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# ASYMPTOTICS OF PARABOLIC GREEN'S FUNCTIONS ON LATTICES

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Abstract. For parabolic spatially discrete equations, we considered the Green functions also known as heat kernels on lattices. Their asymptotic expansions with respect to powers of the time variable $t$ are obrained up to an arbitrary order, the remainders are estimated uniformly on the entire lattice. The spatially discrete (difference) operators under consideration are finite-difference approximations of continuous strongly elliptic differential operators (with constant coefficients) of arbitrary even order in $\mathbb{R}^d$ with arbitrary $d \in \mathbb{N}$. This genericity, besides numerical and deterministic lattice-dynamics applications, makes it possible to obtain higher-order asymptotics of transition probability functions for continuous-time random walks on $\mathbb{Z}^d$ and other lattices.

## §1. Introduction

In this paper, we deal with Green's functions, or heat kernels, of general parabolic equations with constant coefficients that are continuous in time and discrete in space. Applications that we have in mind include spatial discretization of continuous models, lattice dynamical systems, and continuous-time random walks.

We consider parabolic problems on the *grid space*, or *lattice*,

$$\mathbb{R}^d_\varepsilon := \{x \in \mathbb{R}^d \,:\, x_k = s_k\varepsilon,\ s_k \in \mathbb{Z},\ k = 1, \dots, d\}, \quad d \in \mathbb{N},\ \varepsilon > 0,$$

of the form

$$\begin{cases} \dot{\mathbf{u}}^\varepsilon(x,t) + \mathbf{A}_\varepsilon \mathbf{u}^\varepsilon(x,t) = 0, & x \in \mathbb{R}^d_\varepsilon,\ t > 0, \\ \mathbf{u}^\varepsilon(x,0) = \boldsymbol{\delta}^\varepsilon(x), & x \in \mathbb{R}^d_\varepsilon. \end{cases} \tag{1.1}$$

Here $\dot{} = \partial/\partial t$, $\boldsymbol{\delta}^\varepsilon(x)$ is the *grid delta-function* given by

$$\boldsymbol{\delta}^\varepsilon(0) = \varepsilon^{-d}, \quad \boldsymbol{\delta}^\varepsilon(x) = 0 \quad \forall x \in \mathbb{R}^d_\varepsilon \setminus \{0\}, \tag{1.2}$$

and $\mathbf{A}_\varepsilon$ is an elliptic difference operator (with constant coefficients), which is assumed to be an $M$th order approximation of a strongly elliptic (continuous) differential operator $\mathcal{A}(\mathcal{D})$ of even order $\ell$ (see §2 for rigorous definitions). The order $M \in \mathbb{N}$ is defined by the estimate

$$\sup_{y \in \mathbb{R}^d} |\mathbf{A}_\varepsilon u(y) - \mathcal{A}(\mathcal{D})u(y)| \le C(u)\varepsilon^M, \qquad \varepsilon > 0, \tag{1.3}$$

which must be true for any *smooth* (continuous and bounded with all its derivatives) function $u(y)$ and an appropriate constant $C(u) > 0$ not depending on $\varepsilon > 0$.

We call a solution of problem (1.1) the *first discrete Green function*. Given arbitrary initial data instead of $\boldsymbol{\delta}^\varepsilon(x)$, one can represent a solution as a (discrete) convolution

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of these initial data with the first Green function. Alternatively, the first Green function determines a semigroup generated by $-\mathbf{A}_\varepsilon$. Various properties of $\mathbf{A}_\varepsilon$ (in terms of its symbol) in function spaces of grid functions, corresponding semigroup properties, and relationship to spatially continuous problems can be found in [1, 2, 6], see also the references therein.

By using the discrete Fourier transform, one can give an integral representation of the discrete Green function (see (2.22)). However, it is not possible to express it via elementary functions, hence its asymptotic expansions play an important role. Asymptotics of the so-called lattice Green's functions in the stationary case were studied beginning from 1950s, see [5], the subsequent papers [3, 9, 12, 15–17], and the monograph [14, Chapter 8]. For parabolic operators, there is vast literature in the spatially continuous case. For example, large-time behavior of Green's functions was treated in [18] (for small perturbations of the heat operator) and in [19, 22] (for spatially periodic coefficients). A survey on the large time behavior of heat kernels for second-order parabolic operators on Riemannian manifolds can be found in [21]. In the spatially discrete case, the research directions include continuous-time random walks on general graphs (see, e.g., [13, 20] and references therein) and on lattices in a random environment (see, e.g., [4]). In both cases, Gaussian bounds for the heat kernel is an important question. However, higher-order asymptotics of Green's functions is not available in general. We mention [7, 10], where an asymptotic expansion of Green's function for specific parabolic equations on one-dimensional lattices was obtained in terms of the Bessel functions.

In our paper, using the integral Fourier representation of the first Green function $\mathbf{u}^\varepsilon(x,t)$, we obtain a higher-order asymptotic formula of the form

$$\frac{\partial^J \mathbf{u}^\varepsilon(x,t)}{\partial t^J} = \frac{1}{t^{d/\ell+J}}\, H_J\left(\frac{x}{t^{1/\ell}}\right) + \sum_{k=M}^{K} \frac{\varepsilon^k}{t^{(k+d)/\ell+J}}\, H_{Jk}\left(\frac{x}{t^{1/\ell}}\right) + \mathbf{r}^\varepsilon_{\mathbf{u}}(J,K;x,t), \qquad x\in\mathbb{R}^d_\varepsilon,\ \ t\ge t_0\varepsilon^\ell. \tag{1.4}$$

Here $\varepsilon>0$, $t_0>0$, $K\ge M$, $J\ge 0$, $H_J(y)$ and $H_{Jk}(y)$, $y\in\mathbb{R}^d$, are explicitly given smooth functions, and the remainder satisfies

$$|\mathbf{r}^\varepsilon_{\mathbf{u}}(J,K;x,t)| \le \frac{\varepsilon^{K+1} R_{\mathbf{u}}(J,K,t_0)}{t^{(K+d+1)/\ell+J}}, \quad x\in\mathbb{R}^d_\varepsilon,\ \ t\ge t_0\varepsilon^\ell, \tag{1.5}$$

with $R_{\mathbf{u}}(J,K,t_0)\ge 0$ not depending on $\varepsilon>0$, $x\in\mathbb{R}^d_\varepsilon$, and $t\ge t_0\varepsilon^\ell$. We emphasize that the estimate in (1.5) is uniform with respect to $x\in\mathbb{R}^d_\varepsilon$.

Moreover, we show that, for $J=0$, the leading order term $\frac{1}{t^{d/\ell}}\, H_0\big(\frac{x}{t^{1/\ell}}\big)$ in (1.4) coincides with the continuous parabolic Green function (see (2.6)). Hence, (1.4) implies that the first discrete Green function approximates the continuous one, uniformly for $x\in\mathbb{R}^d_\varepsilon$, with an error of $O\big(\frac{1}{t^{(M+d)/\ell}}\big)$ as $t\to\infty$ and with an error of $O(\varepsilon^M)$ as $\varepsilon\to 0$, where $M$ is the order of approximation in (1.3) (see Corollaries 3.1 and 3.2). In the case where $-\mathbf{A}_\varepsilon$ is an approximation of the Laplacian, the first term in (1.4) with $J=0$ equals $\frac{1}{(2\sqrt{\pi t})^d} e^{-|x|^2/4t}$. In particular, it is radially symmetric. The other terms, being nonsymmetric, take into account the radial nonsymmetry of the lattice $\mathbb{R}^d_\varepsilon$.

The asymptotic formula (1.4) is proved in §3 (Theorem 3.1), after rigorous definitions of elliptic differential and difference operators in §2.

We note that the general form of the difference operator $\mathbf{A}_\varepsilon$ (rather than the Laplacian only) allows one to treat continuous-time random walks on $d$-dimensional lattices that are not necessarily cubic, but still represented by a discrete additive subgroup of $\mathbb{R}^d$. This is possible whenever one can linearly transform the vertices of such a lattice to $\mathbb{R}^d_\varepsilon$ (see [13, Section 1]) and obtain an elliptic generator $\mathbf{A}_\varepsilon$ on $\mathbb{R}^d_\varepsilon$. Then, linearly transforming (1.4) back to the original noncubic lattice, one can deduce asymptotics of

the transition probability function of the original random walk. Example 2.1.2 illustrates an elliptic difference operator that is a generator obtained after transforming the random walk on the triangular lattice to $\mathbb{Z}^2$.

Along with the first discrete Green function, we study the *second discrete Green function* in §4 and §5. It is defined as a solution of the problem

$$\begin{cases} \dot{\mathbf{v}}^\varepsilon(x,t) + \mathbf{A}_\varepsilon \mathbf{v}(x,t) = \boldsymbol{\delta}^\varepsilon(x), & x \in \mathbb{R}^d_\varepsilon, \ t > 0, \\ \mathbf{v}^\varepsilon(x,0) = 0, & x \in \mathbb{R}^d_\varepsilon. \end{cases} \tag{1.6}$$

Given an arbitrary time-independent right-hand side instead of $\boldsymbol{\delta}^\varepsilon(x)$, one can represent a solution as a (discrete) convolution of this right-hand side with the second Green function. An interesting application of the second Green function $\mathbf{v}^\varepsilon(x,t)$ occurs in discrete reaction-diffusion equations with hysteretic nonlinearity in the right-hand side. Higher-order asymptotic formulas for $\mathbf{v}^\varepsilon(x,t)$ play a central role in understanding pattern formation mechanisms there (see [8]).

Asymptotics of the second Green function depends on the sign of $\ell - d$, where $\ell$ is the order of the continuous differential operator $\mathcal{A}(\mathcal{D})$ and $d$ is the spatial dimension. If $\ell - d \le -1$, then we prove that the second discrete Green function satisfies

$$\begin{aligned} \mathbf{v}^\varepsilon(x,t) = {} & \frac{1}{t^{d/\ell-1}} F_0\left(\frac{x}{t^{1/\ell}}\right) + \frac{1}{\varepsilon^{d-\ell}} \Omega\left(\frac{x}{\varepsilon}\right) \\ & + \sum_{k=M}^{K} \frac{\varepsilon^k}{t^{(k+d)/\ell-1}} F_k\left(\frac{x}{t^{1/\ell}}\right) + \mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t), \quad x \in \mathbb{R}^d_\varepsilon, \ t \ge t_0 \varepsilon^\ell, \end{aligned} \tag{1.7}$$

where the remainder is estimated as

$$|\mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t)| \le \frac{\varepsilon^{K+1} R_{\mathbf{v}}(K,t_0)}{t^{(K+d+1)/\ell-1}}, \quad x \in \mathbb{R}^d_\varepsilon, \ t \ge t_0 \varepsilon^\ell, \tag{1.8}$$

with $R_{\mathbf{v}}(K,t_0) \ge 0$ not depending on $\varepsilon > 0$, $x \in \mathbb{R}^d_\varepsilon$, and $t \ge t_0\varepsilon^\ell$. Again, the estimate in (1.5) is uniform with respect to $x \in \mathbb{R}^d_\varepsilon$. In §4 (Lemma 4.2), we prove (1.7) with the functions $F_k(y)$ that are given explicitly as solutions of certain first-order PDEs with $H_0(y)$ and $H_{0k}(y)$ on the right-hand side, but a still unknown function $\Omega(y)$. In Subsection 5.1 (Theorem 5.1), we find the function $\Omega(y)$ by simultaneously passing to the limit, as $t \to \infty$, in (1.7) and in the explicit integral representation of $\mathbf{v}^\varepsilon(x,t)$.

If $\ell - d \ge 0$, the situation is subtler. In §4, we prove an analog of (1.7) in the form

$$\begin{aligned} \mathbf{v}^\varepsilon(x,t) = {} & t^{1-d/\ell} F_0\left(\frac{x}{t^{1/\ell}}\right) + \sum_{k=M}^{\ell-d} \varepsilon^k t^{1-(k+d)/\ell} F_k\left(\frac{x}{t^{1/\ell}}\right) + \varepsilon^{\ell-d} \Omega\left(\frac{x}{\varepsilon}\right) \\ & + \sum_{k=\max(M,\ell-d+1)}^{K} \frac{\varepsilon^k}{t^{(k+d)/\ell-1}} F_k\left(\frac{x}{t^{1/\ell}}\right) + \mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t), \\ & \qquad x \in \mathbb{R}^d_\varepsilon \setminus \{0\}, \ t \ge t_0 \varepsilon^\ell, \end{aligned} \tag{1.9}$$

(if $M \ge l - d + 1$, the first sum is absent). As before, the remainder satisfies (1.8), the functions $F_k(y)$ are given explicitly as solutions of certain first-order PDEs with $H_0(y)$ and $H_{0k}(y)$ on the right-hand side, and $\Omega(y)$ is still unknown. However, unlike in the case where $\ell - d \le -1$, formula (1.9) may contain the function $F_{\ell-d}(y)$, which in general turns out to be undefined at $y = 0$ and to have logarithmic growth as $y \to 0$. Furthermore, we cannot pass to the limit as $t \to \infty$ immediately because both the terms with $k = 0, M, \ldots, \ell - d$ in (1.9) and the explicit integral representation of $\mathbf{v}^\varepsilon(x,t)$ tend to infinity. Hence, in Subsection 5.2 (Theorem 5.2), we use the explicit integral representation of $\mathbf{v}^\varepsilon(x,t)$ to deduce another asymptotic representation. It involves a linear combination of nonnegative powers $t^{1-(k+d)/\ell}$ $(k = 0, \ldots, \ell - d)$ and of $\ln t$ with $x$-dependent coefficients and is applicable for each fixed $x \in \mathbb{R}^d_\varepsilon$ (but not uniformly in $\mathbb{R}^d_\varepsilon$).

Comparing it with (1.9) allows one to determine $\Omega(y)$ and obtain uniform asymptotics in $\mathbb{R}^d_\varepsilon$.

In §6, we apply our general results to the $(2N)$th order approximation of the one-dimensional Laplacian:

$$\mathbf{A}_\varepsilon = -\mathbf{\Delta}_\varepsilon, \quad \mathbf{\Delta}_\varepsilon \mathbf{u}(x) := \varepsilon^{-2} \sum_{\nu=1}^{N} a_\nu \big(\mathbf{u}(x-\varepsilon\nu) - 2\mathbf{u}(x) + \mathbf{u}(x+\varepsilon\nu)\big) \tag{1.10}$$

with appropriately chosen coefficients $a_\nu \in \mathbb{R}$. (In this case, $\ell = 2$ and $d = 1$, hence we are in the situation where $\ell - d \geq 0$.) First, we claim that $\mathbf{A}_\varepsilon$ is elliptic for any $N \in \mathbb{N}$ in the sense of Condition 2.3 (the proof is given in Appendix 6.5). Then we explicitly find all the functions in the expansions (1.4) and (1.9) for the first and second Green functions and in the corresponding expansions for the (spatial) gradients of the first and second Green functions.

Interestingly, if $N = 1$, i.e., $\mathbf{\Delta}_\varepsilon \mathbf{u}(x) := \varepsilon^{-2}\big(\mathbf{u}(x-\varepsilon) - 2\mathbf{u}(x) + \mathbf{u}(x+\varepsilon)\big)$, it turns out that $\Omega(x/\varepsilon)$ in (1.9) vanishes for all $x \in \mathbb{R}^d_\varepsilon$, which is proved in Appendix 6.5. In general, $\Omega(x/\varepsilon) \not\equiv 0$ for $N \geq 2$.

To conclude, we collect some general notation that we use throughout the paper. We use variables $x \in \mathbb{R}^d_\varepsilon$ or $x \in \mathbb{Z}^d$ (for $\varepsilon = 1$) and $y \in \mathbb{R}^d$ for the grid and continuous coordinates, respectively, and $\eta \in R_{\pi\varepsilon^{-1}}$, $\theta \in R_\pi$, and $\xi \in \mathbb{R}^d$ for the Fourier coordinates, where

$$R_\lambda := \{\xi \in \mathbb{R}^d \,:\, |\xi_k| \leq \lambda,\ k = 1, \dots, d\}, \quad \lambda > 0. \tag{1.11}$$

By $B_\lambda$, we denote the ball of radius $\lambda > 0$ in $\mathbb{R}^d$. We denote by $(r, \varphi_1, \dots \varphi_{d-1})$ the spherical coordinates of $y \in \mathbb{R}^d$ or $\theta \in R_\pi$ and introduce the function

$$r_\pi(\varphi) := \text{ the distance from the origin to the boundary of } R_\pi \text{ in the direction } \varphi. \tag{1.12}$$

We use the upper-case bold letters for difference operators: $\mathbf{A}_\varepsilon$, $\mathbf{\Delta}_\varepsilon$ and lower-case bold letters for grid functions (i.e., functions defined on $\mathbb{R}^d_\varepsilon$): $\mathbf{u}^\varepsilon(x,t)$, $\mathbf{v}^\varepsilon(x,t)$, etc., indicating by superscript $\varepsilon$ that $x \in \mathbb{R}^d_\varepsilon$. We may omit the superscript if $\varepsilon = 1$ and $x \in \mathbb{Z}^d$. We denote polynomials by calligraphic letters: $\mathcal{A}(\xi)$, $\mathcal{P}(\xi)$, etc. and other (nonpolynomial) functions by usual letters: $A(\theta)$, $H(y)$, $\Omega(y)$, $u(y,t)$, etc.

## §2. Continuous and discrete parabolic Green's functions

In this section, we introduce parabolic equations with general elliptic operators of even order $\ell$ in $\mathbb{R}^d$, $d \in \mathbb{N}$, and the corresponding finite-difference approximation and define their Green functions.

**2.1. Continuous Green function.** Consider a differential operator with constant coefficients in $\mathbb{R}^d$, $d \in \mathbb{N}$:

$$\mathcal{A}(\mathcal{D}) = \sum_{|\alpha|=\ell} b_\alpha \mathcal{D}^\alpha, \tag{2.1}$$

where $\ell \in \mathbb{N}$ is even, $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_d$, $b_\alpha \in \mathbb{C}$, $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_d^{\alpha_d}$, and $\mathcal{D}_j = -i\partial/\partial y_j$, $j = 1, \dots, d$.

**Definition 2.1.** The polynomial

$$\mathcal{A}(\xi) := \sum_{|\alpha|=\ell} b_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^d, \tag{2.2}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \cdot \dots \cdot \xi_d^{\alpha_d}$ (obtained from $\mathcal{A}(\mathcal{D})$ by formally replacing $\mathcal{D}_j$ by $\xi_j$), is called the *symbol* of the differential operator $\mathcal{A}(\mathcal{D})$.

Obviously, there is $C > 0$ such that

$$|\mathcal{A}(\xi)| \le C|\xi|^\ell \quad \forall \xi \in \mathbb{R}^d. \tag{2.3}$$

We assume throughout that the operator $\mathcal{A}(\mathcal{D})$ is *strongly elliptic*, i.e., the following condition is fulfilled.

**Condition 2.1.** *There is $c > 0$ such that*

$$\operatorname{Re}\mathcal{A}(\xi) \ge c|\xi|^\ell \quad \forall \xi \in \mathbb{R}^d. \tag{2.4}$$

We briefly recall the notion of a *continuous Green function* for parabolic equations. We define it as a solution of the problem

$$\begin{cases} \dfrac{\partial u(y,t)}{\partial t} + \mathcal{A}(\mathcal{D})u(y,t) = 0, & y \in \mathbb{R}^d,\ t > 0, \\ u(y,0) = \delta(y), & y \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

with the $\delta$-function in the initial condition. To give an explicit formula for the Green function, we formally use the Fourier transform and deduce from (2.5)

$$u(y,t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-t\mathcal{A}(\xi)}e^{iy\xi}\,d\xi, \quad y \in \mathbb{R}^d,\ t > 0,$$

where the integral converges for each fixed $t > 0$ due to (2.4). Making the change of variables $\xi \mapsto \xi t^{-1/\ell}$ and using the homogeneity of $\mathcal{A}(\xi)$, we obtain

$$u(y,t) = \frac{1}{t^{d/\ell}}H\left(\frac{y}{t^{1/\ell}}\right), \quad y \in \mathbb{R}^d,\ t > 0, \tag{2.6}$$

where

$$H(y) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-\mathcal{A}(\xi)}e^{iy\xi}\,d\xi. \tag{2.7}$$

Due to (2.4), $H(y)$ and all its derivatives decay at infinity faster than any negative power of $|y|$.

**Definition 2.2.** We call the function given by (2.6) the *continuous Green function.*

*Remark* 2.1. If $\mathcal{A}(\mathcal{D}) = -\Delta$, where $\Delta$ is the Laplace operator, then $\mathcal{A}(\xi) = |\xi|^2$, $H(y) = \frac{1}{2^d\pi^{d/2}}e^{-|y|^2/4}$, and (2.6) assumes the well-known form $u(y,t) = \frac{1}{(2\sqrt{\pi t})^d}e^{-|y|^2/4t}$.

**2.2. Discrete Green function.** For $\varepsilon > 0$, we define the *grid space*

$$\mathbb{R}^d_\varepsilon := \left\{x \in \mathbb{R}^d \,:\, x_k = s_k\varepsilon,\ s_k \in \mathbb{Z},\ k = 1,\dots,d\right\}.$$

A function defined on $\mathbb{R}^d_\varepsilon$ is called a *grid function.* We say that a grid function (continuous function) is *rapidly decreasing* if it decays at infinity faster than any negative power of $|x|$, $x \in \mathbb{R}^d_\varepsilon$ (of $|y|$, $y \in \mathbb{R}^d$). For a continuous function $u(y)$, $y \in \mathbb{R}^d$, we use the notation

$$\delta_{\varepsilon,k\pm}u(y) = u(y \pm e_k\varepsilon) - u(y), \tag{2.8}$$

where $e_k$ is the unit vector of the axis $y_k$. The same notation is used for grid functions $\mathbf{u}^\varepsilon(x)$, $x \in \mathbb{R}^d_\varepsilon$.

We fix a natural $\widehat{\ell} \ge \ell$ and consider the following difference operator with constant coefficients:

$$\mathbf{A}_\varepsilon := \varepsilon^{-\ell}\sum_{\ell \le |\nu| \le \widehat{\ell}} a_\nu \delta^{\nu_1}_{\varepsilon,1-}\delta^{\nu_2}_{\varepsilon,1+}\dots\delta^{\nu_{2d-1}}_{\varepsilon,d-}\delta^{\nu_{2d}}_{\varepsilon,d+}, \tag{2.9}$$

where $\nu = (\nu_1,\dots,\nu_{2d})$ is a multi-index, $|\nu| = \nu_1 + \dots + \nu_{2d}$, and $a_\nu \in \mathbb{C}$.

We say that a function $u(y)$, $y \in \mathbb{R}^d$, is *smooth* if it is continuous and bounded together with all its derivatives.

**Definition 2.3.** For $M \in \mathbb{N}$, the difference operator $\mathbf{A}_\varepsilon$ is said to be an *$M$th order approximation of the differential operator $\mathcal{A}(\mathcal{D})$* if, for any smooth function $u(y)$, there is a constant $C(u) > 0$ such that

$$\sup_{y\in\mathbb{R}^d} |\mathbf{A}_\varepsilon u(y) - \mathcal{A}(\mathcal{D})u(y)| \le C(u)\varepsilon^M \quad \forall \varepsilon > 0. \tag{2.10}$$

We assume throughout that the following is fulfilled.

**Condition 2.2.** *There is $M \in \mathbb{N}$ such that $\mathbf{A}_\varepsilon$ is an $M$th order approximation of the differential operator $\mathcal{A}(\mathcal{D})$.*

Next, we introduce a symbol of $\mathbf{A}_\varepsilon$ and relate it to the symbol of $\mathcal{A}(\mathcal{D})$. To motivate the notion of a symbol, we define the Fourier transform of a rapidly decreasing grid function $\mathbf{u}^\varepsilon(x)$ by the formula

$$(\mathbf{F}_\varepsilon \mathbf{u}^\varepsilon)(\eta) := (2\pi)^{-d} \sum_{x\in\mathbb{R}^d_\varepsilon} e^{-ix\eta}\mathbf{u}(x)\varepsilon^d, \quad \eta \in R_{\pi\varepsilon^{-1}}, \tag{2.11}$$

where $R_{\pi\varepsilon^{-1}}$ is given by (1.11). The Fourier transformation $\mathbf{F}_\varepsilon$ establishes an isomorphism between rapidly decreasing grid functions and smooth functions that are $2\pi\varepsilon^{-1}$-periodic with respect to each of the variables $\eta_1, \dots, \eta_d$. The inverse transform is given by

$$(\mathbf{F}_\varepsilon^{-1} v)(x) = \int_{R_{\pi\varepsilon^{-1}}} e^{ix\eta} v(\eta)\, d\eta. \tag{2.12}$$

If $\mathbf{u}^\varepsilon(x)$ is a rapidly decreasing grid function, then $\mathbf{A}_\varepsilon \mathbf{u}^\varepsilon(x)$ is also rapidly decreasing and

$$(\mathbf{F}_\varepsilon \mathbf{A}_\varepsilon \mathbf{u}^\varepsilon)(\eta) = \varepsilon^{-\ell} A(\eta\varepsilon) \cdot (\mathbf{F}_\varepsilon \mathbf{u}^\varepsilon)(\eta), \tag{2.13}$$

where the function $\varepsilon^{-\ell}A(\eta\varepsilon)$ is obtained by replacing the operator $\delta_{\varepsilon,k\pm}$ on the right-hand side of formula (2.9) by the expression $e^{\pm i\eta_k\varepsilon} - 1$.

**Definition 2.4.** The function

$$A(\theta) := \sum_{\ell\le|\nu|\le\widehat{\ell}} a_\nu \big(e^{-i\theta_1} - 1\big)^{\nu_1} \big(e^{i\theta_1} - 1\big)^{\nu_2} \dots \big(e^{-i\theta_d} - 1\big)^{\nu_{2d-1}} \big(e^{i\theta_d} - 1\big)^{\nu_{2d}}, \quad \theta \in R_\pi, \tag{2.14}$$

is called the *symbol of the difference operator $\mathbf{A}_\varepsilon$*.

The following lemma establishes a relationship between the symbols of $\mathcal{A}(\mathcal{D})$ and $\mathbf{A}_\varepsilon$.

**Lemma 2.1.** *If Condition 2.2 is fulfilled, then, for any $K \ge M$,*

$$A(\theta) = \sum_{k=0}^{K} \mathcal{A}_{k+\ell}(\theta) + O(|\theta|^{K+\ell+1}), \tag{2.15}$$

*where*

$$\mathcal{A}_{k+\ell}(\theta) := \begin{cases} \mathcal{A}(\theta) & \text{if } k = 0, \\ 0 & \text{if } k = 1, \dots, M-1, \\ \text{a homogeneous polynomial} & \\ \text{of degree } k+\ell & \text{if } k = M, \dots, K, \end{cases} \tag{2.16}$$

*and $O(\,\cdot\,)$ is understood as $|\theta| \to 0$.*

*If the symbol of $\mathbf{A}_\varepsilon$ satisfies (2.15) and (2.16) with some $K \ge M$, then Condition 2.2 is fulfilled.*

*Proof.* We observe that, for any smooth $u(y)$ and any $J \geq 1$,

$$\delta_{\varepsilon,k\pm}u(y) = \sum_{j=1}^{J} \frac{(\pm 1)^j \varepsilon^j}{j!} \frac{\partial^j u(y)}{\partial y_k^j} + \varepsilon^{J+1} R_{k,J+1}(y),$$

$$e^{\pm i\theta_k} - 1 = \sum_{j=1}^{J} \frac{(\pm 1)^j}{j!} (i\theta_k)^j + O(|\theta|^{J+1}),$$

where $\sup_{y\in\mathbb{R}^d} |R_{k,J+1}(y)| \leq C_1(u)$ with some $C_1(u) \geq 0$. Therefore, taking into account definitions (2.9) and (2.14), we have for any $K \geq M$:

$$\mathbf{A}_\varepsilon u(y) = \sum_{k=0}^{K} \varepsilon^k \mathcal{A}_{k+\ell}(\mathcal{D}) u(y) + \varepsilon^{K+1} R_{K+\ell+1}(y),$$

$$A(\theta) = \sum_{k=0}^{K} \mathcal{A}_{k+\ell}(\theta) + O(|\theta|^{K+\ell+1}),$$

where $\mathcal{A}_{k+\ell}(\,\cdot\,)$ are the same homogeneous polynomials of degree $k+\ell$ in both formulas and $\sup_{y\in\mathbb{R}^d} |R_{K+\ell+1}(y)| \leq C_2(u)$ with some $C_2(u) \geq 0$. Comparing the last two identities and taking (2.10) into account, we conclude the proof. □

Along with Condition 2.2, we assume throughout the following *ellipticity condition.*

**Condition 2.3.** $\operatorname{Re} A(\theta) > 0$ *for all* $\theta \in R_\pi \setminus \{0\}$, *where* $R_\pi$ *is defined in* (1.11).

*Remark* 2.2. Condition 2.3 does not automatically follow from the fact that $\mathbf{A}_\varepsilon$ is an approximation of a strongly elliptic operator $\mathcal{A}(\mathcal{D})$ (only the inequality $\operatorname{Re} A(\theta) > 0$ for small $|\theta| \neq 0$ follows). However, Lemma 6.2 below implies that Condition 2.3 is fulfilled, for example, for any order approximation of the second derivative in $\mathbb{R}$ (see (6.8) and (6.9)) and, hence, for the corresponding approximation of the Laplace operator in $\mathbb{R}^d$, $d \in \mathbb{N}$.

**Example 2.1.** 1. Consider the difference operator

$$\mathbf{A}_\varepsilon := \varepsilon^{-2} \sum_{j=1}^{d} \delta_{\varepsilon,j+}\delta_{\varepsilon,j-}.$$

For $\varepsilon = 1$, the operator $-(2d)^{-1}\mathbf{A}_1$ is a generator of the continuous-time random walk on $\mathbb{Z}^d$ (see [13, Section 1]). The operator $\mathbf{A}_\varepsilon$ is the 2nd order approximation of the strongly elliptic operator $\mathcal{A}(\mathcal{D}) = -\Delta$, where $\Delta$ is the Laplacian. The symbol of $\mathbf{A}_\varepsilon$ is given by

$$A(\theta) := 2\sum_{j=1}^{d} (1 - \cos\theta_j).$$

In §6, we will also consider $(2N)$th order approximations of the Laplacian for $d = 1$.

2. Let $d = 2$. Consider the operator

$$\mathbf{A}_\varepsilon := \frac{2}{3\varepsilon^2}\big(2\delta_{\varepsilon,1+}\delta_{\varepsilon,1-} + 2\delta_{\varepsilon,2+}\delta_{\varepsilon,2-} - \delta_{\varepsilon,1+}\delta_{\varepsilon,2-} - \delta_{\varepsilon,2+}\delta_{\varepsilon,1-}\big). \tag{2.17}$$

The operator $\mathbf{A}_\varepsilon$ is the 2nd order approximation of the strongly elliptic operator

$$\mathcal{A}(\mathcal{D})u := -\frac{4}{3}(u_{y_1y_1} + u_{y_2y_2} - u_{y_1y_2}).$$

The symbol of $\mathbf{A}_\varepsilon$ is given by

$$A(\theta) = 4 - \frac{4}{3}\big(\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 - \theta_2)\big)$$

and the symbol of $\mathcal{A}(\mathcal{D})$ is given by

$$\mathcal{A}(\theta) = \frac{2}{3}(\theta_1^2 + \theta_2^2 + (\theta_1 - \theta_2)^2).$$

On the other hand, $-\mathbf{A}_\varepsilon$ corresponds to the discretization on the triangular lattice of the continuous Laplacian. Indeed, the matrix

$$M := \begin{pmatrix} 1 & -1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{pmatrix}$$

takes the triangular lattice to $\mathbb{Z}^2$ and, at the same time, $\mathcal{A}((M^T)^{-1}\xi) = \xi_1^2 + \xi_2^2$. The latter is the symbol of the differential operator $-\Delta$.

Lemma 2.1, together with Conditions 2.1, 2.2, and 2.3, implies the following.

**Lemma 2.2.** *We have*

$$|A(\theta)| \le C|\theta|^\ell \quad \forall \theta \in R_\pi, \tag{2.18}$$

$$\operatorname{Re} A(\theta) \ge c|\theta|^\ell \quad \forall \theta \in R_\pi, \tag{2.19}$$

*where $C$ and $c$ are positive constants that do not depend on $\theta \in R_\pi$.*

Without loss of generality, we assume that the constants $C$ and $c$ in (2.18) and (2.19) are the same as in (2.3) and (2.4), respectively.

Now we introduce discrete Green functions. Let $\boldsymbol{\delta}^\varepsilon(x)$ be the *grid delta-function*, see (1.2).

**Definition 2.5.** We call the function $\mathbf{u}^\varepsilon(x,t)$, $x \in \mathbb{R}^d_\varepsilon$, $t \ge 0$, the *first discrete Green function* if $\mathbf{u}^\varepsilon(\,\cdot\,,t)$ is a rapidly decreasing grid function for all $t \ge 0$, $\mathbf{u}^\varepsilon(x,\,\cdot\,) \in C^1[0,\infty)$ for all $x \in \mathbb{R}^d_\varepsilon$, and

$$\begin{cases} \dot{\mathbf{u}}^\varepsilon(x,t) + \mathbf{A}_\varepsilon \mathbf{u}^\varepsilon(x,t) = 0, & x \in \mathbb{R}^d_\varepsilon,\ t > 0, \\ \mathbf{u}^\varepsilon(x,0) = \boldsymbol{\delta}^\varepsilon(x), & x \in \mathbb{R}^d_\varepsilon. \end{cases} \tag{2.20}$$

We call the function $\mathbf{v}^\varepsilon(x,t)$, $x \in \mathbb{R}^d_\varepsilon$, $t \ge 0$, the *second discrete Green function* if $\mathbf{v}^\varepsilon(\,\cdot\,,t)$ is a rapidly decreasing grid function for all $t \ge 0$, $\mathbf{v}^\varepsilon(x,\,\cdot\,) \in C^1[0,\infty)$ for all $x \in \mathbb{R}^d_\varepsilon$, and

$$\begin{cases} \dot{\mathbf{v}}^\varepsilon(x,t) + \mathbf{A}_\varepsilon \mathbf{v}^\varepsilon(x,t) = \boldsymbol{\delta}^\varepsilon(x), & x \in \mathbb{R}^d_\varepsilon,\ t > 0, \\ \mathbf{v}^\varepsilon(x,0) = 0, & x \in \mathbb{R}^d_\varepsilon. \end{cases} \tag{2.21}$$

Here $\dot{} = \partial/\partial t$.

In §3, we will establish a relationship between the discrete and continuous Green functions in terms of asymptotic formulas.

Using the discrete Fourier transform (2.11), its inverse (2.12), and relation (2.13), we obtain the explicit representations

$$\mathbf{u}^\varepsilon(x,t) = \dot{\mathbf{v}}^\varepsilon(x,t) = \frac{1}{(2\pi)^d} \int_{R_{\pi\varepsilon^{-1}}} e^{-t\varepsilon^{-\ell}A(\eta\varepsilon)} e^{ix\eta}\, d\eta, \tag{2.22}$$

$$\mathbf{v}^\varepsilon(x,t) = \frac{1}{(2\pi)^d} \int_{R_{\pi\varepsilon^{-1}}} \frac{1 - e^{-t\varepsilon^{-\ell}A(\eta\varepsilon)}}{\varepsilon^{-\ell}A(\eta\varepsilon)}\, e^{ix\eta}\, d\eta. \tag{2.23}$$

Changing the variables in the integrals in (2.22) and (2.23), we obtain for $J = 0, 1, 2, \ldots$:

$$\frac{\partial^J \mathbf{u}^\varepsilon(x,t)}{\partial t^J} \equiv \varepsilon^{-J\ell-d} \frac{\partial^J \mathbf{u}^1(x',\tau)}{\partial \tau^J}\bigg|_{\substack{x'=x/\varepsilon,\\ \tau=t/\varepsilon^\ell}}, \quad \mathbf{v}^\varepsilon(x,t) \equiv \varepsilon^{\ell-d}\mathbf{v}^1\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\ell}\Big). \tag{2.24}$$

## §3. Asymptotics of the first Green function $\mathbf{u}^\varepsilon(x,t)$

**3.1. Formulation of the result.** In the theorems below, we will not explicitly indicate the dependence of the remainders in asymptotic formulas on the number $M$ fixed in Condition 2.2.

In this section, we obtain the following asymptotics for $\mathbf{u}^\varepsilon(x,t)$.

**Theorem 3.1.** *For any $\varepsilon > 0$, $t_0 > 0$, an integer $K \geq M$, an integer $J \geq 0$, and all $x \in \mathbb{R}^d_\varepsilon$ and $t \geq t_0\varepsilon^\ell$, we have*

$$\frac{\partial^J \mathbf{u}^\varepsilon(x,t)}{\partial t^J} = \sum_{k=0}^{K} \frac{\varepsilon^k}{t^{(k+d)/\ell+J}}\, H_{Jk}\Big(\frac{x}{t^{1/\ell}}\Big) + \mathbf{r}^\varepsilon_{\mathbf{u}}(J,K;x,t), \tag{3.1}$$

*where*

$$H_{Jk}(y) := \begin{cases} (-\mathcal{A}(\mathcal{D}))^J H(y) & \text{if } k=0,\\ 0 & \text{if } k=1,\dots,M-1,\\ \text{finite linear combinations} & \\ \text{of derivatives of } H(y) & \text{if } k=M,\dots,K, \end{cases} \tag{3.2}$$

*$H(y)$ is given by (2.7),*

$$|\mathbf{r}^\varepsilon_{\mathbf{u}}(J,K;x,t)| \leq \frac{\varepsilon^{K+1} R_{\mathbf{u}}(J,K,t_0)}{t^{(K+d+1)/\ell+J}}, \tag{3.3}$$

*and $R_{\mathbf{u}}(J,K,t_0) \geq 0$ does not depend on $\varepsilon > 0$, $x \in \mathbb{R}^d_\varepsilon$, and $t \geq t_0\varepsilon^\ell$.*

*Remark* 3.1. It follows from the proof below that $H_{Jk}(y) = \mathcal{R}_{J,k+\ell}(\mathcal{D})H(y)$, where $\mathcal{R}_{J,k+\ell}(\xi)$ are polynomials of degree at least $k+\ell$ defined by (3.15).

Note that the main term in the asymptotic representation of $\mathbf{u}^\varepsilon(x,t)$ coincides with the continuous Green function (2.6). Theorem 3.1 implies that the higher the approximation order $M$ (see Definition 2.3) is, the faster the first discrete Green function converges to the continuous Green function, as $t \to \infty$ or as $\varepsilon \to 0$. More precisely, the following corollaries hold.

**Corollary 3.1.** *Under the assumptions of Theorem* 3.1*, for any $\varepsilon_0 > 0$,*

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \sup_{x\in\mathbb{R}^d_\varepsilon} \left| \mathbf{u}^\varepsilon(x,t) - \frac{1}{t^{d/\ell}}\, H\Big(\frac{x}{t^{1/\ell}}\Big)\right| = O\Big(\frac{1}{t^{(M+d)/\ell}}\Big) \quad \text{as } t \to \infty.$$

**Corollary 3.2.** *Under the assumptions of Theorem* 3.1*, for any $t_1 > 0$,*

$$\sup_{t\geq t_1} \sup_{x\in\mathbb{R}^d_\varepsilon} \left| \mathbf{u}^\varepsilon(x,t) - \frac{1}{t^{d/\ell}}\, H\Big(\frac{x}{t^{1/\ell}}\Big)\right| = O(\varepsilon^M) \quad \text{as } \varepsilon \to 0.$$

### 3.2. Proof of Theorem 3.1.

**Step 1.** First, we assume that $\varepsilon = 1$ and denote $\mathbf{u}(x,t) := \mathbf{u}^1(x,t)$. Note that $|\mathbf{u}(x,t)|$ is bounded for $x \in \mathbb{Z}^d$ and $t \geq 0$ due to (2.22) and (2.19). Therefore, if we show that Theorem 6.1 is true with $\varepsilon = 1$ and some $t_0 > 0$, it will follow that the theorem is also true with $\varepsilon = 1$ and all $t_0 > 0$. Let us fix $t_0 = 1$ and consider $t \geq 1$.

Differentiating (2.22) with $\varepsilon = 1$ and making the change of variables $\xi = t^{1/\ell}\eta$, we write

$$\begin{aligned} \frac{\partial^J \mathbf{u}(x,t)}{\partial t^J} &= \frac{1}{(2\pi)^d\, t^{d/\ell}} \int_{R_{\pi t^{1/\ell}}} (-1)^J \big(A(t^{-1/\ell}\xi)\big)^J e^{-tA(t^{-1/\ell}\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi \\ &= I_1(x,t) + I_2(x,t), \end{aligned} \tag{3.4}$$

where

$$I_1(x,t) := \frac{1}{(2\pi)^d\, t^{d/\ell}} \int_{R_{\pi t^{1/L}}} (-1)^J \big(A(t^{-1/\ell}\xi)\big)^J e^{-tA(t^{-1/\ell}\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi, \tag{3.5}$$

$$I_2(x,t) := \frac{1}{(2\pi)^d\, t^{d/\ell}} \int_{R_{\pi t^{1/\ell}} \setminus R_{\pi t^{1/L}}} (-1)^J \big(A(t^{-1/\ell}\xi)\big)^J e^{-tA(t^{-1/\ell}\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi, \tag{3.6}$$

$$L := (\ell+2)\ell. \tag{3.7}$$

**Step 2.** We estimate $I_1(x,t)$ in (3.5). First, we consider the term $e^{-tA(t^{-1/\ell}\xi)}$. By Lemma 2.1,

$$tA(t^{-1/\ell}\xi) = \mathcal{A}(\xi) + \sum_{k=M}^{K} t^{-k/\ell}\mathcal{A}_{k+\ell}(\xi) + \widehat{A}(t,\xi), \tag{3.8}$$

where

$$|\widehat{A}(t,\xi)| \le c_1 t^{-(K+1)/\ell}|\xi|^{K+1+\ell}, \quad \xi \in R_{\pi t^{1/\ell}},\ \ t \ge 1. \tag{3.9}$$

Here and below $c_1, c_2, \ldots > 0$ do not depend on $\xi$ and $t$.

Further, (3.7) implies for $\xi \in R_{\pi t^{1/L}}$ and $t \ge 1$ that

$$t^{-k/\ell}|\xi|^{k+\ell} \le c_2 t^{-k/\ell} t^{(k+\ell)/L} = c_2 t^{(\ell-(\ell+1)k)/L} \le c_2, \quad k = 1, 2, \ldots.$$

Therefore,

$$\left| \sum_{k=M}^{K} t^{-k/\ell}\mathcal{A}_{k+\ell}(\xi) + \widehat{A}(t,\xi) \right| \le c_3, \quad \xi \in R_{\pi t^{1/L}},\ \ t \ge 1. \tag{3.10}$$

Using the Taylor expansion of the exponential function, we obtain

$$\begin{aligned} &\exp\bigg( -\sum_{k=M}^{K} t^{-k/\ell}\mathcal{A}_{k+\ell}(\xi) - \widehat{A}(t,\xi) \bigg) \\ &\quad = 1 + \sum_{m=1}^{K} \frac{1}{m!}\bigg( \sum_{k=M}^{K} t^{-k/\ell}\mathcal{A}_{k+\ell}(\xi) + \widehat{A}(\xi) \bigg)^m + \widehat{A}_1(\xi,t) \\ &\quad = 1 + \sum_{k=M}^{K} t^{-k/\ell}\mathcal{P}_{k+\ell}(\xi) + \widehat{P}(\xi,t). \end{aligned} \tag{3.11}$$

Here the $\mathcal{P}_{k+\ell}(\xi)$ are polynomials of degree at least $k+\ell$ and the remainders are estimated due to relations (3.9) and (3.10) as follows:

$$\begin{aligned} |\widehat{A}_1(\xi,t)| &\le e^{c_3} \left| \sum_{k=M}^{K} t^{-k/\ell}\mathcal{A}_{k+\ell}(\xi) + \widehat{A}(\xi,t) \right|^{K+1} \\ &\le c_4 e^{c|\xi|^\ell/2} t^{-M(K+1)/\ell}, \quad \xi \in R_{\pi t^{1/L}},\ \ t \ge 1, \end{aligned}$$

$$|\widehat{P}(\xi,t)| \le c_5 e^{c|\xi|^\ell/2} t^{-(K+1)/\ell}, \quad \xi \in R_{\pi t^{1/L}},\ t \ge 1, \tag{3.12}$$

where $c$ is the constant from (2.4).

Now we estimate the term $(A(t^{-1/\ell}\xi))^J$. Using (3.8), we have

$$\big(A(t^{-1/\ell}\xi)\big)^J = t^{-J}\bigg( (\mathcal{A}(\xi))^J + \sum_{k=M}^{K} t^{-k/\ell}\mathcal{D}_{J,k+\ell}(\xi) + \widehat{D}(t,\xi) \bigg), \tag{3.13}$$

where the $\mathcal{D}_{J,k+\ell}(\xi)$ are polynomials of degree at least $k+\ell$ and

$$|\widehat{D}(\xi,t)| \le c_6 e^{c|\xi|^\ell/2} t^{-(K+1)/\ell}, \quad \xi \in R_{\pi t^{1/\ell}}, \ t \ge 1. \tag{3.14}$$

Combining (3.11)–(3.14) with (3.8) yields

$$\begin{aligned} &(-1)^J \big(A(t^{-1/\ell}\xi)\big)^J e^{-tA(t^{-1/\ell}\xi)} \\ &\quad = e^{-\mathcal{A}(\xi)} \Big( t^{-J} \left(-\mathcal{A}(\xi)\right)^J + \sum_{k=M}^{K} t^{-k/\ell - J} \mathcal{R}_{J,k+\ell}(\xi) + \widehat{R}(t,\xi) \Big), \end{aligned} \tag{3.15}$$

where the $\mathcal{R}_{J,k+\ell}(\xi)$ are polynomials of degree at least $k+\ell$ and

$$|\widehat{R}(\xi,t)| \le c_7 e^{c|\xi|^\ell/2} t^{-(K+1)/\ell - J}, \quad \xi \in R_{\pi t^{1/L}}, \ t \ge 1. \tag{3.16}$$

Substituting (3.15) in (3.5), we have

$$\begin{aligned} I_1(x,t) = &\frac{1}{(2\pi)^d\, t^{d/\ell+J}} \int_{R_{\pi t^{1/L}}} \left(-\mathcal{A}(\xi)\right)^J e^{-\mathcal{A}(\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi \\ &+ \sum_{k=M}^{K} \frac{1}{(2\pi)^d\, t^{(k+d)/\ell+J}} \int_{R_{\pi t^{1/L}}} \mathcal{R}_{J,k+\ell}(\xi)\, e^{-\mathcal{A}(\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi \\ &+ \frac{1}{(2\pi)^d\, t^{d/\ell}} \int_{R_{\pi t^{1/L}}} e^{-\mathcal{A}(\xi)} \widehat{R}(\xi,t) \cdot e^{ixt^{-1/\ell}\xi}\, d\xi. \end{aligned} \tag{3.17}$$

Note that, due to (2.4), the polynomials $\mathcal{B}(\xi) = (-\mathcal{A}(\xi))^J$ and $\mathcal{B}(\xi) = \mathcal{R}_{J,k+l}(\xi)$ satisfy

$$\left| \int_{\mathbb{R}^d \setminus R_{\pi t^{1/L}}} \mathcal{B}(\xi) e^{-\mathcal{A}(\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi \right| \le \int_{\mathbb{R}^d \setminus R_{\pi t^{1/L}}} |\mathcal{B}(\xi)| \cdot e^{-c|\xi|^\ell}\, d\xi = O\Big(\frac{1}{t^{(K+1)/\ell}}\Big), \tag{3.18}$$

where $O(\,\cdot\,)$ is taken as $t \to \infty$ (uniformly with respect to $x$). Similarly, due to (3.16), $\widehat{R}(\xi,t)$ satisfies

$$\begin{aligned} \left| \int_{R_{\pi t^{1/L}}} e^{-\mathcal{A}(\xi)} \widehat{R}(\xi,t) \cdot e^{ixt^{-1/\ell}\xi}\, d\xi \right| &\le \frac{c_7}{t^{(K+1)/\ell+J}} \int_{R_{\pi t^{1/L}}} e^{-\operatorname{Re}\mathcal{A}(\xi)} \cdot e^{c|\xi|^\ell/2}\, d\xi \\ &\le \frac{c_7}{t^{(K+1)/\ell+J}} \int_{\mathbb{R}^d} e^{-c|\xi|^\ell/2}\, d\xi. \end{aligned} \tag{3.19}$$

Combining (3.17)–(3.19), we obtain

$$\begin{aligned} I_1(x,t) = &\frac{1}{(2\pi)^d\, t^{d/\ell+J}} \int_{\mathbb{R}^d} (-\mathcal{A}(\xi))^J e^{-\mathcal{A}(\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi \\ &+ \sum_{k=M}^{K} \frac{1}{(2\pi)^d t^{(k+d)/\ell+J}} \int_{\mathbb{R}^d} \mathcal{R}_{J,k+\ell}(\xi)\, e^{-\mathcal{A}(\xi)} \cdot e^{ixt^{-1/\ell}\xi}\, d\xi + O\Big(\frac{1}{t^{(K+d+1)/\ell+J}}\Big). \end{aligned} \tag{3.20}$$

Relations (2.7) and (3.20) and the fact that $\mathcal{R}_{J,k+\ell}(\xi)$ are polynomials imply

$$I_1(x,t) = \frac{1}{t^{d/\ell+J}}\, H_{J0}\Big(\frac{x}{t^{1/\ell}}\Big) + \sum_{k=M}^{K} \frac{1}{t^{(k+d)/\ell+J}}\, H_{Jk}\Big(\frac{x}{t^{1/\ell}}\Big) + O\Big(\frac{1}{t^{(K+d+1)/\ell+J}}\Big), \tag{3.21}$$

where $H_{J0}(y) = (-\mathcal{A}(\mathcal{D}))^J H(y)$ and $H_{Jk}(y) = \mathcal{R}_{J,k+\ell}(\mathcal{D}) H(y)$.

**Step 3.** We estimate $I_2(x,t)$ in (3.6). Using (2.18) and (2.19), we have for $\xi \in R_{\pi t^{1/\ell}} \setminus R_{\pi t^{1/L}}$:

$$\left|\left(A(t^{-1/\ell}\xi)\right)^J e^{-tA(t^{-1/\ell}\xi)}\right| \le c_8 e^{-c|\xi|^\ell/2} \le c_8 e^{-c\pi^\ell t^{\ell/L}/2}.$$

Therefore,

$$|I_2(x,t)| \le c_9 e^{-c\pi^\ell t^{\ell/L}/2}, \quad t \ge 1. \tag{3.22}$$

Relations (3.4)–(3.6), (3.21), and (3.22) yield the assertion of Theorem 3.1 with $\varepsilon = 1$. Using the first relation in (2.24), we obtain the assertion with any $\varepsilon > 0$.

## §4. Preliminary asymptotics of the second Green function $\mathbf{v}^\varepsilon(x,t)$

Throughout this section, we assume that $\varepsilon = 1$ and denote $\mathbf{v}(x,t) := \mathbf{v}^1(x,t)$. First, we obtain an asymptotic formula for $\mathbf{v}(x,t)$ up to some unknown function $\Omega(x)$. In the next section, we will provide an explicit formula for it.

We introduce the spherical coordinates $(r,\varphi)$ with $r > 0$ and $\varphi \in \Phi$, where

$$\Phi := \{\varphi \in \mathbb{R}^{d-1} \,:\, \varphi_1, \dots, \varphi_{d-2} \in [0,\pi],\ \varphi_{d-1} \in [0, 2\pi)\}. \tag{4.1}$$

They are related to the Cartesian coordinates $y \in \mathbb{R}^d$ via

$$\begin{gathered} y_1 = r\cos\varphi_1, \quad y_2 = r\sin\varphi_1\cos\varphi_2, \quad y_3 = r\sin\varphi_1\sin\varphi_2\cos\varphi_3, \dots, \\ y_{d-1} = r\sin\varphi_1\dots\sin\varphi_{d-2}\cos\varphi_{d-1}, \quad y_d = r\sin\varphi_1\dots\sin\varphi_{d-2}\sin\varphi_{d-1}, \end{gathered} \tag{4.2}$$

and the Jacobian is given by

$$dy = r^{d-1} J(\varphi)\, dr\, d\varphi, \quad J(\varphi) := \sin^{d-2}\varphi_1 \sin^{d-3}\varphi_2 \dots \sin\varphi_{d-2}. \tag{4.3}$$

We will denote functions written in Cartesian and spherical coordinates by the same letter. For example $H(r,\varphi)$ stands for $H(y)$, etc.

We set

$$H_k(y) := H_{0k}(y), \quad y \in \mathbb{R}^d,$$

where $H_{0k}(y)$ are the functions from Theorem 3.1 with $J = 0$. For $y \in \mathbb{R}^d \setminus \{0\}$, we introduce the functions

$$F_k(y) := \ell r^{\ell-d-k} \int_r^\infty \frac{H_k(\rho,\varphi)}{\rho^{\ell-d-k+1}}\, d\rho \qquad \text{if } k = 0, \dots, \ell-d, \tag{4.4}$$

$$F_k(y) := -\frac{\ell}{r^{d+k-\ell}} \int_0^r \rho^{d+k-\ell-1} H_k(\rho,\varphi)\, d\rho \qquad \text{if } k \ge \ell-d+1. \tag{4.5}$$

Definition (4.4) will be used only in the case where $\ell - d \ge 0$ and definition (4.5) only for $k \ge 0$.

*Remark* 4.1. Due to (3.2), $F_k(y) \equiv 0$ for $k = 1, \dots, M-1$.

**Lemma 4.1.** *Let a function $F_k(y)$ be defined by* (4.4) *or* (4.5). *Then*

(1) *it satisfies the partial differential equation*

$$(\ell - d - k)F_k - \sum_{j=1}^d y_j \frac{\partial F_k}{\partial y_j} = \ell H_k, \quad y \in \mathbb{R}^d \setminus \{0\}; \tag{4.6}$$

(2) *it tends to* 0 *as* $|y| \to \infty$;

(3) *if $k \ne \ell - d$, it is bounded in $\mathbb{R}^d$ and, for each fixed $\varphi$, continuous with respect to $r$ at the origin*;

(4) *if $k \ge \ell - d + 1$, it is continuous with respect to $y$ at the origin and*

$$F_k(0) := \lim_{y\to 0} F_k(y) = -\frac{\ell}{k+d-\ell} H_k(0). \tag{4.7}$$

*Proof.* Item 1 follows by observing that the equation in (4.6) is equivalent to the following:

$$(\ell - d - k)F_k - r\frac{\partial F_k}{\partial r} = \ell H_k, \quad r > 0.$$

The latter can be solved as an ordinary differential equation for each fixed $\varphi \in \Phi$. It is easy to check that a unique solution that is bounded at infinity is $F_k(y)$ defined by (4.4) or (4.5), respectively.

Items 2, 3, and 4 follow from the fact that the $H_k(y)$ are infinitely differentiable at the origin and are rapidly decreasing functions. □

Now we find a preliminary asymptotic formula for $\mathbf{v}(x,t)$ in terms of the above functions $F_k(y)$. It involves an unknown grid function $\Omega(x)$ and, if $\ell - d \geq 0$, it is not valid at $x = 0$. The reason for the latter is essentially the fact that the functions $F_k(y)$ are not continuous at $y = 0$ for $k = 0, \dots, \ell - d$.

**Lemma 4.2.** *There exists a grid function $\Omega(x)$ such that, for any $t_0 > 0$, integer $K \geq \max(M, \ell - d)$, $x \in \mathbb{Z}^d \setminus \{0\}$, and $t \geq t_0$, we have*

$$\mathbf{v}(x,t) = \Omega(x) + \sum_{k=0}^{K} \frac{1}{t^{(k+d)/\ell - 1}} F_k\Big(\frac{x}{t^{1/\ell}}\Big) + \mathbf{r}_{\mathbf{v}}(K; x, t). \tag{4.8}$$

*Here the $F_k(y)$ are given by* (4.4) *and* (4.5),

$$|\mathbf{r}_{\mathbf{v}}(K; x, t)| \leq \frac{R_{\mathbf{v}}(K, t_0)}{t^{(K+d+1)/\ell - 1}}, \tag{4.9}$$

*and $R_{\mathbf{v}}(K, t_0) \geq 0$ does not depend on $x \in \mathbb{Z}^d$ and $t \geq t_0$.*

*Furthermore, if $\ell - d \leq -1$, then the identity in* (4.8) *and the estimate in* (4.9) *are also valid for $x = 0$ with $F_k(0)$ defined by* (4.7).

*Proof.* We integrate (3.1) with $J = 0$ and $\varepsilon = 1$ from $t_0$ to $t$ and use the identity $\mathbf{u}(x,t) \equiv \dot{\mathbf{v}}(x,t)$:

$$\begin{aligned} \mathbf{v}(x,t) - \mathbf{v}(x,t_0) = \sum_{k=0}^{K} \int_{t_0}^{t} \frac{1}{\tau^{(k+d)/\ell}} H_k\Big(\frac{x}{\tau^{1/\ell}}\Big)\, d\tau \\ + \int_{t_0}^{\infty} \mathbf{r}^1_{\mathbf{u}}(0, K; x, \tau)\, d\tau - \int_{t}^{\infty} \mathbf{r}^1_{\mathbf{u}}(0, K; x, \tau)\, d\tau, \quad x \in \mathbb{Z}^d. \end{aligned} \tag{4.10}$$

From the chain rule and (4.6), it follows that

$$\frac{1}{\tau^{(k+d)/\ell}} H_k\Big(\frac{x}{\tau^{1/\ell}}\Big) = \frac{\partial}{\partial \tau}\left(\frac{1}{\tau^{(k+d)/\ell - 1}} F_k\Big(\frac{x}{\tau^{1/\ell}}\Big)\right), \quad x \in \mathbb{Z}^d \setminus \{0\}. \tag{4.11}$$

Combining (4.10) and (4.11) and taking (3.3) and the inequality $K \geq \ell - d$ into account yield

$$\mathbf{v}(x,t) = \Omega(x) + \sum_{k=0}^{K} \frac{1}{t^{(k+d)/\ell - 1}} F_k\Big(\frac{x}{t^{1/\ell}}\Big) + \mathbf{r}_{\mathbf{v}}(K; x, t), \quad x \in \mathbb{Z}^d \setminus \{0\}, \tag{4.12}$$

with some grid function $\Omega(x)$ and the remainder $\mathbf{r}_{\mathbf{v}}(K; x, t)$ satisfying (4.9). This proves (4.8) in the case where $x \neq 0$. If $\ell - d \leq -1$, the same formula for $x = 0$ follows directly from (4.10) by substituting $x = 0$ therein and using (4.7).

To see that $\Omega(x)$ and $\mathbf{r}_{\mathbf{v}}(K; x, t)$ do not depend on $t_0$, denote by $\Omega^*(x)$ and $\mathbf{r}^*_{\mathbf{v}}(K; x, t)$ the functions corresponding to $t_0$ replaced by some $t_0^* > t_0$. Then (4.12) implies that

$$0 = \Omega(x) - \Omega^*(x) + \mathbf{r}_{\mathbf{v}}(K; x, t) - \mathbf{r}^*_{\mathbf{v}}(K; x, t), \quad t \geq t_0^*.$$

Passing to the limit as $t \to \infty$ yields $\Omega(x) = \Omega^*(x)$. In its turn, this and (4.12) imply that $\mathbf{r}_{\mathbf{v}}(K; x, t) = \mathbf{r}^*_{\mathbf{v}}(K; x, t)$ for $t \geq t_0^*$. □

In the case where $\ell - d \geq 0$, we will need modified versions of the functions in (4.4). To introduce them, we denote by $n_\varphi$ the unit vector originated at 0 and pointing in the direction $\varphi$ and by $f^{(j)}(0,\varphi)$ the $j$th directional derivative of a function $f(y)$ in the direction $n_\varphi$ at the origin. Now, for each fixed $\varphi \in \Phi$, we expand $H_k(\rho,\varphi)$ in the Taylor series around $\rho = 0$:

$$H_k(\rho,\varphi) = \sum_{j=0}^{\ell-d-k} \frac{H_k^{(j)}(0,\varphi)}{j!}\rho^j + \widehat{H}_k(\rho,\varphi), \quad k = 0,\dots,\ell-d, \tag{4.13}$$

where

$$|\widehat{H}_k(\rho,\varphi)| \leq c_1 \rho^{\ell-d-k+1}, \quad \rho \in [0,1], \tag{4.14}$$

and $c_1 > 0$ does not depend on $\rho \in [0,1]$ and $\varphi \in \Phi$.

For $\ell - d \geq 0$ and $y > 0$, we set

$$\widehat{F}_k(y) := \ell r^{\ell-d-k} \int_r^1 \frac{\widehat{H}_k(\rho,\varphi)}{\rho^{\ell-d-k+1}}\, d\rho, \quad k = 0,\dots,\ell-d, \tag{4.15}$$

$$\widehat{F}(y) := \ell \sum_{k=0}^{\ell-d} r^{\ell-d-k} \int_0^1 \frac{\widehat{H}_k(\rho,\varphi)}{\rho^{\ell-d-k+1}}\, d\rho, \tag{4.16}$$

$$G_k(y) := \begin{cases} \dfrac{\ell}{\ell-k-d} \displaystyle\sum_{m=0}^{k} \frac{H_m^{(k-m)}(0,\varphi)\, r^{k-m}}{(k-m)!} & \text{if } k = 0,\dots,\ell-d-1, \\ \displaystyle\sum_{m=0}^{\ell-d} \frac{H_m^{(\ell-d-m)}(0,\varphi)\, r^{\ell-d-m}}{(\ell-d-m)!} & \text{if } k = \ell-d, \end{cases} \tag{4.17}$$

$$\begin{aligned} \Omega_k(y) &:= \ell r^{\ell-d-k} \\ &\times \left( \int_1^\infty \frac{H_k(\rho,\varphi)}{\rho^{\ell-d-k+1}}\, d\rho - \sum_{j=0}^{\ell-d-k-1} \frac{H_k^{(j)}(0,\varphi)}{j!(\ell-j-d-k)} - \frac{H_k^{(\ell-d-k)}(0,\varphi)}{(\ell-d-k)!} \ln r \right), \\ &\qquad k = 0,\dots,\ell-d. \end{aligned} \tag{4.18}$$

*Remark* 4.2.

1. The integral in (4.16) converges due to (4.14).
2. The first formula in (4.17) is used only if $\ell - d \geq 1$.
3. In (4.18) and below, we use the convention that the sum vanishes if the upper summation limit is less than the lower limit.

The following assertion follows from (4.14) and the definition of $\widehat{F}_k(y)$, $\widehat{F}(y)$, and $G_k(y)$.

**Lemma 4.3.** *Let $\ell - d \geq 0$.*

1. *The functions $\widehat{F}_k(y)$, $\widehat{F}(y)$, and $G_k(y)$ are bounded near the origin. For each fixed $\varphi \in \Phi$, they are continuous with respect to $r \geq 0$.*
2. *We have*

$$\lim_{t\to\infty} \sum_{k=0}^{\ell-d} t^{1-(d+k)/\ell} \widehat{F}_k\left(\frac{x}{t^{1/\ell}}\right) = \widehat{F}(x), \quad x \in \mathbb{Z}^d \setminus \{0\}.$$

Now, based on Lemma 4.2 in the case where $\ell - d \geq 0$, we find another preliminary asymptotic formula for $\mathbf{v}(x,t)$ in terms of the above functions $F_k(y)$, $\widehat{F}_k(y)$, $\widehat{F}(y)$, $G_k(y)$, and $\Omega_k(y)$. Now the asymptotic formula is extended to $x = 0$, but still involves an unknown grid function $\Omega(x)$, which will be determined in the next section.

**Lemma 4.4.** *Let $\ell - d \geq 0$. Then, for any $t_0 > 0$, any integer $K \geq \max(M, \ell - d)$, and $t \geq t_0$, the following holds.*

1. *If $x \in \mathbb{Z}^d \setminus \{0\}$, then, along with the asymptotics* (4.8)*, we have*

$$\begin{aligned}\mathbf{v}(x,t) = &\sum_{k=0}^{\ell-d-1} t^{1-(d+k)/\ell}\left(\widehat{F}_k\left(\frac{x}{t^{1/\ell}}\right) + G_k(x)\right) + G_{\ell-d}(x)\ln t \\ &+ \widehat{F}_{\ell-d}\left(\frac{x}{t^{1/\ell}}\right) + \widehat{\Omega}(x) + \sum_{k=\ell-d+1}^{K} \frac{1}{t^{(d+k)/\ell-1}} F_k\left(\frac{x}{t^{1/\ell}}\right) + \mathbf{r}_{\mathbf{v}}(K;x,t),\end{aligned} \tag{4.19}$$

*where*

$$\widehat{\Omega}(y) = \Omega(y) + \sum_{k=0}^{\ell-d} \Omega_k(y). \tag{4.20}$$

2. *If $x = 0$, then we have*

$$\begin{aligned}\mathbf{v}(0,t) = &\sum_{k=0}^{\ell-d-1} t^{1-(d+k)/\ell} \frac{\ell\, H_k(0)}{\ell-d-k} + H_{\ell-d}(0)\ln t \\ &+ \omega + \sum_{k=\ell-d+1}^{K} \frac{1}{t^{(d+k)/\ell-1}} \frac{\ell H_k(0)}{\ell-d-k} + \mathbf{r}_{\mathbf{v}}(K;0,t),\end{aligned} \tag{4.21}$$

*where $\omega \in \mathbb{C}$ does not depend on $t_0$ and $t$ and the $F_k(0)$ are defined in* (4.7)*.*

*In both cases, $\mathbf{r}_{\mathbf{v}}(K;x,t)$ satisfies estimate* (4.9)*.*

*Proof.* First, we fix $x \in \mathbb{Z}^d \setminus \{0\}$. Consider the terms in (4.8) corresponding to $k = 0, \dots, \ell - d$. As we will see below, they (in general) tend to infinity as $t \to \infty$. For each $m \in \{0, \dots, \ell - d\}$, we have

$$t^{1-(m+d)/\ell} F_m\left(\frac{x}{t^{1/\ell}}\right) = \ell r^{\ell-d-m} \int_{rt^{-1/\ell}}^{1} \frac{H_m(\rho,\varphi)}{\rho^{\ell-d-m+1}}\, d\rho + \ell r^{\ell-d-m} \int_1^{\infty} \frac{H_m(\rho,\varphi)}{\rho^{\ell-d-m+1}}\, d\rho. \tag{4.22}$$

Consider the first integral in (4.22). Using (4.13), we represent it as follows:

$$\int_{rt^{-1/\ell}}^{1} \frac{H_m(\rho,\varphi)}{\rho^{\ell-d-m+1}}\, d\rho = \sum_{j=0}^{\ell-d-m} \frac{H_m^{(j)}(0,\varphi)}{j!} \int_{rt^{-1/\ell}}^{1} \rho^{j+d+m-\ell-1}\, d\rho + \int_{rt^{-1/\ell}}^{1} \frac{\widehat{H}_m(\rho,\varphi)}{\rho^{\ell-d-m+1}}\, d\rho,$$

which yields

$$\begin{aligned}\int_{rt^{-1/\ell}}^{1} \frac{H_m(\rho,\varphi)}{\rho^{\ell-d-m+1}}\, d\rho = &\sum_{j=0}^{\ell-d-m-1} \frac{H_m^{(j)}(0,\varphi) r^{j+d+m-\ell}}{j!(\ell-j-d-m)}\, t^{1-(j+d+m)/\ell} \\ &+ \frac{H_m^{(\ell-d-m)}(0,\varphi)}{(\ell-d-m)!} \cdot \frac{\ln t}{\ell} - \sum_{j=0}^{\ell-d-m-1} \frac{H_m^{(j)}(0,\varphi)}{j!(\ell-j-d-m)} \\ &- \frac{H_m^{(\ell-d-m)}(0,\varphi)}{(\ell-d-m)!} \ln r + \int_{rt^{-1/\ell}}^{1} \frac{\widehat{H}_m(\rho,\varphi)}{\rho^{\ell-d-m+1}}\, d\rho.\end{aligned}$$

Combining this with (4.22), we obtain

$$\begin{aligned}t^{1-(m+d)/\ell} F_m\left(\frac{x}{t^{1/\ell}}\right) = \ell &\sum_{j=0}^{\ell-d-m-1} \frac{H_m^{(j)}(0,\varphi)\, r^j}{j!(\ell-j-d-m)}\, t^{1-(d+m+j)/\ell} \\ &+ \frac{H_m^{(\ell-d-m)}(0,\varphi)\, r^{\ell-d-m}}{(\ell-d-m)!} \ln t + \Omega_m(x) + t^{1-(m+d)/\ell} \widehat{F}_m\left(\frac{x}{t^{1/\ell}}\right),\end{aligned} \tag{4.23}$$

where the $\widehat{F}_m(y)$ are given by (4.15) and the $\Omega_m(y)$ by (4.18).

Equalities (4.8) and (4.23) imply (4.19) with $G_k(y)$ given by (4.17) and $\widehat{\Omega}(y)$ by (4.20). For $x = 0$, formula (4.21) directly follows from integrating (4.10) with $x = 0$. □

The advantage of the representation (4.19) is that it allows one to distinguish time-independent coefficients at positive powers of $t$ in the asymptotic formula for $\mathbf{v}(x,t)$. Namely, the following assertion holds.

**Corollary 4.1.** *Let the assumptions of Lemma* 4.4 *be fulfilled. Then, for each* $x \in \mathbb{Z}^d \setminus \{0\}$, *we have*

$$\mathbf{v}(x,t) = \sum_{k=0}^{\ell-d-1} t^{1-(d+k)/\ell} G_k(x) + G_{l-d}(x)\ln t + \widehat{F}(x) + \widehat{\Omega}(x) + o(1) \quad \text{as} \quad t \to \infty, \tag{4.24}$$

*where* $G_k(x)$, $\widehat{F}(x)$, *and* $\widehat{\Omega}(x)$ *are given by* (4.17), (4.16), *and* (4.20), *respectively, and* $o(1)$ *depends on* $x$.

*Proof.* The assertion follows from (4.19) and Lemma 4.3 (part 4.3). □

*Remark* 4.3. The asymptotic formulas (4.8) and (4.19) are uniform with respect to $x \in \mathbb{Z}^d \setminus \{0\}$, in the sense that the constant $R_{\mathbf{v}}(K,t_0)$ in estimate (4.9) does not depend on $x \in \mathbb{Z}^d \setminus \{0\}$. Unlike those formulas, the function $o(1)$ tends to 0 as $t \to \infty$ in relation (4.24) uniformly with respect to $x \neq 0$ from any compact set, but *not* uniformly in $\mathbb{Z}^d \setminus \{0\}$.

However, in the next subsection, we will find an asymptotics analogous to (4.24) directly from the integral representation (2.23) of the Green function. In this new asymptotics, the time-independent term will be explicitly present and thus will coincide with $\widehat{F}(x) + \widehat{\Omega}(x)$ from (4.24). Hence, we will find $\widehat{\Omega}(x)$ and thus $\Omega(x)$ and "finalize" the (uniform) asymptotic formulas (4.8) and (4.19).

## §5. Asymptotics of the second Green function $\mathbf{v}^\varepsilon(x,t)$

**5.1. Case $\ell - d \le -1$.** First, we complete Lemma 4.2 by finding $\Omega(x)$ in the case where $\ell - d \le -1$.

**Theorem 5.1.** *Let* $\ell - d \le -1$. *Then, for any* $\varepsilon > 0$, $t_0 > 0$, *any integer* $K \ge M$, *and all* $x \in \mathbb{R}^d_\varepsilon$ *and* $t \ge t_0\varepsilon^\ell$, *we have*

$$\mathbf{v}^\varepsilon(x,t) = \frac{1}{\varepsilon^{d-\ell}}\Omega\left(\frac{x}{\varepsilon}\right) + \frac{1}{t^{d/\ell-1}} F_0\left(\frac{x}{t^{1/\ell}}\right) + \sum_{k=M}^{K} \frac{\varepsilon^k}{t^{(k+d)/\ell-1}} F_k\left(\frac{x}{t^{1/\ell}}\right) + \mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t), \tag{5.1}$$

*where*

$$\Omega(y) := \frac{1}{(2\pi)^d}\int_{R_\pi} \frac{e^{iy\theta}}{A(\theta)}\, d\theta, \tag{5.2}$$

*the* $F_k(y)$ *are given by* (4.5),

$$|\mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t)| \le \frac{\varepsilon^{K+1} R_{\mathbf{v}}(K,t_0)}{t^{(K+d+1)/\ell-1}},$$

*and* $R_{\mathbf{v}}(K,t_0) \ge 0$ *does not depend on* $\varepsilon > 0$, $x \in \mathbb{R}^d_\varepsilon$, *and* $t \ge t_0\varepsilon^\ell$.

*Proof.* Due to (2.24), it suffices to prove the theorem for $\varepsilon = 1$ and $\mathbf{v}(x,t) = \mathbf{v}^1(x,t)$.

Since $\ell - d \le 0$, from (4.7) it follows that

$$\left|\frac{1}{t^{(k+d)/\ell-1}} F_k\left(\frac{x}{t^{1/\ell}}\right)\right| \to 0 \quad \text{as} \quad t \to \infty, \quad k = 0,1,2,\ldots.$$

Therefore, by Lemma 4.2,

$$\Omega(x) = \lim_{t\to\infty} \mathbf{v}(x,t), \quad x \in \mathbb{Z}^d.$$

On the other hand, the representation (2.23) with $\varepsilon = 1$, estimate (2.19), and Lebesgue's dominated convergence theorem imply that

$$\lim_{t\to\infty} \mathbf{v}(x,t) = \frac{1}{(2\pi)^d} \int_{R_\pi} \frac{e^{iy\theta}}{A(\theta)}\, d\theta, \quad x \in \mathbb{Z}^d,$$

where the integral is absolutely convergent for $\ell - d \le -1$ due to (2.19). The last two identities, combined with Lemma 4.2 and Remark 4.1, complete the proof. □

As in Corollaries 3.1 and 3.2, Theorem 5.1 and Lemma 4.1, item 3, imply that the higher approximation order $M$ (see Definition 2.3) we choose, the faster the second Green function converges to the main term in its asymptotic representation, as $t \to \infty$ or as $\varepsilon \to 0$.

**5.2. Case $\ell - d \ge 0$.** First, we take $\varepsilon = 1$ and fix an arbitrary $x \in \mathbb{Z}^d$. Using the representation (2.23) with $\varepsilon = 1$, we write

$$\mathbf{v}(x,t) = \frac{1}{(2\pi)^d}\big(\mathbf{v}_1(x,t) + \mathbf{v}_2(x,t) - \mathbf{v}_3(x,t)\big), \tag{5.3}$$

$$\begin{aligned} &\mathbf{v}_1(x,t) := \int_{B_{t^{-1/\ell}}} \frac{1-e^{-tA(\theta)}}{A(\theta)} e^{ix\theta}\, d\theta, \quad \mathbf{v}_2(x,t) := \int_{R_\pi \setminus B_{t^{-1/\ell}}} \frac{e^{ix\theta}}{A(\theta)}\, d\theta, \\ &\mathbf{v}_3(x,t) := \int_{R_\pi \setminus B_{t^{-1/\ell}}} \frac{e^{-tA(\theta)}}{A(\theta)} e^{ix\theta}\, d\theta, \end{aligned} \tag{5.4}$$

where $B_\lambda := \{\theta \in \mathbb{R}^d : |\theta| < \lambda\}$ and $R_\pi$ is the cube defined by (1.11).

**1. Estimate of $\mathbf{v}_1(x,t)$.** Making the change of variables $\theta = t^{-1/\ell}\xi$, we obtain

$$\mathbf{v}_1(t) = t^{-d/\ell} \int_{B_1} \frac{1-e^{-tA(t^{-1/\ell}\xi)}}{A(t^{-1/\ell}\xi)} e^{ixt^{-1/\ell}\xi}\, d\xi. \tag{5.5}$$

Using the Taylor expansion for $e^{ix\theta}$ and formula (2.15), we represent

$$\frac{e^{ix\theta}}{A(\theta)} = \sum_{k=0}^{\ell-d} Q_{k-\ell}(\theta,x) + \widehat{Q}(\theta,x), \tag{5.6}$$

where $Q_{-\ell}(\theta,x) = \frac{1}{\mathcal{A}(\theta)}$, $Q_{k-\ell}(\theta,x)$ are homogeneous functions of degree $k-\ell$ with respect to $\theta$ representable as ratios of homogeneous polynomials, and there is $c_1 = c_1(x) > 0$ such that

$$|\widehat{Q}(\theta,x)| \le c_1 |\theta|^{-d+1}, \quad \theta \in R_\pi. \tag{5.7}$$

Relations (5.6) and (5.7) yield

$$\frac{e^{ixt^{-1/\ell}\xi}}{A(t^{-1/\ell}\xi)} = \sum_{k=0}^{\ell-d} t^{-(k-\ell)/\ell} Q_{k-\ell}(\xi,x) + \widehat{Q}(t^{-1/\ell}\xi,x), \tag{5.8}$$

where

$$|\widehat{Q}(t^{-1/\ell}\xi,x)| \le c_1 t^{(d-1)/\ell} |\xi|^{-d+1}, \quad \xi \in R_{\pi t^{1/\ell}}, \ t \ge t_0. \tag{5.9}$$

On the other hand, due to (3.8) and (3.11),

$$1 - e^{-tA(t^{-1/\ell}\xi)} = 1 - e^{-\mathcal{A}(\xi)} - e^{-\mathcal{A}(\xi)}\Big(\sum_{m=M}^{K} t^{-m/\ell}\mathcal{P}_{m+\ell}(\xi) + \widehat{P}(\xi,t)\Big), \tag{5.10}$$

where

$$|\widehat{P}(\xi,t)| \le c_2 t^{-(K+1)/\ell} |\xi|^{K+1+\ell}, \quad \xi \in B_1, \ t \ge t_0. \tag{5.11}$$

Combining (5.5) with (5.8)–(5.11) and recalling that $K \geq \ell - d$, we obtain

$$\mathbf{v}_1(x,t) = \sum_{k=0}^{\ell-d} t^{1-(k+d)/\ell} \int_{B_1} \big(1 - e^{-\mathcal{A}(\xi)}\big) Q_{k-\ell}(\xi, x)\, d\xi$$
$$- \sum_{k=0}^{\ell-d-M} \sum_{m=M}^{K} t^{1-(k+m+d)/\ell} \int_{B_1} e^{-\mathcal{A}(\xi)} Q_{k-\ell}(\xi,x) \mathcal{P}_{m+\ell}(\xi)\, d\xi + o(1) \quad \text{as } t \to \infty.$$

Therefore, explicitly writing a coefficient of the zeroth power of $t$, we have

$$\begin{aligned} \mathbf{v}_1(x,t) = \mathbf{v}_1^*(x,t) &+ \int_{B_1} \big(1 - e^{-\mathcal{A}(\xi)}\big) Q_{-d}(\xi, x)\, d\xi \\ &- \sum_{k=0}^{\ell-d-M} \int_{B_1} e^{-\mathcal{A}(\xi)} Q_{k-\ell}(\xi,x) \mathcal{P}_{2\ell-d-k}(\xi)\, d\xi + o(1) \ \text{ as } t \to \infty, \end{aligned} \tag{5.12}$$

where $\mathbf{v}_1^*(x,t)$ is a linear combination of the positive powers $t^{1-(k+d)/\ell}$ for $k = 0, \dots, \ell - d - 1$ with $x$-dependent coefficients.

**2. Estimate of $\mathbf{v}_2(x,t)$.** We write $Q_{k-\ell}(\theta, x)$ (defined in (5.6)) in the spherical coordinates $(r, \varphi)$ with respect to $\theta$:

$$Q_{k-\ell}(\theta, x) = r^{k-\ell} \widetilde{Q}_{k-\ell}(\varphi, x) \tag{5.13}$$

and note that the $\widetilde{Q}_{k-\ell}(\varphi, x)$ are infinitely differentiable with respect to $\varphi$ and $x$. Recall the function $r_\pi(\varphi)$ defined in (1.12). Then, using (5.4), (5.6), and (5.13), we obtain

$$\mathbf{v}_2(x,t) = \sum_{k=0}^{\ell-d} \int_\Phi \widetilde{Q}_{k-\ell}(\varphi, x) J(\varphi)\, d\varphi \int_{t^{-1/\ell}}^{r_\pi(\varphi)} r^{d+k-\ell-1}\, dr + \int_{R_\pi \setminus B_{t^{-1/\ell}}} \widehat{Q}(\theta, x)\, d\theta.$$

Since $\widehat{Q}(\theta, x)$ is integrable at the origin due to (5.7), we calculate (as $t \to \infty$)

$$\begin{aligned} \mathbf{v}_2(x,t) = \mathbf{v}_2^*(x,t) &+ \frac{\ln t}{\ell} \int_\Phi \widetilde{Q}_{-d}(\varphi, x) J(\varphi)\, d\varphi \\ &- \sum_{k=0}^{\ell-d-1} \frac{1}{\ell - d - k} \int_\Phi (r_\pi(\varphi))^{d+k-\ell} \widetilde{Q}_{k-\ell}(\varphi, x) J(\varphi)\, d\varphi \\ &+ \int_{R_\pi} \widehat{Q}(\theta, x)\, d\theta + \int_\Phi \ln(r_\pi(\varphi)) \widetilde{Q}_{-d}(\varphi, x) J(\varphi)\, d\varphi + o(1), \end{aligned} \tag{5.14}$$

where $\mathbf{v}_2^*(x,t)$ is a linear combination of the positive powers $t^{1-(k+d)/\ell}$ for $k = 0, \dots, \ell - d - 1$ with $x$-dependent coefficients.

**3. Estimate of $\mathbf{v}_3(x,t)$.** Making the change of variables $\theta = t^{-1/\ell}\xi$, from (5.4) we obtain

$$\mathbf{v}_3(t) = t^{-d/\ell} \int_{R_{\pi t^{1/L}} \setminus B_1} \frac{e^{-tA(t^{-1/\ell}\xi)}}{A(t^{-1/\ell}\xi)} e^{ixt^{-1/\ell}\xi} d\xi + t^{-d/\ell} \int_{R_{\pi t^{1/\ell}} \setminus R_{\pi t^{1/L}}} \frac{e^{-tA(t^{-1/\ell}\xi)}}{A(t^{-1/\ell}\xi)} e^{ixt^{-1/\ell}\xi} d\xi,$$

where $L$ is defined by (3.7). Therefore, using (2.19), as in (3.22) we obtain

$$\mathbf{v}_3(t) = t^{-d/\ell} \int_{R_{\pi t^{1/L}} \setminus B_1} \frac{e^{-tA(t^{-1/\ell}\xi)}}{A(t^{-1/\ell}\xi)} e^{ixt^{-1/\ell}\xi}\, d\xi + o(1) \quad \text{as } t \to \infty. \tag{5.15}$$

Using (5.15), (3.8), (3.11), and (5.8), we have

$$\mathbf{v}_3(x,t) = t^{-d/\ell}\int_{R_{\pi t^{1/L}}\setminus B_1} e^{-\mathcal{A}(\xi)}\Big(1+\sum_{m=M}^{K} t^{-m/\ell}\mathcal{P}_{m+\ell}(\xi)+\widehat{P}(\xi,t)\Big)$$
$$\times\Big(\sum_{k=0}^{\ell-d} t^{-(k-\ell)/\ell}Q_{k-\ell}(\xi,x)+\widehat{Q}(t^{-1/\ell}\xi,x)\Big)\,d\xi+o(1).$$

Taking estimates (3.12) and (5.9) into account and recalling that $K\ge \ell-d$, we arrive at

$$\begin{aligned}\mathbf{v}_3(x,t) &= \int_{\mathbb{R}^d\setminus B_1} e^{-\mathcal{A}(\xi)}\Big(1+\sum_{m=M}^{K} t^{-m/\ell}\mathcal{P}_{m+\ell}(\xi)\Big)\Big(\sum_{k=0}^{\ell-d} t^{1-(k+d)/\ell}Q_{k-\ell}(\xi,x)\Big)d\xi+o(1)\\ &= \mathbf{v}_3^*(x,t)+\int_{\mathbb{R}^d\setminus B_1} e^{-\mathcal{A}(\xi)}Q_{-d}(\xi,x)\,d\xi\\ &\quad+\sum_{k=0}^{\ell-d-M}\int_{\mathbb{R}^d\setminus B_1} e^{-\mathcal{A}(\xi)}Q_{k-\ell}(\xi,x)\mathcal{P}_{2\ell-d-k}(\xi)\,d\xi+o(1),\end{aligned}\tag{5.16}$$

where $\mathbf{v}_3^*(x,t)$ is a linear combination of the positive powers $t^{1-(k+d)/\ell}$ for $k=0,\dots,\ell-d-1$ with $x$-dependent coefficients.

**4. Estimate of $\mathbf{v}(x,t)$.** Combining (5.3) with (5.12), (5.14), and (5.16), for each $x\in\mathbb{Z}^d$ we have

$$\mathbf{v}(x,t)=\mathbf{v}^*(x,t)+\frac{\ln t}{(2\pi)^d\ell}\int_\Phi \widetilde{Q}_{-d}(\varphi,x)J(\varphi)\,d\varphi+S(x)+o(1),\tag{5.17}$$

where $\mathbf{v}^*(x,t)$ is a linear combination of the positive powers $t^{1-(k+d)/\ell}$ for $k=0,\dots,\ell-d-1$ with $x$-dependent coefficients and

$$\begin{aligned}S(y) := \frac{1}{(2\pi)^d}\Big(&\int_{B_1}\big(1-e^{-\mathcal{A}(\xi)}\big)Q_{-d}(\xi,y)\,d\xi-\int_{\mathbb{R}^d\setminus B_1} e^{-\mathcal{A}(\xi)}Q_{-d}(\xi,y)\,d\xi\\ &+\int_{R_\pi}\widehat{Q}(\theta,y)\,d\theta-\sum_{k=0}^{\ell-d-M}\int_{\mathbb{R}^d} e^{-\mathcal{A}(\xi)}Q_{k-\ell}(\xi,y)\mathcal{P}_{2\ell-d-k}(\xi)\,d\xi\\ &-\sum_{k=0}^{\ell-d-1}\frac{1}{\ell-d-k}\int_\Phi (r_\pi(\varphi))^{d+k-\ell}\widetilde{Q}_{k-\ell}(\varphi,y)J(\varphi)\,d\varphi\\ &+\int_\Phi \ln(r_\pi(\varphi))\widetilde{Q}_{-d}(\varphi,y)J(\varphi)\,d\varphi\Big).\end{aligned}\tag{5.18}$$

Now we are in a position to formulate our main result in the case where $\ell-d\ge 0$.

**Theorem 5.2.** *Let $\ell-d\ge 0$. Then, for any $\varepsilon>0$, $t_0>0$, any integer $K\ge \max(M,\ell-d)$, and all $t\ge t_0\varepsilon^\ell$, the following holds.*

1. *If $x\in\mathbb{R}^d_\varepsilon\setminus\{0\}$, then*

$$\begin{aligned}\mathbf{v}^\varepsilon(x,t) &= t^{1-d/\ell}\,F_0\left(\frac{x}{t^{1/\ell}}\right)+\sum_{k=M}^{\ell-d}\varepsilon^k t^{1-(k+d)/\ell}\,F_k\left(\frac{x}{t^{1/\ell}}\right)+\varepsilon^{\ell-d}\,\Omega\left(\frac{x}{\varepsilon}\right)\\ &\quad+\sum_{k=\max(M,\ell-d+1)}^{K}\frac{\varepsilon^k}{t^{(k+d)/\ell-1}}\,F_k\left(\frac{x}{t^{1/\ell}}\right)+\mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t),\end{aligned}\tag{5.19}$$

*where the $F_k(y)$ are given by* (4.4) *and* (4.5)*, respectively,*

$$\Omega(y) = S(y) - \widehat{F}(y) - \sum_{k=0}^{\ell-d} \Omega_k(y),$$

*$S(y)$ is given by* (5.18)*, $\widehat{F}(y)$ by* (4.16)*, and $\Omega_k(y)$ by* (4.18)*.*

2. *If $x = 0$, then*

$$\begin{aligned}\mathbf{v}^\varepsilon(0,t) = \sum_{k=0}^{\ell-d-1} \varepsilon^k t^{1-(d+k)/\ell} \frac{\ell\, H_k(0)}{\ell-d-k} + \varepsilon^{\ell-d}\big(H_{\ell-d}(0)\ln t - \ell H_{\ell-d}(0)\ln\varepsilon + S(0)\big) \\ + \sum_{k=\ell-d+1}^{K} \frac{\varepsilon^k}{t^{(d+k)/\ell-1}} \frac{\ell\, H_k(0)}{\ell-d-k} + \mathbf{r}^\varepsilon_{\mathbf{v}}(K;0,t),\end{aligned} \tag{5.20}$$

*where $S(0)$ is given by* (5.18) *with $y = 0$.*

*In both cases,*

$$|\mathbf{r}^\varepsilon_{\mathbf{v}}(K;x,t)| \le \frac{\varepsilon^{K+1} R_{\mathbf{v}}(K,t_0)}{t^{(K+d+1)/\ell-1}}$$

*and $R_{\mathbf{v}}(K,t_0) \ge 0$ does not depend on $\varepsilon > 0$, $x \in \mathbb{R}^d_\varepsilon$, and $t \ge t_0\varepsilon^\ell$.*

*Proof.* The result follows from Lemmas 4.2 and 4.4, Corollary 4.1, and formula (5.17). □

*Remark* 5.1. Using Theorems 3.1 and 5.2, one can easily obtain asymptotic formulas for the first-order "spatial derivatives" $\varepsilon^{-1}\delta_{\varepsilon,k\pm}\mathbf{u}^\varepsilon(x,t)$ and $\varepsilon^{-1}\delta_{\varepsilon,k\pm}\mathbf{v}^\varepsilon(x,t)$, $k = 1,\dots,d$, and for their higher-order analog. To do so, one should use Taylor's expansions around $x/t^{1/\ell}$ of the functions entering formulas (3.1) and (5.19). We will not provide this result in full generality, but rather present it for a one-dimensional example in Section 6.5.

## §6. Higher order approximations of the 1D Laplacian

In this section, we illustrate our general results by applying them to the one-dimensional discrete Laplace operator. Throughout this section, we fix $N \in \mathbb{N}$. In what follows, we will not explicitly indicate the dependence of constants, functions, etc. on $N$.

**6.1. List of constants and functions.** For asymptotic formulas below, we will explicitly find all the coefficients. In this subsection, we collect the constants and the functions which are needed for that.

For an integer $J \ge 0$, set

$$b_n := \frac{2(-1)^{n+1}}{(2(n+1))!} \sum_{\nu=1}^{N} a_\nu \nu^{2(n+1)}, \qquad n \ge N, \tag{6.1}$$

$$c_{nm} := \sum_{\substack{l_1,\dots,l_m \ge N,\\ l_1+\dots+l_m=n}} b_{l_1}\cdot\dots\cdot b_{l_m}, \qquad n \ge N,\ m \ge 1, \tag{6.2}$$

$$\mathcal{P}_n(\xi) := \sum_{1\le m\le n/N} \frac{c_{nm}\xi^{2(n+m)}}{m!}, \qquad n \ge N, \tag{6.3}$$

$$d_{Jn} := \sum_{1\le m\le \min(n/N,J)} (-1)^m \binom{J}{m} c_{nm}, \qquad n \ge N, \tag{6.4}$$

$$\mathcal{Q}_{Jn}(\xi) := 0, \qquad n = N,\dots,2N-1, \tag{6.5}$$

$$\mathcal{Q}_{Jn}(\xi) := \sum_{\substack{l_1,l_2\ge N,\\ l_1+l_2=n}} d_{Jl_1}\xi^{2l_1}\mathcal{P}_{l_2}(\xi), \qquad n \ge 2N, \tag{6.6}$$

$$\mathcal{R}_{Jn}(\xi) := (-1)^J \xi^{2J}(d_{Jn}\xi^{2n} + \mathcal{P}_n(\xi) + \mathcal{Q}_{Jn}(\xi)), \qquad n \ge N. \tag{6.7}$$

As before, we use the convention that the sum vanishes if the set over which the summation is taken is empty.

**6.2. Setting.** Let $\mathbf{u}^\varepsilon(x)$, $x \in \mathbb{R}^1_\varepsilon$, be a grid function. We consider the difference operator

$$\mathbf{A}_\varepsilon = -\boldsymbol{\Delta}_\varepsilon, \quad \boldsymbol{\Delta}_\varepsilon \mathbf{u}(x) := \varepsilon^{-2} \sum_{\nu=1}^{N} a_\nu \big(\mathbf{u}(x-\varepsilon\nu) - 2\mathbf{u}(x) + \mathbf{u}(x+\varepsilon\nu)\big), \tag{6.8}$$

with the coefficients $a_\nu$ satisfying the relations

$$\sum_{\nu=1}^{N} a_\nu \nu^2 = 1, \quad \sum_{\nu=1}^{N} a_\nu \nu^{2m} = 0, \quad m = 2, \dots, N. \tag{6.9}$$

The matrix of system (6.9) is the Vandermonde matrix, which guarantees the existence and uniqueness of the $a_\nu$. Thus, throughout this section, $\ell = 2$ and $d = 1$.

The following lemma shows that Condition 2.2 is fulfilled with $M = 2N$.

**Lemma 6.1.** *The operator $\boldsymbol{\Delta}_\varepsilon$ is the $(2N)$th order approximation of the second derivative (the one-dimensional Laplacian) in the sense of Definition* 2.3.

*Proof.* For any smooth function $u(y)$ and $\varepsilon > 0$, relations (6.9) imply

$$\begin{aligned}
&\varepsilon^{-2} \sum_{\nu=1}^{N} a_\nu \big(u(y-\varepsilon\nu) - 2u(y) + u(y+\varepsilon\nu)\big) \\
&\quad = \varepsilon^{-2} \sum_{\nu=1}^{N} a_\nu \Bigg( \sum_{m=1}^{N} \frac{2(\varepsilon\nu)^{2m}}{(2m)!} u^{(2m)}(y) \\
&\qquad\qquad + \frac{(\varepsilon\nu)^{2N+2}}{(2N+2)!} \big(u^{(2N+2)}(y - \xi_{1\nu}\varepsilon\nu) + u^{(2N+2)}(y + \xi_{2\nu}\varepsilon\nu)\big) \Bigg) \\
&\quad = \varepsilon^{-2} \Bigg( \sum_{m=1}^{N} u^{(2m)}(y) \frac{2\varepsilon^{2m}}{(2m)!} \sum_{\nu=1}^{N} a_\nu \nu^{2m} \Bigg) + \varepsilon^{2N} R_N(y) = u''(y) + \varepsilon^{2N} R_N(y),
\end{aligned}$$

where

$$R_N(y) = \sum_{\nu=1}^{N} \frac{\nu^{2N+2}}{(2N+2)!} \Big( u^{(2N+2)}(y - \xi_{1\nu}\varepsilon\nu) + u^{(2N+2)}(y + \xi_{2\nu}\varepsilon\nu) \Big), \qquad \xi_{1\nu}, \xi_{2\nu} \in [0,1]. \qquad \square$$

Due to Definition 2.4, the symbol of the operator $\mathbf{A}_\varepsilon = -\boldsymbol{\Delta}_\varepsilon$ is given by

$$A(\theta) = 2 \sum_{\nu=1}^{N} a_\nu (1 - \cos \nu\theta). \tag{6.10}$$

The following lemma shows that Condition 2.3 is fulfilled. The proof is given in Appendix A.

**Lemma 6.2.** *Let $A(\theta)$ be given by* (6.10). *Then*

(1) $A'(0) = A'(\pi) = 0$ *and* $A'(\theta) > 0$ *for all* $\theta \in (0, \pi)$;
(2) $A(0) = 0$ *and* $A(\theta) > 0$ *for all* $\theta \in (0, \pi]$;
(3) $A(\theta) = \theta^2 - \sum_{n=N}^{K} b_n \theta^{2n+2} + O(\theta^{2(K+2)})$, *where the $b_n$ are given by* (6.1).

From now on, we confine ourselves to the case of $\varepsilon = 1$. The case of $\varepsilon > 0$ can easily be obtained by applying the scaling rule (2.24). Let $\boldsymbol{\delta}(x) := \boldsymbol{\delta}^1(x)$, $x \in \mathbb{Z}$, be the grid delta-function defined in (1.2) with $\varepsilon = 1$.

**Definition 6.1.** Let $\varepsilon = 1$. We call the function $\mathbf{u}(x,t)$, $x \in \mathbb{Z}$, $t \geq 0$, the *first discrete Green function* if $\mathbf{u}(\,\cdot\,,t)$ is a rapidly decreasing grid function for all $t \geq 0$, $\mathbf{u}(x,\,\cdot\,) \in C^1[0,\infty)$ for all $x \in \mathbb{Z}$, and

$$\begin{cases} \dot{\mathbf{u}}(x,t) - \boldsymbol{\Delta}_1 \mathbf{u}(x,t) = 0, & x \in \mathbb{Z},\ \ t > 0, \\ \mathbf{u}(x,0) = \boldsymbol{\delta}(x), & x \in \mathbb{Z}. \end{cases}$$

We call the function $\mathbf{v}(x,t)$, $x \in \mathbb{Z}$, $t \geq 0$, the *second discrete Green function* if $\mathbf{v}(\,\cdot\,,t)$ is a rapidly decreasing grid function for all $t \geq 0$, $\mathbf{v}(x,\,\cdot\,) \in C^1[0,\infty)$ for all $x \in \mathbb{Z}$, and

$$\begin{cases} \dot{\mathbf{v}}(x,t) - \boldsymbol{\Delta}_1 \mathbf{v}(x,t) = \boldsymbol{\delta}(x), & x \in \mathbb{Z},\ \ t > 0, \\ \mathbf{v}(x,0) = 0, & x \in \mathbb{Z}. \end{cases}$$

**6.3. Asymptotics of the first Green function $\mathbf{u}(x,t)$.** We begin with asymptotic formulas for the first Green function $\mathbf{u}(x,t)$. Consider the function

$$h(y) := \frac{1}{2\sqrt{\pi}}\, e^{-y^2/4}, \tag{6.11}$$

which coincides with $H(y)$ defined in (2.7). Note that $\frac{1}{\sqrt{t}} h(y)$ is the Green function of the operator $u_t(y,t) - u_{yy}(y,t)$ (cf. (2.6)).

**Theorem 6.1.** *For any $t_0 > 0$, integers $J \geq 0$ and $K_1 \geq N$, and all $x \in \mathbb{Z}$ and $t \geq t_0$, we have*

$$\frac{d^J \mathbf{u}(x,t)}{dt^J} = \frac{1}{t^J\sqrt{t}} h_{J0}\Big(\frac{x}{\sqrt{t}}\Big) + \sum_{n=N}^{K_1} \frac{1}{t^{n+J}\sqrt{t}}\, h_{Jn}\Big(\frac{x}{\sqrt{t}}\Big) + \mathbf{r}_{\mathbf{u}}(J,K_1;x,t), \tag{6.12}$$

*where*

$$h_{J0}(x) := \frac{d^{2J} h(x)}{dx^{2J}}, \quad h_{Jn}(x) := \mathcal{R}_{Jn}\Big(-i\frac{d}{dx}\Big) h(x), \tag{6.13}$$

$$|\mathbf{r}_{\mathbf{u}}(J,K_1;x,t)| \leq \frac{R_{\mathbf{u}}(J,K_1,t_0)}{t^{J+K_1+1}\sqrt{t}},$$

*the $\mathcal{R}_{Jn}(\xi)$ are polynomials given by* (6.7)*, and $R_{\mathbf{u}}(J,K_1,t_0) \geq 0$ does not depend on $t \geq t_0$ and $x \in \mathbb{Z}$.*

*Proof.* Due to Lemma 6.2, part 3, the Taylor expansion of the symbol $A(\theta)$ contains only even powers of $\theta$. This fact and Theorem 3.1 with $M = 2N$, $K = 2K_1$, and $k = 2n$ imply the result. □

**6.4. Asymptotics of the second Green function $\mathbf{v}(x,t)$.** For $y \geq 0$, we introduce the functions

$$f_0(y) = 2y \int_y^\infty \rho^{-2} h(\rho)\, d\rho, \tag{6.14}$$

$$f_n(y) = -\frac{2}{y^{2n-1}} \int_0^y \rho^{2n-2} h_{0n}(\rho)\, d\rho, \quad n = 1,2,\ldots. \tag{6.15}$$

Note that

$$f_0''(y) = h(y), \quad y > 0, \tag{6.16}$$

where $h(y)$ is given by (6.11). Furthermore, $f_0(y)$ is positive, real analytic for[1] $y \geq 0$ (use e.g., (6.16)), and vanishes at infinity, together with all its derivatives.

Using (6.11) and (6.13), we see that the $f_n(y)$ are real analytic for $y \geq 0$ and vanish at infinity, together with all their derivatives.

[1]Analyticity at $y = 0$ is understood in the sense that $f(y)$ can be represented as the Taylor series with respect to the powers of $y$ which converges to $f(y)$ in a right-hand neighborhood of $y = 0$.

**Theorem 6.2.** *For any $t_0 > 0$, any integer $K_1 \geq N$, and all $x \in \mathbb{Z}$ and $t \geq t_0$, we have*

$$\begin{aligned} &\mathbf{v}(x,t) = \sqrt{t}\, f_0\Big(\frac{x}{\sqrt{t}}\Big) + \Omega(x) + \sum_{n=N}^{K_1} \frac{1}{t^{n-1}\sqrt{t}}\, f_n\Big(\frac{x}{\sqrt{t}}\Big) + \mathbf{r}_{\mathbf{v}}(K_1; x, t), \quad x \geq 0, \\ &\mathbf{v}(x,t) = \mathbf{v}(-x,t), \quad x \leq -1, \end{aligned} \tag{6.17}$$

*where*

$$\Omega(x) = \frac{1}{2\pi}\bigg( \int_{-\pi}^{\pi} \Big(\frac{\cos(x\xi)}{A(\xi)} - \frac{1}{\xi^2}\Big) d\xi - \frac{2}{\pi}\bigg) + \frac{x}{2}, \tag{6.18}$$

*$f_0(y), f_1(y), \ldots, f_{K_1}(y)$ are given by* (6.14), (6.15),

$$|\mathbf{r}_{\mathbf{v}}(K_1; x, t)| \leq \frac{R_{\mathbf{v}}(K_1, t_0)}{t^{K_1}\sqrt{t}},$$

*and $R_{\mathbf{v}}(K_1, t_0) \geq 0$ does not depend on $t \geq t_0$ and $x \geq 0$.*

*Moreover, if $N = 1$, then $\Omega(x) = 0$ for all integers $x \geq 0$.*

*Proof.* Since $v(x,t) \equiv v(-x,t)$, $x \in \mathbb{Z}$, it remains to prove the first formula in (6.17). Due to Lemma 6.2, part 3, the Taylor expansion of the symbol $A(\theta)$ contains only even powers of $\theta$. Therefore, $H_k(y) \equiv H_{0k}(y) \equiv 0$ for odd $k$. Hence, $F_k(y) \equiv 0$ for odd $k$, where the $F_k(y)$ are given by (4.5). On the other hand, relations (6.14), (6.15) and (4.4), (4.5), imply that

$$f_0(y) \equiv F_0(y), \quad f_n(y) \equiv F_{2n}(y), \quad n \geq 1. \tag{6.19}$$

Thus, applying Theorem 5.2 with $\varepsilon = 1$, $M = 2N$, $K = 2K_1$, and $k = 2n$, we obtain the first formula in (6.17) with $\Omega(x)$ given by

$$\begin{aligned} \Omega(x) = \frac{1}{2\pi}\bigg( &\int_{-\pi}^{\pi} \left(\frac{\cos(x\xi)}{A(\xi)} - \frac{1}{\xi^2}\right) d\xi - \frac{2}{\pi}\bigg) \\ &- 2x \int_0^1 \frac{h(\rho) - h(0)}{\rho^2}\, d\rho - 2x\bigg( \int_1^\infty \frac{h(\rho)}{\rho^2}\, d\rho - h(0)\bigg). \end{aligned} \tag{6.20}$$

Note that

$$-2\int_0^1 \frac{h(\rho) - h(0)}{\rho^2}\, d\rho - 2\bigg( \int_1^\infty \frac{h(\rho)}{\rho^2}\, d\rho - h(0)\bigg) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\rho^2/4} - 1}{\rho^2}\, d\rho = \frac{1}{2},$$

where the last identity can be proved, e.g., by introducing a parameter $a$ as follows: $e^{-a\rho^2/4}$ and differentiating with respect to it. Hence, (6.20) yields (6.18).

Finally, if $N = 1$, then $\Omega(x) = 0$ for all $x \in \mathbb{Z}$ by Lemma B.1. □

**6.5. Asymptotics of the gradient of the Green functions.** Next, we derive asymptotic formulas for $\nabla\mathbf{v}(x,t) := \mathbf{v}(x+1,t) - \mathbf{v}(x,t)$ and its time derivatives, see Remark 5.1. We introduce the function

$$g(x) := f_0'(x), \quad x \geq 0, \tag{6.21}$$

where $f_0(x)$ is given by (6.14). Note that $g(x)$ is negative, real analytic for $x \geq 0$, and vanishes at infinity, together with all its derivatives.

Furthermore, for $n \geq 2$, we set

$$g_n(x) := \frac{1}{(n+1)!} f_0^{(n+1)}(x) + \sum_{\substack{s \geq N,\, j \geq 1, \\ 2s+j-1=n}} \frac{1}{j!} f_s^{(j)}(x), \quad x \geq 0, \tag{6.22}$$

where $f_s(x)$ are given by (6.15). Note that the $f_s(x)$ are real analytic for $x \geq 0$ and vanish at infinity, together with all their derivatives. Therefore, expanding the functions $f_s$ around $x/\sqrt{t}$ in (6.17), we arrive at the following theorem.

**Theorem 6.3.** *For any $t_0 > 0$, any integer $K_1 \geq 2$, and all $x \in \mathbb{Z}$ and $t \geq t_0$, we have*

$$\begin{aligned} \nabla \mathbf{v}(x,t) = {} & \nabla\Omega(x) + g\Big(\frac{x}{\sqrt{t}}\Big) + \frac{1}{2\sqrt{t}}\, h\Big(\frac{x}{\sqrt{t}}\Big) \\ & + \sum_{n=2}^{K_1} \frac{1}{t^{n/2}}\, g_n\Big(\frac{x}{\sqrt{t}}\Big) + \widehat{\mathbf{r}}_{\mathbf{v}}(K_1;x,t), \quad x \geq 0, \\ \nabla \mathbf{v}(x,t) = {} & -\nabla \mathbf{v}(-(x+1),t), \quad x \leq -1, \end{aligned} \tag{6.23}$$

*where $\Omega(x)$ is given by* (6.18), *$g_2(x), \dots, g_{K_1}(x)$ are given by* (6.22),

$$|\widehat{\mathbf{r}}_{\mathbf{v}}(K_1;x,t)| \leq \frac{\widehat{R}_{\mathbf{v}}(K_1,t_0)}{t^{(K_1+1)/2}},$$

*and $\widehat{R}_{\mathbf{v}}(K_1,t_0) \geq 0$ does not depend on $t \geq t_0$ and $x \geq 0$.*

*Moreover, if $N = 1$, then $\nabla\Omega(x) = 0$ for all integers $x \geq 0$.*

Finally, for $n \geq 2$, we set

$$g_{Jn}(x) := \frac{1}{(n+1)!} h_{J0}^{(n+1)}(x) + \sum_{\substack{s \geq N,\, j \geq 1,\\ 2s+j-1=n}} \frac{1}{j!} h_{Js}^{(j)}(x), \quad x \geq 0, \tag{6.24}$$

where $h_{J0}(y)$ and $h_{Js}(y)$ are given by (6.13). Note that $h_{J0}(y)$ and $h_{Js}(y)$ are real analytic for $x \geq 0$ and vanish at infinity, together with all their derivatives. Therefore, expanding the functions $h_{J0}$ and $h_{Js}$ around $x/\sqrt{t}$ in (6.12), we arrive at the following theorem.

**Theorem 6.4.** *For any $t_0 > 0$, any integers $J \geq 0$ and $K_1 \geq 2$, and all $x \in \mathbb{Z}$ and $t \geq t_0$, we have*

$$\begin{aligned} \frac{d^J \nabla \mathbf{u}(x,t)}{dt^J} = {} & \frac{1}{t^{J+1}} h^{(2J+1)}(x) + \frac{1}{2t^{J+1}\sqrt{t}} h^{(2J+2)}(x) \\ & + \sum_{n=2}^{K_1} \frac{1}{t^{J+1+n/2}}\, g_{Jn}\Big(\frac{x}{\sqrt{t}}\Big) + \widehat{\mathbf{r}}_{\mathbf{u}}(J,K_1;x,t), \quad x \geq 0, \\ \frac{d^J \nabla \mathbf{u}(x,t)}{dt^J} = {} & -\frac{d^J \nabla \mathbf{u}(-(x+1),t)}{dt^J}, \quad x < 0, \end{aligned} \tag{6.25}$$

*where $g_{J2}(x), \dots, g_{JK_1}(x)$ are given by* (6.24); *furthermore,*

$$|\widehat{\mathbf{r}}_{\mathbf{u}}(J,K_1;x,t)| \leq \frac{\widehat{R}_{\mathbf{u}}(J,K_1,t_0)}{t^{J+1+(K_1+1)/2}},$$

*and $\widehat{R}_{\mathbf{u}}(J,K_1,t_0) \geq 0$ does not depend on $t \geq t_0$ and $x \geq 0$.*

## A. Proof of Lemma 6.2(Ellipticity)

Let us prove assertion 1.

**Step 1.** Let $\mathcal{T}_\nu(y)$ and $\mathcal{U}_\nu(y)$ be the Chebyshev polynomials of the first and the second kind, respectively. Using the relations

$$\cos\nu\theta = \mathcal{T}_\nu(\cos\theta), \quad \frac{\sin\nu\theta}{\sin\theta} = \mathcal{U}_{\nu-1}(\cos\theta), \quad \nu\mathcal{U}_{\nu-1}(y) = \mathcal{T}_\nu(y), \quad \nu = 1,2,\dots,$$

we obtain

$$A'(\theta) = 2\sin\theta \sum_{\nu=1}^{N} a_\nu \mathcal{T}_\nu'(\cos\theta).$$

Thus, it suffices to prove that

$$\mathcal{B}(y) := \sum_{\nu=1}^{N} a_\nu \mathcal{T}_\nu'(y) > 0, \quad y \in [-1,1]. \tag{A.1}$$

**Step 2.** Suppose we have proved that

$$(-1)^{j-1}\mathcal{B}^{(j-1)}(1) > 0, \quad j = 1,\dots,N. \tag{A.2}$$

Since $\mathcal{T}_\nu(y)$ is a polynomial of degree $\nu$, from (A.2) it follows that

$$(-1)^{N-1}\frac{d^{N-1}\mathcal{B}(y)}{dy^{N-1}} = (-1)^{N-1}\frac{d^{N-1}\mathcal{B}(1)}{dy^{N-1}} > 0, \quad y \in [0,1]. \tag{A.3}$$

Therefore, $(-1)^{N-2}\frac{d^{N-2}\mathcal{B}(y)}{dy^{N-2}}$ is monotone decreasing and, due to (A.2), is positive for $y \in [-1,1]$. Hence, $(-1)^{N-3}\frac{d^{N-3}\mathcal{B}(y)}{dy^{N-3}}$ is monotone decreasing and, due to (A.2), is also positive for $y \in [-1,1]$. Continuing by induction, we conclude that $\mathcal{B}(y)$ is positive for $y \in [-1,1]$.

**Step 3.** It remains to prove (A.2), i.e.,

$$(-1)^{j-1}\sum_{\nu=1}^{N} a_\nu \mathcal{T}_\nu^{(j)}(1) > 0, \quad j = 1,\dots,N. \tag{A.4}$$

Set

$$u(\theta) := \cos(\theta), \quad v(y) := \arccos(y).$$

Then $\mathcal{T}_\nu(y) = u(\nu v(y))$. In what follows, we will use the following representations for $\alpha = 0,1,2,\dots$:

$$v^{(\alpha)}(y) = (-1)^\alpha (1-y)^{-\alpha+1/2}\sum_{n=0}^{\infty} A_{\alpha n}(1-y)^n, \tag{A.5}$$

where $A_{\alpha n} > 0$. This series and all the series below converge in a neighborhood of $y = 1$. To prove (A.5), one can write $\arccos(y) = \int_y^1 (1-z)^{-1/2}(1+z)^{-1/2}dz$ and expand $(1+z)^{-1/2}$ into the Taylor series around $z = 1$.

We fix some $j \in \{1,\dots,N\}$. From now on, we will not explicitly indicate the dependence of emerging coefficients on $j$. Since $\mathcal{T}_\nu(y) = u(\nu v(y))$, we see that

$$\mathcal{T}_\nu^{(j)}(y) = \sum_{k=1}^{j} \nu^k u^{(k)}(\nu v(y)) \sum_{\substack{1\le l_1\le\dots\le l_k,\\ l_1+\dots+l_k=j}} B_{l_1\dots l_k} v^{(l_1)}(y)\cdot\dots\cdot v^{(l_k)}(y), \tag{A.6}$$

where $B_{l_1\dots l_k} > 0$. Therefore, using (A.5), we have

$$\mathcal{T}_\nu^{(j)}(y) = (-1)^j \sum_{k=1}^{j}\sum_{l=0}^{\infty} C_{kl}\nu^k u^{(k)}(\nu v(y))(1-y)^{l-j+k/2}, \tag{A.7}$$

where $C_{kl} > 0$.

Expanding $u^{(k)}(\nu v(y))$ and the powers of $v(y)$ (using (A.5) with $\alpha = 0$), we obtain

$$u^{(k)}(\nu v(y)) = \sum_{m=0}^{\infty} D_{km}\nu^m v^m(y), \quad v^m(y) = (1-y)^{m/2}\sum_{n=0}^{\infty} E_{mn}(1-y)^n, \tag{A.8}$$

where $D_{km} \in \mathbb{R}$, $D_{km} = 0$ for $k+m$ odd, and $E_{mn} > 0$. Therefore,

$$u^{(k)}(\nu v(y)) = \sum_{m=0}^{2j-k}\sum_{n=0}^{\infty} D_{km}E_{mn}\nu^m(1-y)^{n+m/2} + (1-y)^{j-k/2}U_{k\nu}(y), \tag{A.9}$$

where $U_{k\nu}(y)$ is continuous at $y = 1$ and

$$U_{k\nu}(1) = 0. \tag{A.10}$$

Now we substitute (A.9) in (A.7) and take (A.10) into account:

$$\mathcal{T}_\nu^{(j)}(y) = (-1)^j \sum_{k=1}^{j} \sum_{m=0}^{2j-k} \sum_{l,n=0}^{\infty} C_{kl} D_{km} E_{mn} \nu^{k+m} (1-y)^{l+n-j+(k+m)/2} + \mathcal{U}_\nu(y), \tag{A.11}$$

where $U_\nu(y)$ is continuous at $y = 1$ and

$$U_\nu(1) = 0. \tag{A.12}$$

Finally, we can calculate the sum on the left-hand side in (A.4). To do so, we note the following:

(a) in (A.11), there are only terms with $\nu^2, \nu^4, \dots, \nu^{2j}$ (since $D_{km} = 0$ for $k + m$ odd);
(b) in (A.11), the terms with negative powers of $(1 - y)$ cancel because $\mathcal{T}_\nu^{(j)}(y)$ is a polynomial (hence, contains only nonnegative powers of $(1 - y)$);
(c) after summation with respect to $\nu$, the terms with $\nu^4, \dots, \nu^{2j}$ will vanish, while $\sum_{\nu=1}^{N} a_\nu \nu^2$ will yield 1, due to (6.9);
(d) the terms with positive powers of $(1 - y)$ and $\mathcal{U}_\nu(y)$ will vanish at $y = 1$ due to (A.12).

Thus, only the terms corresponding to $(k, m) = (1, 1)$ and $(k, m) = (2, 0)$ remain:

$$(-1)^{j-1} \sum_{\nu=1}^{N} a_\nu \mathcal{T}_\nu^{(j)}(1) = (-1)^{j-1}(-1)^j \Bigg( D_{11} \sum_{\substack{n,l\geq 0,\\ n+l=j-1}} C_{1l} E_{1n} + D_{20} \sum_{\substack{n,l\geq 0,\\ n+l=j-1}} C_{2l} E_{0n} \Bigg). \tag{A.13}$$

Since $u(\theta) = \cos\theta$, it follows from the first equality in (A.8) that $D_{11}$ and $D_{20}$ are the leading order coefficients in the Taylor expansions of $-\sin\theta$ and $-\cos\theta$, respectively. Hence, $D_{11} = D_{20} = -1$ and (A.13) yields

$$(-1)^{j-1} \sum_{\nu=1}^{N} a_\nu \mathcal{T}_\nu^{(j)}(1) = \sum_{\substack{n,l\geq 0,\\ n+l=j-1}} C_{1l} E_{1n} + \sum_{\substack{n,l\geq 0,\\ n+l=j-1}} C_{2l} E_{0n} > 0,$$

which completes the proof of[2] (A.4) and assertion 1 in the lemma.

Assertion 2 follows from assertion 1.

The Taylor expansion in assertion 3 follows from the Taylor expansions of $\cos\nu\theta$ and relations (6.9).

[2]The left-hand side in (A.4) can also be found, by using the following observation. In the proof, we have shown that, in (A.6), only the terms with $u'$ and $u''$ are relevant. Furthermore, in the expansion of $u'(\nu g(y))$ only the term $-\nu g(y)$ is relevant, and in the expansion of $u''(\nu g(y))$ only the term $-1$ is relevant. Therefore, we would have obtained the same result if we replaced $u(\theta)$ by the function $-\theta^2/2$ and deleted all the terms with negative powers of $(1 - y)$ after the respective expansions in the end. Therefore,

$$(-1)^{j-1} \sum_{\nu=1}^{N} a_\nu \mathcal{T}_\nu^{(j)}(1) = (-1)^j \sum_{\nu=1}^{N} a_\nu \frac{d^j}{dy^j} \Big( \frac{\nu^2 \arccos^2 y}{2} \Big) \Big|_{y=1} = \frac{(-1)^j}{2} \frac{d^j}{dy^j} (\arccos^2 y) \Big|_{y=1}$$

because $\frac{d^j}{dy^j}(\arccos^2 y) = 2v(y)v'(y)$ already has no negative powers of $(1-y)$ due to (A.5). Interestingly, the left-hand side in (A.4) does not depend on $N$.

## B. An identity

In this section, we prove a result, which we need for Theorems 6.2 and 6.3 in the case $N = 1$.

**Lemma B.1 1.** *Let*

$$\Omega(x) := \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}\left(\frac{\cos(x\xi)}{2(1-\cos\xi)} - \frac{1}{\xi^2}\right)d\xi - \frac{2}{\pi}\right) + \frac{x}{2}, \quad x \in \mathbb{Z},\ \ x \geq 0.$$

*Then* $\Omega(x) = 0$ *for all integers* $x \geq 0$.

*Proof.* **Step 1.** First, we show that

$$\Omega(x) - \Omega(x+1) = 0 \quad \text{for all integers } x \geq 0. \tag{B.1}$$

We have

$$\Omega(x) - \Omega(x+1) = I(x) - \tfrac{1}{2},$$

where

$$I(x) := \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\cos(x\xi) - \cos((x+1)\xi)}{1-\cos\xi}\,d\xi.$$

If $x = 0$, then, obviously, $I(x) = 1/2$.

Assume that $x \geq 1$. Using the formula

$$\cos(x+1)\xi = \cos x\xi\cos\xi - \sin x\xi\sin\xi,$$

we obtain

$$I(x) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\sin x\xi\sin\xi}{1-\cos\xi}\,d\xi + \frac{1}{4\pi}\int_{-\pi}^{\pi}\cos x\xi\,d\xi = \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{\sin x\xi\sin\xi}{1-\cos\xi}\,d\xi. \tag{B.2}$$

For $x = 1$, we have $I(x) = 1/2$. To prove (B.1), it remains to show that the right-hand side of (B.2) does not depend on $x \geq 1$. Using the formula $\sin(x+1)\xi = \sin x\xi\cos\xi + \cos x\xi\sin\xi$, we obtain

$$\begin{aligned}\frac{(\sin(x+1)\xi - \sin x\xi)\sin\xi}{1-\cos\xi} &= -\sin x\xi\sin\xi + \frac{\cos x\xi\sin^2\xi}{1-\cos\xi}\\ &= -\sin x\xi\sin\xi + \cos x\xi + \cos x\xi\cos\xi.\end{aligned}$$

For $x = 1$, the integral of the right-hand side vanishes by direct calculation, while for $x \geq 2$, it vanishes by the orthogonality of the systems $\{\sin x\xi\}_{x\in\mathbb{Z}}$ and $\{\cos x\xi\}_{x\in\mathbb{Z}}$, respectively, in $L_2(-\pi,\pi)$.

**Step 2.** It remains to show that $\Omega(0) = 0$. We have

$$\Omega(0) = \frac{1}{\pi}\lim_{\sigma\to 0}\Big(I_1(\sigma) + I_2(\sigma) - \frac{1}{\pi}\Big), \tag{B.3}$$

where

$$I_1(\sigma) := \int_{\sigma}^{\pi}\frac{d\xi}{2(1-\cos\xi)} = \frac{1}{4}\int_{\sigma}^{\pi}\frac{d\xi}{\sin^2(\xi/2)} = \frac{1}{2}\cot(\sigma/2) = \frac{1}{\sigma} + O(\sigma), \tag{B.4}$$

$$I_2(\sigma) := -\int_{\sigma}^{\pi}\frac{d\xi}{\xi^2} = \frac{1}{\pi} - \frac{1}{\sigma}. \tag{B.5}$$

Combining (B.3), (B.4), and (B.5) yields $\Omega(0) = 0$. □

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