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COMBINATORIAL IDENTITIES FOR POLYHEDRAL CONES

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Dedicated to Professor Yuriĭ Dmitrievich Burago at the occasion of his 80th birthday

Abstract. Some known relations for convex polyhedral cones, involving angles or conical intrinsic volumes, are superficially of a metric character, but have indeed a purely combinatorial core. This fact is strengthened in some cases, with implications for valuations on polyhedral cones, and is worked out in the case of the extended Klivans–Swartz formula.

§1. Introduction

Let C be a convex polyhedral cone in $\mathbb{R}^d$, and let $\mathcal{F}(C)$ denote the set of its faces of dimensions $0, \dots, \dim C$. For faces $F \subseteq G$ of C, we denote by $\beta(F,G)$ the internal angle of G at F and by $\gamma(F,G)$ the external angle of G at F (see §2). We write o for the face $\{o\}$, where o is the origin of $\mathbb{R}^d$. The angle sum relations

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}\beta(F,C) = (-1)^d\beta(o,C), \tag{1}$$

$$\sum_{F\in\mathcal{F}(C)} \beta(o,F)\gamma(F,C) = 1, \tag{2}$$

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}\beta(o,F)\gamma(F,C) = 0 \tag{3}$$

are well known. Formula (1) is the *Sommerville relation*. Identities equivalent to generalizations of (2) and (3) appeared first in Santaló's work on spherical integral geometry; in particular, a consequence of (2) and (3) is related to the spherical Gauss–Bonnet theorem. McMullen [12] proved these (and more) relations by a combinatorial approach.

All these relations can be obtained, by integration, from purely combinatorial identities. A quite general combinatorial version of (1) appeared in [2]; see also §3. For (2) and (3), let $N(C,F)$ denote the normal cone of C at its face F. The cones $F+N(C,F)$, $F\in\mathcal{F}(C)$, form a tessellation of $\mathbb{R}^d$. In terms of characteristic functions,

$$\sum_{F\in\mathcal{F}(C)} \mathbf{1}_{F+N(C,F)}(x) = 1 \quad \text{for} \quad x\in\mathbb{R}^d\setminus(G_1\cup\dots\cup G_k), \tag{4}$$

where $G_1,\dots,G_k$ are facets of the cones $F+N(C,F)$, $F\in\mathcal{F}(C)$. Integration of this identity over $\mathbb{R}^d$ with the standard Gaussian measure, or over the unit sphere $\mathbb{S}^{d-1}$ with respect to the normalized spherical Lebesgue measure, yields relation (2). Similarly, (3)

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can be obtained from the identity

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}\mathbf{1}_{F-N(C,F)}(x) = 0 \text{ for } x\in\mathbb{R}^d\setminus U, \tag{5}$$

where U is an exceptional set determined by faces of dimensions less than $d-1$. This identity is due to McMullen; at the beginning of §3 in [12] he indicated a proof, which was carried out in [17, Theorem 6.5.5].

It is easily seen that (4) can be strengthened to the identity

$$\sum_{F\in\mathcal{F}(C)} \mathbf{1}_{\operatorname{relint} F+N(C,F)}(x) = 1 \text{ for } x\in\mathbb{R}^d, \tag{6}$$

where relint denotes the relative interior. This improvement, namely to an identity for characteristic functions holding everywhere, is irrelevant for the integration, yet from a combinatorial point of view, it contains considerably more information.

We can also write (6) as a relation for closed cones, using the identity

$$\mathbf{1}_{\operatorname{relint} F+N(C,F)}(x) = \sum_{G\in\mathcal{F}(F)} (-1)^{\dim F-\dim G}\mathbf{1}_{G+N(C,F)}(x) \tag{7}$$

(see §2), which yields

$$\sum_{F\in\mathcal{F}(C)} \mathbf{1}_{F+N(C,F)}(x) + \sum_{\substack{F,G\in\mathcal{F}(C)\\ G\subsetneqq F}} (-1)^{\dim F-\dim G}\mathbf{1}_{G+N(C,F)}(x) = 1. \tag{8}$$

Our first goal in this paper is to strengthen (5), proving it without the exceptional set U. This is in line with some recent efforts, in [9], to remove restrictions for the validity of certain combinatorial identities for polytopes.

Theorem 1.1. *If $C\subset\mathbb{R}^d$ is a polyhedral cone, not a subspace, then*

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}\mathbf{1}_{F-N(C,F)}(x) = 0 \tag{9}$$

for all $x\in\mathbb{R}^d$.

We shall prove Theorem 1.1 in §4, showing that McMullen's approach in [12], which uses the incidence algebra of the face lattice, works also at the level of characteristic functions. For this, we need a version of the Sommerville relation at the same level, which will be provided in §3.

Recall that a *valuation* on the set $\mathcal{P}C^d$ of polyhedral cones in $\mathbb{R}^d$ is a mapping φ from $\mathcal{P}C^d$ into some Abelian group with the property that $\varphi(P\cup Q)+\varphi(P\cap Q) = \varphi(P)+\varphi(Q)$ whenever $P, Q, P\cup Q\in\mathcal{P}C^d$.

Corollary 1.1. *Let φ be a valuation on $\mathcal{P}C^d$, and let $C\in\mathcal{P}C^d$. Then*

$$\sum_{F\in\mathcal{F}(C)} \varphi(F+N(C,F)) + \sum_{\substack{F,G\in\mathcal{F}(C)\\ G\subsetneqq F}} (-1)^{\dim F-\dim G}\varphi(G+N(C,F)) = \varphi(\mathbb{R}^d), \tag{10}$$

and if C is not a subspace, then

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}\varphi(F-N(C,F)) = 0. \tag{11}$$

By $\mathcal{F}_k(C)$ we denote the set of k-dimensional faces of a polyhedral cone C, $k \in \{0, \dots, \dim C\}$. Let Γ_d denote the standard Gaussian probability measure on $\mathbb{R}^d$. The *conical intrinsic volumes* are defined by

$$v_k(C) = \sum_{F \in \mathcal{F}_k(C)} \Gamma_d(F + N(C,F)) = \sum_{F \in \mathcal{F}_k(C)} \beta(o,F)\gamma(F,C)$$

for $C \in \mathcal{PC}^d$ and $k = 0, \dots, d$. The second identity follows from a well-known property of the Gaussian measure (note that $v_d(C) = \Gamma_d(C)$, and that $N(C,o) = C^\circ$, the polar cone of C). We see that for the special valuation $\varphi = v_d$, (10) and (11) reduce to (2) and (3) (note that $\dim(G + N(C,F)) < d$ in the second sum of (10)). Thus, Corollary 1.1 can be viewed as the most general version of the relations (2) and (3).

Another corollary can be regarded as a general version of the spherical Gauss–Bonnet relation.

Corollary 1.2. *Let φ by a valuation on $\mathcal{PC}^d$ invariant under the orthogonal group* $\mathrm{O}(d)$*, and let $C \in \mathcal{PC}^d$. Then*

$$2 \sum_{\substack{F \in \mathcal{F}(C) \\ 2 \mid \dim F}} \varphi(F + N(C,F)) + \sum_{\substack{F,G \in \mathcal{F}(C) \\ G \subsetneqq F}} (-1)^{\dim F - \dim G} \varphi(G + N(C,F)) = \varphi(\mathbb{R}^d). \tag{12}$$

This follows by adding (10) and (11) and by noting that $F - N(C,F)$ is the image of $F + N(C,F)$ under an orthogonal transformation. Applying (12) to the special valuation $\varphi = v_d$ and assuming that C is not a subspace, we obtain

$$2 \sum_{2 \mid k} v_k(C) = 1.$$

For the intersection of the cone C with the unit sphere $\mathbb{S}^{d-1}$, this yields a version of the spherical Gauss–Bonnet relation (see, e.g., [17, p. 258], and compare [15, (17.21), (17.22)]).

Our next topic is the combinatorial core of the extended Klivans–Swartz formula. This refers to a central hyperplane arrangement $\mathcal{A}$, that is, a finite set of subspaces of $\mathbb{R}^d$ of codimension one. Its intersection lattice $\mathcal{L}(\mathcal{A})$ is the set of all intersections of hyperplanes from $\mathcal{A}$, partially ordered by reverse inclusion. Let μ be the Möbius function of $\mathcal{L}(\mathcal{A})$ (see, e.g., Stanley [20, 3.7], or §2). For $j \in \{0, \dots, d\}$, let $\mathcal{L}_j(\mathcal{A}) = \{S \in \mathcal{L}(\mathcal{A}) : \dim S = j\}$. The jth-level characteristic polynomial of $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A},j}(t) = \sum_{L \in \mathcal{L}_j(\mathcal{A})} \sum_{S \in \mathcal{L}(\mathcal{A})} \mu(L,S) t^{\dim S} \tag{13}$$

$$= \sum_{m=0}^{j} a_{jm} t^m, \tag{14}$$

where (14) defines the coefficients a_{jm}, $m = 0, \dots, j$. We denote by $\mathcal{R}_j(\mathcal{A})$ the set of all j-dimensional cones in the tessellation of $\mathbb{R}^d$ induced by $\mathcal{A}$, that is, of all j-faces of the cones in $\mathcal{R}_d(\mathcal{A})$, where the elements of $\mathcal{R}_d(\mathcal{A})$ are the closures of the components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{A}} H$. The extended Klivans–Swartz formula says that

$$\sum_{F \in \mathcal{R}_j(\mathcal{A})} v_k(F) = (-1)^{j-k} a_{jk} \tag{15}$$

for $j \in \{0, \dots, d\}$ and $k \in \{0, \dots, j\}$. For $j = d$, it was proved by Klivans and Swartz [10]; a different proof was given in [8]. The general case is due to Amelunxen and Lotz [1].

The crucial point of (15) is that the left side, which involves the metric functionals v_k, depends only on the partial order of $\mathcal{L}(\mathcal{A})$ and thus is a combinatorial quantity. For some special cases of (15), from [1] it is obvious that they have a combinatorial character. For example, if $j \in \{0,1\}$, then the values $v_k(F)$ $(k \le j)$ are constants, hence (15) follows from [1, (2.16)]. Also the case $k = j$ of (15) is purely combinatorial, since for $L \in \mathcal{L}_j(\mathcal{A})$ we have

$$\sum_{F \in \mathcal{R}_j(\mathcal{A}),\, F \subseteq L} v_j(F) = 1,$$

so that $\sum_{F \in \mathcal{R}_j(\mathcal{A})} v_j(F) = |\mathcal{L}_j(\mathcal{A})|$, as was noted in [1] after Theorem 6.1.

The following theorem reduces the remaining cases of (15) to their combinatorial core. Recall that subspaces $L, M \subset \mathbb{R}^d$ are said to be in general position if $\dim(L \cap M) = \max\{0, \dim L + \dim M - d\}$. A subspace $L \subset \mathbb{R}^d$ is in general position with respect to $\mathcal{L}(\mathcal{A})$ if it is in general position with respect to each element of $\mathcal{L}(\mathcal{A})$.

Theorem 1.2. *Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{R}^d$, and let $\chi_{\mathcal{A},j}(t) = \sum_{m=0}^{j} a_{jm} t^m$ be its jth-level characteristic polynomial. Let $j \in \{1, \dots, d\}$. Let $L \subset \mathbb{R}^d$ be a subspace that is in general position with respect to $\mathcal{L}(\mathcal{A})$.*

If $\dim L = 1$, then

$$\sum_{F \in \mathcal{R}_j(\mathcal{A})} \mathbf{1}\{L \cap F^\circ \ne \{o\}\} = 2(-1)^j a_{j0}. \tag{16}$$

If $\dim L = d - k$ with $k \in \{1, \dots, j-1\}$, then

$$\sum_{F \in \mathcal{R}_j(\mathcal{A})} \mathbf{1}\{L \cap F \ne \{o\}\} = (-1)^{j-k} \left[\sum_{i=0}^{k} a_{ji} + \sum_{i=k+1}^{j} a_{ji} (-1)^{i-k} \right]. \tag{17}$$

Relation (16) can be read off from the proof of Theorem 6.1 in [1]. We shall prove (17) in §5 and show there how Theorem 1.2 yields (15) by integration. This approach extends the proof that Kabluchko, Vysotsky and Zaporozhets [8] gave for the original Klivans–Swartz formula (the case where $j = d$). We think that the combinatorial relation (17) is of independent interest.

§2. Preliminaries

The d-dimensional real vector space $\mathbb{R}^d$ is equipped with its standard scalar product $\langle \cdot\,, \cdot \rangle$ and induced norm $\|\cdot\|$. Its unit sphere is given by $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. A *linear hyperplane* is a linear subspace of codimension one, and a *hyperplane* is a translate of a linear hyperplane. Every hyperplane bounds two closed halfspaces.

By a *polyhedron* in $\mathbb{R}^d$ we understand the intersection of a finite family of closed halfspaces. The family may be empty, so that by convention also $\mathbb{R}^d$ is viewed as a polyhedron. All polyhedra are convex and closed. A nonempty bounded polyhedron is called a *polytope*. A polyhedron $P \ne \varnothing$ is a *polyhedral cone* if $x \in P$ implies $\lambda x \in P$ for all $\lambda \ge 0$. We denote by $\mathcal{Q}^d$ the set of polyhedra (since $\mathcal{P}^d$ is reserved for the set of polytopes) and by $\mathcal{PC}^d$ the set of polyhedral cones in $\mathbb{R}^d$.

The relative interior of a polyhedron (that is, the interior with respect to its affine hull) is called a ro-polyhedron (this is not a polyhedron, as it is not closed, except if it is one-pointed). We denote by $\mathcal{Q}^d_{ro}$ the set of ro-polyhedra in $\mathbb{R}^d$.

The intersection of a nonempty polyhedron P with a supporting hyperplane is again a polyhedron; it is called a *face* of P. The polyhedron P is, by definition, also a face of itself. A polyhedron P has finitely many faces, of dimensions $0, \dots, \dim P$. We denote by $\mathcal{F}_k(P)$ the set of k-dimensional faces of P, for $k = 0, \dots, \dim P$, and by $\mathcal{F}(P)$ the set of all faces of P.

With a polyhedron $P \in \mathcal{Q}^d$, we associate the following types of polyhedral cones. The cone of exterior normal vectors (including the zero vector o) of a polyhedron P at a face

F is denoted by $N(P,F)$. The *angle cone* (also known as tangent cone) of P at a face F of P is defined by

$$A(F,P) = \operatorname{pos}(P - z_0),$$

for any $z_0 \in \operatorname{relint} F$; here pos denotes the positive hull. The *recession cone* of P is defined by

$$\operatorname{rec} P = \{y \in \mathbb{R}^d : x + \lambda y \in P \text{ for all } x \in P \text{ and all } \lambda \geq 0\}.$$

At this point, we recall the internal and external angles. With different notation, they were introduced in [5, Chapter 14] (and generalized in [4]). Let σ_k denote the spherical Lebesgue measure on the unit sphere $\mathbb{S}^k$. Let P be a polyhedron, and let F be a face of P. The *internal angle* of P at F is defined by

$$\beta(F,P) = \frac{\sigma_{k-1}(A(F,P) \cap \mathbb{S}^{d-1})}{\sigma_{k-1}(\mathbb{S}^{k-1})} \text{ with } k = \dim P.$$

The *external angle* of P at F is defined by

$$\gamma(F,P) = \frac{\sigma_{d-m-1}(N(P,F) \cap \mathbb{S}^{d-1})}{\sigma_{d-m-1}(\mathbb{S}^{d-m-1})} \text{ with } m = \dim F.$$

Let $P \in \mathcal{Q}^d$ be a nonempty polyhedron. For $x \in \mathbb{R}^d$, there is a unique point $p(P,x) \in P$ such that $\|x - p(P,x)\| \leq \|x - y\|$ for all $y \in P$. This gives rise to the *metric projection* $p(P,\cdot)\colon \mathbb{R}^d \to P$, also called the *nearest-point* map of P (see, e.g., [16, 1.2]). Since each polyhedron is the disjoint union of the relative interiors of its faces, for each $x \in \mathbb{R}^d$ there is a unique face $F \in \mathcal{F}(P)$ with $p(P,x) \in \operatorname{relint} F$. Since $p(P,x) - x \in N(P,F)$, relation (6) follows immediately.

We recall some known facts about valuations. Let $\mathcal{S}$ be an intersectional family of sets, that is, a family satisfying $A \cap B \in \mathcal{S}$ if $A, B \in \mathcal{S}$. A *valuation* on $\mathcal{S}$ is a function φ from $\mathcal{S}$ into some Abelian group that is *additive* in the sense that $\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B)$ for all $A, B \in \mathcal{S}$ with $A \cup B \in \mathcal{S}$, and satisfies $\varphi(\varnothing) = 0$ if $\varnothing \in \mathcal{S}$. The function φ is *fully additive* if

$$\varphi(A_1 \cup \cdots \cup A_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq m} \varphi(A_{i_1} \cap \cdots \cap A_{i_r})$$

for all $m \in \mathbb{N}$ and $A_1, \ldots, A_m$ with $A_1 \cup \cdots \cup A_m \in \mathcal{S}$. We denote by $\mathsf{U}(\mathcal{S})$ the family of all finite unions of elements from $\mathcal{S}$. Then $(\mathsf{U}(\mathcal{S}), \cup, \cap)$ is a lattice. The following is Groemer's [3] (first) extension theorem. The proof can also be found in [16, Theorem 6.2.1].

Theorem 2.1. *Let φ be a function from an intersectional family $\mathcal{S}$ of sets into an Abelian group such that $\varphi(\varnothing) = 0$ if $\varnothing \in \mathcal{S}$. Then the following conditions* (a)–(c) *are equivalent*:

(a) *φ is fully additive*;

(b) *if $n_1 \mathbf{1}_{A_1} + \cdots + n_m \mathbf{1}_{A_m} = 0$ with $A_i \in \mathcal{S}$ and $n_i \in \mathbb{Z}$ for $i = 1, \ldots, m$, then*

$$n_1 \varphi(A_1) + \cdots + n_m \varphi(A_m) = 0;$$

(c) *φ has an additive extension to the lattice $\mathsf{U}(\mathcal{S})$.*

A function φ on the set $\mathcal{P}C^d$ of polyhedral cones with values in an Abelian group is said to be *weakly additive* if for each $C \in \mathcal{P}C^d$ and each linear hyperplane H with the corresponding halfspaces H^+, H^- the relation

$$\varphi(C) = \varphi(C \cap H^+) + \varphi(C \cap H^-) - \varphi(C \cap H)$$

is satisfied. Every additive function on $\mathcal{P}C^d$ is weakly additive.

Theorem 2.2. *Every weakly additive function on $\mathcal{P}C^d$ with values in an Abelian group is fully additive on $\mathcal{P}C^d$.*

This is similar to the corresponding theorem for polytopes, see [16, Theorem 6.2.3]. It can also be proved in a similar way, replacing polytopes by polyhedral cones and hyperplanes by linear hyperplanes.

If now φ is a valuation on $\mathcal{PC}^d$, then by Theorem 2.2 it is fully additive. By Theorem 2.1, it has an additive extension to $\mathsf{U}(\mathcal{Q}^d_{ro})$. The elements of $\mathsf{U}(\mathcal{Q}^d_{ro})$ are finite unions of ro-polyhedra and are called *generalized ro-polyhedra*. For a valuation φ on $\mathcal{PC}^d$, assertion (b) of Theorem 2.1 with $\mathcal{S} = \mathcal{PC}^d$ is valid. This is the reason why Corollaries 1.1 and 1.2 follow from (8) and (9).

An important example of a valuation is the Euler characteristic. An elementary existence proof was given by Hadwiger [6] for finite unions of convex bodies. The fact that his proof extends to unbounded and to relatively open convex sets was pointed out (and generalized) by Lenz [11]. For completeness, we present here the short extension of Hadwiger's proof to generalized ro-polyhedra.

Theorem 2.3 (and Definition). *There is a unique real valuation χ on $\mathsf{U}(\mathcal{Q}^d_{ro})$, the* Euler characteristic, *with*

$$\chi(Q) = (-1)^{\dim Q} \text{ for } Q \in \mathcal{Q}^d_{ro} \setminus \{\varnothing\}.$$

It satisfies $\chi(P) = 1$ if $P \in \mathcal{Q}^d \setminus \{\varnothing\}$ is compact.

Proof. Existence is proved by induction on the dimension. The zero-dimensional case being trivial, we assume that $d \geq 1$ and that the existence of χ has been proved in affine spaces of dimension less than d. Let $u \in \mathbb{R}^d \setminus \{o\}$, and let $H_\lambda = \{x \in \mathbb{R}^d : \langle u, x\rangle = \lambda\}$ for $\lambda \in \mathbb{R}$. For a generalized ro-polyhedron $Q \in \mathsf{U}(\mathcal{Q}^d_{ro})$, we define

$$\chi(Q) := -\lim_{\mu \to -\infty} \chi(Q \cap H_\mu) + \sum_{\lambda \in \mathbb{R}} \Big[\chi(Q \cap H_\lambda) - \lim_{\mu \downarrow \lambda} \chi(Q \cap H_\mu)\Big]. \tag{18}$$

This definition makes sense for the following reasons. First, each $Q \cap H_\lambda$, $\lambda \in \mathbb{R}$, is a generalized ro-polyhedron in an affine space of dimension $d-1$, so that $\chi(Q \cap H_\lambda)$ is defined. Second, since Q is the disjoint union of finitely many ro-polyhedra $Q_1, \dots, Q_r$, there are finitely many numbers $\lambda_1, \dots, \lambda_s$ such that for λ in any of the components of $\mathbb{R} \setminus \{\lambda_1, \dots, \lambda_s\}$, the dimension of $Q_i \cap H_\lambda$ is independent of λ for $i = 1, \dots, r$ (where $\dim \varnothing = -1$, by definition). Thus, $\lambda \mapsto \chi(Q \cap H_\lambda)$ is constant on each such component. This shows, third, that all limits in (18) exist, and the sum is finite. The induction hypothesis implies that the function χ thus defined on $\mathsf{U}(\mathcal{Q}^d_{ro})$ is a valuation. Now let $Q \in \mathcal{Q}^d_{ro}$. If Q is contained in some H_λ, then $\chi(Q) = (-1)^{\dim Q}$ by the induction hypothesis. If Q is not contained in some H_λ, then the right-hand side of (18) gives $-(-1)^{\dim Q - 1} + 0 = (-1)^{\dim Q}$ if $Q \cap H_\lambda \neq \varnothing$ for all large $-\lambda$, and otherwise it gives $0 + (0 - (-1)^{\dim Q-1}) = (-1)^{\dim Q}$. Similarly, we obtain $\chi(P) = 1$ if $P \in \mathcal{Q}^d \setminus \{\varnothing\}$ is compact. The uniqueness of χ is clear, because each $Q \in \mathsf{U}(\mathcal{Q}^d_{ro})$ is a disjoint union of ro-polyhedra. □

The following consequence is simple, but useful. It was, in fact, the reason for considering ro-polyhedra.

Lemma 2.1. *If a generalized ro-polyhedron $Q \in \mathsf{U}(\mathcal{Q}^d_{ro})$ is the disjoint union of ro-polyhedra $Q_1, \dots, Q_m \in \mathcal{Q}^d_{ro}$, then*

$$\sum_{i=1}^m (-1)^{\dim Q_i} = \chi(Q).$$

In fact, since $Q_i \cap Q_j = \varnothing$ for $i \neq j$, the additivity of χ yields

$$\sum_{i=1}^m (-1)^{\dim Q_i} = \sum_{i=1}^m \chi(Q_i) = \chi\bigg(\bigcup_{i=1}^m Q_i\bigg) = \chi(Q).$$

In particular, since a polyhedron $P \in \mathcal{Q}^d$ is the disjoint union of the relative interiors of its faces, we immediately obtain the *Euler relation*

$$\sum_{F \in \mathcal{F}(P)} (-1)^{\dim F} = \chi(P). \tag{19}$$

For a polyhedral cone $C \in \mathcal{PC}^d$, it is easily seen that

$$\chi(C) = \begin{cases} 0 & \text{if } C \text{ is not a subspace,} \\ (-1)^{\dim C} & \text{if } C \text{ is a subspace.} \end{cases} \tag{20}$$

Applying this to the angle cone of a polyhedron $P \in \mathcal{Q}^d$ at one of its faces $F \neq P$, we obtain the local Euler relation

$$\sum_{F \subseteq J \in \mathcal{F}(P)} (-1)^{\dim J} = 0, \quad F \in \mathcal{F}(P) \setminus \{P\}. \tag{21}$$

Now we recall some facts about posets (partially ordered sets). Let $(\mathcal{S}, \preceq)$ be a finite partially ordered set. The elements of the *incidence algebra* $I(\mathcal{S})$ are the real functions ξ on the ordered pairs (S,T) of elements of $\mathcal{S}$ with the property that $\xi(S,T) = 0$ if $S \not\preceq T$. Addition is the pointwise addition of functions, and multiplication is defined by

$$(\xi \circ \eta)(S,T) = \sum_{J \in I(\mathcal{S})} \xi(S,J)\eta(J,T).$$

This yields an associative algebra. One defines the functions

$$\delta(S,T) := \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{if } S \neq T, \end{cases} \quad \zeta(S,T) := \begin{cases} 1 & \text{if } S \preceq T, \\ 0 & \text{if } S \not\preceq T, \end{cases}$$

so that δ is the unit element of the incidence algebra. The *Möbius function* of $I(\mathcal{S})$ is defined recursively by

$$\mu(S,S) = 1, \quad \mu(S,T) = 0 \ \text{ if } S \not\preceq T,$$
$$\mu(S,T) = -\sum_{S \preceq J \precneqq T} \mu(S,J) \ \text{ if } S \precneqq T$$

for $S, T \in \mathcal{S}$. Then $\mu \circ \zeta = \delta = \zeta \circ \mu$.

If $P \in \mathcal{Q}^d$ is a nonempty polyhedron, the Möbius function of its face lattice $(\mathcal{F}(P), \subseteq)$, partially ordered by inclusion, is given by

$$\mu(F,G) = (-1)^{\dim G - \dim F}, \quad F, G \in \mathcal{F}(P), \ F \subseteq G, \tag{22}$$

(and $\mu(F,G) = 0$ if $F \not\subseteq G$). This follows immediately from (21).

To prove (7), let $C \in \mathcal{PC}^d$ and $F \in \mathcal{F}(C)$. We fix $x \in \mathbb{R}^d$ and write $\psi(M) := \mathbf{1}_{M+N(C,F)}(x)$ for subsets $M \subseteq F$. Let $G \in \mathcal{F}(F)$. Since G is the disjoint union of the relative interiors of its faces, and since F and $N(C,F)$ are totally orthogonal, we have

$$\psi(G) = \sum_{J \in \mathcal{F}(G)} \psi(\operatorname{relint} J).$$

This yields

$$\begin{aligned} \sum_{G \in \mathcal{F}(F)} (-1)^{\dim G} \psi(G) &= \sum_{G \in \mathcal{F}(F)} (-1)^{\dim G} \sum_{J \in \mathcal{F}(G)} \psi(\operatorname{relint} J) \\ &= \sum_{J \in \mathcal{F}(F)} \psi(\operatorname{relint} J) \sum_{J \subseteq G \in \mathcal{F}(F)} (-1)^{\dim G} \\ &= \psi(\operatorname{relint} F)(-1)^{\dim F}, \end{aligned}$$

where (21) was used. This is relation (7).

It remains to prove Theorems 1.1 and 1.2. For the first, we need a combinatorial version of the Sommerville relation. We prove a more general version, for arbitrary polyhedra, in the next section.

§3. The combinatorial Brianchon–Gram–Sommerville relation

The *combinatorial Brianchon–Gram–Sommerville relation*, which we now derive, extends, at the level of characteristic functions, the classical angle sum relations of Gram (or Brianchon–Gram) for bounded polyhedra and of Sommerville for polyhedral cones. Both angle sum relations were unified and extended to arbitrary polyhedra by McMullen in [13], to which we also refer for historical remarks and references. A formulation at the level of scissors congruence (less general than (23)) can be found in McMullen [14, Theorem 4.15]. The result can also be deduced from investigations of Chen [2]. For the reader's convenience, we give a shorter proof, extending the approach of McMullen [13], which in its turn was motivated by a simple proof of Gram's relation due to Shephard [18].

Theorem 3.1. *For a polyhedron $P \in \mathcal{Q}^d$,*

$$\sum_{F \in \mathcal{F}(P)} (-1)^{\dim F} \mathbf{1}_{\operatorname{relint} A(F,P)}(x) = (-1)^{\dim P} \mathbf{1}_{-\operatorname{rec} P}(x) \tag{23}$$

for $x \in \mathbb{R}^d \setminus \{o\}$.

Proof. Let $P \in \mathcal{Q}^d$ and $x \in \mathbb{R}^d \setminus \{o\}$. If $x \notin \operatorname{lin}(P - P)$, then both sides of (23) are zero. Therefore, we need only consider points in $\operatorname{lin}(P-P)$. This means that without loss of generality we can (and will) assume that $\dim P = d$. For $x \in \mathbb{R}^d \setminus \{o\}$, let H_x be a hyperplane orthogonal to x, let Π_x be the orthogonal projection to H_x, and let $P_x = \Pi_x(P)$. Let

$$\mathcal{F}(P,x) := \{F \in \mathcal{F}(P) \,:\, \dim F \le d-1,\, x \in \operatorname{int} A(F,P)\}.$$

First, suppose that $x \notin -\operatorname{rec} P$. For each $F \in \mathcal{F}(P,x)$, the projection $\Pi_x(F)$ is a polyhedron in H_x, whose relative interior is contained in the relative interior of P_x. The ro-polyhedra

$$\Pi_x(\operatorname{relint} F) \text{ with } F \in \mathcal{F}(P,x)$$

form a disjoint decomposition of $\operatorname{relint} P_x$. Therefore, Lemma 2.1 gives

$$\sum_{\substack{F \in \mathcal{F}(P,x)\\ \dim F \le d-1}} \chi\big(\Pi_x(\operatorname{relint} F)\big) = \chi(\operatorname{relint} P_x).$$

For $F \in \mathcal{F}(P)$ with $\dim F \le d-1$ we have

$$F \in \mathcal{F}(P,x) \Leftrightarrow x \in \operatorname{int} A(F,P) \Leftrightarrow \mathbf{1}_{\operatorname{int} A(F,P)}(x) = 1,$$

whence

$$\sum_{\substack{F \in \mathcal{F}(P)\\ \dim F \le d-1}} (-1)^{\dim F} \mathbf{1}_{\operatorname{int} A(F,P)}(x) = (-1)^{d-1}.$$

This holds if $x \notin -\operatorname{rec} P$. If $x \in -\operatorname{rec} P \setminus \{o\}$, then $\mathcal{F}(P,x) = \emptyset$, hence $\mathbf{1}_{\operatorname{int} A(F,P)}(x) = 0$ for all $F \in \mathcal{F}_j(P)$, $j \in \{0, \dots, d-1\}$. Thus, for arbitrary $x \in \mathbb{R}^d \setminus \{o\}$ we have

$$\sum_{\substack{F \in \mathcal{F}(P)\\ \dim F \le d-1}} (-1)^{\dim F} \mathbf{1}_{\operatorname{int} A(F,P)}(x) = (-1)^{d-1}\big(1 - \mathbf{1}_{-\operatorname{rec} P}(x)\big),$$

which can be written in the form (23) because $\operatorname{int} A(P,P) = \mathbb{R}^d$. □

Clearly, integrating (23) with the Gaussian measure Γ_d, we obtain an angle sum relation, which reduces to (1) in the case of a polyhedral cone.

§4. Proof of Theorem 1.1

We prove a more general relation, for an arbitrary nonempty polyhedron $P \in \mathcal{Q}^d$. Let $E \neq P$ be a face of P. Then we claim that

$$\sum_{E \subseteq F \in \mathcal{F}(P)} (-1)^{\dim F} \mathbf{1}_{A(E,F)-N(P,F)}(x) = 0 \tag{24}$$

for $x \in \mathbb{R}^d$. For $x = o$, this follows from (21), hence in the following we may assume that $x \in \mathbb{R}^d \setminus \{o\}$. Theorem 1.1 is a special case of (24). In fact, if $P = C$ is a polyhedral cone and $E = \{o\}$, then (9) follows from (24), because $A(o, F) = F$ for the cones $F \in \mathcal{F}(C)$.

The proof of (24) requires a few preparations. If F is a face of the polyhedron P, we denote by $L(F) := \operatorname{lin}(F - F)$ the linear subspace that is parallel to the affine hull of F, and by $F^\perp$ the orthogonal complement of $L(F)$.

Applying (6) to the angle cone $A(E, P)$, we obtain

$$\sum_{F' \in \mathcal{F}(A(E,P))} \mathbf{1}_{\operatorname{relint} F' + N(A(E,P),F')} = 1.$$

Let $z_0 \in \operatorname{relint} E$. Since the faces F' of $A(E, P)$ are in one-to-one correspondence with the faces F of P satisfying $E \subseteq F$ and such that $F' = \operatorname{pos}(F - z_0)$, we have

$$A(E, F) = F' \text{ and } N(A(E, P), F') = N(P, F).$$

It follows that

$$\sum_{E \subseteq F \in \mathcal{F}(P)} \mathbf{1}_{\operatorname{relint} A(E,F)+N(P,F)} = 1. \tag{25}$$

We have $A(E, F) \subseteq L(F)$ and $N(P, F) \subseteq F^\perp$. Hence, if $x = x_1 + x_2$ with $x_1 \in L(F)$ and $x_2 \in F^\perp$, then

$$\begin{aligned} x \in \operatorname{relint} A(E, F) + N(P, F) &\Leftrightarrow x_1 \in \operatorname{relint} A(E, F) \wedge x_2 \in N(P, F) \\ &\Leftrightarrow x \in \operatorname{relint} A(E, F) + F^\perp \wedge x \in N(P, F) + L(F). \end{aligned}$$

Therefore, (25) is equivalent to

$$\sum_{E \subseteq F \in \mathcal{F}(P)} \mathbf{1}_{\operatorname{relint} A(E,F)+F^\perp} \mathbf{1}_{N(P,F)+L(F)} = 1. \tag{26}$$

Next, applying (23) to the angle cone $A(E, P)$ and observing that $\operatorname{rec} A(E, P) = A(E, P)$, we obtain

$$\sum_{E \subseteq F \in \mathcal{F}(P)} (-1)^{\dim F} \mathbf{1}_{\operatorname{relint} A(F,P)}(x) = (-1)^{\dim P} \mathbf{1}_{-A(E,P)}(x) \tag{27}$$

for $x \in \mathbb{R}^d \setminus \{o\}$. Let G be a face of P with $E \subseteq G$. Relation (27) for $P = G$ reads

$$\sum_{E \subseteq F \subseteq G} (-1)^{\dim F} \mathbf{1}_{\operatorname{relint} A(F,G)}(x) = (-1)^{\dim G} \mathbf{1}_{-A(E,G)}(x) \tag{28}$$

for $x \in \mathbb{R}^d \setminus \{o\}$. For $x \notin L(G)$, both sides of (28) are zero. If we write $x = x_1 + x_2$ with $x_1 \in L(G)$ and $x_2 \in G^\perp$, we have

$$\begin{aligned} \mathbf{1}_{\operatorname{relint} A(F,G)+G^\perp}(x) = 1 &\Leftrightarrow \mathbf{1}_{\operatorname{relint} A(F,G)}(x_1) = 1, \\ \mathbf{1}_{-A(F,G)+G^\perp}(x) = 1 &\Leftrightarrow \mathbf{1}_{-A(F,G)}(x_1) = 1. \end{aligned}$$

Therefore, (28) can equivalently be written as

$$\sum_{E\subseteq F\subseteq G}(-1)^{\dim F}\mathbf{1}_{\operatorname{relint} A(F,G)+G^\perp}(x)=(-1)^{\dim G}\mathbf{1}_{-A(E,G)+G^\perp}(x) \tag{29}$$

for $x\in\mathbb{R}^d\setminus\{o\}$.

Now we are in a position to complete the proof of (24). Following McMullen [12], we use the incidence algebra of the face lattice of P, for which the functions δ,ζ,μ were defined in §2. We fix a vector $x\in\mathbb{R}^d\setminus\{o\}$ and define the following functions of the incidence algebra:

$$\begin{aligned}B(F,G)&=\mathbf{1}_{\operatorname{relint} A(F,G)+G^\perp}(x),\\ \overline{B}(F,G)&=(-1)^{\dim G-\dim F}\mathbf{1}_{-A(F,G)+G^\perp}(x),\\ \Gamma(F,G)&=\mathbf{1}_{N(G,F)+L(F)}(x)\end{aligned}$$

for $F,G\in\mathcal{F}(P)$. Then relations (29) and (26) (for $P=G$) say that

$$\mu\circ B=\overline{B},\quad B\circ\Gamma=\zeta.$$

Therefore,

$$\overline{B}\circ\Gamma=(\mu\circ B)\circ\Gamma=\mu\circ(B\circ\Gamma)=\mu\circ\zeta=\delta.$$

In particular, for $F\in\mathcal{F}(P)\setminus\{P\}$ this gives

$$(\overline{B}\circ\Gamma)(F,P)=0.$$

Explicitly, this reads

$$\sum_{E\subseteq F\in\mathcal{F}(P)}(-1)^{\dim F-\dim E}\mathbf{1}_{-A(E,F)+F^\perp}(x)\mathbf{1}_{N(P,F)+L(F)}(x)=0.$$

It holds for all $x\in\mathbb{R}^d\setminus\{o\}$ and can equivalently be written in the form (24).

§5. Proof of Theorem 1.2

The notation in the following is as in Theorem 1.2 and in §2 in general. Let $\mathcal{A}$ be a central hyperplane arrangement in $\mathbb{R}^d$, and let $L\subset\mathbb{R}^d$ be a subspace of dimension $k\in\{2,\dots,d-1\}$ that is in general position with respect to $\mathcal{A}$. Then $\mathcal{A}^L$ denotes the central arrangement in L given by the $(k-1)$-subspaces $H\cap L$, $H\in\mathcal{A}$.

We write $r_j(\mathcal{A})=|\mathcal{R}_j(\mathcal{A})|$. As mentioned, relation (16) was essentially proved in [1]. For $j=1$, it follows from [1, (2.16)], so let $j\geq 2$. Let H be a linear hyperplane which is in general position with respect to $\mathcal{A}$. Deleting the expectations in the displayed formula before (6.2) in [1], we see that

$$r_{j-1}(\mathcal{A}^H)=\sum_{F\in\mathcal{R}_j(\mathcal{A})}\mathbf{1}\{H\cap F\neq\{o\}\}=\sum_{F\in\mathcal{R}_j(\mathcal{A})}\big[1-\mathbf{1}\{H\cap F=\{o\}\}\big].$$

Since H is in general position with respect to $\mathcal{A}$, we have $H\cap F=\{o\}\Leftrightarrow H^\perp\cap F^\circ\neq\{o\}$ (see [1, Lemma 2.4]). Thus,

$$r_{j-1}(\mathcal{A}^H)=r_j(\mathcal{A})-\sum_{F\in\mathcal{R}_j(\mathcal{A})}\mathbf{1}\{H^\perp\cap F^\circ=\{o\}\}.$$

In [1, Lemma 6.2] it was shown that $r_{j-1}(\mathcal{A}^H)=r_j(\mathcal{A})-(-1)^j2a_{j0}$. With $H^\perp:=L$, this gives (16).

Turning to (17), suppose that $L\subset\mathbb{R}^d$ is a subspace in general position with respect to $\mathcal{L}(\mathcal{A})$. The case of (17) $\dim L=1$ (and hence $j=d$, $k=d-1$) is trivial, because then the left-hand side of (17) is equal to 2, and $\sum_{i=0}^d a_{di}=0$ and $a_{dd}=1$ (by the definition of the Möbius function).

Let $\dim L \geq 2$. Let μ_L denote the Möbius function of $\mathcal{L}(\mathcal{A}^L)$. From the definition of the Möbius function one can deduce that $\mu(S,T) = \mu_L(S \cap L, T \cap L)$ for $S,T \in \mathcal{L}(\mathcal{A})$ (see the details in the proof of Lemma 6.2 in [1]). By (13),

$$\begin{aligned}\chi_{\mathcal{A}^L,j-k}(t) &= \sum_{L' \in \mathcal{L}_{j-k}(\mathcal{A}^L)} \sum_{S \in \mathcal{L}(\mathcal{A}^L)} \mu_L(L',S) t^{\dim S} \\ &= \sum_{r=0}^{j-k} \sum_{L' \in \mathcal{L}_{j-k}(\mathcal{A}^L)} \sum_{S \in \mathcal{L}_r(\mathcal{A}^L)} \mu_L(L',S) t^r.\end{aligned}$$

Writing

$$\chi_{\mathcal{A}^L,j-k}(t) = \sum_{r=0}^{j-k} c_{jr} t^r,$$

we have

$$c_{jr} = \sum_{L' \in \mathcal{L}_{j-k}(\mathcal{A}^L)} \sum_{S \in \mathcal{L}_r(\mathcal{A}^L)} \mu_L(L',S) = \sum_{\overline{L} \in \mathcal{L}_j(\mathcal{A})} \sum_{\overline{S} \in \mathcal{L}_{r+k}(\mathcal{A})} \mu(\overline{L},\overline{S}) = a_{j(r+k)}.$$

Therefore,

$$\chi_{\mathcal{A}^L,j-k}(t) = c_{j0} + \sum_{i=k+1}^{j} a_{ji} t^{i-k}.$$

From

$$\chi_{\mathcal{A}^L,j-k}(1) = \sum_{L' \in \mathcal{L}_{j-k}(\mathcal{A}^L)} \sum_{S \in \mathcal{L}(\mathcal{A}^L)} \mu_L(L',S) = \sum_{\overline{L} \in \mathcal{L}_j(\mathcal{A})} \sum_{\overline{S} \in \mathcal{L}(\mathcal{A})} \mu(\overline{L},\overline{S}) = \chi_{\mathcal{A},j}(1)$$

we get

$$c_{j0} + \sum_{i=k+1}^{j} a_{ji} = \sum_{r=0}^{j} a_{jr},$$

which gives

$$\chi_{\mathcal{A}^L,j-k}(t) = \sum_{i=0}^{k} a_{ji} + \sum_{i=k+1}^{j} a_{ji} t^{i-k}. \tag{30}$$

A result of Zaslavsky [21] (see also [19, Theorem 2.6]) says that

$$r_j(\mathcal{A}) = (-1)^j \chi_{\mathcal{A},j}(-1).$$

This gives

$$\sum_{F \in \mathcal{R}_j(\mathcal{A})} \mathbf{1}\{L \cap F \neq \{o\}\} = r_{j-k}(\mathcal{A}^L) = (-1)^{j-k} \chi_{\mathcal{A}^L,j-k}(-1). \tag{31}$$

Now (31) and (30) yield (17). This completes the proof of Theorem 1.2.

From the combinatorial result of Theorem 1.2, the extended Klivans–Swartz formula (15) can now be obtained by integration. Let $G(d,k)$ be the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$, and let ν_k denote its rotation invariant probability measure. For cones $C \in \mathcal{PC}^d$, one defines

$$U_j(C) = \frac{1}{2} \int_{G(d,d-j)} \mathbf{1}\{L \cap C \neq \{o\}\} \, \nu_{d-j}(\mathrm{d}L), \quad j = 1, \dots, d,$$

and $U_d(C) = U_{d+1}(C) = 0$. It follows from the spherical (or conical) kinematic formula of integral geometry (see, e.g., [17, 6.63], but observe that the present v_j are there denoted by v_{j-1}) that

$$v_j(C) = U_{j-1}(C) - U_{j+1}(C) \text{ for } j = 1, \dots, d.$$

Now integration of (16) with ν_1 over $G(d,1)$ gives on the left-hand side

$$\sum_{F\in\mathcal{R}_j(\mathcal{A})} 2U_{d-1}(F^\circ) = \sum_{F\in\mathcal{R}_j(\mathcal{A})} 2v_d(F^\circ) = \sum_{F\in\mathcal{R}_j(\mathcal{A})} 2v_0(F)$$

(see [17, 6.5]), so that (15) for $k=0$ results. For $k\geq 1$ we obtain (using that ν_m-almost all $L\in G(d,m)$ are in general position with respect to $\mathcal{L}(\mathcal{A})$ and that ν_m is normalized),

$$\begin{aligned}
\sum_{F\in\mathcal{R}_j(\mathcal{A})} v_k(F) &= \sum_{F\in\mathcal{R}_j(\mathcal{A})} \big[U_{k-1}(F)-U_{k+1}(F)\big]\\
&= \sum_{F\in\mathcal{R}_j(\mathcal{A})} \bigg[\frac{1}{2}\int_{G(d,d-k+1)} \mathbf{1}\big\{L\cap F\neq\{o\}\big\}\,\nu_{d-k+1}(\mathrm{d}L)\\
&\qquad -\frac{1}{2}\int_{G(d,d-k-1)} \mathbf{1}\big\{L\cap F\neq\{o\}\big\}\,\nu_{d-k-1}(\mathrm{d}L)\bigg]\\
&= \frac{1}{2}(-1)^{j-k-1}\left[\sum_{i=0}^{k-1} a_{ji} + \sum_{i=k}^{j} a_{ji}(-1)^{i-k+1} - \sum_{i=0}^{k+1} a_{ji} - \sum_{i=k+2}^{j} a_{ji}(-1)^{i-k-1}\right]\\
&= (-1)^{j-k}a_{jk},
\end{aligned}$$

which is (15) for $k\geq 1$.

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