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ON THE CHROMATIC NUMBER OF AN INFINITESIMAL PLANE LAYER

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Abstract. This paper is devoted to a natural generalization of the problem on the chromatic number of the plane. The chromatic number of the spaces $\mathbb{R}^n \times [0,\varepsilon]^k$ is considered.

It is proved that $5 \le \chi(\mathbb{R}^2 \times [0,\varepsilon]) \le 7$ and $6 \le \chi(\mathbb{R}^2 \times [0,\varepsilon]^2) \le 7$ for $\varepsilon > 0$ sufficiently small.

Also, some natural questions arising from these considerations are posed.

§1. Introduction

Consider the graph the vertex set of which coincides with the points of the plane and the edges connect all pairs of points at Euclidean distance of 1. Nelson stated the problem to determine the chromatic number of this graph (we denote this quantity by $\chi(\mathbb{R}^2)$). Then this problem was popularized by M. Gardner, P. Erdős, H. Hadwiger, and A. Soifer. Now it is known as the *Nelson–Hadwiger problem*. We are grateful to A. Soifer for pointing out our historical mistakes.

The following theorem is well known.

Theorem 1.1. $4 \le \chi(\mathbb{R}^2) \le 7$.

These bounds are relatively easy. Unfortunately, no improvements have appeared over the last 65 years. On the other hand, attempts to solve the initial problem gave birth to a lot of interesting questions and fruitful results. For a detailed list of references on the problem see §2.

A natural weakening is in finding points at a distance arbitrary close to 1. In this case there are such points for any plane 5-coloring [8] (some weaker statements are in [9] and [11]); this statement is a straightforward corollary to Theorem 9. A more interesting result is that if the colors are measurable, then an arbitrary 4-coloring produces 1-distant points [10].

We consider the "almost planar" case, or the case of dimension "$2+\varepsilon$". Our result is that if an arbitrary layer between two parallel planes in $\mathbb{R}^3$ is colored in 4 colors, then there are 1-distant points (Theorem 3.1).

Moreover, we show that in the case of 2 infinitesimal dimensions the same is true for 5 colors. In other words, if the Cartesian product of a plane and an arbitrary small square is 5-colored, then there are 1-distant points (see Theorem 3.3).

The case of Cartesian product looks much harder and more interesting than the case of forbidden distance interval $(1-\varepsilon,\ 1+\varepsilon)$. For instance, we can find 1-distant points in a layer only for 4 colors. Thereby, the following question arises.

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Question 1.1. Let the Cartesian product of a plane and a segment be colored in 5 colors. Are there 1-distant points?

Another branch of research is the following question.

Question 1.2. Consider an n-coloring of plane. Is there a color containing all possible distances?

In other words, is there a coloring such that the first color has no distance d_1, the second has no d_2 and so on? In the case where $d_i = 1$, we have the problem on the plane's chromatic number. Surprisingly, the problem is nontrivial even for $n = 3$, and there are some examples of colorings for $n = 6$.

Similar questions arise for layers and almost-distances.

§2. On the chromatic number of spaces

There is a lot of activity around the problem; a lot of results were obtained in several related settings. In particular, the chromatic numbers of spaces have been studied in the research group of A. M. Raigorodskii. We mention only a short list of main results. More detailed information on the Nelson–Hadwiger and related problems can be found in the following surveys: P. K. Agarwal and J. Pach [20], P. Brass, W. Moser and J. Pach [5], M. Benda and M. Perles [4], K. B. Chilakamarri [6], V. Klee and S. Wagon [15], A. M. Raigorodskii [24–28], A. Soifer [32, 33], and L. A. Székely [34].

2.1. On the chromatic number of a plane. We start with some weakenings. If all the connected components of every color are connected regions bounded by Jordan curves, then one need at least 6 colors, see D. R. Woodall [35].

K. J. Falconer showed in 1981 that if we demand that the colors be measurable, then we need at least 5 colors for a proper plane coloring [10]. Surely, the example of 7-coloring still works (see Figure 1).

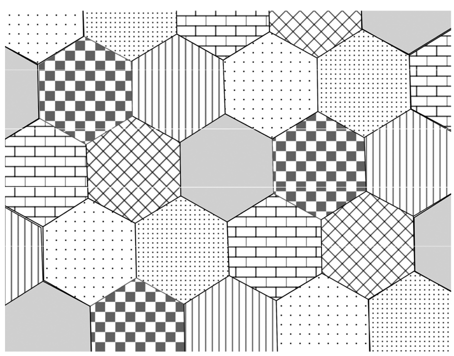

FIGURE 1. A proper coloring of plane in 7 colors. The sides of regular hexagons have length $\frac{1}{\sqrt{7}}$.

One of the main difficulties is that the answer may depend on set theory axiomatics, as was shown by Shelah and Soifer [30]. If we assume the axiom of choice, then by the Erdős–de Bruijn theorem the chromatic number of an infinite graph is realized on a finite subgraph. But computer simulations give no subgraphs with chromatic number at least 5, so one can conjecture that the chromatic number of the plane in the standard axiomatics is equal to 4. If we consider the ZF plus the axiom of dependent choice instead of ZFC, and in addition demand the Lebesgue measurability of all plane subsets in question, then one can repeat the proof of Falconer and the chromatic number lies between 5 and 7.

The case of a bounded subset of the plane can also be treated, see for example [17].

2.2. The case of an arbitrary metric space. Now we consider several generalizations. For an arbitrary metric space (X, d) and real number $a > 0$, define a graph $G_a(X, d)$ in the following way: its vertex set coincides with the points of the metric space, and the edges connect all pairs of points at the distance a. We are still interested in the chromatic number $\chi(X, d, a)$ of the graph. Usually the role of (X, d) is played by $\mathbb{R}^n$ or $\mathbb{Q}^n$ with the Euclidean metrics. We shall restrict ourselves to $a = 1$. It should be noted that all the graphs G_a are isomorphic in the real case.

2.2.1. *Chromatic numbers of real spaces.* The line case is obvious: $\chi(\mathbb{R}) = 2$. For $n = 2$ we have exactly the initial problem on the chromatic number of the plane. In the case where $n = 3$, the problem is even harder than the classical Nelson–Hadwiger problem; the latest bounds are dated this century.

Theorem 2.1.
$$6 \leq \chi(\mathbb{R}^3) \leq 15.$$

The lower bound is due to O. Nechushtan [19]; the upper bound is due to D. Coulson [7].

In asymptotics the following holds true.

Theorem 2.2.
$$(1.239\ldots + o(1))^n \leq \chi\left(\mathbb{R}^n\right) \leq (3 + o(1))^n.$$

The lower bound belongs to A. M. Raigorodskiĭ [23]; the upper bound belongs to D. Larman and A. Rogers [18]. It should be noted that asymptotical lower bounds were obtained by a *linear algebraic method* interesting for its own sake; moreover, J. Kahn and G. Kalai [14] used these bounds to provide a counterexample to Borsuk's conjecture [14], which had been open at the moment for more than 50 years. More information on the Borsuk conjecture and the linear algebraic method can be found in [26].

2.2.2. *Chromatic numbers of rational spaces.* The line case is still trivial: $\chi(\mathbb{Q}) = 2$.

It is somewhat surprising that the exact value of the chromatic number of $\mathbb{Q}^n$ is known not only in dimension 2, but also in dimensions 3 and 4 (see D. R. Woodall [35], P. D. Johnson [13], and M. Benda–M. Perles [4]).

Theorem 2.3. $\chi(\mathbb{Q}^2) = \chi(\mathbb{Q}^3) = 2$, $\chi(\mathbb{Q}^4) = 4$.

The best asymptotic lower bound at this time is due to E. I. Ponomarenko and A. M. Raigorodskiĭ [21, 22], and the best current upper bound belongs to D. Larman and A. Rogers [18].

Theorem 2.4.
$$(1.199\ldots + o(1))^n \leq \chi\left(\mathbb{Q}^n\right) \leq (3 + o(1))^n.$$

We mention that the mixed case $\mathbb{R} \times \mathbb{Q}$ was considered in [2].

2.3. Polychromatic numbers. In the book [12], H. Hadwiger and H. Debrunner (inspired by P. Erdős) formulated a natural question on finding *the polychromatic number of the plane*, that is the minimal number of colors one needs to construct a plane coloring such that for every color i there is a distance d such that i has no points at the distance d. We use the notation χ_p for this quantity, introduced by A. Soifer in [31]. The best current bounds were discovered by Raiskiĭ and Stechkin in [29]. (Stechkin's example was published in the same paper with his permission.)

Theorem 2.5.

$$4 \le \chi_p \le 6.$$

One can find different proofs of the same bounds in the paper [35] by D. R. Woodall.

§3. Main results

We are interested in the chromatic numbers of metric spaces of the type $\mathbb{K}^n \times [0,\varepsilon]^k$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$, $n, k \ge 1$, with the Euclidean metrics. We call such metric spaces "layers", and in the case where $n = k = 1$ we call them "strips". The main focus of this paper is on the case of $n = 2$.

3.1. The chromatic numbers of one-dimensional layers. We start with a simple observation. The lower bound can be found in [3] and [6], but we prove it for the completeness of presentation.

Proposition 3.1. *Let* $0 < h \le \sqrt{\frac{3}{4k}}$. *Then*

$$\chi(\mathbb{R} \times [0,h]^k) = 3.$$

Let $\sqrt{\frac{3}{4k}} < h \le \sqrt{\frac{8}{9k}}$. *Then*

$$\chi(\mathbb{R} \times [0,h]^k) = 4.$$

The upper bound. Let $0 < h \le \sqrt{\frac{3}{4k}}$. We color $\mathbb{R}$ in 3 colors by iterating monochromatic half-intervals of length $1/2$ (with periodic colors $1, 2, 3, 1, 2, 3$ and so on). Then to a point of $\mathbb{R} \times [0,h]^k$ we assign the color of its projection to the real line. Then the diameter of the monochromatic parallelepiped $[0; \frac{1}{2}] \times [0,h]^k$ is at most 1; if it is equal to 1, then its ends have different colors.

Similarly, one can properly color the strip in 4 colors if $\sqrt{\frac{3}{4k}} < h \le \sqrt{\frac{8}{9k}}$; in this case half-open intervals have length $1/3$.

The lower bound. We use a trivial example to show the outline of the proof that will be used later in dimensions 3 and 4.

Suppose that the strip $\mathbb{R} \times [0,\varepsilon]$ is properly colored in finitely many colors. Let $l \in \mathbb{N}$ be such that $1/l = \delta \le \varepsilon^2$. On the lower boundary $\mathbb{R} \times \{0\}$ we choose differently colored points $u = (x, 0)$, $v = (x+\delta, 0)$, at the distance δ apart. Such a choice is possible because the points $(0,0), (\delta,0), \dots, (1,0)$ cannot be all of the same color. Denote $w = (x+\delta/2, \varepsilon)$. One of the pairs u, w and v, w meets two different colors. Let it be u, w. Then one needs to use an additional color for the point ξ that is at the distance 1 of u and w and lies inside the strip. $\square$

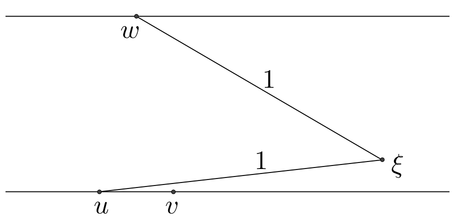

Figure 2. The lower bound on $\chi(\mathbb{R} \times [0,\varepsilon])$.

In the case where $\sqrt{\frac{3}{4k}} < h \le \sqrt{\frac{8}{9k}}$, the set $\mathbb{R} \times [0,h]^k$ contains the strip $\mathbb{R} \times [0,h_1]$, $h_1 > \sqrt{3}/2$, which is the product of $\mathbb{R}$ and a k-dimensional hypercube with the edge length h. In the strip we can embed the distance graph depicted in Figure 2, and moreover,

$d(y, z)$ can be an arbitrary number in $[0, 3 - 2\sqrt{3 - h_1^2}]$. We choose an embedding of the graph such that the points x, y, z lie on the boundary of the strip and $d(y, z) = 1/m$, $m \in \mathbb{N}$. Copying the construction m times, we get a distance graph with the chromatic number 4. $\square$

Remark. The number of vertices tends to infinity as h tends to the value corresponding to a gap of $\chi(\mathbb{R} \times h)$ ($h = 0$ and $h = \sqrt{\frac{3}{4k}}$), but the graph can be embedded to a bounded domain not depending on h.

Obviously, the function $\xi_{n,k}(h) = \chi(\mathbb{R}^n \times [0, h]^k)$ defined for $h \geq 0$ is monotone nondecreasing. For every fixed n, k, the number of possible values of $\xi_{n,k}(h)$ is finite because $\chi(\mathbb{R}^n) \leq \xi_{n,k}(h) \leq \chi(\mathbb{R}^{n+k})$, and so the number of discontinuity points is finite. But it seems plausible that in the case where $n > 1$, one cannot find a discontinuity point of $\xi_{n,k}(h)$ without refinements of the current bounds on $\chi(\mathbb{R}^n)$.

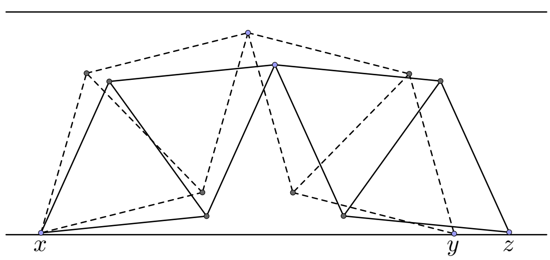

FIGURE 3. A chain of θ-graphs in the strip.

In fact, the lower bound from Proposition 3.1 can be extended.

Proposition 3.2. *Let ε be a positive real number, and let Q be an ε-neighborhood of some curve ξ of diameter at least* 2. *Then* $\chi(Q) \geq 3$.

3.2. Chromatic numbers of 2-dimensional layers. Let us consider an intermediate case between plane and space, i.e., $\mathbb{R}^2 \times [0, \varepsilon]$ (a layer of height ε).

Although this metric space is still 7-colorable, the following lower bound is more difficult than in the plane's case.

Theorem 3.1. *Let ε be a positive number less than $\sqrt{3/7}$. Then*

$$5 \leq \chi(\mathbb{R}^2 \times [0, \varepsilon]) \leq 7.$$

Unlike the strip case ($n = 1$), we cannot even show that the function $\chi(\mathbb{R}^2 \times [0, \varepsilon])$ is discontinuous at the point $\varepsilon = 0$.

Now we consider a "blow-up" of the plane in a higher-dimensional space. Since the standard plane 7-coloring does not contain distances from some interval, the upper bound preserves when the dimension grows.

Theorem 3.2. *Let k be an integer and let $\varepsilon < \varepsilon_0(k)$ be a positive number. Then*

$$\chi(\mathbb{R}^2 \times [0, \varepsilon]^k) \leq 7.$$

The lower bound can be refined even for $k = 2$.

Theorem 3.3. *Let ε be a positive number. Then*

$$\chi(\mathbb{R}^2 \times [0, \varepsilon]^2) \geq 6.$$

Note that in fact we again consider colorings of a bounded domain, with diameter independent of ε.

In the proof of Theorem 2.4 we use the following lemma, which is interesting by itself.

Lemma 3.1. *Suppose that the Euclidean plane is properly k-colored. Then for an arbitrary $\varepsilon > 0$ there is an ε-ball containing points of at least 3 colors.*

Corollary 3.1. *Suppose that the Euclidean plane is properly k-colored. Then for an arbitrary $\varepsilon > 0$ there is a circle of radius ε containing points of at least 3 colors.*

We prove a generalization of Lemma 3.1.

Theorem 3.4. *Suppose $\mathbb{R}^n$ is properly m-colored. In other words, let C_i be the set of points of color i; we have*

$$\bigcup_{i=1}^{m} C_i = \mathbb{R}^n,$$

and every C_i has no pair of points at the distance 1. Then there is a point lying in $n+1$ different sets $\overline{C}_i$.

This statement is obvious if the connected components of $\overline{C}_i$ are polytopes, but it is also valid in the case of a general covering with one forbidden distance.

3.3. Chromatic numbers of rational spaces. The following theorem holds true in the rational case.

Theorem 3.5. *For a sufficiently small positive ε we have*

$$\chi(\mathbb{Q} \times [0,\varepsilon]^3_{\mathbb{Q}}) = 3.$$

Obviously $[0,\varepsilon]^3_{\mathbb{Q}}$ cannot be replaced by $[0,\varepsilon]^2_{\mathbb{Q}}$ because $\chi(\mathbb{Q}^3) = 2$.

§4. Proofs

We start with a construction widely used in the proofs.

Definition 4.1. Let ω_r be a circle with radius r. We call $r > 0$ a *forbidden radius* if $G_1(\omega_r)$ contains an odd cycle.

Proposition 4.1. *The forbidden radii are dense in* $[1/2, \infty)$.

Proof. Indeed, every $q \in \mathbb{Q} \cap (0, \frac{1}{2})$ such that $q = \frac{l}{2k+1}$ with $k, l \in \mathbb{N}$ gives a forbidden radius

$$r = \frac{1}{2\sin \pi q}. \qquad \square$$

4.1. Proof of the lower bound in Theorem 3.1. Denote the Euclidean metrics by ρ, and the sphere (of maximal dimension) with radius r and center u by $S(u;r)$. Suppose that a layer $\mathbb{R}^2 \times [0,\varepsilon]$ is properly colored. Let $0 < \delta < \varepsilon^2$. Choose points u, v of different colors on the boundary of the layer ($\mathbb{R}^2 \times \{0\}$) such that $\rho(u,v) = \delta$.

We pick $\varepsilon_1 > 0$ such that $\sqrt{\delta} \le \varepsilon_1 < \varepsilon$ and $r = \sqrt{1 - \varepsilon_1^2/4}$ is a forbidden radius. Let us construct an isosceles triangle uvw such that the altitude ww_1 is perpendicular to the boundary of the layer and the lateral sides are of length ε_1. Since u and v have different colors, at least one of the pairs u, w and v, w has different colors. Without loss of generality, suppose that the points u, w are of colors 1 and 2. Then the circle

$$\omega = S(u;1) \cap S(w;1)$$

lies in the layer, has no points of colors 1 and 2, and has forbidden radius r, so it demands at least 3 additional colors. $\square$

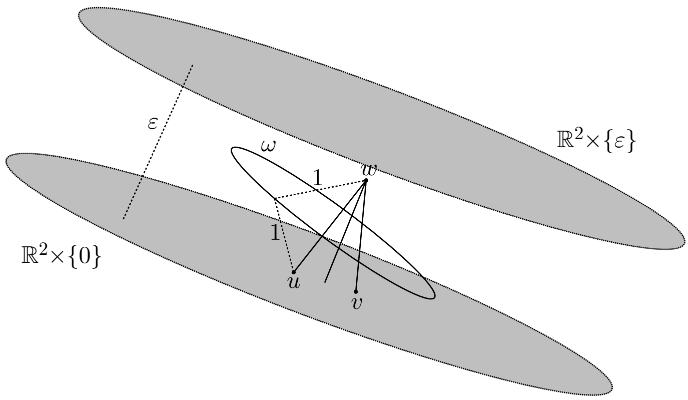

FIGURE 4. A circle with forbidden radius in the layer $\mathbb{R}^2 \times [0, \varepsilon]$.

4.2. Proof of the upper bound in Theorems 3.1 and 3.2. Consider the standard proper coloring of the plane in 7 colors (see Figure 1). It has no pair of points of the same color at a distance between $2/\sqrt{7}$ and 1. Let us color the layer in the following way: every hypercube $(x, y) \times [0, \varepsilon]^k$ gets the color of the point (x, y) in the plane coloring. This coloring is proper if $(2/\sqrt{7})^2 + k\varepsilon^2 < 1$, which is equivalent to the inequality in the statement. $\square$

4.3. Proof of Proposition 3.2. Without loss of generality assume that $\varepsilon < 1$. Suppose the contrary, i.e., there is a proper 2-coloring of Q. Denote by $G(Q)$ the corresponding graph and find an odd cycle.

Consider a point $u \in \xi$. The intersection of $S(u; 1)$ and ξ is nonempty because $\operatorname{Diam} \xi \geq 2$. Let $v \in S(u; 1) \cap \xi$, $\|u - v_1\| = 1$, and $\|v_i - v_{i+1}\| = 1$, $i = 1, 2, 3$. If all the edges between neighbor unit elements of the broken line $vuv_1v_2v_3v_4$ do not exceed $\frac{\varepsilon}{2}$, then $\|v - v_1\| < \frac{\varepsilon}{2}$, $\|u - v_2\| < \frac{\varepsilon}{2}$, $\|v - v_3\| < \varepsilon$, $\|u - v_4\| < \varepsilon$, and then $v_i \in Q$, $i = 1, 2, 3, 4$.

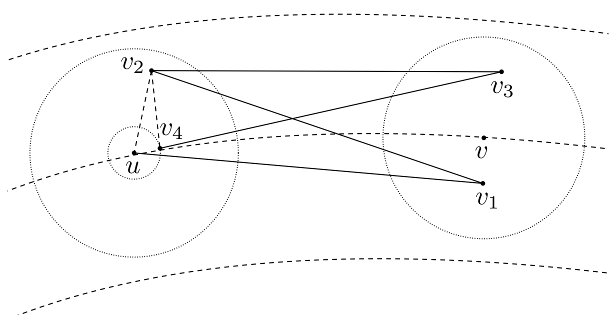

FIGURE 5. A path of length 4 connecting u and v_4.

Moreover, $l_1 = \|u - v_2\| \in \left[0; 2\sin\frac{\varepsilon}{4}\right]$ and $l_2 = \|v_2 - v_4\| \in \left[0; 2\sin\frac{\varepsilon}{4}\right]$ can be chosen arbitrarily, and the oriented angle between the vectors $\overrightarrow{v_2u}$ and $\overrightarrow{v_2v_4}$ can be chosen independently in the interval $[-\frac{\varepsilon}{4}; \frac{\varepsilon}{4}]$. Fix the line containing the vector $\overrightarrow{v_2u}$, let it be orthogonal to uv. Then all possible positions of v_4 form a figure containing a rhombus centered at u with the length of side $2\sin\frac{\varepsilon}{4}$ and the angle $\frac{\varepsilon}{2}$. Therefore, there exists a path of length 4 between u and an arbitrary point in the γ-neighborhood of u, where $\gamma = \sin\frac{\varepsilon}{2}\sin\frac{\varepsilon}{4}$.

Walking in this way along the curve ξ with steps of size γ from u to v, we construct an even path connecting u and v, and so an odd cycle in $G(Q)$. $\square$

4.4. Proof of Lemma 3.1. We show the existence of a ball of an arbitrarily small radius containing points of at least 3 colors. Suppose the contrary: there is a proper plane coloring and $\varepsilon > 0$ such that every ε-ball contains points of at most 2 colors. Divide plane into squares with side

$$\delta \leq \frac{2}{\sqrt{10}}\varepsilon.$$

Then every such square is 2-colored.

By Proposition 3.2, the outer boundary of any connected 2-colored domain forms some finite figure (otherwise, connecting centers of adjacent squares one can get a sufficiently large broken line). Every 2-colored connected domain consist of monochromatic squares, so it has finite diameter. So we can consider a 2-colored connected domain such that its outer boundary forms a figure of the maximal area. Addition of an arbitrary outer adjacent square gives a contradiction. $\square$

4.5. Proof of Corollary 3.1. By Lemma 3.1, for every $\varepsilon > 0$ there is a ball of radius ε containing points of three different colors. We show that there is a circle of radius at most ε such that it has points of at least 3 colors. Consider a triangle ABC in an ε-ball with differently colored vertices. It has an obtuse angle (otherwise the circumscribed circle of the triangle fits); without loss of generality it is angle A. Consider the point D such that $\angle ADB = \angle ADC = \pi/3$. Then $\angle BDC = 2\pi/3$. Note that at least one of the triangles ABD, ACD, BCD has all vertices of different colors. The radii of the circumscribed circles of these triangles are at most

$$\frac{\varepsilon}{2\sin\angle D} = \frac{\varepsilon}{\sqrt{3}} < \varepsilon.$$

The proof is complete, because $\varepsilon > 0$ can be chosen arbitrarily. $\square$

4.6. Proof of Theorem 3.3. The proof is based on the following construction: if there is a triangle with vertices v_1, v_2, v_3 of different colors lying in the layer and having circumscribed circle centered at u_0, then the layer contains a circle that is the intersection of three unit spheres with centers at v_1, v_2, v_3. Under a proper choice of the vertices of that triangle, this circle contains an odd cycle, so it demands at least three additional colors, giving the required bound.

Consider a proper coloring of a layer

$$\mathbb{R}^2 \times [0,\varepsilon]^2 = \{(x,y,z,t) \mid x,y \in \mathbb{R},\ z,t \in [0,\varepsilon]\}$$

in some colors. We need the following auxiliary statement.

Proposition 4.2. *Let $\phi(v_1, v_2, v_3)$ be the angle between the 2-dimensional plane containing v_1, v_2, v_3 and the plane $\{(0,0,z,t)\}$. For arbitrary $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, the layer contains a triangle with vertices v_1, v_2, v_3 of three different colors and with angles α_1, α_2, α_3 satisfying the following conditions*:

$$\phi(v_1, v_2, v_3) \leq \varepsilon_2; \tag{1}$$

$$\alpha_i \geq \frac{\pi}{5} - \varepsilon_3, \quad i = 1,2,3. \tag{2}$$

Proof. Choose $\varepsilon_1 < \varepsilon/2$. Let $M = \{(z_1,t_1),(z_2,t_2),\dots(z_5,t_5)\}$ be the vertex set of a regular pentagon inscribed into the circle with radius ε_1 centered at $(\varepsilon/2, \varepsilon/2)$. For a given point $(x, y) \in \mathbb{R}^2$, define $Q_{x,y}$ as the vertices of the corresponding pentagon in the infinitesimal square:

$$Q_{x,y} = \{(x, y)\} \times M.$$

If $Q_{x,y}$ has at least 3 colors for some x, y, then a triangle with vertices of $Q_{x,y}$ of different colors is proper. Suppose the contrary, i.e., for every x, y the set $Q_{x,y}$ has at most 2 colors. Then $Q_{x,y}$ has at least 3 points of some color; denote this color by $c(x, y)$.

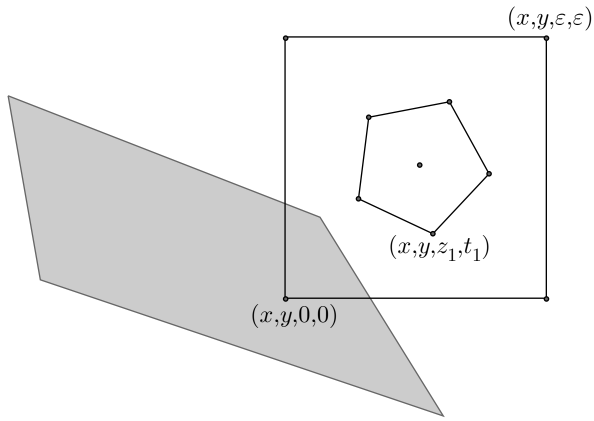

FIGURE 6. Five points $Q_{x,y} = \{(x, y)\} \times M$.

For a given coloring of the strip, consider an auxiliary coloring of an auxiliary plane $P = \mathbb{R}^2$ defined in the following way: a point (x, y) has the color $c(x, y)$. Note that this coloring is proper. Indeed, if the points (x_1, y_1) and (x_2, y_2) are at the distance of 1 and have the same color (say, 1), then the corresponding fives in the layer $Q_{x_1,y_1} = \{q_{1i}\}$, $Q_{x_2,y_2} = \{q_{2i}\}$ each have at least 3 points of color 1. But since

$$\rho(q_{1i}, q_{2i}) = 1, \quad i = 1, \dots, 5,$$

the set $Q_{x_1,y_1} \cup Q_{x_2,y_2}$ contains at most 5 points of each color.

Applying Lemma 3.1 to P, we see that for every $\delta > 0$ there are points u, v, w of different colors $c(u)$, $c(v)$, $c(w)$ with pairwise distances of at most δ. This means that the fives Q_u, Q_v, Q_w have at least 3 points of colors $c(u)$, $c(v)$, $c(w)$, respectively. We can choose a point in each set with different projections to the plane $(0, 0, z, t)$ and of colors $c(u)$, $c(v)$, $c(w)$. It is easily seen that the conditions

$$16\left(\frac{\delta}{\varepsilon_1} + 2\frac{\delta^2}{\varepsilon_1^2}\right) \le \sin\varepsilon_2;$$
$$\delta \le \frac{\varepsilon_1}{2}\sin\frac{\varepsilon_3}{2}$$

imply inequalities (1), (2). □

Now we are ready to prove Theorem 3.3. Consider points v_1, v_2, v_3, satisfying the conditions of Proposition 4. Let u_0 be the center of the circumscribed circle of the triangle $v_1v_2v_3$, let $\overline{n}$ be some unit vector orthogonal to the 2-dimensional triangle's plane, let $u_1 = u_0 + \delta_1\overline{n}$, and let $L(u_1, v_1, v_2, v_3)$ be the hyperplane containing u_1, v_1, v_2, v_3. Let $B(u_1; \delta_2) \subset L(u_1, v_1, v_2, v_3)$ be the open 3-dimensional ball of radius $\delta_2 > 0$ centered at u_1.

For a given point $w \in B(u_1; \delta_2)$, we define

$$T_1(w) = S(v_2; 1) \cap S(v_3; 1) \cap S(w; 1),$$

where $S(v;1)$ is the unit sphere, centered at v.

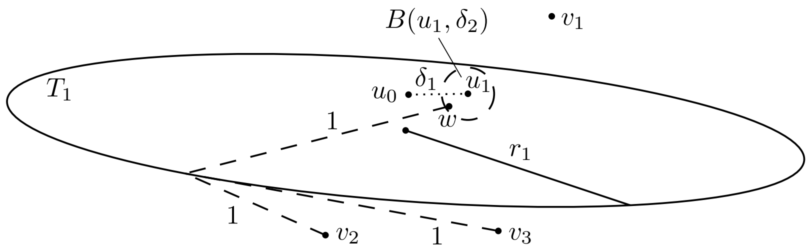

FIGURE 7. Construction of a circle of forbidden radius.

Let the radius of the circle $T_1(w)$ be $r_1(w)$; the circles $T_2(w)$, $T_3(w)$ and so their radii $r_2(w), r_3(w)$ are defined similarly.

Note that the vertices of wv_1v_2, wv_2v_3, wv_1v_3 are at the distance of at most $\delta+\delta_1+\delta_2$ from the vertices of the triangle, lying in $\{(0,0)\} \times [0,\varepsilon]^2$; therefore, if δ, δ_1, δ_2 are sufficiently small, then the corresponding circles lie inside the layer.

For the 3-dimensional ball $B(u_1;\delta_2)$ defined above, we introduce the function

$$r\colon B(u_1;\delta_2) \to \mathbb{R}^3;$$
$$r(w) = (r_1(w), r_2(w), r_3(w)).$$

Observe that in the case where $w = u_1$, the gradients of $r_i(w)$ are collinear to the medians of the isosceles triangle $u_1v_2v_3$, $u_1v_1v_3$, $u_1v_1v_2$ corresponding to u_1:

$$\nabla r_1(u_1) = \lambda_1 \left(u_1 - (v_2+v_3)/2\right); \quad \nabla r_2(u_1) = \lambda_2 \left(u_1 - (v_1+v_3)/2\right);$$
$$\nabla r_3(u_1) = \lambda_3 \left(u_1 - (v_1+v_2)/2\right),$$

and also $\lambda_i \neq 0$ and the simplex $u_1v_1v_2v_3$ is nondegenerate. Hence, for $w = u_1$ the Jacobian $\partial r/\partial w$ is not 0, and in a neighborhood of u_1 the function $r(\,\cdot\,)$ satisfies the conditions of the inverse function theorem. But the forbidden radii are dense in a neighborhood of every image $r_1(w)$, $r_2(w)$, $r_3(w)$, so that there is a triple of forbidden radii r_1^*, r_2^*, r_3^* with a preimage u^* in $B(u_1;\delta_2)$.

Then for every color of u^* at least one of the triangles $u^*v_1v_2$, $u^*v_2v_3$, $u^*v_1v_3$ has vertices of different colors, and the corresponding circle of forbidden radius has at least 3 additional colors. Summarizing, we get $\chi(\mathbb{R}^2 \times [0,\varepsilon]^2) \geq 6$. $\square$

4.7. Proof of Theorem 3.4. The idea of the proof is to construct a family of closed sets of diameter at most 2 such that they cover $\mathbb{R}^n$. Then the statement follows from the definition of topological dimension (we use the standard topology on $\mathbb{R}^n$).

Recall that C_i is the set of points of $\mathbb{R}^n$ of the ith color, $1 \leq i \leq m$, and put

$$C_i^* := \overline{\operatorname{Int} \overline{C}_i} \qquad \text{(the closure of the interior of the closure).}$$

Split every C_i^* in the connected components (in the sense of the standard topology):

$$C_i^* = \bigcup_{\alpha \in A_i} D_\alpha.$$

For brevity, denote $\{D_\alpha\} = \bigcup_{i=1}^m \bigcup_{\alpha \in A_i} D_\alpha$.

(i). *The sets C_i^* cover $\mathbb{R}^n$.*

Suppose the contrary: there exists v so that $v \notin C_i^*$ for all i. Then there is an open ball $B(v;\varepsilon)$ with

$$B(v;\varepsilon) \cap C_i^* = \varnothing, \quad B(v;\varepsilon) \subset \bigcup C_i.$$

Consider a ball

$$B^1 \subset B(v;\varepsilon) \setminus \overline{C}_1.$$

Obviously, B^1 cannot be a subset of $\overline{C}_i$, otherwise the intersection of $\overline{C}_i$ and $B(v;\varepsilon)$ is nonempty. Define a nested sequence of balls

$$B^{k+1} \subset B^k \setminus \overline{C}_k.$$

The points of B^{m+1} belong to none of the $\overline{C}_i$, which is a contradiction.

(ii). *If the unit sphere S centered at a point v contains inner points of $\overline{C}_i$, $1 \le i \le k \le n$, then v belongs to at least one of the sets C_j^*, $k+1 \le j \le m$.*

One can choose points $x_1, \dots, x_n$ such that

$$x_i \in S \cap \operatorname{Int} \overline{C}_i, \qquad 1 \le i \le k,$$
$$x_i \in S, \quad k+1 \le i \le n,$$

and $\{v, x_1, \dots, x_n\}$ are in a general position (in the sense on nondegenerate simplices). Consider $\varepsilon > 0$ such that $B(x_i;\varepsilon) \subset \overline{C}_i$, $1 \le i \le k$. The color of the point

$$w = w(q_1, \dots, q_k) = \bigcup_{1 \le i \le k} S(q_i; 1)$$

(if it is defined) differ from all the colors $q_1, \dots, q_n$. Let

$$z \in B(0;\varepsilon), \quad y_i = x_i + z.$$

In a sufficiently small neighborhood of the set of points $\{y_i\}$, the function $w(\cdot)$ is defined and continuous in each argument. Choose points

$$y_i' \in C_i, \quad 1 \le i \le k,$$
$$y_i' = y_i, \quad 1 \le k+1 \le n,$$

such that $w(y_1', \dots, y_n')$ exists. Then

$$w(y_1', \dots, y_k') \in \bigcup_{j=k+1}^{m} C_j.$$

Moreover,

$$\delta(y_1', \dots, y_k') = \max_{1 \le i \le k} \|y_i' - y_i\|$$

can be arbitrarily small, whence

$$w(y_1, \dots, y_k) \in \bigcup_{j=k+1}^{m} \overline{C}_j.$$

Since $z \in B(0;\varepsilon)$ can be chosen arbitrarily, we have

$$B(v;\varepsilon) \subset \bigcup_{j=k+1}^{m} \overline{C}_j.$$

Hence, at least one of the sets $\overline{C}_j$, $j = k+1, \dots, m$, is dense in some neighborhood of v.

(iii). *If a point $v \in \mathbb{R}^n$ is covered by at most n sets from $\{D_\alpha\}$, then the diameter of at least one of these sets is at most 2.*

Otherwise every set in $\{D_\alpha\}$ that covers v has a nonempty intersection with the sphere S radius of 1 centered at v. Without loss of generality we may assume that v is covered by the sets $D_1, \dots, D_n$ that are the connected components of $C_1^*, \dots, C_n^*$, respectively. Let $\min\{\operatorname{Diam} D_i\} = 2 + \delta$.

Let $w \in \mathbb{R}^n$ with $\|w\| = 1$ be a direction, and let S_η be the unit sphere centered at the point $u(\eta) = v + \eta w$, $\eta \in \mathbb{R}_+$. Then the set

$$T_i = \{\eta \in \mathbb{R}_+ \,:\, S_\eta \cap \operatorname{Int} D_i \neq \varnothing;\ 1 \leq i \leq n\}$$

is dense in $[0, \delta]$. Hence, for all $\eta \in [0, \delta]$ we have

$$u(\eta) \in \bigcup_{n+1}^{m} \overline{C}_j.$$

The same argument is valid for an arbitrary unit vector w. But then every neighborhood of v contains a ball that is a subset of $\bigcup_{n+1}^{m} \overline{C}_j$, and hence, contains an inner point of at least one set $\overline{C}_j$, $j > n$. This contradiction proves (iii).

(iv). *If every point of $\mathbb{R}^n$ is covered by at most n sets from $\{D_\alpha\}$, then the family of sets $\Delta = \{D_\alpha |\ \operatorname{Diam}(D_\alpha) \leq 2\}$ covers $\mathbb{R}^n$.*

This is an obvious consequence of (iii).

(v). *There are sets $D_1', D_2', \ldots, D_{n+1}' \in \Delta$ with nonempty intersection.*

Consider the ball $B(0; R) \subset \mathbb{R}^n$ and its covering by a family of sets from Δ. A suitable definition of the topological dimension (see P. S. Aleksandrov and B. A. Pasynkov [1]) shows that if $B(0; R)$ is covered by closed sets of diameter at most 2 and R is sufficiently large, then there are $n + 1$ sets with nonempty intersection.

(vi). *There are $n + 1$ sets from $\{C_i\}$ with nonempty intersection of closures.*

Suppose that sets $D_1', D_2', \ldots, D_{n+1}' \in \Delta$ satisfy

$$\bigcap_{i=1}^{n+1} D_i' \neq \varnothing,$$

$$D_i' \subset C_{l_i}^*, \quad i = 1, 2, \ldots, n + 1.$$

Note that the indices l_i are pairwise different, otherwise pairwise intersecting sets D_i' are not different connected components of C_i^*. Hence,

$$\varnothing \neq \bigcap_{i=1}^{n+1} D_i' \subset \bigcap_{i=1}^{n+1} C_{l_i}^* \subset \bigcap_{i=1}^{n+1} \overline{C}_{l_i},$$

and $\{C_{l_i}\}$ is the desired subfamily. □

Remark. By using the Sperner lemma, one can get a bound on the radius of a ball containing at least one point from $n + 1$ sets $\overline{C}_i$.

4.8. Proof of Theorem 3.5. Let x be the valid coordinate and y, z, t infinitesimal coordinates.

The upper bound. Color a point (x, y, z, t) such that $\frac{2k}{3} < x \leq \frac{2(k+1)}{3}$ in color k mod 3 (k is integer).

The lower bound. We show the existence of an odd cycle in the distance graph $G_1(\mathbb{Q} \times [0, \varepsilon]_{\mathbb{Q}}^3)$.

Consider an even n with $n > 2\varepsilon^{-2}$ and a vector $e = (1 - n^{-1}, bn^{-1}, cn^{-1}, dn^{-1})$ such that $b^2 + c^2 + d^2 = 2n - 1$. This vector has unit length. Note that e lies in the strip in question because

$$\max\left(|b|n^{-1},\ |c|n^{-1},\ |d|n^{-1}\right) < \sqrt{\frac{2}{n}} < \varepsilon.$$

Also consider the vector $e' = (1 - n^{-1}, -bn^{-1}, -cn^{-1}, -dn^{-1})$ and the sequence of points A_i defined in the following way:

$$A_0 := (0, 0, 0, 0); \quad A_{2k+1} := A_{2k} + e; \quad A_{2k+2} := A_{2k+1} + e'.$$

Easily, $A_n = (n-1, 0, 0, 0)$ because n is even. So the points $A_0, \ldots, A_n$ and the points $(1, 0, 0, 0), \ldots, (n-2, 0, 0, 0)$ form a desired odd cycle. Finally, observe that for every $\varepsilon > 0$ there exist integers n, b, c, d satisfying our conditions (for example, $b = c = d = 2l + 1$, $n = 6l^2 + 6l + 2$, where l is sufficiently large). $\square$

§5. Conclusion and further questions

We have shown that $5 \leq \chi(\mathbb{R}^2 \times [0, \varepsilon]) \leq 7$ and $6 \leq \chi(\mathbb{R}^2 \times [0, \varepsilon]^2) \leq 7$, and also that $\chi(\mathbb{R}^2 \times [0, \varepsilon]^k) \leq 7$ for a sufficiently small $\varepsilon > 0$. The question on the existence of k such that $\chi(\mathbb{R}^2 \times [0, \varepsilon]^k) = 7$ for an arbitrarily small $\varepsilon > 0$ arises naturally.

One can observe that we have discrete continuity in the cases where we can compute the chromatic number of a real layer. Is this true in general? In other words, is the function $\chi(\mathbb{K}^n \times [0, \varepsilon]^m)$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{Q}\}$, discrete continuous in ε?

In the 1-dimensional and 2-dimensional cases, the additional infinitesimal dimension increases the lower bound on the chromatic number of the space. General arguments show that the inequality $\chi(\mathbb{R}^3 \times [0, \varepsilon]) \geq 7$ should be valid, where ε is an arbitrary positive number, but we cannot prove it. Moreover, we conjecture that $\chi(\mathbb{R}^n \times [0, \varepsilon]) > \chi(\mathbb{R}^n)$, but it seems much harder.

In the paper [16], A. Kupavskiĭ asked about the maximal guaranteed number of colors in an m-dimensional sphere of radius r over all proper colorings of $\mathbb{R}^n$ in finitely many colors. In the same paper some bounds for r separated away from 0 were given. Lemma 3.1 complements these results in the case of infinitesimal r, but only for $n = 2$, $m = 1$. It seems plausible that Theorem 3.4 can be used to get the same result for $n = m + 1$ and arbitrary $n > 2$.

A dual problem is also of interest, i.e., to construct a "reasonable" metric space with chromatic number exactly k for a given integer k. For instance, such a problem is interesting for a space with large affine subspace, in particular, for $[0, h_1] \times \cdots \times [0, h_m] \times [0, \varepsilon]^l \times \mathbb{R}^s$, $s > 0$.

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References

[1] P. S. Aleksandrov and B. A. Pasynkov, *Introduction to dimension theory*, Nauka, Moscow, 1973. (Russian) MR0365524 (51:1776)

[2] M. Axenovich, J. Choi, M. Lastrina, T. McKay, J. Smith, and B. Stanton, *On the chromatic number of subsets of the Euclidean plane*, Graphs Combin. **30** (2014), no. 1, 71–81. MR3143859

[3] B. L. Bauslaugh, *Tearing a strip off the plane*, J. Graph Theory **29** (1998), no. 1, 17–33. MR1633912 (99j:05063)

[4] M. Benda and M. Perles, *Colorings of metric spaces*, Geombinatorics **9** (2000), no. 3, 113–126. MR1746078 (2001e:52031)

[5] P. Brass, W. O. J. Moser, and J. Pach, *Research problems in discrete geometry*, Springer, New York, 2005. MR2163782 (2006i:52001)

[6] K. B. Chilakamarri, *The unit-distance graph problem: a brief survey and some new results*, Bull. Inst. Combin. Appl. **8** (1993) 39–60. MR1217358 (94e:05003)

[7] D. Coulson, *A* 15-*colouring of* 3-*space omitting distance one*, Discrete Math. **256** (2002), no. 1–2, 83–90. MR1927057 (2003c:05073)

[8] J. D. Currie and R. B. Eggleton, *Chromatic properties of the Euclidean plane*, `arXiv:1509.03667`, 2015.

[9] G. Exoo, *ε-unit distance graphs*, Discrete Comput. Geom. **33** (2005), no. 1, 117–123. MR2105753 (2005h:05075)

[10] K. J. Falconer, *The realization of distances in measurable subsets covering* $\mathbb{R}^n$, J. Combin. Theory Ser. A **31** (1981), no. 2, 184–189. MR0629593 (82m:05031)

[11] J. Grytczuk, K. Junosza-Szaniawski, J. Sokół, and K. Węsek, *Fractional and* j*-fold coloring of the plane*, Discrete Comput. Geom. **55** (2016), no. 3, 594–609. MR3473669

[12] H. Hadwiger and H. Debrunner, *Kombinatorische Geometrie in der Ebene*, Monographica de "L'Enseignement Mathematique," Vol. 2, Inst. Math., Univ. Geneve, 1960. MR120559

[13] P. D. Johnson, *Coloring abelian groups*, Discrete Math. **40** (1982), no. 2, 219–233. MR0676727 (83k:20059)

[14] J. Kahn and G. Kalai, *A counterexample to Borsuk's conjecture*, Bull. Amer. Math. Soc. **29** (1993), no. 1, 60–62. MR1193538 (94a:52007)

[15] V. Klee and S. Wagon, *Old and new unsolved problems in plane geometry and number theory*, Dolciani Math. Expos., vol. 11, Math. Assoc. Amer., Washington, DC, 1991. MR1133201 (92k:00014)

[16] A. B. Kupavskii, *Colorings of spheres embedded in* $\mathbb{R}^n$, Mat. Sb. **202** (2011), no. 6, 83–110; English transl., Sb. Math. **202** (2011), no. 5–6, 859–886. MR2849314 (2012e:52044)

[17] A. B. Kupavskii and A. M. Raigorodskii, *On the chromatic numbers of small-dimensional Euclidean spaces*, Electron. Notes Discrete Math., vol. 34, Elsevier Sci., Amsterdam, 2009, pp. 435–439.

[18] D. G. Larman and A. C. Rogers, *The realization of distances within sets in Euclidean space*, Mathematika **19** (1972), no. 1, 1–24. MR0319055 (47:7601)

[19] O. Nechushtan, *On the space chromatic number*, Discrete Math. **256** (2002), no. 1, 499–507. MR1927575 (2003c:05084)

[20] J. Pach and P. K. Agarwal, *Combinatorial geometry*, John Wiley & Sons, New York, 1995. MR1354145 (96j:52001)

[21] E. I. Ponomarenko and A. M. Raigorodskii, *A new lower bound for the chromatic number of the rational space*, Uspekhi Mat. Nauk **68** (2013), no. 5, 183–184; English transl., Russian Math. Surveys **68** (2013), no. 5, 960–962. MR3155166

[22] ———, *New lower bound for the chromatic number of a rational space with one and two forbidden distances*, Mat. Zametki **97** (2015), no. 2, 255–261; English transl., Math. Notes **97** (2015), no. 1–2, 249–254. MR3370511

[23] A. M. Raigorodskii, *On the chromatic number of space*, Uspekhi Mat. Nauk **55** (2000), no. 2, 147–148; English transl., Russian Math. Surveis **55** (2000), no. 2, 351–352. MR1781075 (2001i:05072)

[24] ———, *The Borsuk problem and the chromatic numbers of some metric spaces*, Uspekhi Mat. Nauk **56** (2001), no. 1, 107–146; English transl., Russian Math. Surveys **56** (2001), no. 1, 103–139. MR1845644

[25] ———, *Chromatic numbers*, MTsNMO, Moscow, 2003. (Russian)

[26] ———, *Linear-algebraic method in combinatorics*, MTsNMO, Moscow, 2007. (Russian)

[27] ———, *Coloring distance graphs and graphs of diameters*, Thirty essays on geometric graph theory, Springer, New York, 2013, pp. 429–460.

[28] ———, *Cliques and cycles in distance graphs and graphs of diameters*, Discrete geometry and algebraic combinatorics, Contemp. Math, vol. 625, Amer. Math. Soc., Providence, RI, 2014, pp. 93–109.

[29] D. E. Raiskii, *The realization of all distances in a decomposition of the space* $\mathbb{R}^n$ *into* $n+1$ *parts*, Mat. Zametki **7** (1970), no. 3, 319–323; English transl., Math. Notes **7** (1970), no. 3, 194–196. MR0261446 (41:6059)

[30] S. Shelah and A. Soifer, *Axiom of choice and chromatic number of the plane*, J. Combin. Theory Ser. A **103** (2003), no. 2, 387–391. MR1996076 (2004g:03079)

[31] A. Soifer, *Relatives of chromatic number of the plane* I, Geombinatorics **1** (1992), no. 4, 13–15. MR1208441

[32] ———, *The mathematical coloring book: mathematics of coloring and the colorful life of its creators*, Springer, New York, 2009. MR2458293 (2010a:05005)

[33] ———, *The Hadwiger–Nelson problem*, Open problems in mathematics, Springer, Cham, 2016, pp. 439–457. MR3526945

[34] L. A. Székely, *Erdős on unit distances and the Szemerédi–Trotter theorems*, Paul Erdős and his mathematics, II (Budapest, 1999), Bolyai Math. Stud., vol. 11, János Bolyai Math. Soc., Budapest, 2002, pp. 649–666. MR1954746 (2004a:52028)

[35] D. R. Woodall, *Distances realized by sets covering the plane*, J. Combin. Theory Ser. A **14** (1973), no. 2, 187–200. MR0310770 (46:9868)

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