Алгебра и анализ
Том 29 (2017), № 5

St. Petersburg Math. J.
Vol. 29 (2018), No. 5, Pages 749–759
https://doi.org/10.1090/spmj/1514
Article electronically published on July 26, 2018

PROBLEM OF IDEALS IN THE ALGEBRA H^∞ FOR SOME SPACES OF SEQUENCES

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Abstract. Metric aspects of the problem of ideals are studied. Let h be a function in the class $H^\infty(\mathbb{D})$ and f a vector-valued function in the class $H^\infty(\mathbb{D}; E)$, i.e., f takes values in some lattice of sequences E. Suppose that $|h(z)| \le \|f(z)\|_E^\alpha \le 1$ for some parameter α. The task is to find a function g in $H^\infty(\mathbb{D}; E')$, where E' is the order dual of E, such that $\sum f_j g_j = h$. Also it is necessary to control the value of $\|g\|_{H^\infty(E')}$. The classical case with $E = l^2$ was investigated by V. A. Tolokonnikov in 1981. Recently, the author managed to obtain a similar result for the space $E = l^1$. In this paper it is shown that the problem of ideals can be solved for any q-concave Banach lattice E with finite q; in particular, $E = l^p$ with $p \in [1, \infty)$ fits.

§1. Introduction and the main result

The problem of ideals has an intimate relationship with the corona theorem. The classical corona problem was formulated by Kakutani and came from the study of the space of maximal ideals of the algebra H^∞. L. Carleson proved the following statement and solved the corona problem in 1968.

Theorem (Carleson). *Assume that $\delta > 0$. Let $f_1, \dots, f_n$ be functions of class $H^\infty(\mathbb{D})$ that satisfy the following condition*:

$$\sum_{j=1}^{n} |f_j(z)| \ge \delta \quad \textit{and} \quad \|f_j\|_{H^\infty} \le 1 \quad \textit{for} \quad 1 \le j \le n.$$

Then there exist functions $g_1, \dots, g_n$ in $H^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^{n} f_j(z) g_j(z) = 1, \quad z \in \mathbb{D},$$

and $\|g_j\|_{H^\infty} \le C(\delta, n)$.

We note the dependence on n in the last inequality. In 1979 T. Wolff proposed another approach to the solution of the corona problem. It was based on the idea of L. Hörmander to involve the $\overline{\partial}$ equation.

Wolff's proof turned out to admit an extension to infinite-dimensional spaces. The following definition makes the statements shorter. All associated notions will be defined in the next section.

Definition 1.1. Let X be a Banach lattice of functions on the set $\mathbb{N}$ of positive integers. We denote by X' the order dual of X. We say that the corona problem is solvable for

2010 *Mathematics Subject Classification.* Primary 30H80.
Key words and phrases. Problem of ideals, corona theorem, Kakutani fixed point theorem.
Supported by RFBR (grant no. 17-01-00607) and by "Native towns", a social investment program of PJSC "Gazprom Neft".

the lattice X if the following statement is true. Let δ be a positive parameter, and let f be a vector-valued function in $H^\infty(\mathbb{D};X)$ satisfying the condition

$$\delta \le \|f(z)\|_X \le 1, \quad z \in \mathbb{D};$$

then one can find a vector-valued function g in $H^\infty(\mathbb{D};X')$ such that:

$$1 = \sum_{i=1}^{\infty} f(z,i)g(z,i) = \langle f(z), g(z)\rangle, \quad z \in \mathbb{D},$$

and moreover, the value $\|g\|_{H^\infty(\mathbb{D};X')}$ is bounded by a constant $C_{X,\delta}$ depending only on the parameter δ and the lattice X.

In this connection, we mention the papers of Tolokonnikov [7] and Uchiyama [6]. In both papers the corona theorem was established for $X = l^2$, but Uchiama also treated the case of $X = l^\infty$. Kislyakov and Rutsky (see [10]) showed that the corona problem has a solution for $X = l^p$, with $2 \le p < \infty$. Using interpolation, Kislyakov proved the corona theorem in the paper [1] for all l^p spaces (and even for q-concave lattices with an additional condition of BMO-regularity). Finally, in [2] Rutsky showed by using the Kakutani fixed-point theorem that the corona problem has a solution for all order-continuous lattices of sequences. His proof was based on Uchiyama's theorem for the lattice l^∞. By imitating the pattern of Definition 1.1, the problem of ideals can be formulated in the following way.

Definition 1.2. Let X be a Banach lattice of sequences on the set $\mathbb{N}$ and X' its order dual. We say the problem of ideals for the lattice X has a solution with exponent α and estimate $C_{X,\alpha}$ if the following statement is true. Let a function h in $H^\infty(\mathbb{D})$ and a vector-valued function f in $H^\infty(\mathbb{D};X)$ satisfy the conditions

$$|h(z)| \le \|f(z)\|_X^\alpha \le 1$$

for all z in $\mathbb{D}$ and some fixed parameter α. Then there exists a function g in $H^\infty(\mathbb{D};X')$ such that

$$h(z) = \sum_{i=1}^{\infty} f(z,i)g(z,i) = \langle f(z), g(z)\rangle, \quad z \in \mathbb{D},$$

and moreover, the value $\|g\|_{H^\infty(\mathbb{D};X')}$ is bounded by the constant $C_{X,\alpha}$ depending only on the parameter α and the lattice X.

In Tolokonnikov's paper mentioned above it was also shown that the problem of ideals has a solution for the space l^2 with the exponent 4 and constant 57. This exponent was improved by many authors, see [8]. For us it suffices to know that for any positive ε the problem of ideals has a solution for the space l^2 with the exponent $2+\varepsilon$ and a constant C depending only on ε. We denote this constant by $C_{l^2,2+\varepsilon}$. In the paper [3] by using the fixed point theorem and the result for the space l^2 it was shown that the problem of ideals has a solution with the exponent $2+\varepsilon$ for arbitrary positive ε and the same constant $C_{l^2,2+\varepsilon}$.

We formulate the main results.

Theorem A. *Let E and F be finite-dimensional Banach lattices* (*viewed as lattices of functions on some finite subset of* $\mathbb{N}$). *Assume that the problem of ideals for E has a solution with exponent α_E and estimate C_E. Let X denote the product of E and F. If X is a Banach lattice, then the problem of ideals has a solution for X with the exponent α_E and the estimate $C_E 2^{\alpha_E}(1+\delta)$ for an arbitrary positive δ.*

Let X be a Banach lattice of sequences and N a positive integer. We denote by X_N the finite-dimensional lattice obtained by restriction of X to the set $\{1 \dots N\}$.

Theorem B. *Let X be an infinite-dimensional Banach lattice of sequences such that for any $N \in \mathbb{N}$ and an arbitrarily small ϵ the problem of ideals has a solution for all finite-dimensional lattices X_N with exponent α and estimate $C_X(1+\epsilon)$ independently of N. Assume also that X is q-concave with constant $M_{q,X}$. Then the problem of ideals has a solution for X with exponent α and estimate $C_X M_{q,X}^{\alpha}$.*

Theorem C. *Let X be a q-concave lattice with Fatou property on the set $\mathbb{N}$. Then the problem of ideals has a solution for X.*

The proof of these theorems is based on the method suggested by Rutsky in [2]. Unlike the corona problem, in Theorem B we are still unable to lift the q-concavity condition for E. The main reason is that the problem of ideals for the space $E = l^{\infty}$ has not been solved yet. See §2 for more details.

§2. The main definitions and the reduction of Theorem C to Theorems A and B

We remind the reader several basic definitions and results from the lattice theory. They are formulated not in the full generality but only in the form sufficient for us. We refer the reader to [4] and [5] for more details.

Definition 2.1. Let (S, μ) be a space with measure and X a linear space of measurable functions supplied with a quasinorm $\|\cdot\|$. We say that X is a lattice of measured functions if it has the following property. Assume that g is a measurable function and in the space X there exists a function f with $|g| \le |f|$ a.e. Then g belongs to X and $\|g\|_X \le \|f\|_X$. The lattice X is called a Banach lattice if it is complete and its quasinorm is in fact a norm (more generally, if such a situation occurs after renorming the space).

For us, it suffices to consider the space $S = \mathbb{N}$ with the counting measure μ. In this case we shall talk of Banach lattices of sequences, and, for brevity, simply of Banach lattices. We need the notions of the product of lattices and a power of a lattice of measurable functions.

Definition 2.2. Let X and Y be the Banach lattices of measurable functions. The product of X and Y is the lattice $XY = \{h = fg,\ f \in X,\ g \in Y\}$ with the usual order and equipped with the quasinorm $\|h\|_{XY} = \inf \|f\|_X \|g\|_Y$, where the greatest lower bound is taken over all representations $h = fg$.

Definition 2.3. Let X be a lattice of measurable functions and α a positive parameter. We denote by X^{α} the lattice $\{f : |f|^{1/\alpha} \in X\}$ with the usual order and equipped with the quasinorm

$$\|f\|_{X^{\alpha}} = \||f|^{1/\alpha}\|_X^{\alpha}.$$

Generally speaking, the lattices XY and Y may fail to be *Banach* lattices. We recall the following important definition.

Definition 2.4. Let X be a Banach lattice of measurable functions, and let $p \in [1, \infty)$. X is said to be p-convex if X^p is a Banach lattice.

Definition 2.5. Let X be a Banach lattice of sequences. The order dual X' consists of all sequences $y = \{y_n\}$ such that for all sequences $x \in X$ we have $\sum_n |x_n y_n| < \infty$.

In what follows, we shall tacitly assume all lattices of measurable functions to satisfy $X'' = X$. In the case of Banach lattices this condition is equivalent to the Fatou property. Now we give the exact definition.

Definition 2.6. Let X be a quasi-Banach lattice of sequences. It has the Fatou property if for any sequence $\{x_n\}$ such that the norms $\|x_n\|$ are uniformly bounded by some constant C the following is true: if the elements x_n converge componentwise to an element x, then $x \in X$ and $\|x\| \leq C$.

We remark that the product and a power of Banach lattices inherit the Fatou property.

Definition 2.7. Suppose that the parameters q and p are conjugate. The lattice X is said to be q-concave if the lattice X' is p-convex.

We recall a direct definition of q-concave lattices. For more details, see [5]. X is a q-concave lattice if and only if there exists a universal constant $M_{q,X}$ such that for any finite sequence of elements x_i in X we have

$$\left(\sum_{i=1}^{N} \|x_i\|_X^q\right)^{1/q} \leq M_{q,X} \left\| \left(\sum_{i=1}^{N} |x_i|^q\right)^{1/q} \right\|_X.$$

Here the constant $M_{q,X}$ does not depend on the number of elements in the sequence, but only on the lattice X and the parameter q.

Definition 2.8. Let X be a Banach lattice. It has an order continuous norm if for any sequence $\{x_n\}$ of elements of X such that $x_n \geq 0, x_n \to 0$ componentwise, and $\sup_n x_n \in X$ we have $\|x_n\|_X \to 0$.

The space l^∞ is an important example of a lattice that does not have an order continuous norm. On the other hand, in finite-dimensional lattices the order continuity of the norm and the Fatou property are fulfilled automatically.

Two following lemmas are well known. See references in [1, 2] (where the proof of Lemma 2.10 can also be found).

Lemma 2.9. *Let X and Y be Banach lattices, and let $p, q \in (1, \infty)$ be mutually conjugate. Then the $(X^{1/p}Y^{1/q})' = (X')^{1/p}(Y')^{1/q}$.*

Lemma 2.10. *Let X, Y, and XY be Banach lattices. Then $X' = (XY)'Y$.*

Below in this section we show that Theorem C is a consequence of Theorems A and B. We need the following well-known assertion. For completeness, we give a proof, because it is important in the structure of our exposition.

Lemma 2.11. *Let X be a q-concave Banach lattice with the Fatou property. Then X can be represented as a product of some Banach lattice with the lattice l^q.*

Proof. From the q-concavity of the lattice X it follows that the order dual lattice X' is p-convex with $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a Banach lattice Z such that $X' = Z^{1/p}$. It is clear that multiplication by the lattice l^∞ in an arbitrary power does not change the lattice. Thus, we have $X' = Z^{1/p}(l^\infty)^{1/q}$. Applying Lemma 2.9 and taking into the account that $(l^\infty)' = l^1$, we see that $X'' = (Z')^{1/p}(l^1)^{1/q} = (Z')^{1/p}l^q$. It remains to recall that the lattice X has a Fatou property, and hence $X'' = X$. $\square$

The following extrapolation lemma enables us to extend the well-known result on the problem of ideals for the space l^2 to the space l^p. Apparently, the resulting exponent α is far from sharp, but a similar drawback of extrapolation occurred already in [10].

Lemma 2.12. *Assume that $p \in [2, \infty)$. The problem of ideals has a solution for all spaces l^p and for all $\varepsilon > 0$ with the exponent $(1+\varepsilon)p$ and a constant depending only on ε.*

Proof. The assumption of the lemma means that the functions h and f satisfy the following relation:

$$|h(z)| \leq \left(\sum_{i=1}^{\infty} |f(z,i)|^p \right)^{1+\varepsilon} \leq 1.$$

Consider the inner-outer factorization of the function $f(z)$. There exists an inner function θ and an outer function F such that $f(z,i) = \theta(z,i)F(z,i)$, moreover, $|\theta(z,i)| \leq 1$ for $i \in \mathbb{N}$. We note that on $\mathbb{T}$ we have

$$\|F(\xi)\|_{l^p} = \|f(\xi)\|_{l^p} \leq 1, \quad \xi \in \mathbb{T},$$

whence, applying the maximum principle, we see that $\|F(z)\|_{l^p} \leq 1$ in the entire disk $\mathbb{D}$.

Now we put $\varphi(z,i) = \theta(z,i)F(z,i)^{p/2}$. Since $|\theta(z,i)| \leq 1$ for all z in the disk $\mathbb{D}$, we obtain

$$|h(z)| \leq \left(\sum_{i=1}^{\infty} |\theta(z,i)|^2 |F(z,i)|^p \right)^{1+\varepsilon} = \|\varphi(z)\|_{l^2}^{2\varepsilon+2} \leq \|F(z)\|_{l^p}^{p(1+\varepsilon)} \leq 1.$$

Applying the theorem on the problem of ideals for the space l^2 (see §1 and the paper [8]), we find a function $g(z)$ in $H^\infty(\mathbb{D}; l^2)$ such that

$$\begin{aligned} h(z) &= \langle \varphi(z), g(z) \rangle \\ &= \sum_{i=1}^{\infty} \theta(z,i)F(z,i)g(z,i)F(z,i)^{p/2-1} = \langle f(z), g(z)F(z)^{p/2-1} \rangle \end{aligned}$$

and the quantity $\|g\|_{H^\infty(l^2)}$ is bounded by the constant $C_{l^2,\varepsilon}$.

Thus, the function

$$g_1(z,i) = g(z,i)F(z,i)^{p/2-1}$$

solves the problem of ideals for the space l^p. It remains to apply the Hölder inequality with the exponents $2(p-1)/p$ and $2(p-1)/(p-2)$ to get the estimate

$$\|g_1\|_{H^\infty(l^{p'})} \leq \|g\|_{H^\infty(l^2)} \|F\|_{H^\infty(l^p)}^{p/2-1} \leq C_{l^2,\varepsilon}.$$

Thereby, we have finished the proof for the space l^p with $p > 2$. $\square$

We note that the problem of ideals for the space l^p with $p \in [1,2)$ can be solved by using Theorems A and B. It suffices to observe that there exists $q \in [2; +\infty)$ such that $l^p = l^2 l^q$. We denote by X_N the lattice of functions in X supported on the set $\{1 \dots N\}$. Clearly, from the solution of the problem of ideals for l^2 we deduce that the problem can be solved for the space l^2_N with the constant that does not depend on the dimension. Hence, the problem of ideals has a solution for the space $l^p_N = l^2_N l^q_N$. It remains to apply the q-concavity of the lattices and Theorem B.

In particular, we have reproved the main result of [3].

For proving Theorem C, it remains to apply the results obtained and Theorems A and B. Indeed, every q-concave lattice (with the Fatou property) can be represented as $X = l^q F$, and without loss of generality we may assume that $q \geq 2$. Consider the sequence $X_N = l^q_N F_N$ (where the notation X_N was introduced in the preceding paragraph). The problem of ideals has a solution for the spaces l^q_N with a constant that does not depend on the dimension. Using theorem A, we deduce that the problem has a solution in the lattice X_N with the corresponding exponent and estimate. We finish the proof by applying Theorem B.

To conclude this section we make a remark about the case of $p = \infty$. In the investigation of the corona theorem it is still an open question whether it is possible to reduce the corona problem for the case of $p = \infty$ to the well-known theorem for $p = 2$. However, for the space l^∞ there exists an individual proof of A. Uchiyama (see [6]) based on the original idea of Carleson. It is not clear, however, whether the problem of ideals has a solution in case of the exponent $p = \infty$.

§3. Proof of Theorem B

In this section we shall prove Theorem B assuming that the claim of Theorem A is true.

We recall that the bounded $*$-weak topology on the space H^∞ coincides with the topology induced by the uniform convergence on the compact subsets of $\mathbb{D}$. For the first time, this quite simple statement was noted, apparently, in [9].

We may assume that for each $j \in \operatorname{supp} X$ there exists a point $z \in \mathbb{D}$ such that $f(z,j) \neq 0$. Otherwise, we can simply take the identical zero as $g(\,\cdot\,, j)$.

Now we fix parameters $\delta > 0$ and $0 < \nu < 1$. We define the set

$$K = \{z \in \nu\overline{\mathbb{D}} \,:\, |h(z)| \geq \delta\},$$

which is obviously compact. We recall that in the case of the finite-dimensional lattice X_N (with the support on a finite subset of $\mathbb{N}$) the problem of ideals is assumed to have a solution with exponent α and estimate $C_X(1+\epsilon)$, where the constant C_X does not depend on the dimension N, and the positive number ϵ can be chosen arbitrarily small. Set $\epsilon = \delta$.

Denote by $f_N(z)$ the function $f(z)\chi_{\mathbb{D}\times\{1\dots N\}}(z)$. We note that f_N takes its values in a finite-dimensional lattice.

For every point $z \in K$ one can find a number $N(z)$ such that

$$(1-\delta/2)|h(z)| \leq \|f_{N(z)}(z)\|_X^\alpha \leq 1.$$

Indeed, since the norm in the lattice X is order continuous, for a fixed z the sequence $\|f_N(z)\|_X$ converges to $\|f(z)\|_X$ as $N \to \infty$. Note that on the set K we have

$$(1-\delta/2)|h(z)| \leq |h(z)| - \delta^2/2 \leq \|f(z)\|_X^\alpha - \delta^2/2 \leq \|f_{N(z)}(z)\|_X^\alpha \leq 1$$

whenever N is so large that the following estimates are satisfied simultaneously:

$$\|f(z)\|_X^\alpha - \|f_{N(z)}(z)\|_X^\alpha \leq \delta^2/2,$$
$$\|f(z) - f_{N(z)}(z)\|_X \leq \delta^2/2.$$

Now for every point $z \in K$ we consider a neighborhood $U(z)$ such that for all $z_1 \in U(z)$ we have

$$(1-\delta)|h(z_1)| < \|f_{N(z)}(z_1)\|_X^\alpha < 1 + \delta^{1/\alpha}.$$

These neighborhoods form an open covering of the compact set K. Hence, we can choose a number N such that for all $z \in K$ we have

$$(1-\delta)|h(z)| \leq \|f_N(z)\|_X^\alpha \leq 1 + \delta^{1/\alpha}.$$

Let t be a positive parameter to be specified later. Now we define a vector-valued function φ with values in the lattice with support $\{1\dots N+1\}$ componentwise: $\varphi(z) = (f_N(z), t)$. We observe the obvious relations

$$\varphi(z) = f_N(z) + te_{N+1} \quad\text{and}\quad |\varphi(z)| = |f_N(z)| + te_{N+1},$$
$$\text{where}\quad e_{N+1} = (0,\dots,0,\underset{N+1}{1},0,\dots).$$

We recall the property of q-concave lattices that was mentioned in §2. If X is a q-concave lattice, then there exists a universal constant $M_{q,X}$ such that for any finite sequence of elements x_i in X we have

$$\Big(\sum_{i=1}^N \|x_i\|_X^q\Big)^{1/q} \leq M_{q,X}\Big\|\Big(\sum_{i=1}^N |x_i|^q\Big)^{1/q}\Big\|_X,$$

moreover, the constant $M_{q,X}$ does not depend on the number of elements in the sequence, but only on the lattice X and the parameter q.

Then in our situation we have the inequality

$$\|\varphi(z)\|_X \geq M_{q,X}^{-1}\left(\|f_N(z)\|_X^q + t^q\|e_{N+1}\|_X^q\right)^{1/q}.$$

Now we choose the parameter t so that $\delta^{1/\alpha} = t\|e_{N+1}\|_X$. For all $z \in \nu\mathbb{D}$ we have

$$\|\varphi(z)\|_X^\alpha \geq M_{q,X}^{-\alpha}\max\{\delta, \|f_N(z)\|_X^\alpha\} \geq |h(z)|M_{q,X}^{-\alpha}(1-\delta).$$

On the other hand,

$$\|\varphi(z)\|_X^\alpha \leq \left(\|f_N(z)\|_X + t\|e_{N+1}\|_X\right)^\alpha \leq \left(1+2\delta^{1/\alpha}\right)^\alpha.$$

We introduce the functions

$$\varphi_1(z) = \frac{\varphi(\nu z)}{1+2\delta^{1/\alpha}} \quad \text{and} \quad h_1(z) = \frac{h(\nu z)(1-\delta)}{M_{q,X}^\alpha(1+2\delta^{1/\alpha})^\alpha}; \quad z \in \mathbb{D}.$$

It is clear that the conditions of the problem of ideals are satisfied for h_1 and φ_1:

$$|h_1(z)| \leq \|\varphi_1(z)\|_X^\alpha \leq 1, \quad z \in \mathbb{D},$$

and moreover, φ_1 takes its values in a finite-dimensional lattice in which the problem of ideals has a solution by assumption. Hence, there exists a function $g_{\delta,\nu}$ such that

$$h_1(z) = \langle \varphi_1(z), g_{\delta,\nu}(z)\rangle, \quad \|g_{\delta,\nu}\|_{H^\infty(X')} \leq C_X(1+\delta).$$

Set $K_\delta = \frac{M_{q,X}^\alpha(1+2\delta^{1/\alpha})^{\alpha-1}}{(1-\delta)}$. Observe that $K_\delta \to M_{q,X}^\alpha$ as $\delta \to 0$.

We denote by I_N the set $\{1\dots N\}$ (recall that N depends on δ and ν), and by χ_{I_N} the corresponding characteristic function. Next, we rewrite the finite-dimensional solution of the problem of ideals in a more convenient way:

$$h(\nu z) = K_\delta\langle f_N(\nu z), g_{\delta,\nu}(z)\chi_{I_N}(z)\rangle + K_\delta t g_{\delta,\nu}(z, N+1).$$

Now we start to vary the parameters δ and ν. Let $\delta \to 0$ and $\nu \to 1$. Applying the compactness of balls in the *-weak topology, we may assume by passing to a subsequence that functions $g_{\delta,\nu}$ converge uniformly on the compact subsets of $\mathbb{D}$ to some function g_1 such that its norm is bounded by C_X. Now we show that the function $g = g_1 M_{q,X}^\alpha$ is a solution of the original problem of ideals. Indeed, for a fixed $z \in \mathbb{D}$ we estimate the quantity

$$\begin{aligned}|\langle f(z), g(z)\rangle - h(z)| &\leq |\langle f(z) - f(\nu z), g(z)\rangle| + |\langle f(\nu z) - f_N(\nu z), g(z)\rangle| \\ &\quad + |\langle f_N(\nu z), g(z) - M_{q,X}^\alpha g_{\delta,\nu}(z)\rangle| + |\langle f_N(\nu z), M_{q,X}^\alpha g_{\delta,\nu}(z)\rangle - h(z)|.\end{aligned}$$

Below, it is convenient to apply the inequality $|\langle x, y\rangle| \leq \|x\|_X\|y\|_{X'}$ and then evaluate each of the factors.

We consider the first summand. From the continuity of the function f inside the disk $\mathbb{D}$ and the finiteness of $\|g\|_{H^\infty(X')}$ it follows that the first term tends to 0 as $\nu \to 1$.

The second summand also tends to 0 as $\delta \to 0$, because the norm of g is finite and

$$\|f(\nu z) - f_N(\nu z)\|_X \leq \frac{\delta^2}{2}.$$

Similarly we infer that the third summand also converges to 0, by using the facts that $\|f_N(\nu z)\|_X \leq 1$ and $M_{q,X}^\alpha g_{\delta,\nu}(z) \to g(z)$ on the compact subsets of $\mathbb{D}$.

Finally, we estimate the last summand:

$$\begin{aligned}&|\langle f_N(\nu z), M_{q,X}^\alpha g_{\delta,\nu}(z)\rangle - h(z)| \\ &\quad \leq \left|K_\delta\langle f_N(\nu z), g_{\delta,\nu}(z)\rangle + K_\delta t g_{\delta,\nu}(z, N+1) - h(\nu z)\right| \\ &\qquad + |h(z) - h(\nu z)| + \left|(M_{q,X}^\alpha - K_\delta)\langle f_N(\nu z), g_{\delta,\nu}(z)\rangle - K_\delta t g_{\delta,\nu}(z, N+1)\right|.\end{aligned}$$

In this estimate the first summand is identically zero, the second converges to zero because h is a continuous function in the disk $\mathbb{D}$, the third also converges to zero because $K_\delta \to M^\alpha_{q,X}$, the value $|\langle f_N(\nu z), g_{\delta,\nu}(z)\rangle|$ is bounded by a universal constant and $|tg_{\delta,\nu}(z, N+1)| \le \delta \|g\|_{X'} \to 0$. Thus, we have shown that $\langle f(z), g(z)\rangle = h(z)$ and the norm of g satisfies $\|g\|_{H^\infty} \le C_X M^\alpha_{q,X}$.

Remark. We note that, in fact, the above proof can also be applied in a more general situation. Instead of the q-concavity for the Banach lattice of sequences X, one may require that there exist a continuous strictly increasing function θ, $\theta(0) = 0$, such that for the function $\varphi(z) = (f(z), t)$, $t > 0$, $z \in \mathbb{D}$, where f takes its values in the finite-dimensional lattice X_N and t occupies the $(N + 1)$st place, we have

$$\|\varphi(z)\|_X \ge \|f(z)\|_X + \theta(t),$$

provided N is sufficiently large.

§4. Proof of Theorem A

Before starting the proof of Theorem A, we formulate several important additional statements. The next fixed-point theorem is the main ingredient of the proof.

Theorem (Ky Fan–Kakutani). *Let K be a compact convex subset of a locally convex linear topological space. Consider a map Φ defined on K and taking values in the set of nonempty compact convex subsets of K. If the graph*

$$\Gamma(\Phi) = \big\{(x, y) \in K \times K\colon\ y \in \Phi(x)\big\}$$

is closed in $K \times K$, then Φ has a fixed point, i.e., a point x with $x \in \Phi(x)$.

We need the following lemma concerning continuous selection of representatives for a function belonging to the product of two lattices. Its proof is based on Michael's selection theorem and can be found in [2, Proposition 5]. Here we only present the statement.

Lemma 4.1. *Assume that F_0 and F_1 are finite-dimensional Banach lattices of measurable functions defined on the same finite set with the counting measure. Then for each $\varepsilon > 0$ there exists a continuous map $\Delta\colon F_0F_1 \setminus \{0\} \to F_1$ taking positive values and such that $\|\Delta f\|_{F_1} \le 1$, and moreover, $\|f(\Delta f)^{-1}\|_{F_0} \le (1 + \varepsilon)\|f\|_{F_0F_1}$.*

We note that the condition of finite dimension is essential. As far as the author knows, a similar claim for the case of infinite-dimensional lattices is still open even when $F_0' = F_1$.

We again recall that the bounded *-weak topology on the space H^∞ coincides with the topology induced by the uniform convergence on the compact subsets of $\mathbb{D}$.

Now we prove the main result of this paper, Theorem A.

As it has already been mentioned, we apply the method by Rutsky. Now we recall the statement we are going to prove. Let E and F be finite-dimensional Banach lattices. Assume the problem of ideals has a solution for the lattice E. We need to show that if the finite-dimensional lattice $X = EF$ is a Banach lattice, then the problem of ideals has a solution for X with exponent α_E and constant $C_E 2^{\alpha_E}(1 + \delta)$ for an arbitrarily small positive δ.

In what follows, by z we denote the variable ranging over the disk $\mathbb{D}$ and by ξ the variable ranging over the circle $\mathbb{T}$.

Consider a measurable representation $f(\xi) = f_E(\xi)f_F(\xi)$. Without loss of generality, we assume that the functions f_E, f_F lie in the spaces $L^\infty(E)$ and $L^\infty(F)$, respectively, and that their norms are bounded by 1.

From Lemma 2.10 it follows that $E' = (EF)'F$. Now we apply Lemma 4.1. We take $F_0 := (EF)'$ and $F_1 := F$ as the lattices in that lemma. Then there exists a continuous map $\Delta\colon (EF)'F(=E') \to F$ satisfying the conditions in the claim of the lemma. We recall that by C_E we have denoted the constant corresponding to the solution of the problem of ideals of the lattice E. Define the ball

$$B = \bigl\{\gamma \in H^\infty(E'),\ \|\gamma\|_{H^\infty(E')} \le C_E\bigr\}.$$

Applying the Banach–Alaoglu theorem, we see that this ball is compact in the $*$-weak topology. Let $r_k \in (0,1]$ and $\delta > 0$. Now we fix k and δ. We denote by σ the element of the finite-dimensional lattice F such that all its components are strictly separated away from zero and its norm is equal to δ. We also fix a function γ in the ball B. All further constructions are performed for this function γ. Now we introduce the function

$$\varphi(\xi,j) = \log\bigl(|\bigl(\Delta\gamma(r_k\xi)\bigr)(j)| + |f_F(\xi,j)| + \sigma_j\bigr).$$

In the last formula by $\gamma(r_k\xi,j)$ we mean the convolution of the function γ with the Poisson kernel corresponding to the radius r_k. The operator of harmonic conjugation is denoted by H. We construct the outer function

$$\Phi(\xi,j) = e^{\varphi(\xi,j)+iH(\varphi(\xi,j))}$$

and define the function $\Psi(z) = \frac{f(z)}{\Phi(z)}$. The functions constructed above have the following useful properties:

$$|\Phi(\xi,j)| = |f_F(\xi,j)| + |\Delta\gamma(r_k\xi,j)| + \sigma_j,$$

and, since $\|f_F\|_{L^\infty(F)} \le 1$ and by Lemma 4.1 we have

$$\|\Delta\gamma(z)\|_F \le 1,$$

it follows that $\|\Phi\|_{H^\infty(F)} \le 2+\delta$.

Recall that by α_E we have denoted the exponent corresponding to the problem of ideals for the Banach lattice E. We complete the construction by defining a map T that takes the initial function γ to the set of functions

$$\bigl\{u(z) \in B \mid \langle u(z), \Psi(z)\rangle = h(z)/(2+\delta)^{\alpha_E}\bigr\}.$$

We want to apply the Ky Fan–Kakutani fixed point theorem to the map T. It is necessary to check that the requirements of that theorem are satisfied. First, we show that for all functions γ belonging to the ball B the images $T\gamma$ are nonempty sets. It suffices to check that the function Ψ satisfies the condition of the problem of ideals for the lattice E. Indeed, for every fixed z we have

$$|h(z)| \le \|f(z)\|_{EF}^{\alpha_E} \le \|\Psi(z)\|_E^{\alpha_E}\|\Phi(z)\|_F^{\alpha_E} \le (2+\delta)^{\alpha_E}\|\Psi(z)\|_E^{\alpha_E}.$$

Also, the formula

$$|\Psi(\xi,j)| = \left|\frac{f(\xi,j)}{\Phi(\xi,j)}\right| = \frac{|f_E(\xi,j)|\,|f_F(\xi,j)|}{|f_F(\xi,j)| + |\Delta\gamma(r_k\xi,j)| + \sigma_j} \le |f_E(\xi,j)|,$$

is valid for each index $j \in \mathbb{N}$, whence we get the estimate

$$\|\Psi(\xi)\|_E \le \|f_E(\xi)\|_E \le 1.$$

Applying the maximum principle, we see that

$$\|\Psi(z)\|_E \le 1.$$

Thus, we have checked that the requirements of the problem of ideals for the lattice E are satisfied, and, hence, the images of T are nonempty sets. It is clear that these images are also convex. Since compactness in our situation coincides with sequential compactness and, as it will be shown, the graph of T is closed, the images of T are compact sets.

It remains to verify that the graph of T is closed. We need to check that for every sequence γ_n that converges uniformly on the compact subsets of $\mathbb{D} \times \operatorname{supp} E$ to some function γ and every sequence of functions u_n lying in the images $T\gamma_n$ and converging uniformly on the compact subsets of $\mathbb{D} \times \operatorname{supp} E$ to some function u, we have $u \in T\gamma$. The last formula means that for the function Ψ constructed above in terms of γ (and belonging to $H^\infty(E)$) we have $\langle u(z), \Psi(z)\rangle = h(z)/(2+\delta)^{\alpha_E}$. We mark with index n the functions Φ and Ψ arising during the construction made for γ_n:

$$\varphi_n(\xi, j) = \log(|\Delta\gamma_n(r_k\xi, j)| + |f_F(\xi, j)| + \sigma_j).$$

$$\Phi_n(\xi, j) = e^{\varphi_n(\xi,j)+iH(\varphi_n(\xi,j))}$$

$$\Psi_n(z) = \frac{f(z)}{\Phi_n(z)}.$$

Now we use the fact that $r_k < 1$. Then for a fixed index j and as $n \to \infty$, the functions $\varphi_n(\,\cdot\,, j)$ converge uniformly on the circle $\mathbb{T}$ to the function φ that was constructed by γ. *A fortiori*, they converge in L^2. Applying the continuity of the operator of harmonic conjugation H in L^2, we see that the functions $\Phi_n(\,\cdot\,, j)$ converge uniformly on the compact subsets of $\mathbb{D}$ to the function $\Phi(\,\cdot\,, j)$ constructed by γ. Since the $\Phi_n(\,\cdot\,, j)$ are separated away from zero by σ_j and all functions take values in finite-dimensional lattices, we may pass to the limit in the relation

$$\left\langle u_n(z), \frac{f(z)}{\Phi_n(z)} \right\rangle = h(z)/(2+\delta)^{\alpha_E}.$$

That is what was to be shown.

Thus, we have checked all requirements of Ky Fan–Kakutani theorem. Hence, there exists a function γ in B such that $\gamma \in T\gamma$. Recall that all our constructions depend on the parameter r_k. Our final aim in the proof of the main theorem is to pass to the limit as $r_k \to 1$ for the functions-solutions and show that the limit function solves the original problem. We denote by γ_k the functions-solutions for the parameter k and by φ_k,Φ_k,Ψ_k the corresponding functions (note that these functions differ from those appeared during the proof of the closedness of the graph T and denoted by similar symbols). By construction, for each $z \in \mathbb{D}$ we have $\langle \gamma_k(z), \Psi_k(z)\rangle = h(z)/(2+\delta)^{\alpha_E}$. Immediately from the definition of the function Ψ_k, it follows that

$$\left\langle f(z), \frac{\gamma_k(z)}{\Phi_k(z)}(2+\delta)^{\alpha_E} \right\rangle = h(z).$$

Applying Lemma 4.1, we see that the values $\left\| \frac{\gamma_k(z,j)}{\Delta\gamma_k(z,j)} \right\|_{(EF)'}$ are uniformly bounded by the constant $(1+\varepsilon)C_E$. We recall that

$$|\Phi_k(\xi, j)| = |\left(\Delta\gamma_k\left(r_k\xi\right)\right)(j)| + \sigma_j + |f_F(\xi, j)|.$$

Since B is compact in the $*$-weak topology, we may assume that for a fixed j the functions γ_k converge uniformly on the compact subsets of $\mathbb{D} \times \operatorname{supp} E$ to some bounded analytic function $u(\,\cdot\,, j)$. Then, taking into account the definition of the functions φ_k and the continuity of the map Δ, we may also assume that for every fixed index $j \in \operatorname{supp} E \subset \mathbb{N}$ the functions $\varphi_k(\,\cdot\,, j)$ converge weakly in L^2 to some function $v(\,\cdot\,, j)$. Applying the weak continuity of the operator of harmonic conjugation, we infer that the functions $\Phi_k(\,\cdot\,, j)$ converge uniformly on the compact subsets of $\mathbb{D} \times \operatorname{supp} E$ and are separated away from zero. Thus,

$$\frac{\gamma_k(r_k z)}{\Phi_k(z, j)} \to w(z, j), \quad k \to \infty, \;\; r_k \to 1,$$

where the $w(z, j)$ are bounded analytic functions for every index j. From these functions we compose a vector-function $w \in H^\infty(E')$. Applying the estimate

$$\left\| \frac{\gamma_k(r_k\xi, j)}{\Phi_k(\xi, j)} \right\| \leq \left\| \frac{\gamma_k(r_k\xi, j)}{\Delta\gamma_k(r_k\xi, j)} \right\| \leq (1+\varepsilon)C_E,$$

we get $\|w(z)\|_{(EF)'} \le (1+\varepsilon)C_E$ for every $z \in \mathbb{D}$. Now we can pass to the limit in the corresponding identity:

$$|\langle f(z), w(z)(2+\delta)^{\alpha_E}\rangle - h(z)| \\ \le \|f(z)\|_{EF}(2+\delta)^{\alpha_E}\left(\left\|w(z) - \frac{\gamma_k(r_k z)}{\Phi_k(z)}\right\|_{(EF)'} + \left\|\frac{\gamma(z)}{\Phi_k(z)} - \frac{\gamma_k(r_k z)}{\Phi_k(z)}\right\|_{(EF)'}\right) \longrightarrow 0.$$

It remains to pass to the limit as $\delta \to 0$ to conclude the proof of the theorem.

Acknowledgments

The author is very much grateful to his scientific advisor, Prof. S. V. Kislyakov, for his guidance and permanent support during the development of this paper.

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Received 01/MAY/2017

Translated by THE AUTHOR