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AFFINE HEMISPHERES OF ELLIPTIC TYPE

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Dedicated to Yuri Burago
at the occasion of his 80th birthday

Abstract. We find that for any n-dimensional, compact, convex set $K \subseteq \mathbb{R}^{n+1}$ there is an affinely-spherical hypersurface $M \subseteq \mathbb{R}^{n+1}$ with center in the relative interior of K such that the disjoint union $M \cup K$ is the boundary of an $(n+1)$-dimensional, compact, convex set. This so-called affine hemisphere M is uniquely determined by K up to affine transformations, it is of elliptic type, is associated with K in an affinely-invariant manner, and it is centered at the Santaló point of K.

§1. Introduction

Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected hypersurface which is locally strongly-convex, i.e., the second fundamental form is a definite symmetric bilinear form at any point $y \in M$. There are several ways to define the affine normal line $\ell_M(y)$ at a point $y \in M$. One possibility is to define $\ell_M(y)$ via the following procedure:

(i) Let $H = T_yM$ be the tangent space to M at the point $y \in M$, viewed as a linear subspace of codimension one in $\mathbb{R}^{n+1}$. Select a vector $v \notin H$ pointing to the convex side of M at the point $y \in M$, and denote $M_t = M \cap (H + y + tv)$ for $t > 0$. Here, $H + y + tv = \{x + y + tv;\ x \in H\}$.

(ii) For a sufficiently small $t > 0$, the section M_t encloses an n-dimensional convex body $\Omega_t \subseteq H + y + tv$. The barycenters $b_t = \mathrm{bar}(\Omega_t)$ depend smoothly on t. The affine normal line $\ell_M(y) \subseteq \mathbb{R}^{n+1}$ is defined to be the line passing through y in the direction of the nonzero vector $\frac{d}{dt} b_t \big|_{t=0}$.

We say that M is *affinely-spherical* with center at a point $p \in \mathbb{R}^{n+1}$ if all of the affine normal lines of M meet at p. In the case where all of the affine normal lines are parallel, we say that M is affinely-spherical with center at infinity. An *affine sphere* is an affinely-spherical hypersurface which is *complete*, i.e., it is a closed subset of $\mathbb{R}^{n+1}$. This definition is clearly affinely-invariant, hence the term "affine sphere". In §5 below we explain that M is affinely-spherical with center at the origin if and only if the cone measure on M is mapped to a measure proportional to the cone measure on the polar hypersurface M^* via the polarity map.

Affine spheres were introduced by the Romanian geometer Tzitzéica [24, 25]. All convex quadratic hypersurfaces in $\mathbb{R}^{n+1}$ are affine spheres, as well as the hypersurface

$$M = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n;\ \forall i, x_i > 0,\ \prod_{i=1}^n x_i = 1 \right\},$$

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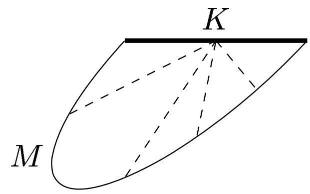

FIGURE 1. Half of an ellipse, which is an affine one-dimensional hemisphere in $\mathbb{R}^2$.

found by Tzitzéica [24, 25] and Calabi [10]. See Loftin [18] for a survey on affine spheres. At any point $y \in M$, the punctured line $\ell_M(y) \setminus \{y\}$ is naturally divided into two rays: one pointing to the convex side of M and the other to the concave side. These two rays are referred to as the convex side and the concave side of $\ell_M(y)$, respectively. An affinely-spherical hypersurface M is called *elliptic* if its center lies on the convex side of all of the affine normal lines. It is *hyperbolic* if its center lies on the concave side of all of the affine normal lines. There are also parabolic affine spheres, whose affine normal lines are all parallel.

Ellipsoids in $\mathbb{R}^{n+1}$ are elliptic affine spheres, while elliptic paraboloids are parabolic affine spheres. There are no other examples of complete affine spheres of elliptic or parabolic type. This nontrivial theorem is the culmination of the works of Blaschke [4], Calabi [9], Pogorelov [21], and Trudinger and Wang [23].

While affine spheres of elliptic or parabolic type are quite rare, there are many hyperbolic affine spheres in $\mathbb{R}^{n+1}$. From the work of Calabi [10] and Cheng-Yau [11] we learn that for any nonempty, open, convex cone $C \subseteq \mathbb{R}^{n+1}$ that does not contain a full line, there exists a hyperbolic affine sphere which is asymptotic to the cone. This hyperbolic affine sphere is determined by the cone C up to homothety, and all hyperbolic affine spheres in $\mathbb{R}^{n+1}$ arise this way. Why are there so few elliptic affine spheres, compared to the abundance of hyperbolic affine spheres? Perhaps completeness is too strong a requirement in the elliptic case. We propose the following:

Definition 1.1. Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly-convex hypersurface. We say that M is an "affine hemisphere" if

1. There exist compact, convex sets $K, \widetilde{K} \subseteq \mathbb{R}^{n+1}$, with $\dim(K) = n$ and $\dim(\widetilde{K}) = n+1$, such that M does not intersect the affine hyperplane spanned by K and
$$K \cup M = \partial \widetilde{K}.$$
2. The hypersurface M is affinely-spherical with center in the relative interior of K.

We say that K is the "anchor" of the affine hemisphere M.

In Definition 1.1, the dimension $\dim(K)$ is the maximal number N such that K contains $N+1$ affinely-independent vectors. Note that when $M \subseteq \mathbb{R}^{n+1}$ is an affine hemisphere, its anchor K is the compact, convex set enclosed by $\overline{M} \setminus M$, where $\overline{M}$ is the closure of M. In particular, $K = \mathrm{Conv}(\overline{M} \setminus M)$ where Conv denotes convex hull. It is clear that an affine hemisphere is always of elliptic type.

Theorem 1.2. *Let $K \subseteq \mathbb{R}^{n+1}$ be an n-dimensional, compact, convex set. Then there exists an affine hemisphere $M \subseteq \mathbb{R}^{n+1}$ with anchor K, uniquely determined up to affine transformations. The affine hemisphere M is centered at the Santaló point of K.*

Thus, with any n-dimensional, compact, convex set $K \subseteq \mathbb{R}^{n+1}$ we associate an $(n+1)$-dimensional, compact, convex set $\widetilde{K} \subseteq \mathbb{R}^{n+1}$ whose boundary consists of two parts: the convex set K itself is a facet, and the rest of the boundary is an affine hemisphere M centered at the Santaló point of K. We refer the reader to Loftin [18] and to Nomizu and Sasaki [20] for information about the rich geometric structure associated with affinely-spherical hypersurfaces. Let us just observe here that by [20, Theorem 6.5], any affine function in $\mathbb{R}^{n+1}$ that vanishes on K is an eigenfunction of the affine-metric Laplacian of M with Dirichlet boundary conditions, corresponding to the first eigenvalue.

The proof of Theorem 1.2 is basically a variant of the *moment measure* construction by Cordero-Erausquin and the author [12] which is in turn influenced by Berman and Berndtsson [3] and is also analogous to the classical Minkowski problem. Let us now present a few questions about affine hemispheres.

1. Other than half-ellipsoids, we are not aware of any affine hemisphere that may be described by a simple formula. Is there a closed form for the affine hemisphere associated with the n-dimensional simplex or the n-dimensional cube? For moment measures, the solutions in the case of the simplex and the cube are given by explicit formulas, see [12].
2. Calabi [10] found a composition rule for hyperbolic affine spheres, allowing one to construct a hyperbolic affine sphere of dimension $n+m+1$ from two hyperbolic affine spheres of dimensions n and m. Is there an analogous construction for affine hemispheres?
3. An intriguing question is whether an affine hemisphere M can be extended beyond its anchor K, to an affinely-spherical hypersurface $\widetilde{M} \supsetneq M$. When the anchor K is an ellipsoid, the affine hemisphere M with anchor K is half an ellipsoid, and may clearly be extended to the surface of a full ellipsoid. On the other hand, if K is a polytope, then the affine hemisphere M cannot be smoothly extended beyond the vertices of K.
4. Finally, is there a theory similar to that of affine hemispheres that is related to *parabolic* affinely-spherical hypersurfaces? See Ferrer, Martínez and Milán [14], Milán [19] and Remark 5.12 below for partial results in this direction.

Throughout this paper, by smooth we always mean C^∞-smooth. We write $|\cdot|$ for the usual Euclidean norm in $\mathbb{R}^n$, and $S^n = \{x \in \mathbb{R}^{n+1}\,;\, |x| = 1\}$ is the Euclidean unit sphere centered at the origin. The standard scalar product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y\rangle$. We write log for the natural logarithm. For a Borel measure μ in $\mathbb{R}^n$ we denote by $\mathrm{Supp}(\mu)$ the support of μ, which is the intersection of all closed sets of full μ-measure. A hypersurface in $\mathbb{R}^{n+1}$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$. A submanifold $M \subseteq \mathbb{R}^{n+1}$ *encloses* a convex set $K \subseteq \mathbb{R}^{n+1}$ if M is the boundary of K relative to the affine subspace spanned by K.

§2. A variational problem

In this section we analyze a variational problem related to affine hemispheres. Similar variational problems were considered by Berman and Berndtsson [3] and by Cordero-Erausquin and the author [12]. For a function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ denote

$$\mathrm{Dom}(\psi) = \{x \in \mathbb{R}^n;\ \psi(x) < +\infty\}\,.$$

The Legendre transform of ψ is the convex function

$$\psi^*(y) = \sup_{x \in \mathrm{Dom}(\psi)} \left[\langle x, y\rangle - \psi(x)\right] \quad (y \in \mathbb{R}^n),$$

where $\sup \varnothing = -\infty$. The function ψ^* is always convex and lower semicontinuous. A convex function $\psi\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is *proper* if it is lower semicontinuous with $\mathrm{Dom}(\psi) \neq \varnothing$. When ψ is convex and proper, the Legendre transform ψ^* is again convex and proper, and $\psi^{**} = \psi$. We will frequently use the formula $\psi^*(0) = -\inf \psi$, as well as the relation

$(\lambda\psi)^*(x) = \lambda\psi^*(x/\lambda)$, which is valid for any $x \in \mathbb{R}^n$ and $\lambda > 0$. It is also well known that for any $v \in \mathbb{R}^n$, denoting $\psi_1(x) = \psi(x) + \langle x, v\rangle$, we have

$$\psi_1^*(y) = \psi^*(y - v) \quad (y \in \mathbb{R}^n). \tag{1}$$

See Rockafellar [26] for a thorough discussion of the Legendre transform. For $p > 0$ and a function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi(0) < 0$ we define

$$\mathcal{I}_p(\psi) = \left(\int_{\mathbb{R}^n} \frac{dx}{(\psi^*(x))^{n+p}}\right)^{-1/p} \in [0, +\infty]. \tag{2}$$

Two remarks are in order. First, note that $\inf \psi^* \geq -\psi(0) > 0$, and that the integral in (2) is a well-defined element of $[0, +\infty]$. Second, for the purpose of definition (2) let us agree that $0^{-\alpha} = +\infty$ and $(+\infty)^{-\alpha} = 0$ for $\alpha > 0$. The functional $\mathcal{I}_p$ is closely related to the Borell–Brascamp–Lieb inequality [5, 6]. This inequality, which is a variant of Brunn–Minkowski, states the following: For any $0 < \lambda < 1$ and three convex functions $\varphi_\lambda, \varphi_0, \varphi_1\colon \mathbb{R}^n \to (0, +\infty]$ such that

$$\varphi_\lambda\left((1-\lambda)x + \lambda y\right) \leq (1-\lambda)\varphi_0(x) + \lambda\varphi_1(y) \quad (x, y \in \mathbb{R}^n), \tag{3}$$

we have

$$\left(\int_{\mathbb{R}^n} \frac{dx}{\varphi_\lambda(x)^{n+p}}\right)^{-1/p} \leq (1-\lambda)\left(\int_{\mathbb{R}^n} \frac{dx}{\varphi_0(x)^{n+p}}\right)^{-1/p} + \lambda\left(\int_{\mathbb{R}^n} \frac{dx}{\varphi_1(x)^{n+p}}\right)^{-1/p}. \tag{4}$$

The Borell–Brascamp–Lieb inequality, sometimes called the dimensional Prékopa inequality, implies the convexity of $\mathcal{I}_p$ as is stated in the following lemma.

Lemma 2.1. *Let $p, \lambda > 0$, and let $\psi, \psi_0, \psi_1\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be functions that are negative at zero. Denote $\varphi = \psi^*, \varphi_0 = \psi_0^*$ and $\varphi_1 = \psi_1^*$. Then the following statements hold.*

(i) $\mathcal{I}_p(\lambda\psi) = \lambda\mathcal{I}_p(\psi)$.

(ii) $\mathcal{I}_p(\psi_0 + \psi_1) \leq \mathcal{I}_p(\psi_0) + \mathcal{I}_p(\psi_1)$.

(iii) *Assume that* $\mathrm{Dom}(\varphi_0) = \mathrm{Dom}(\varphi_1) = \mathbb{R}^n$. *Then equality in* (ii) *occurs if and only if there exists $x_0 \in \mathbb{R}^n$ and $\lambda > 0$ such that*

$$\varphi_1(x) = \lambda\varphi_0(x_0 + x/\lambda) \textit{ for all } x \in \mathbb{R}^n.$$

Proof. By using the formula $(\lambda\psi)^*(x) = \lambda\varphi(x/\lambda)$, which is valid for any $x \in \mathbb{R}^n$, we obtain

$$\mathcal{I}_p(\lambda\psi) = \left(\int_{\mathbb{R}^n} \frac{dx}{(\lambda\varphi(x/\lambda))^{n+p}}\right)^{-1/p} = \lambda^{\frac{n+p}{p}} \cdot \lambda^{-\frac{n}{p}} \left(\int_{\mathbb{R}^n} \frac{dx}{\varphi(x)^{n+p}}\right)^{-1/p} = \lambda\mathcal{I}_p(\psi).$$

Thus (i) is proven. Next, denote $\varphi_{1/2} = \left[(\psi_0+\psi_1)/2\right]^*$. Then $\varphi_0, \varphi_1, \varphi_{1/2}\colon \mathbb{R}^n \to (0, +\infty]$ are convex functions, and for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
\varphi_{1/2}\left(\frac{x+y}{2}\right) &= \sup_{z \in \mathrm{Dom}(\psi_0) \cap \mathrm{Dom}(\psi_1)} \left[\left\langle \frac{x+y}{2}, z\right\rangle - \frac{\psi_0(z) + \psi_1(z)}{2}\right] \\
&\leq \frac{1}{2}\left\{\sup_{z \in \mathrm{Dom}(\psi_0)} \left[\langle x, z\rangle - \psi_0(z)\right] + \sup_{z \in \mathrm{Dom}(\psi_1)} \left[\langle y, z\rangle - \psi_1(z)\right]\right\} = \frac{\varphi_0(x) + \varphi_1(y)}{2}.
\end{aligned}$$

Hence condition (3) is satisfied, with $\lambda = 1/2$. The case $\lambda = 1/2$ of the Borell–Brascamp–Lieb inequality (4) implies that

$$\mathcal{I}_p\left(\frac{\psi_0 + \psi_1}{2}\right) \leq \frac{\mathcal{I}_p(\psi_0) + \mathcal{I}_p(\psi_1)}{2}$$

and (ii) now follows from (i). According to Dubuc [13], equality occurs in (4), with $\varphi_0, \varphi_1 \colon \mathbb{R}^n \to (0, +\infty)$ being convex functions, if and only if there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that $\varphi_1(x) = \lambda \varphi_0(x_0 + x/\lambda)$ for all $x \in \mathbb{R}^n$. This proves (iii). □

The next lemma describes a lower semicontinuity property of the functional $\mathcal{I}_p$.

Lemma 2.2. *Let $p > 0$ and let $K \subseteq \mathbb{R}^n$ be a convex, open set containing the origin. Let $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function with $\psi(0) < 0$ such that $K \subseteq \mathrm{Dom}(\psi) \subseteq \overline{K}$. Assume that for any $\ell \geq 1$ we are given a function $\psi_\ell \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi_\ell(0) < 0$ and such that $\psi_\ell \longrightarrow \psi$ pointwise in the set K as $\ell \to \infty$. Then,*

$$\mathcal{I}_p(\psi) \leq \liminf_{\ell \to \infty} \mathcal{I}_p(\psi_\ell).$$

Proof. The convex function ψ is finite and hence continuous in the convex, open set K. Since $0 \in K$ and $\psi(0) < 0$, we may find $\varepsilon > 0$ and linearly independent vectors $v_1, \ldots, v_n \in K$ such that

$$\psi(\pm v_i) < -\varepsilon \quad \text{for} \quad i = 1, \ldots, n.$$

By the pointwise convergence in K, there exists ℓ_0 such that $\psi_\ell(\pm v_i) < -\varepsilon$ for all $\ell \geq \ell_0$ and $i = 1, \ldots, n$. The convex hull of the $2n$ points $\{\pm v_i;\, i = 1, \ldots, n\}$ contains a Euclidean ball of radius $\delta > 0$ centered at the origin. Consequently, for $\ell \geq \ell_0$ and $x \in \mathbb{R}^n$,

$$\psi_\ell^*(x) = \sup_{y \in \mathrm{Dom}(\psi_\ell)} [\langle x, y \rangle - \psi_\ell(x)] \geq \sup_{i=1,\ldots,n} [|\langle x, v_i \rangle| + \varepsilon] \geq \varepsilon + \delta |x|. \tag{5}$$

Next, we claim that for any $x_0 \in \mathbb{R}^n$,

$$\psi^*(x_0) \leq \liminf_{\ell \to \infty} \psi_\ell^*(x_0). \tag{6}$$

Indeed, since ψ is convex, its restriction to any line segment in the convex set $\mathrm{Dom}(\psi)$ is upper semicontinuous (see, e.g., [15]). From the inclusion $\mathrm{Dom}(\psi) \subseteq \overline{K}$ we thus learn that

$$\psi^*(x_0) = \sup_{y \in \mathrm{Dom}(\psi)} [\langle x_0, y \rangle - \psi(y)] = \sup_{y \in K} [\langle x_0, y \rangle - \psi(y)].$$

Hence, for any $\varepsilon > 0$ there exists $y_0 \in K$ such that $\psi^*(x_0) \leq \varepsilon + \langle x_0, y_0 \rangle - \psi(y_0)$. By the pointwise convergence in K, for a sufficiently large ℓ we observe that $\psi_\ell(y_0) \leq \psi(y_0) + \varepsilon$. Therefore, for a sufficiently large ℓ,

$$\psi_\ell^*(x_0) \geq \langle x_0, y_0 \rangle - \psi_\ell(y_0) \geq -\varepsilon + \langle x_0, y_0 \rangle - \psi(y_0) \geq -2\varepsilon + \psi^*(x_0)$$

and (6) is proven. The function $(\varepsilon + \delta|x|)^{-(n+p)}$ is integrable in $\mathbb{R}^n$. Thanks to (5) and (6) we may use the dominated convergence theorem, and conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{dx}{(\psi^*(x))^{n+p}} &\geq \int_{\mathbb{R}^n} \left[\lim_{\ell \to \infty} \sup_{k \geq \ell} \frac{1}{(\psi_k^*(x))^{n+p}} \right] dx = \lim_{\ell \to \infty} \int_{\mathbb{R}^n} \left[\sup_{k \geq \ell} \frac{1}{(\psi_k^*(x))^{n+p}} \right] dx \\ &= \limsup_{\ell \to \infty} \int_{\mathbb{R}^n} \left[\sup_{k \geq \ell} \frac{1}{(\psi_k^*(x))^{n+p}} \right] dx \geq \limsup_{\ell \to \infty} \int_{\mathbb{R}^n} \frac{dx}{(\psi_\ell^*(x))^{n+p}}. \qquad \square\end{aligned}$$

The next theorem is our main result in this section. It is essentially a theorem about the Legendre transform of the functional $\mathcal{I}_p^2$, viewed as a convex functional on an infinite-dimensional cone.

Theorem 2.3. *Let $p > 0$ and let μ be a Borel probability measure on $\mathbb{R}^n$ with*

$$\int_{\mathbb{R}^n} |x| \, d\mu(x) < +\infty$$

such that the barycenter of μ lies at the origin. Assume that the origin belongs to the interior of $\mathrm{Conv}(\mathrm{Supp}(\mu))$*. Then there exists a μ-integrable, proper, convex function $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi(0) < 0$ such that*

$$\int_{\mathbb{R}^n} \psi \, d\mu + \left(\int_{\mathbb{R}^n} \frac{dx}{(\psi^*(x))^{n+p}} \right)^{-2/p} \leq \int_{\mathbb{R}^n} \psi_1 \, d\mu + \left(\int_{\mathbb{R}^n} \frac{dx}{(\psi_1^*(x))^{n+p}} \right)^{-2/p} \tag{7}$$

for any μ-integrable, proper, convex function $\psi_1 \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi_1(0) < 0$. Moreover, the expression on the left-hand side of (7) *is a finite, negative number, and $\psi(x) = +\infty$ for any $x \in \mathbb{R}^n \setminus \overline{K}$ where K is the interior of* $\mathrm{Conv}(\mathrm{Supp}(\mu))$*.*

Note that the origin belongs to the interior of $\mathrm{Conv}(\mathrm{Supp}(\mu))$ if and only if $\mathrm{Supp}(\mu)$ spans $\mathbb{R}^n$. The remainder of this section is dedicated to the proof of Theorem 2.3. Let us fix a number $p > 0$ and a Borel probability measure μ satisfying the requirements of Theorem 2.3. For a μ-integrable, proper convex function $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi(0) < 0$ we denote

$$\mathcal{I}_{\mu,p}(\psi) = \int_{\mathbb{R}^n} \psi \, d\mu + \mathcal{I}_p^2(\psi) = \int_{\mathbb{R}^n} \psi \, d\mu + \left(\int_{\mathbb{R}^n} \frac{dx}{(\psi^*(x))^{n+p}} \right)^{-2/p}.$$

Since the barycenter of μ is at the origin, we learn from (1) that $\mathcal{I}_{\mu,p}(\psi) = \mathcal{I}_{\mu,p}(\psi_1)$ whenever $\psi_1(x) = \psi(x) + \langle x, v \rangle$ for some $v \in \mathbb{R}^n$. The first step in the proof of Theorem 2.3 is the following proposition.

Proposition 2.4. *Let $p > 0$ and let μ be as in Theorem* 2.3*. Then,*

$$\inf_{\psi} \mathcal{I}_{\mu,p}(\psi) > -\infty$$

where the infimum is taken over all μ-integrable, proper convex functions $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi(0) < 0$.

The proof of Proposition 2.4 relies on several lemmas.

Lemma 2.5. *There exist $c_1, c_2 > 0$, depending on μ, with the following property: for any $\theta \in S^{n-1}$,*

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle 1_{\{\langle x, \theta \rangle > c_1\}} \, d\mu(x) \geq c_2,$$

where $1_{\{\langle x, \theta \rangle > c_1\}}$ equals one when $\langle x, \theta \rangle > c_1$ and it vanishes elsewhere.

Proof. The origin belongs to the interior of $\mathrm{Conv}(\mathrm{Supp}(\mu))$. Therefore, for any $\theta \in S^{n-1}$,

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle 1_{\{\langle x, \theta \rangle > 0\}} \, d\mu(x) > 0. \tag{8}$$

For $t > 0$ consider the nonnegative function

$$f_t(\theta) = \int_{\mathbb{R}^n} \langle x, \theta \rangle 1_{\{\langle x, \theta \rangle > t\}} \, d\mu(x) \quad (\theta \in S^{n-1}).$$

We claim that f_t is lower semicontinuous. Indeed, if $\theta_j \longrightarrow \theta$, then by Fatou's lemma,

$$\begin{aligned} f_t(\theta) &= \int_{\mathbb{R}^n} \langle x, \theta \rangle 1_{\{\langle x, \theta \rangle > t\}} \, d\mu(x) \\ &\leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \langle x, \theta_j \rangle 1_{\{\langle x, \theta_j \rangle > t\}} \, d\mu(x) = \liminf_{j \to \infty} f_t(\theta_j). \end{aligned}$$

Denote by m_t the minimum of the function f_t on S^{n-1}, and let $\theta_t \in S^{n-1}$ be a point such that $f_t(\theta_t) = m_t$. Since S^{n-1} is compact, there exists a sequence $t_j \to 0^+$ such that $\theta_{t_j} \to \theta$ for a certain unit vector $\theta \in S^{n-1}$. By (8) and Fatou's lemma,

$$0 < \int_{\mathbb{R}^n} \langle x, \theta \rangle 1_{\{\langle x, \theta \rangle > 0\}} \, d\mu(x) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} \langle x, \theta_{t_j} \rangle 1_{\{\langle x, \theta_j \rangle > t_j\}} \, d\mu(x) = \liminf_{j \to \infty} m_{t_j}.$$

Consequently there exists $j \geq 1$ such that $m_{t_j} > 0$. The lemma follows with $c_1 = t_j$ and $c_2 = m_{t_j}$. ☐

Lemma 2.6. *There exists $c > 0$, depending on μ, with the following property. Let $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex function that is μ-integrable. Denote $\alpha = -\psi(0)$. Assume that $\psi(0) = \inf \psi$ and that $\int_{\mathbb{R}^n} \psi \, d\mu < 0$. Then for any $x \in \mathbb{R}^n$,*

$$\psi(x) \leq -\alpha/2 \textit{ when } |x| < c.$$

Proof. We will prove the lemma with $c = \min\{c_1, c_2/4\}$ where c_1, c_2 are the positive constants from Lemma 2.5. Assume by contradiction that the conclusion of the lemma fails. Then the convex set $A = \{x \in \mathbb{R}^n \,;\, \psi(x) \leq -\alpha/2\}$ does not contain an open ball of radius c around the origin. By the convexity of A, there exists $\theta \in S^{n-1}$ such that $\langle x, \theta \rangle < c$ for all $x \in A$. By the convexity of the function ψ, for any $x \in \mathbb{R}^n$ with $\langle x, \theta \rangle \geq c$,

$$-\frac{\alpha}{2} < \psi\left(\frac{cx}{\langle x, \theta \rangle}\right) \leq \frac{c}{\langle x, \theta \rangle}\psi(x) + \left(1 - \frac{c}{\langle x, \theta \rangle}\right)\psi(0) = \frac{c}{\langle x, \theta \rangle}\psi(x) - \alpha \cdot \left(1 - \frac{c}{\langle x, \theta \rangle}\right).$$

Consequently, $\psi(x) \geq \alpha\langle x, \theta \rangle/(2c) - \alpha$ for any $x \in \mathbb{R}^n$ with $\langle x, \theta \rangle \geq c$. Since $\inf \psi = -\alpha$ and $c \leq c_1$, by Lemma 2.5 we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi \, d\mu &= \int_{\mathbb{R}^n} \psi(x) 1_{\{\langle x, \theta \rangle \leq c_1\}} \, d\mu(x) + \int_{\mathbb{R}^n} \psi(x) 1_{\{\langle x, \theta \rangle > c_1\}} \, d\mu(x) \\
&\geq -\alpha + \int_{\mathbb{R}^n} \left[\frac{\alpha}{2c} \cdot \langle x, \theta \rangle - \alpha\right] \cdot 1_{\{\langle x, \theta \rangle > c_1\}} \, d\mu(x) \\
&\geq -2\alpha + \frac{\alpha}{2c} \cdot c_2 \geq -2\alpha + 2\alpha = 0,
\end{aligned}$$

in contradiction to our assumption that $\int_{\mathbb{R}^n} \psi \, d\mu < 0$. ☐

Lemma 2.7. *There exists $\widetilde{c} > 0$, depending on μ and p, with the following property. Let $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex function that is μ-integrable. Denote $\alpha = -\psi(0)$. Assume that $\psi(0) = \inf \psi$ and that $\int_{\mathbb{R}^n} \psi \, d\mu < 0$. Then,*

$$\mathcal{I}_{\mu,p}(\psi) \geq -\alpha + \widetilde{c}\alpha^2.$$

Proof. From Lemma 2.6, for any $y \in \mathbb{R}^n$,

$$\psi^*(y) = \sup_{x \in \mathrm{Dom}(\psi)} [\langle x, y \rangle - \psi(x)] \geq \sup_{x \in \mathbb{R}^n, |x| < c} [\langle x, y \rangle + \alpha/2] = \frac{\alpha}{2} + c|y|.$$

Since $\inf \psi = -\alpha$, we deduce that

$$\begin{aligned}\mathcal{I}_{\mu,p}(\psi) &= \int_{\mathbb{R}^n} \psi \, d\mu + \left(\int_{\mathbb{R}^n} \frac{dy}{(\psi^*(y))^{n+p}} \right)^{-2/p} \\ &\geq -\alpha + \left(\int_{\mathbb{R}^n} \frac{dy}{(\alpha/2 + c|y|)^{n+p}} \right)^{-2/p} \\ &= -\alpha + \alpha^2 \left(\int_{\mathbb{R}^n} \frac{dy}{(1/2 + c|y|)^{n+p}} \right)^{-2/p} = -\alpha + \widetilde{c}\alpha^2. \qquad \square\end{aligned}$$

Lemma 2.8. *Assume that $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a μ-integrable, convex function. Then* $\mathrm{Dom}(\psi)$ *contains the interior of* $\mathrm{Conv}(\mathrm{Supp}(\mu))$. *In particular,* $\mathrm{Dom}(\psi)$ *contains the origin in its interior.*

Proof. Otherwise, we could use a hyperplane and separate the convex set $\mathrm{Dom}(\psi)$ from an open ball intersecting $\mathrm{Supp}(\mu)$. This would imply that ψ is not μ-integrable, a contradiction. $\square$

Proof of Proposition 2.4. Let $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex function with $\psi(0) < 0$ that is μ-integrable. We will show that

$$\mathcal{I}_{\mu,p}(\psi) \geq -\frac{1}{4\widetilde{c}} \tag{9}$$

where $\widetilde{c} > 0$ is the constant from Lemma 2.7. In the case where $\int \psi \, d\mu \geq 0$ we have $\mathcal{I}_{\mu,p}(\psi) \geq 0$, and (9) is trivial. We may thus assume that

$$\int_{\mathbb{R}^n} \psi \, d\mu < 0. \tag{10}$$

The origin is in the interior of $\mathrm{Dom}(\psi)$, according to Lemma 2.8. From Rockafellar [26, Theorem 23.4] we learn that there exists $w \in \mathbb{R}^n$ such that

$$\psi(x) \geq \psi(0) + \langle x, w \rangle \quad (x \in \mathbb{R}^n). \tag{11}$$

Recall that $\mathcal{I}_{\mu,p}(\psi) = \mathcal{I}_{\mu,p}(\psi_1)$ whenever $\psi_1(x) = \psi(x) + \langle x, v \rangle$ for some $v \in \mathbb{R}^n$. By adding an appropriate linear functional to ψ, we may assume that $w = 0$ in (11) and hence $\psi(0) = \inf \psi$. Denote $\alpha = -\psi(0)$, which is a positive number, as follows from (10). We may now apply Lemma 2.7 and obtain the inequality

$$\mathcal{I}_{\mu,p}(\psi) \geq -\alpha + \widetilde{c}\alpha^2 \geq -\frac{1}{4\widetilde{c}},$$

completing the proof of (9). The proposition is thus proven. $\square$

The next proposition is the second step in the proof of Theorem 2.3.

Proposition 2.9. *The infimum in Proposition* 2.4 *is attained.*

Again, the proof of Proposition 2.9 relies on a few small lemmas.

Lemma 2.10. *There exists a μ-integrable, proper convex function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi(0) < 0$ such that $\mathcal{I}_{\mu,p}(\psi) < 0$.*

Proof. Let $\delta > 0$ and denote $\psi_\delta(x) = -\delta + \varepsilon|x|$ for $\varepsilon = \delta^{1+p/(4n)}$. Then,

$$\left(\int_{\mathbb{R}^n} \frac{dx}{(\psi_\delta^*(x))^{n+p}} \right)^{-2/p} = \left(\int_{B(0,\varepsilon)} \frac{dx}{\delta^{n+p}} \right)^{-2/p} = A\delta^{3/2}$$

where $B(0,\varepsilon) = \{x \in \mathbb{R}^n \,;\, |x| < \varepsilon\}$ and $A = \mathrm{Vol}_n(B(0,1))^{-2/p} > 0$. Consequently,

$$\mathcal{I}_{\mu,p}(\psi_\delta) = A\delta^{3/2} + \int_{\mathbb{R}^n} (-\delta + \varepsilon|x|) \, d\mu(x) = A\delta^{3/2} - \delta + \delta^{1+p/(4n)} \cdot \int_{\mathbb{R}^n} |x| \, d\mu(x).$$

By our assumptions on the measure μ, we know that $\int |x| \, d\mu(x) < \infty$. For a small, positive δ, the leading term in $\mathcal{I}_{\mu,p}(\psi_\delta)$ is $-\delta$. Consequently, $\mathcal{I}_{\mu,p}(\psi_\delta) < 0$ for a sufficiently small $\delta > 0$. $\square$

In order to prove Proposition 2.9, we select a minimizing sequence

$$\{\psi_\ell\}_{\ell=1,2,\ldots,\infty}.$$

In other words, for any $\ell \geq 1$ the function $\psi_\ell \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a μ-integrable, proper, convex function with $\psi_\ell(0) < 0$ and

$$\mathcal{I}_{\mu,p}(\psi_\ell) \xrightarrow{\ell \to \infty} \inf_{\psi} \mathcal{I}_{\mu,p}(\psi)$$

where the infimum is taken over all μ-integrable, proper, convex functions $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi(0) < 0$. Thanks to Lemma 2.10, we may select the sequence $\{\psi_\ell\}$ so that

$$\sup_{\ell \geq 1} \mathcal{I}_{\mu,p}(\psi_\ell) < 0. \tag{12}$$

Moreover, we know that $\mathcal{I}_{\mu,p}(\psi_\ell)$ remains intact when we add a linear functional to ψ_ℓ. Arguing as in the proof of Proposition 2.4, we may add appropriate linear functionals to ψ_ℓ and assume that

$$\inf_{x \in \mathbb{R}^n} \psi_\ell(x) = \psi_\ell(0) \text{ for } \ell \geq 1. \tag{13}$$

Lemma 2.11. *We have* $\sup_\ell \psi_\ell(0) < 0$ *and* $\inf_\ell \psi_\ell(0) > -\infty$.

Proof. By (13), for any $\ell \geq 1$,

$$\psi_\ell(0) = \inf_{x \in \mathbb{R}^n} \psi_\ell(x) \leq \int_{\mathbb{R}^n} \psi_\ell \, d\mu \leq \mathcal{I}_{\mu,p}(\psi_\ell).$$

Inequality (12) thus implies that $\sup_\ell \psi_\ell(0) < 0$. Moreover, from (12) it follows that $\int \psi_\ell \, d\mu < 0$ for all ℓ. From (12), (13) and Lemma 2.7,

$$\psi_\ell(0) + \widetilde{c}(\psi_\ell(0))^2 \leq \mathcal{I}_{\mu,p}(\psi_\ell) < 0 \quad (\ell \geq 1).$$

Hence $\inf_\ell \psi_\ell(0) \geq -1/\widetilde{c} > -\infty$. $\square$

Write $K \subseteq \mathbb{R}^n$ for the interior of $\mathrm{Conv}(\mathrm{Supp}(\mu))$. Then K is an open, convex set containing the origin. Lemma 16 in [12] states that for any nonnegative, μ-integrable, convex function $f \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and any point $x \in K$ we have

$$f(x) \leq C_\mu(x) \int_{\mathbb{R}^n} f \, d\mu, \tag{14}$$

where $C_\mu(x) > 0$ depends solely on x and μ.

Lemma 2.12. *There exists a sequence of integers* $\{\ell_j\}_{j=1,2,\ldots}$ *such that* ψ_{ℓ_j} *converges pointwise on* K *to a certain convex function* $\psi \colon K \to \mathbb{R}$.

Proof. Fix a point $x_0 \in K$. We claim that

$$\sup_{\ell \geq 1} |\psi_\ell(x_0)| < +\infty. \tag{15}$$

Indeed, the fact that the sequence $\{\psi_\ell(x_0)\}_{\ell=1,2,\ldots}$ is bounded from below follows from (13) and Lemma 2.11. In order to show that this sequence is bounded from above, we denote

$$\beta = -\inf\{\psi_\ell(x);\, x \in \mathbb{R}^n, \ell \geq 1\} = -\inf\{\psi_\ell(0);\, \ell \geq 1\} \tag{16}$$

which is a finite, positive number thanks to Lemma 2.11. Apply (14) for the nonnegative, μ-integrable, convex function $f_\ell = \psi_\ell + \beta$, and obtain

$$\begin{aligned} f_\ell(x_0) &\leq C_\mu(x_0) \int_{\mathbb{R}^n} f_\ell(x)\, d\mu(x) = C_\mu(x_0) \int_{\mathbb{R}^n} (\psi_\ell + \beta)\, d\mu \\ &\leq C_\mu(x_0)\, (\beta + \mathcal{I}_{\mu,p}(\psi_\ell)) \leq C_\mu(x_0)\beta, \end{aligned}$$

where we have used (12) in the last passage. This shows that $\sup_\ell f_\ell(x_0) < \infty$, and consequently $\sup_\ell \psi_\ell(x_0) < \infty$. The proof of (15) is complete. We may now invoke Theorem 10.9 from Rockafellar [26], thanks to (15), and conclude that there exists a subsequence $\{\psi_{\ell_j}\}$ satisfying the conclusion of the lemma. □

Proof of Proposition 2.9. We use the convergent subsequence $\{\psi_{\ell_j}\}$ from Lemma 2.12. The function $\psi = \lim_j \psi_{\ell_j}$ is finite and convex on the open, convex set K. Moreover, $\psi(0) \in (-\infty, 0)$ as follows from Lemma 2.11. Since $\psi_\ell(x) \geq \psi_\ell(0)$ for any $x \in \mathbb{R}^n$ and $\ell \geq 1$, also

$$\psi(0) = \inf_{x \in K} \psi(x) \in (-\infty, 0). \tag{17}$$

The function ψ is currently defined only on the set K. In order to have a globally defined function on $\mathbb{R}^n$, we set $\psi(x) = +\infty$ for $x \in \mathbb{R}^n \setminus \overline{K}$. For $x \in \partial K$, define

$$\psi(x) = \lim_{t \to 1^-} \psi(tx). \tag{18}$$

Since ψ is convex on K, from (17) it follows that the function $t \mapsto \psi(tx)$ is monotone nondecreasing in $t \in (0,1)$, hence the limit in (18) is well defined. Moreover, the function $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex function, since on $\overline{K}$ we have $\psi = \sup_{t \in (0,1)} f_t$ where $f_t(x) = \psi(tx)$ is finite, convex and continuous on $\overline{K}$. The measure μ is supported on the closure $\overline{K}$. From the pointwise convergence on K, it follows that $\psi_{\ell_j}(tx) \longrightarrow \psi(tx)$ for any $0 < t < 1$ and $x \in \overline{K}$. We claim that by Fatou's lemma, for any $0 < t < 1$,

$$\int_{\overline{K}} \psi(tx)\, d\mu(x) \leq \liminf_{j \to \infty} \int_{\overline{K}} \psi_{\ell_j}(tx)\, d\mu(x) \leq \liminf_{j \to \infty} \int_{\overline{K}} \psi_{\ell_j}(x)\, d\mu(x). \tag{19}$$

Indeed, the use of Fatou's lemma is legitimate according to (13) and Lemma 2.11, because $\inf_{x,\ell} \psi_\ell(x) > -\infty$. Relation (13) also implies that $\psi_\ell(tx) \leq \psi_\ell(x)$ for any $x \in \overline{K}, \ell \geq 1$ and $0 < t < 1$, completing the justification of (19). Next, we use the fact that $\psi(tx) \nearrow \psi(x)$ as $t \to 1^-$ for any $x \in \overline{K}$. Since ψ is bounded from below, we may use the monotone convergence theorem, and upgrade (19) to the bound

$$\begin{aligned} \int_{\mathbb{R}^n} \psi\, d\mu = \int_{\overline{K}} \psi\, d\mu &= \lim_{t \to 1^-} \int_{\overline{K}} \psi(tx)\, d\mu(x) \\ &\leq \liminf_{j \to \infty} \int_{\overline{K}} \psi_{\ell_j}\, d\mu = \liminf_{j \to \infty} \int_{\mathbb{R}^n} \psi_{\ell_j}\, d\mu. \end{aligned} \tag{20}$$

Recall from (12) that $\sup_j \int \psi_{\ell_j}\, d\mu < 0$. From (17) and (20) it follows that ψ is a μ-integrable, proper, convex function with $\psi(0) < 0$. All that remains is to prove that

$$\mathcal{I}_{\mu,p}(\psi) \leq \liminf_{j \to \infty} \mathcal{I}_{\mu,p}(\psi_{\ell_j}). \tag{21}$$

The convex function ψ satisfies $K \subseteq \mathrm{Dom}(\psi) \subseteq \overline{K}$, and $\psi_{\ell_j} \longrightarrow \psi$ pointwise on K as $j \to \infty$. From Lemma 2.2,

$$\mathcal{I}_p(\psi) \leq \liminf_{j \to \infty} \mathcal{I}_p(\psi_{\ell_j}) \text{ and hence } \mathcal{I}_p^2(\psi) \leq \liminf_{j \to \infty} \mathcal{I}_p^2(\psi_{\ell_j}). \tag{22}$$

Now (21) follows from (20), (22) and the definition of $\mathcal{I}_{\mu,p}$. □

From the proof of Proposition 2.9 we see that the minimizer ψ may be selected so that $\psi(x) = +\infty$ for any $x \in \mathbb{R}^n \setminus \overline{K}$. Theorem 2.3 now follows from Proposition 2.4, Proposition 2.9 and Lemma 2.10.

§3. q-MOMENT MEASURES

Let $q > 0$ and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a positive, convex function such that $Z_\varphi := \int_{\mathbb{R}^n} \varphi^{-(n+q)} < \infty$. The function φ is differentiable almost everywhere in $\mathbb{R}^n$ because it is convex. We define the *q-moment measure of φ* to be the push-forward of the probability measure on $\mathbb{R}^n$ with density $Z_\varphi^{-1}/\varphi^{n+q}$ under the measurable map $x \mapsto \nabla\varphi(x)$. In other words, a Borel probability measure μ on $\mathbb{R}^n$ is the q-moment measure of φ if for any bounded, continuous function $b\colon \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} b(y)\, d\mu(y) = \int_{\mathbb{R}^n} \frac{b(\nabla\varphi(x))}{\varphi^{n+q}(x)} \frac{dx}{Z_\varphi}. \tag{1}$$

The moment measure of φ is a well-defined probability measure on $\mathbb{R}^n$, whenever φ is a positive, convex function on $\mathbb{R}^n$ such that $\varphi^{-(n+q)}$ is integrable.

Lemma 3.1. *Let $q > 0$ and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a positive, convex function. Then the function $\varphi^{-(n+q)}$ is integrable if and only if $\lim_{|x|\to\infty} \varphi(x) = +\infty$. Moreover, in this case there exist $\alpha, \beta > 0$ such that $\varphi(x) \geq \alpha + \beta|x|$ for all $x \in \mathbb{R}^n$.*

Proof. Assume that $\varphi^{-(n+q)}$ is integrable. Then for any $R > 0$, the open convex set $\{x \in \mathbb{R}^n\,;\, \varphi(x) < R\}$ has a finite volume and hence it is bounded. Therefore $\lim_{|x|\to\infty} \varphi(x) = +\infty$. Conversely, assume that $\varphi(x)$ tends to infinity as $|x| \to \infty$. Then there exists $R > 0$ such that $\varphi(x) \geq \varphi(0) + 1$ whenever $|x| \geq R$. By convexity, for any $|x| > R$,

$$\varphi(0) + 1 \leq \varphi\left(\frac{R}{|x|}x\right) \leq \left(1 - \frac{R}{|x|}\right)\varphi(0) + \frac{R}{|x|}\varphi(x).$$

Therefore $\varphi(x) \geq \varphi(0) + |x|/R$ for all $|x| > R$. By continuity, $c = \min_{|x|\leq R} \varphi(x)$ is positive. Hence $\varphi(x) \geq c/2 + \min\{1/R, c/(2R)\} \cdot |x|$ for all $x \in \mathbb{R}^n$, and $\varphi^{-(n+q)}$ is integrable. □

Lemma 3.1 demonstrates that if $\varphi^{-(n+q)}$ is integrable for some $q > 0$, then it is integrable for all $q > 0$. The moment measures from [12] correspond in a sense to the case of $q = \infty$, because in [12] we push forward the measure on $\mathbb{R}^n$ with density $\exp(-\varphi)$ via the map $x \mapsto \nabla\varphi(x)$. For a convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ and for $\lambda > 0$ we say that

$$(\lambda \times \varphi)(x) = \lambda\varphi(x/\lambda) \quad (x \in \mathbb{R}^n)$$

is the λ-dilation of φ. Note that the q-moment measure of φ is exactly the same as the q-moment measure of its dilation $\lambda \times \varphi$, assuming that one of these q-moment measures exists. It is also clear that replacing $\varphi(x)$ by its translation $\varphi(x - x_0)$, for some $x_0 \in \mathbb{R}^n$, does not have any effect on the resulting q-moment measure.

Theorem 3.2. *Let $q > 1$ and let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$ whose barycenter lies at the origin. Assume that the origin is in the interior of* $\mathrm{Conv}(\mathrm{Supp}(\mu))$.

Then there exists a positive, convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ whose q-moment measure is μ. This convex function φ is uniquely determined up to translation and dilation.

Theorem 3.2 is a variant for q-moment measures of a result proven in [12] in the case of moment measures. The case where μ is not compactly supported will not be discussed in this paper, although we expect that, like in [12], essential-continuity will play a role in the analysis of this case. We also restrict our attention to the case of $q > 1$. The necessity of the barycenter condition in Theorem 3.2 follows from the next statement.

Proposition 3.3. *Let $q > 1$ and let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$. Assume that μ is the q-moment measure of a positive, convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$. Then the barycenter of μ lies at the origin, which belongs to the interior of* $\mathrm{Conv}(\mathrm{Supp}(\mu))$.

Proof. We may substitute $b(x) = x_i$ in (1), because b is bounded on $\mathrm{Supp}(\mu)$. This shows that for $i = 1, \dots, n$,

$$\int_{\mathbb{R}^n} x_i \, d\mu(x) = \int_{\mathbb{R}^n} \frac{\partial_i \varphi}{\varphi^{n+q}} = -\frac{1}{n+q-1} \int_{\mathbb{R}^n} \partial_i \left(\frac{1}{\varphi^{n+q-1}} \right) = 0,$$

along the lines of [12, Lemma 4]. Therefore the barycenter of μ lies at the origin. Assume by contradiction that the origin is not in the interior of $\mathrm{Conv}(\mathrm{Supp}(\mu))$. Since the barycenter of μ lies at the origin, necessarily μ is supported in a hyperplane of the form $H = \theta^\perp$ for some $\theta \in S^{n-1}$. Since μ is the q-moment measure of φ, we see that

$$\partial_\theta \varphi(x) = \langle \nabla \varphi(x), \theta \rangle = 0 \text{ for almost all } x \in \mathbb{R}^n. \tag{2}$$

The function φ is locally Lipschitz in $\mathbb{R}^n$, being a finite, convex function. The relation (2) shows that φ is constant on almost any line parallel to θ, contradicting the integrability of $\varphi^{-(n+q)}$. □

The proof of Theorem 3.2 occupies most of the remainder of this section. We begin the proof with the following claim.

Lemma 3.4. *Let $q > 1$ and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a positive, convex function with*

$$\int_{\mathbb{R}^n} \varphi^{-(n+q)} < \infty.$$

Write μ for the q-moment measure of φ, and assume that μ is compactly supported. Set $\psi = \varphi^*$. Then,

$$\int_{\mathbb{R}^n} |\psi| \, d\mu < \infty.$$

Proof. From the definition of the Legendre transform, it follows that for any point $x \in \mathbb{R}^n$ at which φ is differentiable,

$$\langle x, \nabla \varphi(x) \rangle = \psi(\nabla \varphi(x)) + \varphi(x).$$

For almost any $x \in \mathbb{R}^n$ we have $\nabla \varphi(x) \in \mathrm{Supp}(\mu)$. Since μ is compactly supported, we see that $|\nabla \varphi(x)|$ is an L^∞-function in $\mathbb{R}^n$. Consequently,

$$\int_{\mathbb{R}^n} \varphi^{-(n+q)} \int_{\mathbb{R}^n} |\psi| \, d\mu = \int_{\mathbb{R}^n} \frac{|\psi(\nabla \varphi(x))|}{\varphi^{n+q}(x)} dx \le \int_{\mathbb{R}^n} \frac{|\langle x, \nabla \varphi(x) \rangle| + \varphi(x)}{\varphi^{n+q}(x)} dx < \infty,$$

by Lemma 3.1, since $q > 1$. This completes the proof. □

Lemma 3.5. *Let $A, p > 0$ and let μ be as in Theorem 3.2. Let $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a μ-integrable, proper, convex function such that $\mathrm{Dom}(\psi)$ is bounded. For $t \in \mathbb{R}$ denote $\psi_t = \psi + t$ and $\varphi_t = \psi_t^*$. Then for any $t < -\psi(0)$, the function $\varphi_t\colon \mathbb{R}^n \to \mathbb{R}$ is a positive, convex function with $\int_{\mathbb{R}^n} \varphi_t^{-(n+p)} \in (0, \infty)$. Moreover, there exists $t < -\psi(0)$ with*

$$\int_{\mathbb{R}^n} \varphi_t^{-(n+p)}(x) \, dx = A.$$

Proof. The set $\mathrm{Dom}(\psi)$ is assumed to be bounded. Set

$$L = 1 + \sup_{x \in \mathrm{Dom}(\psi)} |x| < \infty.$$

Denoting $\varphi = \psi^*$, we learn from Corollary 13.3.3 in Rockafellar [26] that the convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is an L-Lipschitz function. Lemma 2.8 implies that ψ is finite on an open neighborhood of the origin. Fix $t < -\psi(0)$. By the continuity of ψ near the origin, there exists $\varepsilon_t > 0$, depending on ψ and t, such that

$$\psi_t(x) < -\varepsilon_t \text{ when } |x| < \varepsilon_t.$$

Hence, for any $y \in \mathbb{R}^n$ and $t < -\psi(0)$,

$$\varphi_t(y) = \sup_{x \in \mathrm{Dom}(\psi_t)} [\langle x, y\rangle - \psi_t(x)] \geq \sup_{|x|<\varepsilon_t} [\langle x, y\rangle + \varepsilon_t] = \varepsilon_t + \varepsilon_t |y|. \tag{3}$$

Set $t_0 = -\psi(0)$, and for $t \in (-\infty, t_0)$ define

$$I(t) = \int_{\mathbb{R}^n} \frac{dx}{(\varphi_t(x))^{n+p}} = \int_{\mathbb{R}^n} \frac{dx}{(\varphi(x) - t)^{n+p}}. \tag{4}$$

From (3) it follows that the function $\varphi_t^{-(n+p)}$ is integrable on $\mathbb{R}^n$. The positive function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is L-Lipschitz, hence the integral of $\varphi_t^{-(n+p)}$ is positive. The function I is clearly monotone nondecreasing in $t \in (-\infty, t_0)$, and by the monotone convergence theorem, I is continuous on $(-\infty, t_0)$. In order to conclude the lemma by the mean value theorem, it suffices to prove that

$$\lim_{t\to-\infty} I(t) = 0, \qquad \lim_{t\to t_0^-} I(t) = +\infty.$$

The fact that $I(t) \to 0$ as $t \to -\infty$ is evident from (4) and the monotone convergence theorem. It remains to show that $I(t) \to +\infty$ as $t \to t_0^-$. With any $t < t_0$ we associate a point $x_0(t) \in \mathbb{R}^n$ that satisfies

$$\varphi(x_0(t)) < \frac{t_0 - t}{2} + \inf_{x\in\mathbb{R}^n} \varphi(x) = \frac{t_0 - t}{2} - \psi(0) = \frac{t_0 - t}{2} + t_0.$$

For any $t < t_0$, denoting $r = (t_0 - t)/(2L)$, we see that $\varphi(x) \leq \varphi(x_0(t)) + (t_0 - t)/2$ for any x in the ball $B(x_0(t), r)$. Therefore, for any $t < t_0$,

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^n} \frac{dx}{(\varphi(x) - t)^{n+p}} \\ &\geq \int_{B(x_0(t),r)} \frac{dx}{(\varphi(x) - t)^{n+p}} \geq \frac{\kappa_n r^n}{(2t_0 - 2t)^{n+p}} = \frac{\kappa_n 2^{-2n-p} L^{-n}}{(t_0 - t)^p} \end{aligned}$$

where $\kappa_n = \mathrm{Vol}_n(B(0, 1))$ is the volume of the Euclidean unit ball. Since $p > 0$,

$$\lim_{t\to t_0^-} I(t) \geq \lim_{t\to t_0^-} \frac{\kappa_n 2^{-2n-p} L^{-n}}{(t_0 - t)^p} = +\infty$$

and the lemma is proven. □

Lemma 3.6. *Let $q > 1$ and let μ be as in Theorem 3.2. Let $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the μ-integrable, proper, convex function whose existence is guaranteed by Theorem 2.3 with $p = q - 1$.*

Denote $\varphi = \psi^*$. Then $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a positive function and the probability measure ν on $\mathbb{R}^n$ with density $Z_\varphi^{-1}/\varphi^{n+q}$ is well defined. Moreover, for any function $\psi_1\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of the form $\psi_1 = \psi + b$, with $b\colon \mathbb{R}^n \to \mathbb{R}$ being a bounded function, we have

$$\int_{\mathbb{R}^n} \psi \, d\mu + \int_{\mathbb{R}^n} \psi^* \, d\nu \leq \int_{\mathbb{R}^n} \psi_1 \, d\mu + \int_{\mathbb{R}^n} \psi_1^* \, d\nu. \tag{5}$$

Proof. Write $\overline{K}$ for the closure of Conv(Supp(μ)), a compact set in $\mathbb{R}^n$. Theorem 2.3 states that $\psi(0) < 0$ and that $\mathrm{Dom}(\psi) \subseteq \overline{K}$. Therefore, by Lemma 3.5, the function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a positive, convex function with

$$\int_{\mathbb{R}^n} \varphi^{-(n+p)} \in (0, +\infty). \tag{6}$$

It thus follows from Lemma 3.1 that the probability measure ν is well defined. The function ψ_1^{**} is proper, convex, and it satisfies $\psi - C \le \psi_1^{**} \le \psi_1 \le \psi + C$ for some $C > 0$. It suffices to prove (5) under the additional assumption that ψ_1 is proper and convex: otherwise, replace ψ_1 with the smaller ψ_1^{**}, and observe that the right-hand side of (5) cannot increase under such a replacement.

Hence we may assume that ψ_1 is a μ-integrable, proper, convex function. Moreover, the convex set $\mathrm{Dom}(\psi_1) = \mathrm{Dom}(\psi)$ is bounded according to Theorem 2.3. The right hand-side of (5) is not altered if we add a constant to the function ψ_1, because μ and ν are probability measures. By adding an appropriate constant to ψ_1 and by using Lemma 3.5 and (6), we may assume that the convex function ψ_1 satisfies $\psi_1(0) < 0$ and

$$\int_{\mathbb{R}^n} \frac{dx}{\varphi_1^{n+p}(x)} = \int_{\mathbb{R}^n} \frac{dx}{\varphi^{n+p}(x)} \tag{7}$$

where $\varphi_1 = \psi_1^*\colon \mathbb{R}^n \to \mathbb{R}$ is a positive function. Since $\psi_1(0) < 0$, by Theorem 2.3,

$$\int_{\mathbb{R}^n} \psi \, d\mu + \left(\int_{\mathbb{R}^n} \frac{1}{\varphi^{n+p}} \right)^{-2/p} \le \int_{\mathbb{R}^n} \psi_1 \, d\mu + \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}} \right)^{-2/p}. \tag{8}$$

From (7) and (8),

$$\int_{\mathbb{R}^n} \psi \, d\mu \le \int_{\mathbb{R}^n} \psi_1 \, d\mu. \tag{9}$$

Note the elementary inequality

$$\frac{n+p}{t^{n+p+1}}(t-s) \le \frac{1}{s^{n+p}} - \frac{1}{t^{n+p}} \quad (s, t > 0),$$

which follows from the convexity of the function $t \mapsto t^{-(n+p)}$ on $(0, \infty)$. The last inequality implies that

$$\int_{\mathbb{R}^n} (\varphi - \varphi_1) \frac{n+p}{\varphi^{n+p+1}} \le \int_{\mathbb{R}^n} \left[\frac{1}{\varphi_1^{n+p}} - \frac{1}{\varphi^{n+p}} \right] = 0 \tag{10}$$

where we have used (7) in the last passage. Since $\varphi_1 - \varphi$ is a bounded function, all integrals in (10) converge. From (10) and the definition of the measure ν,

$$\int_{\mathbb{R}^n} \varphi \, d\nu \le \int_{\mathbb{R}^n} \varphi_1 \, d\nu. \tag{11}$$

The desired inequality (5) follows from (9) and (11). $\square$

Proof of the existence part in Theorem 3.2. Lemma 3.6 is the variational problem associated with *optimal transportation*, see Brenier [7] and Gangbo and McCann [16]. Let $\psi, \varphi = \psi^*$ and ν be as in Lemma 3.6. Then $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a positive, convex function on $\mathbb{R}^n$. A standard argument from [7, 16] leads us from (5) to the conclusion that $\nabla\varphi$ pushes forward the measure ν to the measure μ.

Let us provide some details. The idea of this standard argument is to apply (5) with the function $\psi_1 = \psi + \varepsilon b$, where $\varepsilon > 0$ is a small number and $b\colon \mathbb{R}^n \to \mathbb{R}$ is a bounded, continuous function. Denoting $\psi_\varepsilon = \psi + \varepsilon b$ for $0 \le \varepsilon < 1$ and $\varphi_\varepsilon = \psi_\varepsilon^*$, one verifies that

$$\frac{d\varphi_\varepsilon(x)}{d\varepsilon}\Big|_{\varepsilon=0} = -b(\nabla\varphi(x))$$

at any point $x \in \mathbb{R}^n$ at which φ is differentiable (see, e.g., Berman and Berndtsson [3, Lemma 2.7] for a short proof). Consequently, by the bounded convergence theorem,

$$\frac{d}{d\varepsilon}\left(\int_{\mathbb{R}^n} \psi_\varepsilon \, d\mu + \int_{\mathbb{R}^n} \varphi_\varepsilon \, d\nu\right)\Big|_{\varepsilon=0} = \int_{\mathbb{R}^n} b(x) \, d\mu(x) - \int_{\mathbb{R}^n} b(\nabla\varphi(x)) \, d\nu(x). \tag{12}$$

However, the expression in (12) must vanish according to (5). Recalling that the density of ν is proportional to $\varphi^{-(n+q)}$, we conclude that (1) is valid for any bounded, continuous function b. Therefore μ is the q-moment measure of φ. □

Our next inequality is analogous to Theorem 8 in [12], and may be viewed as an "above tangent" version of the Borell–Brascamp–Lieb inequality.

Proposition 3.7. *Let $q > 1$ and let μ be as in Theorem 3.2. Suppose that $\varphi_0 \colon \mathbb{R}^n \to (0, \infty)$ is a convex function whose q-moment measure is μ. Denote $p = q - 1$ and $\psi_0 = \varphi_0^*$. Then ψ_0 is μ-integrable, and for any μ-integrable, proper, convex function $\psi_1 \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\psi_1(0) < 0$, denoting $\varphi_1 = \psi_1^*$,*

$$\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}}\right)^{-2/p} \geq \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}}\right)^{-2/p} + \frac{2(n+p)\int_{\mathbb{R}^n} \varphi_0^{-(n+p+1)}}{p\left(\int_{\mathbb{R}^n} \varphi_0^{-(n+p)}\right)^{\frac{p+2}{p}}} \int_{\mathbb{R}^n} (\psi_0 - \psi_1) \, d\mu.$$

We begin the proof of Proposition 3.7 with two reductions.

Lemma 3.8. *It suffices to prove Proposition 3.7 under the additional requirements that $\mathrm{Dom}(\psi_1) \subseteq \mathrm{Dom}(\psi_0)$ and that $\psi_1 - \psi_0$ is bounded from below on $\mathrm{Dom}(\psi_0)$.*

Proof. From Lemma 3.1, it follows that $\psi_0(0) < 0$. For $N > 0$ and $x \in \mathbb{R}^n$ define $f_N(x) = \max\{\psi_1(x), \psi_0(x) - N\}$. The functions ψ_0 and ψ_1 are negative at zero, and hence f_N is a proper, convex function on $\mathbb{R}^n$ with $f_N(0) < 0$ and $\mathrm{Dom}(f_N) \subseteq \mathrm{Dom}(\psi_0)$. The function ψ_0 is μ-integrable according to Lemma 3.4. The μ-integrability of ψ_0 and ψ_1 implies that f_N is μ-integrable. Assuming that Proposition 3.7 is proven under the additional requirement in the formulation of the lemma, we may assert that

$$\left(\int_{\mathbb{R}^n} \frac{1}{(f_N^*)^{n+p}}\right)^{-2/p} \geq \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}}\right)^{-2/p} + \frac{2(n+p)\int_{\mathbb{R}^n} \varphi_0^{-(n+p+1)}}{p\left(\int_{\mathbb{R}^n} \varphi_0^{-(n+p)}\right)^{\frac{p+2}{p}}} \int_{\mathbb{R}^n} (\psi_0 - f_N) \, d\mu. \tag{13}$$

All that remains is to prove that

$$\int_{\mathbb{R}^n} \psi_1 \, d\mu = \lim_{N\to\infty} \int_{\mathbb{R}^n} f_N \, d\mu \tag{14}$$

and

$$\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}} \leq \liminf_{N\to\infty} \int_{\mathbb{R}^n} \frac{1}{(f_N^*)^{n+p}}. \tag{15}$$

Since $f_N \geq \psi_1$, we have $f_N^* \leq \varphi_1$ and $(f_N^*)^{-(n+p)} \geq \varphi_1^{-(n+p)}$. Hence (15) is trivial. Note that $f_N \searrow \psi_1$ as $N \to \infty$ pointwise in $\mathrm{Dom}(\psi_0)$. Since ψ_0 is μ-integrable, the set $\mathrm{Dom}(\psi_0)$ has a full μ-measure. Consequently, $f_N(x) \searrow \psi_1(x)$ as $N \to \infty$ for μ-almost any $x \in \mathbb{R}^n$. The monotone convergence theorem implies (14). □

Lemma 3.9. *It suffices to prove Proposition 3.7 under the additional requirement that $\mathrm{Dom}(\psi_1) = \mathrm{Dom}(\psi_0)$ and that $\psi_1 - \psi_0$ is bounded on $\mathrm{Dom}(\psi_0)$.*

Proof. According to Lemma 3.8, we may assume that for some $C > 0$,

$$\psi_1(x) + C \geq \psi_0(x) \quad (x \in \mathbb{R}^n). \tag{16}$$

From (16), it follows that for any $N > 0$,

$$\varphi_0 - N \leq \max\{\varphi_1, \varphi_0 - N\} \leq \varphi_0 + C. \tag{17}$$

For $N > 0$, let us define

$$g_N = \left(\max\{\varphi_1, \varphi_0 - N\}\right)^* . \tag{18}$$

Since φ_0 is a proper, convex function, from (17) it follows that $g_N : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex function as well. It also follows from (17) that $\mathrm{Dom}(g_N) = \mathrm{Dom}(\psi_0)$ and that $g_N - \psi_0$ is a bounded function on $\mathrm{Dom}(\psi_0)$. The μ-integrability of ψ_0, proved in Lemma 3.4, implies that g_N is μ-integrable. We learn from (18) that $g_N(0) \leq \psi_1(0) < 0$. Assuming that Proposition 3.7 is proven under the additional requirement in the formulation of this lemma, we may assert that (13) holds true when f_N is replaced by g_N. All that remains to prove is that

$$\int_{\mathbb{R}^n} \psi_1 \, d\mu \geq \limsup_{N\to\infty} \int_{\mathbb{R}^n} g_N \, d\mu \tag{19}$$

and

$$\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}} \leq \liminf_{N\to\infty} \int_{\mathbb{R}^n} \frac{1}{(g_N^*)^{n+p}}. \tag{20}$$

Since $\psi_1 \geq g_N$, inequality (19) is fulfilled trivially. Since $\mathrm{Dom}(\varphi_0) = \mathbb{R}^n$,from (18) it follows that

$$g_N^* = \max\{\varphi_1, \varphi_0 - N\} \xrightarrow{N\to\infty} \varphi_1$$

pointwise in $\mathbb{R}^n$. Now (20) follows from Fatou's lemma. □

Proof of Proposition 3.7. The μ-integrability of ψ_0 follows from Lemma 3.4, while Lemma 3.1 implies that $\inf \varphi_0 > 0$. According to Lemma 3.9, we may assume that $\mathrm{Dom}(\psi_0) = \mathrm{Dom}(\psi_1)$, and that

$$M = \sup_{\mathrm{Dom}(\psi_0)} |\psi_1 - \psi_0| < \infty. \tag{21}$$

Denote $f(x) = \psi_0(x) - \psi_1(x)$ for $x \in \mathrm{Dom}(\psi_0)$ and $f(x) = +\infty$ for $x \notin \mathrm{Dom}(\psi_0)$. Set $\psi_t = (1-t)\psi_0 + t\psi_1$ and $\varphi_t = \psi_t^*$. Thus $\mathrm{Dom}(\psi_t) = \mathrm{Dom}(\psi_0)$ while $\psi_t = \psi_0 - tf$ in the set $\mathrm{Dom}(\psi_0)$. At any point $x \in \mathbb{R}^n$ at which φ_0 is differentiable, for any $0 \leq t \leq 1$ we have

$$\varphi_t(x) = \psi_t^*(x) = \sup_{y \in \mathrm{Dom}(\psi_0)} \left[\langle x, y\rangle - \psi_0(y) + tf(y)\right] \overset{\text{“}y=\nabla\varphi_0(x)\text{”}}{\geq} \varphi_0(x) + tf(\nabla\varphi_0(x)). \tag{22}$$

Denote $m = \inf \varphi_0$, which is a finite, positive number, thanks to the integrability of $\varphi_0^{-(n+q)}$ and to Lemma 3.1. By the Lagrange mean-value theorem from calculus, for any $a, b, t \in \mathbb{R}$ with $0 < t < m/(2M), a \geq m$ and $|b| \leq M$,

$$\frac{1}{t}\left[\frac{1}{(a+tb)^{n+p}} - \frac{1}{a^{n+p}}\right] = -\frac{n+p}{\xi^{n+p+1}} b \leq -\frac{n+p}{a^{n+p+1}} b + \frac{C_{n,p,m,M}}{a^{n+p+1}} \cdot t \tag{23}$$

for some ξ between a and $a + tb$, where $C_{n,p,m,M} > 0$ depends only on n, p, m and M. From (22) and (23) it follows that for any $t \in (0, m/(2M))$,

$$\begin{aligned} \frac{1}{t}\int_{\mathbb{R}^n}\left[\frac{1}{\varphi_t^{n+p}} - \frac{1}{\varphi_0^{n+p}}\right] &\leq \frac{1}{t}\int_{\mathbb{R}^n}\left[\frac{1}{(\varphi_0(x) + tf(\nabla\varphi_0(x)))^{n+p}} - \frac{1}{\varphi_0^{n+p}(x)}\right] dx \\ &\leq -(n+p)\int_{\mathbb{R}^n} \frac{f\circ\nabla\varphi_0}{\varphi_0^{n+p+1}} + Ct\int_{\mathbb{R}^n}\frac{1}{\varphi_0^{n+p+1}} \xrightarrow{t\to 0^+} -(n+p)\int_{\mathbb{R}^n}\frac{f\circ\nabla\varphi_0}{\varphi_0^{n+p+1}}, \end{aligned} \tag{24}$$

where $C = C_{n,p,m,M}$ and we have used the facts that $\varphi_0^{-(n+p+1)}$ is integrable and that $f \circ \nabla\varphi_0$ is an L^∞-function. Relation (21) implies that $|\varphi_0(x) - \varphi_1(x)| \le M$ for all $x \in \mathbb{R}^n$. Hence $\mathrm{Dom}(\varphi_0) = \mathrm{Dom}(\varphi_1) = \mathbb{R}^n$. Consequently, the function

$$I(t) = \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_t^{n+p}} \right)^{-2/p} \qquad (0 \le t \le 1)$$

satisfies $I(0), I(1) \in [0, +\infty)$. By Lemma 2.1, the function I is the square of a nonnegative, convex funtion on the interval $[0,1]$. Therefore I is a convex function. Consequently, the function I is finite and upper semicontinuous on $[0,1]$, being a convex function on the interval $[0,1]$ which is finite at the endpoints of the interval. The lower semicontinuity of I at the origin follows from (24). Hence I is continuous at the origin, and by convexity,

$$\begin{aligned} I(1) - I(0) &\ge \liminf_{t \to 0^+} \frac{I(t) - I(0)}{t} \\ &= -\frac{2}{p} \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}} \right)^{-\frac{p+2}{p}} \cdot \limsup_{t \to 0^+} \frac{1}{t} \int_{\mathbb{R}^n} \left[\frac{1}{\varphi_t^{n+p}} - \frac{1}{\varphi_0^{n+p}} \right] \\ &\ge \frac{2(n+p)}{p} \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}} \right)^{-\frac{p+2}{p}} \int_{\mathbb{R}^n} \frac{f \circ \nabla\varphi_0}{\varphi_0^{n+p+1}}, \end{aligned} \tag{25}$$

where we have used (24) in the last passage. The proposition follows from (25) and from the definition of μ as the q-moment measure of φ_0. □

The proof of Proposition 3.7 looks rather different from the transportation proof of Theorem 8 in [12]. The main difference is that above we apply the Borell–Brascamp–Lieb inequality in the form of Lemma 2.1, while in [12] we essentially reprove the Prékopa theorem.

Proof of the uniqueness part in Theorem 3.2. Assume that

$$\varphi_0, \varphi_1 \colon \mathbb{R}^n \to (0, +\infty)$$

are convex functions whose q-moment measure is μ. Our goal is to prove that there exists $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$\varphi_0(x) = \lambda\varphi_1(x_0 + x/\lambda) \text{ for } x \in \mathbb{R}^n. \tag{26}$$

By Lemma 3.1, the integrals $\int_{\mathbb{R}^n} \varphi_i^{-(n+r)}$ converge for all $r > 0$ and $i = 0, 1$, because φ_0 and φ_1 possess q-moment measures. Replacing $\varphi_0(x)$ by its dilation $(\lambda \times \varphi_0)(x) = \lambda\varphi_0(x/\lambda)$, we may assume that

$$\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}} \right)^{-\frac{p+2}{p}} \int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p+1}} = \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}} \right)^{-\frac{p+2}{p}} \int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p+1}}. \tag{27}$$

Indeed, replacing φ_0 by $\lambda \times \varphi_0$ has the effect of multiplying the left-hand side of (27) by λ, hence we may select an appropriate dilation of φ_0 and assume that (27) holds true. Denote $\psi_i = \varphi_i^*$ for $i = 0, 1$ and set

$$\psi_{1/2} = (\psi_0 + \psi_1)/2.$$

From Lemma 3.1 it follows that $\inf \varphi_i > 0$ for $i = 0, 1$. Therefore $\psi_i(0) = -\inf \varphi_i < 0$ for $i = 0, 1$ and consequently $\psi_{1/2}(0) < 0$. Denote $\varphi_{1/2} = \psi_{1/2}^*$. Lemma 2.1 implies that

$$\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_{1/2}^{n+p}} \right)^{-1/p} \le \frac{1}{2} \left[\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}} \right)^{-1/p} + \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}} \right)^{-1/p} \right]. \tag{28}$$

According to Lemma 2.1(iii), when equality occurs in (28), there exists $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ for which (26) holds true. All that remains to show is that equality does occur in (28). The functions ψ_0 and ψ_1 are μ-integrable, according to Lemma 3.4. Hence also $\psi_{1/2} = (\psi_0 + \psi_1)/2$ is μ-integrable. Denote by α the quantity in (27). Applying Proposition 3.7 for ψ_0 and $\psi_{1/2}$, we obtain

$$\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_{1/2}^{n+p}}\right)^{-2/p} \geq \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}}\right)^{-2/p} + \frac{2(n+p)}{p}\alpha \int_{\mathbb{R}^n} (\psi_0 - \psi_{1/2})\, d\mu.$$

Applying Proposition 3.7 for ψ_1 and $\psi_{1/2}$, we obtain

$$\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_{1/2}^{n+p}}\right)^{-2/p} \geq \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}}\right)^{-2/p} + \frac{2(n+p)}{p}\alpha \int_{\mathbb{R}^n} (\psi_1 - \psi_{1/2})\, d\mu.$$

Adding these two inequalities, and using the relation $2\psi_{1/2} = \psi_0 + \psi_1$, we have

$$\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_{1/2}^{n+p}}\right)^{-2/p} \geq \frac{1}{2}\left[\left(\int_{\mathbb{R}^n} \frac{1}{\varphi_0^{n+p}}\right)^{-2/p} + \left(\int_{\mathbb{R}^n} \frac{1}{\varphi_1^{n+p}}\right)^{-2/p}\right]. \tag{29}$$

From (29) we deduce that equality occurs in (28), because $\sqrt{(a^2+b^2)/2} \geq (a+b)/2$ for all $a, b \geq 0$. This completes the proof. □

For a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ we write $\nabla^2 f(x)$ for the Hessian matrix of f at the point $x \in \mathbb{R}^n$. A smooth function $f\colon L \to \mathbb{R}$ is *strongly convex*, where $L \subseteq \mathbb{R}^n$ is a convex, open set, if $\nabla^2 f(x)$ is positive definite for any $x \in L$. Suppose that $L \subseteq \mathbb{R}^n$ is a nonempty, open, bounded, convex set. We are interested in smooth, convex solutions $\varphi\colon \mathbb{R}^n \to (0,\infty)$ to the equation with the constraint

$$\begin{cases} \det \nabla^2 \varphi = C/\varphi^{n+2} & \text{in } \mathbb{R}^n, \\ \nabla\varphi(\mathbb{R}^n) = L, \end{cases} \tag{30}$$

where $C > 0$ is a positive number. Here, of course, $\nabla\varphi(\mathbb{R}^n) = \{\nabla\varphi(x)\,;\, x \in \mathbb{R}^n\}$. Thanks to the regularity theory for optimal transportation developed by Caffarelli [8] and Urbas [27], Theorem 3.2 admits the following corollary.

Theorem 3.10. *Let $L \subseteq \mathbb{R}^n$ be a nonempty, open, bounded, convex set. Then there exists a smooth, positive, convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ solving* (30) *if and only if the barycenter of L lies at the origin. Moreover, this convex function φ is uniquely determined up to translation and dilation.*

Proof. Let μ be the uniform measure on L, normalized to be a probability measure. Assume first that the barycenter of L lies at the origin. Then the origin belongs to the interior of $\mathrm{Supp}(\mu)$. Applying Theorem 3.2 with $q = 2$, we obtain a positive convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ whose q-moment measure is μ. That is, for any bounded, continuous function $b\colon L \to \mathbb{R}$,

$$\int_L b(y)\, dy = C_{L,\varphi} \int_{\mathbb{R}^n} \frac{b(\nabla\varphi(x))}{\varphi^{n+2}(x)}\, dx, \tag{31}$$

where $C_{L,\varphi} = \mathrm{Vol}_n(L)/\int_{\mathbb{R}^n} \varphi^{-(n+2)}$. Caffarelli's regularity theory for optimal transportation (see [8] and the Appendix in [1]) implies that φ is C^∞-smooth in $\mathbb{R}^n$. From (31) and from the change-of-variables formula, it follows that for any $x \in \mathbb{R}^n$,

$$\det \nabla^2 \varphi(x) = \frac{C_{L,\varphi}}{\varphi^{n+2}(x)}. \tag{32}$$

In particular, the Hessian $\nabla^2\varphi(x)$ is invertible and hence positive definite for any $x \in \mathbb{R}^n$. Since $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth, strongly convex function, the set $\nabla\varphi(\mathbb{R}^n)$ is convex and open, according to Theorem 26.5 in Rockafellar [26] or to Section 1.2 in Gromov [17]. From (31) we obtain that $\nabla\varphi(\mathbb{R}^n) = L$, thus φ solves (30).

Moreover, we claim that the smooth, positive, convex solution φ to (30) is uniquely determined up to translation and dilation. Indeed, any such solution φ is strongly convex, and consequently $\nabla\varphi$ is a diffeomorphism between $\mathbb{R}^n$ and the convex, open set $\nabla\varphi(\mathbb{R}^n) = L$. From (30) and the change-of-variables formula we thus learn that μ is the q-moment measure of φ with $q = 2$. By Theorem 3.2, the function φ is uniquely determined up to translation and dilation.

In order to prove the other direction of the theorem, assume that φ is a smooth, positive, convex solution to (30). As explained in the preceding paragraphs, μ is the q-moment measure of φ, with $q = 2$. Proposition 3.3 now shows that the barycenter of μ lies at the origin. $\square$

§4. The affine hemisphere equations

In this section we review the partial differential equations for affinely-spherical hypersurfaces described by Tzitzéica [24, 25], Blaschke [4] and Calabi [10]. Recall from §1 the definition of the *affine normal line* $\ell_M(y)$ which is a line in $\mathbb{R}^{n+1}$ passing through the point y of the smooth, connected, locally strongly convex hypersurface $M \subset \mathbb{R}^{n+1}$. We use $y = (x,t) \in \mathbb{R}^n \times \mathbb{R}$ as coordinates in $\mathbb{R}^{n+1}$. For a set $L \subseteq \mathbb{R}^n$ and a function $\psi\colon L \to \mathbb{R}$ denote

$$\mathrm{Graph}_L(\psi) = \{(x, \psi(x))\,;\, x \in L\} \subseteq \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}.$$

The affine normal line $\ell_M(y)$ depends on the third order approximation to M near y, as shown in the following lemma.

Lemma 4.1. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly convex hypersurface. Let $L \subseteq \mathbb{R}^n$ be an open, convex set containing the origin. Assume that $U \subseteq \mathbb{R}^{n+1}$ is an open set such that*

$$M \cap U = \mathrm{Graph}_L(\psi)$$

where $\psi\colon L \to \mathbb{R}$ is a smooth, strongly convex function with $\psi(0) = 0, \nabla\psi(0) = 0$ and $\nabla^2\psi(0) = \mathrm{Id}$. Here, Id is the identity matrix.

Then for $y_0 = (0,0) \in M$, the line $\ell_M(y_0)$ is the line passing through the point y_0 in the direction of the vector

$$\big(-(\nabla^2\psi(0))^{-1} \cdot \nabla(\log\det\nabla^2\psi)(0), n+2\big) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}. \tag{1}$$

Proof. The vector $v = (0,1) \in \mathbb{R}^n \times \mathbb{R}$ is pointing to the convex side of M at the point y_0. The tangent space to M at the point y_0 is $H = T_{y_0}M = \{(x,0)\,;\, x \in \mathbb{R}^n\}$. For a sufficiently small $t > 0$, the section $M_t = M \cap (H + tv)$ encloses an n-dimensional convex body $\Omega_t \subset H + tv$ given by

$$\Omega_t = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\,;\, \psi(x) \le t\}.$$

Denote

$$a_{ijk} = \partial^{ijk}\psi(0) = \frac{\partial^3\psi}{\partial x_i \partial x_j \partial x_k}(0).$$

By Taylor's theorem, for a sufficiently small $t > 0$,

$$\Omega_t = \left\{(x,t) \in \mathbb{R}^n \times \mathbb{R};\ \frac{|x|^2}{2} + \frac{1}{6}\sum_{i,j,k=1}^n a_{ijk}x_ix_jx_k + O(|x|^4) \le t\right\},$$

where $O(|x|^4)$ is an abbreviation for an expression that is bounded in absolute value by $C|x|^4$, where C depends only on M. By using the spherical-coordinates representation of Ω_t, we see that for a sufficiently small $t > 0$,

$$\frac{\Omega_{t/2}}{\sqrt{t}} = \left\{ (r\theta, \sqrt{t}/2);\ \theta \in S^{n-1},\ 0 \leq r \leq r_t(\theta) = 1 - \frac{\sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k}{6}\sqrt{t} + O(t) \right\},$$

where $t^{-1/2} \cdot \Omega_{t/2} = \{y/\sqrt{t};\ y \in \Omega_{t/2}\}$. Consequently, the barycenter satisfies $\mathrm{bar}(\Omega_{t/2}) = (x_t, t/2)$ for

$$x_t = \sqrt{t}\frac{n\int_{S^{n-1}} \theta\, r_t(\theta)^{n+1}\, d\theta}{(n+1)\int_{S^{n-1}} r_t(\theta)^n\, d\theta} = -t \cdot \frac{n}{6} \cdot \int_{S^{n-1}} \theta \Big(\sum_{i,j,k=1}^n a_{ijk}\theta_i\theta_j\theta_k \Big)\, d\sigma_{n-1}(\theta) + O(t^{3/2}),$$

where σ_{n-1} is the uniform probability measure on S^{n-1}. Let $X = (X_1, \ldots, X_n)$ be a standard Gaussian random vector in $\mathbb{R}^n$, and recall that $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}X_i^4 = 3$ for all i. For any homogenous polynomial p of degree 4 in n real variables, we know that $\mathbb{E}p(X) = n(n+2)\int_{S^{n-1}} p(\theta)\, d\sigma_{n-1}(\theta)$. Hence,

$$\mathrm{bar}(\Omega_{t/2}) = \Big(-t\frac{n}{6n(n+2)}\mathbb{E}X\Big[\sum_{i,j,k=1}^n a_{ijk}X_iX_jX_k \Big] + O(t^{3/2}), t/2 \Big).$$

Consequently, the line $\ell_M(y_0)$ is in the direction of the vector

$$\Big(-\mathbb{E}X\Big[\sum_{i,j,k=1}^n \partial^{ijk}\psi(0)X_iX_jX_k \Big], 3(n+2) \Big) = \left(-3\nabla(\Delta\psi)(0), 3(n+2) \right),$$

where $\Delta\psi = \sum_{i=1}^n \partial^{ii}\psi$. Since $\nabla^2\psi(0) = \mathrm{Id}$, we see that

$$\nabla(\Delta\psi)(0) = \left(\nabla^2\psi(0)\right)^{-1} \cdot \nabla(\log\det\nabla^2\psi)(0),$$

and the lemma is proven. □

Suppose that V is a finite-dimensional linear space over $\mathbb{R}$, and let $\psi\colon V \to \mathbb{R}$ be a smooth, strongly convex function. In general it is impossible to identify a specific vector in V as the gradient of the function ψ at the origin, unless we introduce additional structure such as a scalar product. Nevertheless, a simple and useful observation is that the vector

$$\left(\nabla^2\psi(0)\right)^{-1} \cdot \nabla\left(\log\det\nabla^2\psi\right)(0) \tag{2}$$

is a well-defined vector in V. This means that for any scalar product that one may introduce in V, we may compute the expression in (2) relative to this scalar product, and the result will always be the same vector in V.

Lemma 4.2. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface and let $L \subseteq \mathbb{R}^n$ be a nonempty, open, convex set. Suppose that $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex function whose restriction to the set L is finite, smooth and strongly convex. Denote $\Lambda(x) = \log\det\nabla^2\psi(x)$ for $x \in L$. Assume that*

$$M = \mathrm{Graph}_L(\psi).$$

Let $x_0 \in L$ and denote $y_0 = (x_0, \psi(x_0)) \in M$. Then the affine normal line $\ell_M(y_0) \subseteq \mathbb{R}^{n+1}$ is the line passing through the point $y_0 \in \mathbb{R}^{n+1}$ in the direction of the vector

$$\left(-(\nabla^2\psi)^{-1}\nabla\Lambda, n+2 - \left\langle(\nabla^2\psi)^{-1}\nabla\Lambda, \nabla\psi\right\rangle \right) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}, \tag{3}$$

where all expressions are evaluated at the point x_0.

Proof. Translating, we may assume that $x_0 = 0$ and $\psi(0) = 0$. Consider first the case where also $\nabla\psi(0) = 0$. In this case, the vector in (3) does not depend on the choice of the Euclidean structure in $\mathbb{R}^n$, hence we may switch to a Euclidean structure for which $\nabla^2\psi(0) = \mathrm{Id}$. Thus (3) follows from Lemma 4.1 in this case. In the case where $v := \nabla\psi(0)$ is a nonzero vector, we apply the linear map in $\mathbb{R}^{n+1}$,

$$(x, t) \mapsto (x, t - \langle x, v\rangle)\,.$$

This linear map transforms M to the graph of the convex function $\psi_1(x) = \psi(x) - \langle x, v\rangle$, and it transforms the vector in (3) to the vector

$$\big(-(\nabla^2\psi_1(0))^{-1} \cdot \nabla(\log\det\nabla^2\psi_1)(0), n+2\big) \in \mathbb{R}^{n+1}.$$

Since $\nabla\psi_1(0) = 0$, we have reduced matters to the case already proven. □

Remark 4.3. The affine normal lines considered in this paper are closely related to the *affine normal field* which was discussed, e.g., by Nomizu and Sasaki [20, Section II.3]. The affine normal field is a certain map $\xi\colon M \to \mathbb{R}^{n+1}$ that is well defined whenever $M \subseteq \mathbb{R}^{n+1}$ is a smooth, connected, locally strongly convex hypersurface. The relationship between the affine normal field and the affine normal line is simple: for any $y \in M$, the affine normal field ξ_y is pointing in the direction of the affine normal line $\ell_M(y)$. Indeed, using affine invariance, it suffices to verify this in the case where $M = \mathrm{Graph}_L(\psi)$. Example 3.3 in [20, Section II.3] demonstrates that when $M = \mathrm{Graph}_L(\psi)$, for any $x \in L$ and $y = (x, \psi(x)) \in M$,

$$\xi_y = \frac{(\det\nabla^2\psi)^{1/(n+2)}}{n+2} \cdot \big(-(\nabla^2\psi)^{-1}\nabla\Lambda, n+2 - \big\langle(\nabla^2\psi)^{-1}\nabla\Lambda, \nabla\psi\big\rangle\big) \in \mathbb{R}^n \times \mathbb{R}, \tag{4}$$

where $\Lambda = \log\det\nabla^2\psi$ and all expressions involving ψ and Λ are evaluated at the point x. The vector in (4) is proportional to the vector described in Lemma 4.2, and hence ξ_y is pointing in the direction of the line $\ell_M(y)$.

Proposition 4.4. *Let M, L and ψ be as in Lemma 4.2. Denote $\varphi = \psi^*$ and $\Omega = \nabla\psi(L) = \{\nabla\psi(x)\,;\, x \in L\}$. Then the following statements hold.*

(i) *The set $\Omega \subseteq \mathbb{R}^n$ is open and the function φ is smooth in Ω.*

(ii) *The hypersurface M is affinely spherical with center at the origin if and only if there exists $C \in \mathbb{R} \setminus \{0\}$ such that*

$$\varphi^{n+2} \cdot \det\nabla^2\varphi = C \text{ in the entire set } \Omega. \tag{5}$$

Proof. The function ψ is smooth and strongly convex in the open, convex set L. By strong convexity, the smooth map $\nabla\psi\colon L \to \Omega$ is one-to-one (see, e.g., [26, Theorem 26.5]). Moreover, the differential of the smooth map $\nabla\psi\colon L \to \Omega$ is nonsingular, and by the inverse function theorem from calculus, the set $\Omega = \nabla\psi(L)$ is open and the map $\nabla\psi\colon L \to \Omega$ is a diffeomorphism. According to [26, Corollary 23.5.1], the inverse of the map $\nabla\psi$ is the smooth map $\nabla\varphi\colon \Omega \to L$, and hence

$$\nabla^2\varphi = (\nabla^2\psi)^{-1} \circ \nabla\varphi. \tag{6}$$

Thus (i) is proven. We move on to the proof of (ii). Assume first that M is affinely spherical with center at the origin. Then for any $x \in L$, the vector in (3) is proportional to $(x, \psi(x))$. That is, for any $x \in L$,

$$-\psi(x)(\nabla^2\psi)^{-1}\nabla(\log\det\nabla^2\psi) = \big[n+2 - \big\langle(\nabla^2\psi)^{-1}\nabla(\log\det\nabla^2\psi), \nabla\psi\big\rangle\big]x. \tag{7}$$

By using the shorter Einstein notation, we may repharse (7) as follows: for $x \in L$ and $i = 1, \ldots, n$,

$$-\psi\psi^{ik}_k = \big(n+2 - \psi^{jk}_k\psi_j\big)x^i. \tag{8}$$

Let us briefly explain this standard notation. We denote $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, $\nabla^2\psi(x) = (\psi_{ij}(x))_{i,j=1,\dots,n}$ and $(\nabla^2\psi)^{-1}(x) = (\psi^{ij}(x))_{i,j=1,\dots,n}$. We abbreviate $\psi_{ij}^k = \sum_{\ell=1}^n \psi^{\ell k}\psi_{ij\ell}$ and $\psi_k^{ij} = \sum_{\ell,m=1}^n \psi^{i\ell}\psi^{jm}\psi_{\ell mk}$, where $\psi_{ijk} = \partial^{ijk}\psi$. The sums are usually implicit in the Einstein notation: an index which appears twice in an expression, once as a superscript and once as a subscript, is being summed upon from 1 to n. The Legendre transform fits well with the Einstein notation, thanks to identities such as

$$\psi^{ijk}(x) = -\varphi_{ijk}(y) \text{ and } \psi_k^{ij}(x) = -\varphi_{ij}^k(y),$$

where expressions involving ψ are evaluated at the point $x \in L$ and expressions involving φ are evaluated at the point $y = \nabla\psi(x) \in \Omega$. Here, $(\nabla^2\varphi)^{-1}(y) = (\varphi^{ij}(y))_{i,j=1,\dots,n}$ and $\varphi_{ij}^k = \sum_\ell \varphi^{\ell k}\varphi_{ij\ell}$. We may thus change the variables $y = \nabla\psi(x)$, and translate (8) to the equation: for any $y \in \Omega$ and $i = 1, \dots, n$,

$$\left(y^j\varphi_j - \varphi\right)\varphi_{ik}^k = \left(n + 2 + \varphi_{jk}^k y^j\right)\varphi_i. \tag{9}$$

The function ψ is smooth and strongly convex, hence the set $\{x \in L; \psi(x) \neq 0\}$ is an open, dense set in L. Denote $U = \{y \in \Omega\,;\, \psi(\nabla\varphi(y)) \neq 0\}$, an open, dense set in Ω. For any $y \in U$ we may define

$$A(y) = \frac{n + 2 + \varphi_{jk}^k y^j}{\left(\sum_\ell y^\ell\varphi_\ell\right) - \varphi}.$$

Thus $\varphi_{ik}^k = A\varphi_i$ throughout the set U, according to (9). Moreover, the following holds in the set U, for $i = 1, \dots, n$:

$$y^j\varphi_j\varphi_{ik}^k = Ay^j\varphi_j\varphi_i = \varphi_{jk}^k y^j\varphi_i. \tag{10}$$

From (9) and (10), we have

$$-\varphi\varphi_{ik}^k = (n+2)\varphi_i. \tag{11}$$

The validity of (11) in the dense set $U \subseteq \Omega$ implies by continuity that (11) holds true in the entire open set Ω. By multiplying (11) by $\varphi^{n+1} \cdot \det\nabla^2\varphi$ we obtain that in all of Ω,

$$\nabla(\varphi^{n+2} \cdot \det\nabla^2\varphi) = 0. \tag{12}$$

The set Ω is connected, being the image of the connected set L under a smooth map. Hence $\det\nabla^2\varphi \cdot \varphi^{n+2} \equiv C$ in Ω. This constant C cannot be zero according to (6), because $\det\nabla^2\varphi$ never vanishes in Ω and φ is not the zero function. This completes the verification of (5). We have thus proven one direction of (ii). However, all of our manipulations in this proof are reversible: The validity of (5) implies the validity of (11), which in turn leads to (9) and eventually to (7). Hence (5) implies that M is affinely spherical with center at the origin. □

The following proposition is close to the original definition of affinely-spherical hypersurfaces given by Tzitzéica [24, 25].

Proposition 4.5. *Let $M \subset \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly convex hypersurface. For $y \in M$ write $K_y > 0$ for the Gauss curvature of M at the point y and denote*

$$\rho_y = \langle y, N_y \rangle$$

where $N_y \in \mathbb{R}^{n+1}$ is the Euclidean unit normal to M at the point y, pointing to the concave side of M. Then M is affinely spherical with center at the origin if and only if there exists $C \in \mathbb{R} \setminus \{0\}$ such that $\rho_y^{n+2}/K_y = C$ for all $y \in M$.

Proof. See Nomizu and Sasaki [20, Section II.5] for a proof of this proposition, or alternatively argue as follows. Since M is connected, it suffices to show that M is affinely spherical with center at the origin if and only if the function $y \mapsto \rho_y^{n+2}/K_y$ is locally constant in M and it never vanishes.

Fix $y_0 \in M$. By applying a rotation in $\mathbb{R}^{n+1}$, we may assume that in a neighborhood of y_0, the hypersurface M looks like the graph of a strongly-convex function. That is, we may assume that there exist an open set $U \subseteq \mathbb{R}^{n+1}$ with $y_0 \in U$, a convex, open set $L \subseteq \mathbb{R}^n$ and a proper, convex function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ which is finite, smooth and strongly convex in L, such that

$$M \cap U = \mathrm{Graph}_L(\psi).$$

A standard exercise in differential geometry is to show that for any $x \in L$, at the point $y = (x, \psi(x))$,

$$\rho_y = \frac{\langle x, \nabla\psi(x)\rangle - \psi(x)}{\sqrt{1 + |\nabla\psi(x)|^2}}, \tag{13}$$

and

$$K_y = \det \nabla^2\psi(x) \cdot (1 + |\nabla\psi(x)|^2)^{-n/2-1}. \tag{14}$$

Denote $\varphi = \psi^*$. From (13) and (14) we obtain

$$\frac{\rho_y^{n+2}}{K_y} = \frac{(\langle x, \nabla\psi(x)\rangle - \psi(x))^{n+2}}{\det \nabla^2\psi(x)} = \varphi^{n+2}(z) \cdot \det \nabla^2\varphi(z)$$

where $z = \nabla\psi(x)$. The desired conclusion now follows from Proposition 4.4. □

§5. The polar affinely spherical hypersurface

In this section we prove Theorem 1.2. We begin with a variant of a construction in convexity considered by Artstein-Avidan and Milman [2] and by Rockafellar [26, Section 15]. Fix a dimension n, and denote

$$\mathcal{H}^+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R};\ t > 0\} \subseteq \mathbb{R}^{n+1},$$
$$cH^- = \{(x,t) \in \mathbb{R}^n \times \mathbb{R};\ t < 0\} \subseteq \mathbb{R}^{n+1}.$$

Consider the fractional-linear transformations $I^+\colon \mathcal{H}^+ \to \mathcal{H}^-$ and $I^-\colon \mathcal{H}^- \to \mathcal{H}^+$ defined via

$$I^+(x,t) = \left(\frac{x}{t}, -\frac{1}{t}\right), \quad I^-(y,s) = \left(-\frac{y}{s}, -\frac{1}{s}\right).$$

Then I^+ is a diffeomorphism whose inverse is I^-. A subset $V \subseteq \mathcal{H}^\pm$ is a *relative half-space* if $V = A \cap \mathcal{H}^\pm$ where $A \subseteq \mathbb{R}^{n+1}$ takes the form

$$A = \{(x,t) \in \mathbb{R}^n \times \mathbb{R};\ \langle x, \theta\rangle + bt + c \geq 0\} \subseteq \mathbb{R}^{n+1}$$

for some $\theta \in \mathbb{R}^n, b, c \in \mathbb{R}$. Note that a relative half-space $V \subseteq \mathcal{H}^\pm$ is a relatively-closed subset of $\mathcal{H}^\pm$. We say that a relative half-space $V \subseteq \mathcal{H}^\pm$ is proper if V and $\mathcal{H}^\pm \setminus V$ are nonempty.

Lemma 5.1. *The maps I^+ and I^- transform relative half-spaces to relative half-spaces.*

Proof. Let $\theta \in \mathbb{R}^n, b, c \in \mathbb{R}$. Then for any subset $V \subseteq \mathcal{H}^+$,

$$V = \left\{(x,t) \in \mathcal{H}^+;\ \langle x, \theta\rangle + bt + c \geq 0\right\} \Longleftrightarrow I^+(V) = \left\{(y,s) \in \mathcal{H}^-;\ \langle y, \theta\rangle - cs + b \geq 0\right\}.$$

Hence $V \subseteq \mathcal{H}^+$ is a relative half-space if and only if $I^+(V) \subseteq \mathcal{H}^-$ is a relative half-space. □

Any relatively closed subset $A \subseteq \mathcal{H}^\pm$ which is convex is the intersection of a family of relative half-spaces in $\mathcal{H}^\pm$. From Lemma 5.1 we conclude the following.

Corollary 5.2. *The maps I^+ and I^- transform relatively-closed, convex sets to relatively-closed, convex sets.*

Similarly to Rockafellar [26, Section 15], we say that the set $I^\pm(A)$ is the *obverse* of the set $A \subseteq \mathcal{H}^\pm$. See Figure 2 for an example of a convex set and its obverse. The *polar body* of a convex subset $S \subseteq \mathbb{R}^d$ is defined via

$$S^\circ = \left\{ x \in \mathbb{R}^d \, ; \, \forall y \in S, \, \langle x, y \rangle \leq 1 \right\}.$$

The set S° is always convex, closed and contains the origin. If $S \subseteq \mathbb{R}^d$ is convex, closed and contains the origin, then $(S^\circ)^\circ = S$. For a subset $S \subseteq \mathbb{R}^n$ and for a function $F \colon S \to \mathbb{R} \cup \{+\infty\}$ we write

$$\mathrm{Epigraph}_S(F) = \{(x,t) \in S \times \mathbb{R} \, ; \, F(x) \leq t\} \subseteq \mathbb{R}^{n+1}.$$

When $S = \mathbb{R}^n$ we abbreviate $\mathrm{Epigraph}(F) = \mathrm{Epigraph}_{\mathbb{R}^n}(F)$. Note that a function $F \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper and convex if and only if $\mathrm{Epigraph}(F)$ is convex, closed and nonempty. The obverse operation interchanges between the Legendre transform and the polarity transform, see the next statement.

Proposition 5.3. *Let $\varphi \colon \mathbb{R}^n \to (0, +\infty]$ be a proper, convex function and denote $\psi = \varphi^*$. Then,*

$$I^+(\mathrm{Epigraph}(\varphi)) = \mathrm{Epigraph}(\psi)^\circ \cap \mathcal{H}^-. \tag{1}$$

Moreover, if $\psi(0) < \infty$, then $\mathrm{Epigraph}(\psi)^\circ \setminus \mathcal{H}^- = \{(x, 0) \, ; \, x \in \mathrm{Dom}(\psi)^\circ\} \subseteq \mathbb{R}^n \times \mathbb{R}$.

Proof. Denote $A = \mathrm{Epigraph}(\varphi)$ and note that $A \subseteq \mathcal{H}^+$ because φ is positive. For any $(y, -s) \in \mathcal{H}^-$,

$$(y, -s) \in I^+(A) \iff (y/s, 1/s) \in A \iff \varphi(y/s) \leq 1/s. \tag{2}$$

Recall that $(s\psi)^*(y) = s\varphi(y/s)$ for any $y \in \mathbb{R}^n$ and $s > 0$. By (2), for any $(y, -s) \in \mathcal{H}^-$,

$$(y, -s) \in I^+(A) \iff (s\psi)^*(y) \leq 1 \iff \forall x \in \mathrm{Dom}(\psi), \ \langle x, y \rangle - s\psi(x) \leq 1.$$

Consequently,

$$\begin{aligned} I^+(A) &= \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \, ; \, s < 0, \, \langle x, y \rangle + s\psi(y) \leq 1 \ \text{ for all } \ x \in \mathrm{Dom}(\psi)\} \\ &= \{(y, s) \in \mathbb{R}^n \times \mathbb{R} \, ; \, s < 0, \, \langle x, y \rangle + ts \leq 1 \ \text{ for all } \ (x, t) \in \mathrm{Epigraph}(\psi)\}. \end{aligned} \tag{3}$$

Hence $I^+(A) = \mathrm{Epigraph}(\psi)^\circ \cap \mathcal{H}^-$, and (1) is proven. Next, assume that $\psi(0) < \infty$. Then $\mathrm{Epigraph}(\psi)$ contains all points of the form $(0, t)$ for $t \geq \psi(0)$. Therefore, for any $(y, s) \in \mathrm{Epigraph}(\psi)^\circ$,

$$\langle 0, y \rangle + ts \leq 1 \ \text{ for all } \ t \geq \psi(0),$$

and hence $s \leq 0$. We conclude that $\mathrm{Epigraph}(\psi)^\circ \setminus \mathcal{H}^- \subseteq \{(y, 0) \, ; \, y \in \mathbb{R}^n\}$. Consequently,

$$\begin{aligned} &\mathrm{Epigraph}(\psi)^\circ \setminus \mathcal{H}^- \\ &\quad = \{(y, 0) \, ; \, y \in \mathbb{R}^n, \, \langle x, y \rangle + t \cdot 0 \leq 1 \text{ for all } (x, t) \in \mathrm{Epigraph}(\psi)\} \\ &\quad = \{(y, 0) \, ; \, y \in \mathbb{R}^n, \, \langle x, y \rangle \leq 1 \text{ for all } x \in \mathrm{Dom}(\psi)\} \\ &\quad = \{(y, 0) \, ; \, y \in \mathrm{Dom}(\psi)^\circ\}. \end{aligned}$$

□

For a subset $A \subseteq \mathcal{H}^\pm \subseteq \mathbb{R}^{n+1}$ we write $\overline{A} \subseteq \mathbb{R}^{n+1}$ and $\partial A \subseteq \mathbb{R}^{n+1}$ for the usual closure and boundary of the set A, viewed as a subset of $\mathbb{R}^{n+1}$. Similarly, when $A \subseteq \mathcal{H}^\pm \subseteq \mathbb{R}^{n+1}$ is convex, we write A° for its polar body, where again A is viewed as a convex subset of $\mathbb{R}^{n+1}$. When $A \subseteq \mathcal{H}^\pm$ is relatively closed, its closure $\overline{A}$ is contained in $\overline{\mathcal{H}^\pm}$, and $\overline{A} \cap \mathcal{H}^\pm = A$. Note that the relative boundary of a subset $A \subseteq \mathcal{H}^\pm$ equals $(\partial A) \cap \mathcal{H}^\pm$.

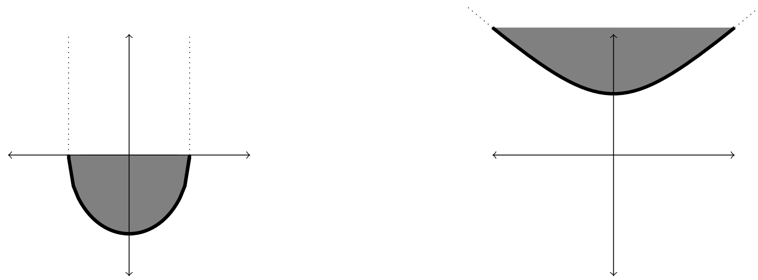

FIGURE 2. A semicircle and its obverse, which is a branch of a hyperbola.

Lemma 5.4. *The two diffeomorphisms $I^{\pm}$ transform smooth, connected, locally strongly convex hypersurfaces to smooth, connected, locally strongly convex hypersurfaces.*

Proof. Let $M \subseteq \mathcal{H}^{\pm}$ be a smooth, connected hypersurface. A locally supporting relative half-space at the point $y \in M$ is a proper, relative half-space $A \subseteq \mathcal{H}^{\pm}$ with $y \in \partial A$ such that $A \supseteq M \cap U$ for some open neighborhood $U \subseteq \mathcal{H}^{\pm}$ of the point y.

A smooth, connected hypersurface $M \subseteq \mathcal{H}^{\pm}$ is locally strongly convex if and only if for any $y \in M$ there is a unique locally supporting relative half-space at the point y that varies smoothly in $y \in M$ and without critical points.

The diffeomorphisms $I^{\pm}$ induce a diffeomorphism between the space of proper, relative half-spaces of $\mathcal{H}^{+}$ and the space of proper, relative half-spaces of $\mathcal{H}^{-}$, as we see from the proof of Lemma 5.1. Thus, if $M \subseteq \mathcal{H}^{\pm}$ is a smooth, connected, locally strongly convex hypersurface, then the same is true for $I^{\pm}(M)$. The lemma is thus proven. $\square$

We say that a subset $A \subseteq \mathcal{H}^{\pm}$ is bounded from below if there exists $(x_0, t_0) \in \mathcal{H}^{\pm}$ such that

$$t > t_0 \text{ for all } (x,t) \in A.$$

It is easy to verify that if $A \subseteq \mathcal{H}^{\pm}$ is bounded from below, then its obverse is also bounded from below.

Lemma 5.5. *Let $L \subseteq \mathbb{R}^n$ be a bounded, open, convex set containing the origin. Let $B \subseteq \mathcal{H}^{-}$ be a relatively closed, convex set that is bounded from below. Assume that the set $(\partial B) \cap \mathcal{H}^{-}$ is a smooth, connected, locally strongly convex hypersurface, while $(\partial B) \setminus \mathcal{H}^{-} = \{(x, 0)\,;\, x \in L^{\circ}\}$.*

Then there exists a proper, convex function $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\overline{\mathrm{Dom}(\psi)} = \overline{L}$ that is smooth and strongly convex in L, with $\nabla\psi(L) = \mathbb{R}^n$, $R\psi(0) < 0$ and $\overline{B} = \mathrm{Epigraph}(\psi)^{\circ}$. Moreover, $I^{-}(B) = \mathrm{Epigraph}(\varphi)$ where $\varphi = \psi^*$.

Proof. Since $B \subseteq \mathcal{H}^{-}$, for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$,

$$(x,t) \in B^{\circ} \quad \Longrightarrow \quad (x, t+r) \in B^{\circ}.$$

Therefore, the closed set B° satisfies $B^{\circ} = \mathrm{Epigraph}(\psi)$ where $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined via

$$\psi(x) = \inf\{t \in \mathbb{R}\,;\, (x,t) \in B^{\circ}\}.$$

Here, $\inf \varnothing = +\infty$. Since $B^{\circ} \subseteq \mathbb{R}^{n+1}$ is closed, convex and it contains the origin, the function ψ is necessarily proper and convex. The set $\overline{B}$ is closed, convex and it contains the origin, as follows from our assumptions. Since $\overline{B}^{\circ} = B^{\circ} = \mathrm{Epigraph}(\psi)$ while $B \subseteq \mathcal{H}^{-}$ is relatively closed,

$$\overline{B} = \mathrm{Epigraph}(\psi)^{\circ} \text{ and } B = \overline{B} \cap \mathcal{H}^{-} = \mathrm{Epigraph}(\psi)^{\circ} \cap \mathcal{H}^{-}. \tag{4}$$

The set $B \subseteq \mathcal{H}^-$ is bounded from below, hence there exists $t_0 < 0$ such that $t > t_0$ for all $(x,t) \in B$. Therefore $(0, 1/t_0) \in B^\circ$ and thus $\psi(0) < 0$. Denote $\varphi = \psi^*$. Then $\varphi\colon \mathbb{R}^n \to (0, +\infty]$ is proper and convex. By (4) and Proposition 5.3,

$$A := I^-(B) = I^-(\mathrm{Epigraph}(\psi)^\circ \cap \mathcal{H}^-) = \mathrm{Epigraph}(\varphi) \tag{5}$$

and moreover,

$$(\partial B) \setminus \mathcal{H}^- = \overline{B} \setminus \mathcal{H}^- = \mathrm{Epigraph}(\psi)^\circ \setminus \mathcal{H}^- = \{(x,0)\,;\, x \in \mathrm{Dom}(\psi)^\circ\}. \tag{6}$$

However, $(\partial B) \setminus \mathcal{H}^- = \{(x,0)\,;\, x \in L^\circ\}$ according to our assumptions. From (6) we thus deduce that $L^\circ = \mathrm{Dom}(\psi)^\circ$ and $\overline{L} = \overline{\mathrm{Dom}(\psi)}$. Since $\mathrm{Dom}(\psi) \subseteq \mathbb{R}^n$ is bounded and $\varphi = \psi^*$, necessarily

$$\mathrm{Dom}(\varphi) = \mathbb{R}^n \tag{7}$$

by [26, Corollary 13.3.3]. The map $\mathcal{I}^-$ is a homeomorphism, and hence it transforms the relative boundary of $B \subseteq \mathcal{H}^-$, which is the set $(\partial B) \cap \mathcal{H}^-$, to the relative boundary of $A \subseteq \mathcal{H}^+$, which is the set $(\partial A) \cap \mathcal{H}^+$. Since the relative boundary $(\partial B) \cap \mathcal{H}^-$ is a smooth, connected, locally strongly convex hypersurface, Lemma 5.4 implies that also the hypersurface

$$(\partial A) \cap \mathcal{H}^+ = I^-((\partial B) \cap \mathcal{H}^-)$$

is smooth, connected and locally strongly convex. Since $\inf \varphi = -\psi(0) > 0$, the relations (5) and (7) imply that

$$(\partial A) \cap \mathcal{H}^+ = \partial A = \mathrm{Graph}_{\mathbb{R}^n}(\varphi).$$

Hence $\mathrm{Graph}_{\mathbb{R}^n}(\varphi)$ is a smooth, connected, locally strongly convex hypersurface. Consequently $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is smooth and strongly convex. This implies that the set $\nabla\varphi(\mathbb{R}^n)$ is the interior of $\mathrm{Dom}(\varphi^*)$ (see, e.g., [26, Theorem 26.5] or [17, Section 1.2]). We conclude that $\nabla\varphi(\mathbb{R}^n) = L$, and [26, Theorem 26.5] shows that the function $\psi = \varphi^*$ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$. We have thus verified all of the conclusions of the lemma. □

There are two convex epigraphs that are associated with the convex set $B \subseteq \mathcal{H}^-$ from Lemma 5.5: the obverse of B is $\mathrm{Epigraph}(\varphi)$ while the polar of B is $\mathrm{Epigraph}(\psi)$. We may think about this triplet of convex sets as three different "coordinate systems" for describing the affine hemisphere equation. We will shortly see that $\partial B \cap \mathcal{H}^-$ is an affine hemisphere centered at the origin if and only if $\mathrm{Epigraph}_L(\psi)$ is affinely spherical with center at the origin, which happens if and only if φ satisfies $\det \nabla^2\varphi = C/\varphi^{n+2}$. Recall that for a smooth hypersurface $M \subseteq \mathbb{R}^{n+1}$ and $y \in M$, we view the tangent space T_yM as an n-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Definition 5.6. Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly convex hypersurface. Assume that $y \notin T_yM$ for all $y \in M$. For $y \in M$ define the vector $\nu_y \in \mathbb{R}^{n+1}$ via the requirements that

$$\langle \nu_y, y\rangle = 1, \quad \nu_y \perp T_yM.$$

We refer to $\nu\colon M \to \mathbb{R}^{n+1}$ as the "polarity map". We define the "polar hypersurface" M^* via

$$M^* := \nu(M) = \{\nu_y\,;\, y \in M\}.$$

What is the relationship between polar hypersurfaces and polar bodies? If $S \subseteq \mathbb{R}^{n+1}$ is a convex set and if $M \subseteq \partial S$ is a smooth, connected, locally strongly convex hypersurface for which the polarity map is well defined, then $M^* \subseteq \partial S^\circ$. Thus, Definition 5.6 provides a local version of the theory of convex duality: a piece of the boundary of S is polar to a certain piece of the boundary of S°.

Suppose that $M \subseteq \mathbb{R}^{n+1}$ is a smooth, connected, locally strongly convex hypersurface such that $y \notin T_y M$ for all $y \in M$. It is well known that M^* is always a smooth, connected, locally strongly convex hypersurface such that $y \notin T_y M^*$ for all $y \in M^*$. Furthermore, the polarity map $\nu\colon M \to M^*$ is a diffeomorphism, and its inverse is the polarity map associated with M^*. In particular, $(M^*)^* = M$.

Lemma 5.7. *Let $L \subseteq \mathbb{R}^n$ be an open, bounded, convex set containing the origin. Let $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex function with $\psi(0) < 0$ such that $\overline{L} = \overline{\mathrm{Dom}(\psi)}$. Assume that ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$. Denote*

$$M = \mathrm{Graph}_L(\psi) \text{ and } \widetilde{K} = \mathrm{Epigraph}(\psi)^\circ.$$

Then M^* is well defined, the convex set $\widetilde{K}$ is compact with $\dim(\widetilde{K}) = (n+1)$, and

$$(\partial\widetilde{K}) \cap \mathcal{H}^- = M^* \text{ while } (\partial\widetilde{K}) \setminus \mathcal{H}^- = \{(x,0)\,;\, x \in L^\circ\}\,. \tag{8}$$

Proof. Define $\varphi = \psi^*$. Since $\nabla\psi(L) = \mathbb{R}^n$, necessarily $\mathrm{Dom}(\varphi) = \mathbb{R}^n$ by [26, Corollary 13.3.3]. Since $\psi(0) < 0$, the function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is positive and convex. Denote $A = \mathrm{Epigraph}(\varphi) \subseteq \mathcal{H}^+$ and $B = \widetilde{K} \cap \mathcal{H}^-$. By Proposition 5.3,

$$B = \widetilde{K} \cap \mathcal{H}^- = \mathrm{Epigraph}(\psi)^\circ \cap \mathcal{H}^- = I^+(\mathrm{Epigraph}(\varphi)) = I^+(A). \tag{9}$$

Since $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is convex and positive, we may assert that $\partial A \cap \mathcal{H}^+ = \partial A = \mathrm{Graph}_{\mathbb{R}^n}(\varphi)$. Consequently

$$\partial\widetilde{K} \cap \mathcal{H}^- = \partial B \cap \mathcal{H}^- = I^+(\partial A \cap \mathcal{H}^+) = I^+(\mathrm{Graph}_{\mathbb{R}^n}(\varphi)). \tag{10}$$

Since ψ is smooth in L, the identity $\psi(x) + \varphi(\nabla\psi(x)) = \langle x, \nabla\psi(x)\rangle$ is fulfilled for all $x \in L$. The fact that $\nabla\psi(L) = \mathbb{R}^n$ thus implies

$$\mathrm{Graph}_{\mathbb{R}^n}(\varphi) = \{(\nabla\psi(x), \langle x, \nabla\psi(x)\rangle - \psi(x)) \in \mathbb{R}^n \times \mathbb{R}\,;\, x \in L\}\,. \tag{11}$$

Note that $\langle x, \nabla\psi(x)\rangle - \psi(x) = \varphi(\nabla\psi(x)) > 0$ for all $x \in L$, and hence ν_y is well defined indeed. From Definition 5.6, it follows that for $x \in L$ and $y = (x, \psi(x)) \in \mathrm{Graph}_L(\psi)$,

$$\nu_y = \frac{(\nabla\psi(x), -1)}{\langle x, \nabla\psi(x)\rangle - \psi(x)} = I^+\left\{(\nabla\psi(x), \langle x, \nabla\psi(x)\rangle - \psi(x))\right\}. \tag{12}$$

Since $M = \mathrm{Graph}_L(\psi)$ and $M^* = \nu(M)$, by (10), (11) and (12),

$$M^* = \nu(\mathrm{Graph}_L(\psi)) = I^+(\mathrm{Graph}_{\mathbb{R}^n}(\varphi)) = \partial\widetilde{K} \cap \mathcal{H}^-. \tag{13}$$

Proposition 5.3 shows that $\widetilde{K} = \mathrm{Epigraph}(\psi)^\circ \subseteq \overline{\mathcal{H}^-}$. In fact, according to Proposition 5.3,

$$(\partial\widetilde{K}) \setminus \mathcal{H}^- = \widetilde{K} \setminus \mathcal{H}^- = \{(x,0)\,;\, x \in \mathrm{Dom}(\psi)^\circ\} = \{(x,0)\,;\, x \in L^\circ\}. \tag{14}$$

Now (8) follows from (13) and (14). From (8) it follows that $\dim(\widetilde{K}) = n+1$, because the convex set $\widetilde{K}$ affinely spans the hyperplane $\partial\mathcal{H}^-$ while it also contains points outside this hyperplane. Moreover, since $0 \in L$ and $\psi(0) < 0$, the convex set $\mathrm{Epigraph}(\psi)$ contains a neighborhood of the origin in $\mathbb{R}^{n+1}$. Therefore the closed set $\widetilde{K} = \mathrm{Epigraph}(\psi)^\circ$ is bounded, and hence it is compact. □

Recall from Proposition 4.5 that N_y is the Euclidean unit normal to M at the point y that is pointing to the concave side of M. Recall also that we denote $\rho_y = \langle N_y, y\rangle$. It follows from Definition 5.6 that if $\rho_y \neq 0$ for all $y \in M$, then the polarity map is well defined, and

$$\nu_y = \frac{N_y}{\rho_y} \text{ for all } y \in M. \tag{15}$$

The map $N\colon M \to S^n$ is the Gauss map associated with M, and we see that the polarity map is proportional to the Gauss map. We define the *cone measure* on a smooth hypersurface $M \subseteq \mathbb{R}^{n+1}$ to be the measure μ_M supported on M whose density with respect to the surface area measure on M is the function $y \mapsto |\rho_y|/(n+1)$. The reason for the term "cone measure" is that for any Borel subset $S \subseteq M$ that does not contain two distinct points on the same ray from the origin,

$$\mu_M(S) = \mathrm{Vol}_{n+1}\left(\{tx;\, 0 \le t \le 1,\, x \in S\}\right).$$

Proposition 5.8. *Let $M \subseteq \mathbb{R}^{n+1}$ be a smooth, connected, locally strongly convex hypersurface. Then M is affinely spherical with center at the origin if and only if the following is true: the polarity map $\nu : M \to M^*$ is well defined, and it pushes forward the cone measure μ_M to a measure proportional to the cone measure μ_{M^*}.*

Proof. If M is affinely spherical with center at the origin, then the polarity map of M is well defined, because $\rho_y \neq 0$ for all $y \in M$ according to Proposition 4.5. For $y \in M$ let $S_y\colon T_yM \to T_yM$ be the shape operator associated with the Euclidean unit normal N. Then $\det(S_y)$ is the Gauss curvature $K_y > 0$. For any vector field X tangent to M we have

$$D_X\nu = D_X\left(N/\rho\right) = \frac{S(X)}{\rho} - \frac{D_X\rho}{\rho^2}N, \tag{16}$$

where $D_X\nu \in \mathbb{R}^{n+1}$ is the derivative of ν in the direction of X. Write $D\nu\colon TM \to TM^*$ for the differential of the smooth polarity map ν. Then for any $y \in M$, the map $(D\nu)_y$ is a linear map from the tangent space $T_y\mathcal{M} = \nu_y^\perp$ to the tangent space $T_{\nu_y}M^* = y^\perp$. Here, $y^\perp$ is the hyperplane orthogonal to y in $\mathbb{R}^{n+1}$. From (16), for any $y \in M$ and $u \in T_yM$,

$$S_y(u) = \rho_y \cdot \mathrm{Proj}_{\nu_y^\perp}\left((D\nu)_y(u)\right), \tag{17}$$

where $\mathrm{Proj}_{\nu_y^\perp}$ is the orthogonal projection operator onto $\nu_y^\perp$ in $\mathbb{R}^{n+1}$. The operator $\mathrm{Proj}_{\nu_y^\perp}\colon y^\perp \to \nu_y^\perp$ distorts n-dimensional volumes by a factor of $|\langle y, \nu_y\rangle|/(|y||\nu_y|)$. The linear map $(D\nu)_y\colon \nu_y^\perp \to y^\perp$ distorts volumes by a factor of $|\det(D\nu)_y|$. Hence, by (17), for any $y \in M$,

$$K_y = \det(S_y) = |\rho_y|^n \cdot \frac{|\langle y, \nu_y\rangle|}{|y|\,|\nu_y|} \cdot |\det(D\nu)_y| = \frac{|\det(D\nu)_y|}{|y|\,|\nu_y|^{n+1}}, \tag{18}$$

where we have used (15) in the last passage. In fact, according to (15), the cone measure μ_M has density $y \mapsto 1/((n+1)|\nu_y|)$ with respect to the surface area measure on M. Denote by θ the measure on M whose density with respect to the surface area measure is $K_y|\nu_y|^{n+1}/(n+1)$.

Recalling that the polarity map of M^* is inverse to that of M, we deduce from (18) that ν pushes forward θ to the cone measure μ_{M^*}. Consequently, ν pushes forward μ_M to a measure proportional to μ_{M^*} if and only if θ is proportional to μ_M, i.e., if and only if there exists $C > 0$ such that

$$K_y|\nu_y|^{n+1}/(n+1) = C/((n+1)|\nu_y|) \text{ for all } y \in M. \tag{19}$$

Recall that $1/|\nu_y| = |\rho_y|$, and that ν and ρ are continuous on the connected manifold M. By Proposition 4.5, the hypersurface M is affinely spherical with center at the origin if and only if there exists $C > 0$ such that (19) holds true. This completes the proof. □

Since the polarity map of M^* is the inverse to the polarity map of M, Proposition 5.8 has the following well-known corollary.

Corollary 5.9. *Let $M \subseteq \mathbb{R}^{n+1}$ be an affinely spherical hypersurface with center at the origin. Then the polar hypersurface M^* is well defined, and it is again affinely spherical with center at the origin.*

Theorem 5.10. *Let $L \subseteq \mathbb{R}^n$ be an open, bounded, convex set containing the origin. Then the following are equivalent:*

(i) *the barycenter of L lies at the origin;*
(ii) *there exists a proper, convex function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\overline{\mathrm{Dom}(\psi)} = \overline{L}$ such that $\mathrm{Graph}_L(\psi)$ is affinely spherical with center at the origin, and such that ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$ and $\psi(0) < 0$.*

Moreover, assuming (i) *or* (ii), *the function ψ from* (ii) *is uniquely determined up to multiplication by a positive scalar $\lambda > 0$ and addition of a linear function $\ell(x) = \langle x, v\rangle$, for some $v \in \mathbb{R}^n$.*

Proof. Assume (i). According to Theorem 3.10, there exists a smooth, positive, convex function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ with $\nabla\varphi(\mathbb{R}^n) = L$ such that

$$\det \nabla^2\varphi = \frac{C}{\varphi^{n+2}} \quad \text{in} \quad \mathbb{R}^n, \tag{20}$$

for some constant $C > 0$. Denote $\psi = \varphi^*$. From [26, Theorem 26.5] we know that $\overline{\mathrm{Dom}(\psi)} = \overline{L}$ and that ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$. According to Proposition 4.4, equation (20) implies that $\mathrm{Graph}_L(\psi)$ is affinely spherical with center at the origin. The infimum of φ is attained and is positive because $0 \in L$. Hence $\psi(0) < 0$, and we have verified all conclusions in (ii).

Next, assume (ii) and let us prove (i). Denote $\varphi = \psi^*$. Since $\overline{L} = \overline{\mathrm{Dom}(\psi)}$ is a bounded set, necessarily $\mathrm{Dom}(\varphi) = \mathbb{R}^n$ by [26, Corollary 13.3.3]. Since ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$ and $\psi(0) < 0$, necessarily φ is a positive, smooth, strongly convex function in $\mathbb{R}^n$ with $\nabla\varphi(\mathbb{R}^n) = L$. Since $\mathrm{Graph}_L(\psi)$ is affinely spherical with center at the origin, Proposition 4.4 shows that (20) holds true. Theorem 3.10 now implies (i). Moreover, Theorem 3.10 states that φ is uniquely determined up to translations and dilations, implying that ψ is determined up to the transformation described above. $\square$

Let $K \subseteq \mathbb{R}^n$ be an n-dimensional, nonempty, bounded, convex set. The *Santaló point* of K is a unique point $z(K) \in \mathbb{R}^n$ such that

$$\mathrm{Vol}_n((K - z(K))^\circ) = \inf_{z \in \mathbb{R}^n} \mathrm{Vol}_n(K - z)^\circ$$

where $K - z = \{x - z\,;\, x \in K\}$. The Santaló point of K is well defined and it belongs to the interior of K, see [22, Section 7.4]. The Santaló point of K satisfies $z(K) = 0$ if and only if the barycenter of K° is well defined and it lies at the origin. The Santaló point is affinely invariant: for any invertible, affine transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ we know that $z(T(K)) = T(z(K))$. Hence the Santaló point is well defined for any nonempty, bounded, convex set embedded in some finite-dimensional real linear space.

Proof of the existence part of Theorem 1.2. Applying an affine transformation in $\mathbb{R}^{n+1}$, we may assume that the Santaló point of K lies at the origin, and that

$$K \subseteq \{(x, 0);\, x \in \mathbb{R}^n\}.$$

Write $K_1 \subseteq \mathbb{R}^n$ for the interior of the set $\big\{x \in \mathbb{R}^n;\, (x,0) \in K\big\}$. Then $K_1 \subseteq \mathbb{R}^n$ is an open, convex set whose Santaló point lies at the origin. Hence $K_1^\circ \subseteq \mathbb{R}^n$ is a compact, convex set containing zero in its interior such that the barycenter of K_1° lies at the origin.

Write $L \subseteq \mathbb{R}^n$ for the interior of K_1°. From Theorem 5.10 it follows that there exists a proper, convex function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\overline{\mathrm{Dom}(\psi)} = \overline{L}$ such that

$$M := \mathrm{Graph}_L(\psi)$$

is affinely spherical with center at the origin. Moreover, $\nabla\psi(L) = \mathbb{R}^n$ and $\psi(0) < 0$. Denote

$$\widetilde{K} = \mathrm{Epigraph}(\psi)^\circ.$$

According to Corollary 5.9, the hypersurface M^* is affinely spherical with center at the origin. Furthermore, Lemma 5.7 shows that $\widetilde{K} \subseteq \mathbb{R}^{n+1}$ is an $(n+1)$-dimensional, compact convex set and

$$M^* = (\partial\widetilde{K}) \cap \mathcal{H}^- \quad \text{while} \quad (\partial\widetilde{K}) \setminus \mathcal{H}^- = L^\circ \times \{0\} = K.$$

Consequently $M^* \subseteq \mathcal{H}^-$ does not intersect the hyperplane $\partial\mathcal{H}^-$ that contains K, while $\partial\widetilde{K} = M^* \cup K$. According to Definition 1.1, the hypersurface M^* is an affine hemisphere with anchor K, which is centered at the Santaló point of K. $\square$

Proposition 5.11. *Let $L \subseteq \mathbb{R}^n$ be a bounded, open, convex set containing the origin. Let $M \subseteq \mathcal{H}^-$ be an affine hemisphere with anchor $L^\circ \times \{0\} \subseteq \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and center at the origin. Then M^* is well defined, and there exists a function ψ as in Theorem* 5.10 (ii) *such that $M^* = \mathrm{Graph}_L(\psi)$.*

Proof. The hypersurface $M \subseteq \mathcal{H}^-$ is an affine hemisphere with anchor $K = L^\circ \times \{0\}$ which is centered at the origin. Let $\widetilde{K}$ be as in Definition 1.1. Denote $B = \widetilde{K} \cap \mathcal{H}^-$ which is a convex, relatively closed subset of $\mathcal{H}^-$ with $\overline{B} = \widetilde{K}$. The convex set B is bounded from below in $\mathcal{H}^-$ because $\widetilde{K}$ is compact. Moreover, by Definition 1.1 the set

$$M = (\partial\widetilde{K}) \cap \mathcal{H}^- = (\partial B) \cap \mathcal{H}^- \tag{21}$$

is a smooth, connected, locally strongly convex hypersurface. Additionally, from Definition 1.1 it follows that

$$(\partial B) \setminus \mathcal{H}^- = (\partial\widetilde{K}) \setminus \mathcal{H}^- = K = L^\circ \times \{0\}. \tag{22}$$

Thus the relatively closed, convex set $B \subseteq \mathcal{H}^-$ satisfies all of the requirements of Lemma 5.5. From the conclusion of Lemma 5.5, there exists a proper, convex function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$\mathrm{Epigraph}(\psi)^\circ = \overline{B} = \widetilde{K} \tag{23}$$

and such that $\psi(0) < 0, \overline{\mathrm{Dom}(\psi)} = \overline{L}$ while ψ is smooth and strongly convex in L with $\nabla\psi(L) = \mathbb{R}^n$. Thanks to (21) and (23), Lemma 5.7 shows that

$$\mathrm{Graph}_L(\psi) = M^*.$$

Since M is affinely spherical with center at the origin, Corollary 5.9 gives that $\mathrm{Graph}_L(\psi)$ is also affinely spherical with center at the origin. Hence the function ψ satisfies all of the conditions of Theorem 5.10 (ii), and the proposition is proven. $\square$

Proof of the uniqueness part of Theorem 1.2. Suppose that M is an affine hemisphere with anchor K, and let $\widetilde{K}$ be as in Definition 1.1. By applying an affine transformation in $\mathbb{R}^{n+1}$, we may assume that M is affinely spherical with center at the origin, and that

$$K \subseteq \{(x, 0)\,;\, x \in \mathbb{R}^n\} \text{ while } \widetilde{K} \subseteq \overline{\mathcal{H}^-}. \tag{24}$$

Definition 1.1 implies that the origin belongs to the relative interior of the n-dimensional, compact, convex set K. Hence there exists a bounded, open, convex set $L \subseteq \mathbb{R}^n$ containing the origin such that $K = L^\circ \times \{0\}$. From (24) and Definition 1.1 we conclude that $M = \partial\widetilde{K} \cap \mathcal{H}^- \subseteq \mathcal{H}^-$. Proposition 5.11 shows that $M^* = \mathrm{Graph}_L(\psi)$ for a certain convex function $\psi\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ satisfying the requirements of Theorem 5.10 (ii).

Theorem 5.10 now implies that the barycenter of L lies at the origin, and hence the affine hemisphere M is centered at the Santaló point of K. According to Theorem 5.10, the function ψ is uniquely determined by L, up to multiplication by a positive scalar and addition of a linear function. It thus follows that the affine hemisphere $M = \mathrm{Graph}_L(\psi)^*$ with anchor $L^\circ \times \{0\}$ is uniquely determined by L, up to a linear transformation. Therefore M is determined by K up to an affine transformation, and the proof is complete. □

Remark 5.12. Let M be an affine hemisphere in $\mathbb{R}^{n+1}$ with center at the origin and anchor $K \subseteq \mathbb{R}^n \times \{0\}$. Let $\widetilde{K} \subseteq \mathbb{R}^n \times [0,\infty)$ be the convex body from Definition 1.1, so that $\partial\widetilde{K} = M \cup K$. For $(x,t) \in \mathbb{R}^n \times [0,\infty)$ set

$$\|(x,t)\|_{\widetilde{K}} = \inf\left\{\lambda > 0 \,:\, (x,t)/\lambda \in \widetilde{K}\right\},$$

the Minkowski functional of $\widetilde{K}$. Denote also $F(x,t) = \|(x,t)\|_{\widetilde{K}}^2/2$. Since the origin belongs to the relative interior of K, the function F is a finite, 2-homogenous, convex function in the half-space $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Note that the closure of the affine hemisphere M is a level set of the function F. It was noted by Bo Berndtsson that the function F satisfies

$$\begin{cases} \det \nabla^2 F(x,t) = C & \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty), \\ F(x,0) = \|x\|_K^2/2 & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{25}$$

where $C > 0$ is a positive constant and $\|x\|_K = \inf\{\lambda > 0\,;\, x/\lambda \in K\}$ is the Minkowski functional of K. Thus F solves the parabolic affine sphere equation $\det \nabla^2 F \equiv \mathit{Const}$ in a half-space, with boundary values that are 2-homogenous and convex. In order to prove the equation in (25), we argue as follows. The map ∇F restricted to M is precisely the polarity map of the affine hemisphere M. Since ∇F is 1-homogenous, for any measurable subset $A \subseteq M$ and $0 < \alpha < \beta$,

$$\{\nabla F(ty);\, y \in A,\, \alpha < t < \beta\} = \{tz;\, z \in \nu(A),\, \alpha < t < \beta\} \tag{26}$$

where $\nu\colon M \to M^*$ is the polarity map associated with M. Proposition 5.8 states that ν pushes forward the cone volume measure on M to a constant multiple of the cone volume measure on M^*. It thus follows from (26) that ∇F pushes forward the Lebesgue measure on $\widetilde{K}$ to a constant multiple of the Lebesgue measure on $\{ty\,;\, y \in M^*, t \in [0,1]\}$. Therefore the Jacobian of the map $y \mapsto \nabla F(y)$ has a constant determinant, and (25) is proved.

Added in proofs. An STL file for 3D-printing an affine hemisphere with a square base, courtesy of Quentin Merigot and Filippo Santambrogio , is available here: `http://www.weizmann.ac.il/math/klartag/sites/math/klartag/files/uploads/square_affine_hemisphere/stl`.

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