big(y_j^{(\nu)}\big)^l = S_{(0,\dots,l,\dots,0)},$$

i.e., for separate coordinates of the solution.

§2. Linearization of system (1.2)

Following the paper [14] by Antipova and Tsikh, we linearize system (1.2). For this, we regard (1.2) as a system of equations in the space $\mathbb{C}^N_x \times \mathbb{C}^n_y$ with the coordinates $x = (x_\lambda)_{\lambda\in\Lambda}$ and $y = (y_1, \dots, y_n)$ and make the following change of variables $(\xi, W) \to (x, y)$:

$$\begin{aligned} x^{(j)}_\lambda &= \xi^{(j)}_\lambda \prod_{k=1}^n W_k^{\frac{\lambda_k}{m_k}-\delta^j_k}, \\ y_j &= W_j^{-\frac{1}{m_j}}. \end{aligned} \tag{2.1}$$

Observe that, under (2.1), for each $\lambda \in \Lambda^{(j)}$ the monomial $x^{(j)}_\lambda y^\lambda$ in (1.2) will turn into $\xi^{(j)}_\lambda W_j^{-1}$, and each $y_j^{m_j}$ into W_j^{-1}. Therefore, (1.2) will become a system of linear equations:

$$W_j = 1 + |\xi^{(j)}|, \quad j = 1, \dots, n,$$

where $|\xi^{(j)}| = \sum_{\lambda\in\Lambda^{(j)}\setminus\{0\}} \xi^{(j)}_\lambda$. It follows that after the variable change $\xi \to x$ in the coefficient space $\mathbb{C}^N$, where

$$x^{(j)}_\lambda = \xi^{(j)}_\lambda \prod_{k=1}^n \left(1 + |\xi^{(k)}|\right)^{\frac{\lambda_k}{m_k}-\delta^j_k}, \quad \lambda \in \Lambda^{(j)}, \ j = 1, \dots, n, \tag{2.2}$$

the solution $y(x)$ takes the form

$$y_j(x(\xi)) = \left(1 + |\xi^{(j)}|\right)^{-\frac{1}{m_j}}, \quad j = 1, \dots, n. \tag{2.3}$$

Note that the solution $y(x(\xi))$ is analytic in the domain

$$G = \left\{ \xi \in \mathbb{C}^N \ : \ \prod_{j=1}^n \left(1 + |\xi^{(j)}|\right) \neq 0 \right\}.$$

As a result, we see that to compute the solution $y(x)$ of system (1.2) it suffices to invert the change (2.2), which we call the *linearization of system* (1.2), and to plug the inversion in (2.3). This procedure can be done with the help of the generalized logarithmic residue formula, which we realize in the next section.

§3. Proof of Theorem 1

Proof. We represent the inversion $\xi(x)$ of the linearization (2.2) in the form of an implicit function (implicit mapping) given by the family of equations

$$F^{(j)}_\lambda(x, \xi) = \xi^{(j)}_\lambda \prod_{k=1}^n \left(1 + |\xi^{(k)}|\right)^{\frac{\lambda_k}{m_k}-\delta^j_k} - x^{(j)}_\lambda = 0, \tag{3.1}$$

where $\lambda \in \Lambda^{(j)}, j = 1, \dots, n$.

Note that (2.2) and (3.1) are multivalued maps analytic near $\xi = 0$. We are interested in the branch determined by the requirement that the functions $\left(1 + |\xi^{(k)}|\right)^{\frac{1}{m_k}}$ be equal to 1 at $\xi^{(k)} = 0$. Then the solution (2.3) will satisfy $y_j(x(0)) = y_j(0) = 1$, i.e., it will correspond to the principal solution $\widehat{y}(x)$ of system (1.2).

To compute the monomial $\widehat{y}^\mu(x)$ of the principal solution of system (1.2), we need to know the value taken by the monomial $\widehat{y}^\mu(\xi)$ for the vector (2.3) at the point $\xi(x)$ defined implicitly by (3.1). This can be done via Yuzhakov's logarithmic residue formula (see [17] and also [16, Theorems 20.1 and 20.2]). By that formula,

$$\widehat{y}^\mu(x) = \frac{1}{(2\pi i)^{N-n}} \int_{\Gamma_\varepsilon} \frac{y^\mu(\xi)\Delta(\xi)\, d\xi}{\prod_{j=1}^n \prod_{\lambda\in\Lambda^{(j)}} F^{(j)}_\lambda(x, \xi)},$$

where $\Gamma_\varepsilon = \{\xi \in \mathbb{C}^N \,:\, |\xi_\lambda^{(j)}| = \varepsilon, \lambda \in \Lambda^{(j)}, j = 1, \dots, n\}$, and $\Delta = \frac{\partial F}{\partial \xi}$ is the Jacobian of the system of equations $F_\lambda^{(j)}(x, \xi) = 0$ relative to the variables ξ. The radius ε in the definition of Γ_ε is chosen sufficiently small (e.g., to ensure that the polydisk of radius ε lie off the set of zeros of the Jacobian Δ).

Observe that, by the specifics of the definition of $F_\lambda^{(j)}$, the Jacobian Δ coincides with the Jacobian of the linearization $x(\xi)$ given by (2.2).

Lemma 1. *The Jacobian of the linearization* (2.2) *looks like this*:

$$\Delta = \prod_{k=1}^{n} \prod_{\lambda \in \Lambda^{(k)}} \frac{y_k^{m_k}(\xi)}{y^\lambda(\xi)} \det \left\| \frac{\delta_j^i + \frac{1}{m_j}\langle \varphi_i^{(j)}, \xi^{(j)} \rangle}{1 + |\xi^{(j)}|} \right\|_{i,j=1}^{n}. \tag{3.2}$$

Proof. The Jacobian Δ has a block structure with n^2 blocks. In the ith diagonal block, the diagonal entries are of the form

$$\frac{\partial F_\lambda^{(j)}}{\partial \xi_\lambda^{(j)}} = \frac{y_j^{m_j}}{y^\lambda} \left(1 - \frac{m_j - \lambda_j}{m_j} \frac{\xi_\lambda^{(j)}}{1 + |\xi^{(j)}|}\right).$$

The off-diagonal entries in this diagonal block are equal to

$$\frac{\partial F_\lambda^{(j)}}{\partial \xi_\tau^{(j)}} = -\frac{y_j^{m_j}}{y^\lambda} \frac{m_j - \lambda_j}{m_j} \frac{\xi_\lambda^{(j)}}{1 + |\xi^{(j)}|}.$$

The off-diagonal blocks consist of the entries

$$\frac{\partial F_\lambda^{(j)}}{\partial \xi_\tau^{(k)}} = -\frac{y_j^{m_j}}{y^\lambda} \frac{-\lambda_k}{m_k} \frac{\xi_\lambda^{(j)}}{1 + |\xi^{(k)}|}.$$

Summation of the rows and columns of the resulting determinant within each block allows us to reduce its size down to $n \times n$. The determinant obtained in this way will be of the form (3.2). $\square$

We write each $F_\lambda^{(j)}$ in the form

$$F_\lambda^{(j)} = \frac{\xi_\lambda^{(j)} y_j^{m_j}(\xi)}{y^\lambda(\xi)} \left(1 - \frac{x_\lambda^{(j)} y^\lambda(\xi)}{\xi_\lambda^{(j)} y_j^{m_j}(\xi)}\right).$$

There exists a number δ such that for all $\xi \in \Gamma_\varepsilon$ and all x with $\|x\| < \delta$ we have

$$\left| \frac{x_\lambda^{(j)} y^\lambda(\xi)}{\xi_\lambda^{(j)} y_j^{m_j}(\xi)} \right| < 1.$$

Thus, the integrand can be represented as a multiple geometric progression. Using (3.2), we get the formula

$$\widehat{y}^\mu = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^N} \frac{x^\alpha}{(2\pi i)^N} \int_{\Gamma_\varepsilon} \frac{\prod_{k=1}^n y_k^{(l_k(\alpha) - |\alpha^{(k)}|) \cdot m_k}}{\prod_{j=1}^n \prod_{\lambda \in \Lambda^{(j)}} (\xi_\lambda^{(j)})^{\alpha_\lambda^{(j)} + 1}} \det \left\| \frac{\delta_j^i + \frac{1}{m_j}\langle \varphi_i^{(j)}, \xi^{(j)} \rangle}{1 + |\xi^{(j)}|} \right\| d\xi.$$

We bring the monomial factors in y under the det sign:

$$\widehat{y}^\mu = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^N} \frac{x^\alpha}{(2\pi i)^{N-n}} \int_{\Gamma_\varepsilon} \frac{\det \left\| y_j^{(l_j(\alpha) - |\alpha^{(j)}|) \cdot m_j} \left(\frac{\delta_j^i + \frac{1}{m_j}\langle \varphi_i^{(j)}, \xi^{(j)} \rangle}{1 + |\xi_\lambda^{(j)}|} \right) \right\|}{\prod_{j=1}^n \prod_{\lambda \in \Lambda^{(j)}} (\xi_\lambda^{(j)})^{\alpha_\lambda^{(j)} + 1}} d\xi.$$

By the Cauchy formula, the integrals occurring under the summation sign coincide with the Taylor coefficients for the determinant function written in the numerator. Therefore, $\widehat{y}^{\mu}(x)$ is equal to

$$\sum_{\alpha\in\mathbb{Z}^N_{\geq 0}} \frac{x^\alpha}{\alpha!}\left[D^{(\alpha)}_{(\xi)} \det\left\| y_j^{m_j(l_j(\alpha)-|\alpha^{(j)}|)}\left(\frac{\delta^i_j + \frac{1}{m_j}\langle\varphi_i^{(j)},\xi^{(j)}\rangle}{1+|\xi^{(j)}|}\right)\right\|\right]_{\xi=0},$$

where $D^{(\alpha)}_{(\xi)}$ stands for the derivative of order α with respect to the variables ξ.

Observe that each row in the above determinant depends of its own collection of variables. Therefore, we can use the polylinearity property to obtain the following expression for $y^\mu(x)$:

$$\sum_{\alpha\in\mathbb{Z}^N_{\geq 0}} \frac{x^\alpha}{\alpha!} \det\left\| D^{(\alpha^{(j)})}_{(\xi^{(j)})}\left[y_j^{m_j(l_j(\alpha)-|\alpha^{(j)}|)}\left(\frac{\delta^i_j + \frac{1}{m_j}\langle\varphi_i^{(j)},\xi^{(j)}\rangle}{1+|\xi^{(j)}|}\right)\right]_{\xi=0}\right\|.$$

Calculating the derivatives, we get the final result:

$$y^\mu = \sum_{\alpha\in\mathbb{Z}^N_{\geq 0}} \frac{x^\alpha(-1)^{|\alpha|}}{\alpha!}\frac{\prod_{j=1}^n \Gamma(l_j(\alpha)+1)}{\prod_{j=1}^n \Gamma(l_j(\alpha)-|\alpha^{(j)}|+1)} \det\left\|\delta^j_i - \frac{\langle\varphi_i^{(j)},\alpha^{(j)}\rangle}{m_j l_j(\alpha)}\right\|_{(i,j)\in P\times P}. \tag{3.3}$$

□

Let $\widehat{x}$ denote the point in the space of coefficients of system (1.4) that has the coordinates $x_\lambda^{(j)} = 0$, $\lambda \neq 0$, and $x_0^{(j)} = -1$. Like for system (1.2), we distinguish the principal branch of the solution of (1.4) by the condition $y(\widehat{x}) = (1,\dots,1)$.

Corollary 1. *The monomial $\widehat{y}^\mu = \widehat{y}^\mu(x)$ of the principal solution of system* (1.4) *can be presented in the form of a hypergeometric type series*:

$$\begin{aligned}\widehat{y}^\mu = \sum_{\alpha\in\mathbb{Z}^N_{\geq 0}} \frac{x^\alpha(-1)^{|\alpha|}}{\alpha!}\frac{\prod_{j=1}^n (-x_0^{(j)})^{l_j(\alpha)-|\alpha^{(j)}|}\Gamma(l_j(\alpha)+1)}{\prod_{j=1}^n \Gamma\big(l_j(\alpha)-|\alpha^{(j)}|+1\big)} \\ \times \det\left\|\delta^j_i - \frac{\langle\varphi_i^{(j)},\alpha^{(j)}\rangle}{m_j\cdot l_j(\alpha)}\right\|_{(i,j)\in P\times P}.\end{aligned} \tag{3.4}$$

Proof. We divide each equation in (1.4) by its constant term taken with the minus sign. Then we introduce the new variables equal to the ratios $\frac{y_j}{(-x_0^{(j)})^{\frac{1}{m_j}}}$. Passing from the original coefficients $\big((x_\lambda^{(i)}), x_0^{(1)},\dots,x_0^{(n)}\big)$ to

$$\bar{x}_\lambda^{(j)} = x_\lambda^{(j)}\frac{(x_0^{(1)})^{\frac{\lambda_1}{m_1}}\dots(x_0^{(n)})^{\frac{\lambda_n}{m_n}}}{(-x_0^{(j)})},$$

we obtain a system of the form (1.2). Now, to get (3.4), it suffices to apply Theorem 1 and to perform the inverse change. □

We consider several examples that illustrate the application of the formula obtained.

Example 1. Consider the following system of three linear equations:

$$\begin{cases} y_1 + ay_2 = 1; \\ y_2 + by_3 = 1; \\ y_3 + cy_1 = 1. \end{cases}$$

In this case, the matrix Φ looks like this:

$$\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By formula (1.3), the coordinate y_1 of the solution can be written as a series:

$$\begin{aligned} y_1 &= \sum_{k=0}^{\infty} \frac{(-1)^{|k|}\Gamma(1+k_c+1)\Gamma(k_a+1)\Gamma(k_b+1)}{k_a!k_b!k_c!\Gamma(1+k_c-k_a+1)\Gamma(k_a-k_b+1)\Gamma(k_b-k_c+1)} \begin{vmatrix} 1 & -\frac{k_a}{1+k_c} & 0 \\ 0 & 1 & -\frac{k_b}{k_a} \\ -\frac{k_c}{k_b} & 0 & 1 \end{vmatrix} a^{k_a} b^{k_b} c^{k_c} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{|k|} a^{k_a} b^{k_b} c^{k_c}}{\Gamma(2+k_c-k_a)\Gamma(1+k_a-k_b)\Gamma(1+k_b-k_c)}. \end{aligned}$$

The coefficients of this series can be nonzero only if one of the following conditions is fulfilled:

$$\begin{aligned} k_a &= k_b = k_c; \\ k_a &= k_b = k_c + 1; \\ k_a &= k_b + 1 = k_c + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 &= \sum_{t=0}^{\infty} \frac{(-1)^{3t}}{\Gamma(2)\Gamma(1)\Gamma(1)}(abc)^t + \sum_{t=0}^{\infty} \frac{(-1)^{3t+2}ab}{\Gamma(1)\Gamma(1)\Gamma(2)}(abc)^t + \sum_{t=0}^{\infty} \frac{(-1)^{3t+1}a}{\Gamma(1)\Gamma(2)\Gamma(1)}(abc)^t \\ &= (1-a+ab)\sum_{t=1}^{\infty}(-1)^t(abc)^t = \frac{1-a+ab}{1+abc}, \end{aligned}$$

which is in agreement with Cramer's rule.

Example 2. Consider the following system of quadratic equations:

$$\begin{cases} y_1^2 + ay_2 - 1 = 0; \\ y_2^2 + by_1 - 1 = 0. \end{cases}$$

Then

$$\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The monomial function $y^\mu = y_1^{\mu_1} y_2^{\mu_2}$ of the solution, written as in (1.3), is representable by the series

$$y^\mu = \sum_{s,t=0}^{\infty} \frac{(-1)^{s+t}\Gamma(\frac{\mu_1}{2}+\frac{t}{2}+1)\Gamma(\frac{\mu_2}{2}+\frac{s}{2}+1)}{s!t!\Gamma(\frac{\mu_1}{2}+\frac{t}{2}-s+1)\Gamma(\frac{\mu_2}{2}+\frac{s}{2}-t+1)} \begin{vmatrix} 1 & -\frac{s}{\mu_1+t} \\ -\frac{t}{\mu_2+s} & 1 \end{vmatrix} a^s b^t.$$

The above system of quadratic equations can be solved in radicals with the help of computer algebra; the solution looks fairly bulky. The first 15 Taylor coefficients of this solution, calculated, again, via computer algebra, coincide with the corresponding coefficients of the series indicated above.

§4. Waring's formulas

For polynomials, the coefficients and the power sums of roots are related by the Newton recurrence formula or the Waring formulas. For systems of algebraic equations, a generalization of Newton's recurrence formulas was found in [18].

We present a generalization of Waring's formulas to the case of a system of algebraic equations. Recall that, under condition (1.5), system (1.4) had $M = m_1 \cdot \dots \cdot m_n$ roots $y^{(\nu)}(x) = \bigl(y_1^{(\nu)}(x), \dots, y_n^{(\nu)}(x)\bigr)$.

Theorem 2. *Under condition* (1.5)*, for any* $\mu \in \mathbb{Z}^n_{\geq 0}$*, the power sum*

$$S_\mu = \sum_{\nu=1}^{M} (y^{(\nu)}(x))^\mu$$

of roots of system (1.4) *can be written in the form of a polynomial in the coefficients* $x = (x_\lambda)$ *of the system by the formula*

$$\sum_{\substack{\alpha \in \mathbb{Z}^{N+n}_{\geq 0} \\ (D_m\chi - \Phi)\alpha = \mu}} \frac{(-1)^{|\alpha|}}{\alpha!} \prod_{j=1}^{n} |\alpha^{(j)}|! \det \left\| m_j \delta_j^i - \frac{\langle \varphi_j^{(i)}, \alpha^{(i)} \rangle}{l_j(\alpha)} \right\|_{(i,j) \in P \times P} x^\alpha, \tag{4.1}$$

where $P = \bigl\{ j \in \{1, \dots, n\} \,:\, l_j(\alpha) \neq 0 \bigr\}$.

Proof. Recall that we denoted by $\widehat{x}$ the point in the space of coefficients of system (1.4) for the coordinates of which we have $x_\lambda^{(j)} = 0$, $\lambda \neq 0$, and $x_0^{(j)} = -1$. For $x = \widehat{x}$, the system takes the form $y_1^{m_1} = 1, \dots, y_n^{m_n} = 1$, and the set of solutions acquires a "lattice" form: the jth coordinate of the solution runs, independently of the other coordinates, through m_j values. In accordance with this observation, we enumerate all solutions (for $x = \widehat{x}$) $y_J = \varepsilon_J = (\varepsilon_{j_1}, \dots, \varepsilon_{j_n})$, $\varepsilon_{j_s} = e^{\frac{2\pi i}{m_s} j_s}$ by multiindices $J = (j_1, \dots, j_n)$, running over the parallelepiped

$$\Pi_m = \bigl\{ (j_1, \dots, j_n) \in \mathbb{Z}^n \,:\, 0 \leq j_s \leq m_s - 1, s = 1, \dots, n \bigr\}.$$

Since the solution $y(x)$ depends continuously on the coefficients x, and the roots $y_J = \varepsilon_J$ are simple, all $M = m_1 \cdot \dots \cdot m_n$ branches of $y(x)$ are close to ε_J, when x varies in a small neighborhood of $\widehat{x}$. Therefore, near $\widehat{x}$ we may enumerate the branches of $y(x)$ as $y_J(x)$, $J \in \Pi_m$.

Note that all solutions of system (1.4) can be expressed in terms of the principal solution: $y_J = \varepsilon_J \widehat{y}(\varepsilon_J^\lambda x_\lambda)$. This is true because all $y_J(\widehat{x}) = \varepsilon_J$ are distinct, and the fact that $y_J(x)$ will nullify the equations of system (1.4) can be checked easily by substitution.

Using formula (1.3) for the principal solution $\widehat{y}$, we write a formula for a monomial y_J^μ of the solution $y_J(x)$:

$$\sum_{\alpha \in \mathbb{Z}^N_{\geq 0}} \frac{(-1)^{|\alpha|}}{\alpha!} \frac{\prod_{j=1}^n \Gamma(l_j(\alpha) + 1)}{\prod_{j=1}^n \Gamma\bigl(l_j(\alpha) - |\alpha^{(j)}| + 1\bigr)} \det \left\| \delta_j^i - \frac{\langle \varphi_j^{(i)}, \alpha^{(i)} \rangle}{m_j l_j(\alpha)} \right\|_{(i,j) \in P \times P}$$
$$\times \prod_{j=1}^{n} \varepsilon_{j_s}^{m_j l_j(\alpha)} \prod_{j=1}^{n} e^{i\pi l_j(\alpha)} \bigl(x_0^{(j)}\bigr)^{l_j(\alpha)} \prod_{\lambda \in \Lambda^{(i)}} \bigl(x_\lambda^{(i)}\bigr)^{\alpha_\lambda^{(i)}}.$$

In the adopted enumeration of the branches, the power sum S_μ is written as

$$S_\mu = \sum_{J \in \Pi_m} y_J^\mu(x).$$

Therefore,

$$
\begin{aligned}
S_\mu = \sum_{\alpha\in\mathbb{Z}^N_{\ge 0}} & \frac{(-1)^{|\alpha|}\prod_{j=1}^n \Gamma(l_j(\alpha)+1)}{\alpha!\prod_{j=1}^n \Gamma\bigl(l_j(\alpha)-|\alpha^{(j)}|+1\bigr)} \det\left[\delta^i_j - \frac{\langle\varphi^{(i)}_j,\alpha^{(i)}\rangle}{m_j l_j(\alpha)}\right]_{(i,j)\in P\times P} \\
& \times \prod_{j=1}^n e^{i\pi(l_j(\alpha))} \prod_{j=1}^n \bigl(x^{(j)}_0\bigr)^{(l_j(\alpha)-|\alpha^{(j)}|)} \prod_{\lambda\in\Lambda^{(i)}} \bigl(x^{(i)}_\lambda\bigr)^{\alpha^{(i)}_\lambda} \sum_{J\in\Pi_m}\prod_{s=1}^n \varepsilon_{j_s}^{m_j l_j(\alpha)}.
\end{aligned}
\tag{4.2}
$$

Consider the power sum of the primitive roots ε_{j_s} of the unity in the powers m_s:

$$
\sum_{J\in\Pi_m}\prod_{s=1}^n \varepsilon_{j_s}^{m_j l_j(\alpha)}.
$$

This sum is equal to $m_1\cdot\dots\cdot m_n$ in the case where all summands equal 1, i.e., if all $m_j l_j(\alpha)$ are divisible by m_j, and to 0 otherwise.

It follows that for all the nonzero terms of the series (4.2) we have $l_j(\alpha)\in\mathbb{Z}$, i.e., the Γ-functions have only integral arguments.

Denote $\beta_j = l_j(\alpha) - |\alpha^{(j)}|$ ($\beta_j\in\mathbb{Z}$). Since the Γ-function of β_j+1 occurs in the denominators of the terms in (4.2), such a form can be nonzero only if $\beta_j\ge 0$ for all $j=1,\dots,n$.

On the other hand, for $\langle\beta,m\rangle = \beta_1 m_1+\dots+\beta_n m_n$ we have

$$
\langle\beta,m\rangle = \sum_{j=1}^n \Bigl(\mu_j + \langle\varphi_j,\alpha\rangle - m_j|\alpha^{(j)}|\Bigr) = |\mu| + \sum_{i=1}^n\sum_{\lambda\in\Lambda^{(i)}} \bigl(m_i-|\lambda|\bigr)\alpha_\lambda \le |\mu|.
$$

Thus, the nonzero terms of the series form the finite sum

$$
\begin{aligned}
S_\mu = \sum_{\substack{\beta\ge 0:\\ \langle\beta,m\rangle\le|\mu|}} \sum_{\substack{\alpha\in\mathbb{Z}^N_{\ge 0}\\ \langle\varphi_j,\alpha\rangle-|\alpha^{(j)}|=\mu_j-\beta_j m_j\\ j=1,\dots,n}} & \frac{(-1)^{|\alpha|}(-1)^{|\beta|}}{\alpha!\beta!}\prod_{j=1}^n \Gamma\bigl(\beta_j+|\alpha^{(j)}|\bigr) \\
& \times M\det\left\|\delta^i_j - \frac{\langle\varphi^{(i)}_j,\alpha^{(i)}\rangle}{m_j l_j(\alpha)}\right\|_{(i,j)\in P\times P} \prod_{j=1}^n \bigl(x^{(j)}_0\bigr)^{\beta_j} \prod_{j=1}^n\prod_{\lambda\in\Lambda^{(j)}\setminus\{0\}} \bigl(x^{(j)}_\lambda\bigr)^{\alpha_\lambda}.
\end{aligned}
$$

Completing the vector $\alpha\in\mathbb{Z}^N_{\ge 0}$ by the coordinates $\alpha^{(j)}_0 = \beta_j$, $j=1,\dots,n$, up to a vector in $\mathbb{Z}^{N+n}_{\ge 0}$ and keeping the notation α for the new vector, we arrive at formula (4.1). □

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