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ON OPERATOR-TYPE HOMOGENIZATION ESTIMATES FOR ELLIPTIC EQUATIONS WITH LOWER ORDER TERMS

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Abstract. In the space $\mathbb{R}^d$, a divergent-type second order elliptic equation in a nonselfadjoint form is studied. The coefficients of the equation oscillate with a period $\varepsilon\to 0$. They can be unbounded in lower order terms. In this case, they are subordinate to some integrability conditions over the unit periodicity cell. An L^2-estimate of order of $O(\varepsilon)$ is proved for the difference of solutions of the original and homogenized problems. The estimate is of operator type. It can be stated as an estimate for the difference of the corresponding resolvents in the operator $(L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d))$-norm. Also, an H^1-approximation is found for the original solution with error estimate of order of $O(\varepsilon)$. This estimate, also of operator type, implies that an appropriate approximation of order of $O(\varepsilon)$ is found for the original resolvent in the operator $(L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d))$-norm.

The results are obtained with the help of the so-called shift method, first suggested by V. V. Zhikov.

§1. Introduction

In homogenization theory, which emerged more than fifty years ago, from the very beginning there have been an increasing interest in error estimates in different norms, such as the energy norms or Lebesgue norms naturally related to the problem under consideration (see, for example, [1–3]). For second order elliptic equations, these are estimates in H^1- and L^2-norms, while for similar parabolic equations, these are also estimates in L^∞- and L^1-norms. In the early results, the majorants in the error estimates depended on data of the problem (e.g., on the right-hand side function in the case of elliptic equations or on the Cauchy data in the case of parabolic equations) in such a way that the estimates could not be given an operator meaning. In particular, such results cannot be restated as estimates in the operator norms for the difference of resolvents or semigroups (according to another terminology, operator exponentials) of the corresponding operators in elliptic or parabolic cases, respectively.

Possibly, for the first time, operator-type homogenization estimates arose in Zhikov's paper [4] (see also Chapter II in [3]). These were estimates for parabolic equations in L^∞-norms, and they were aimed at applications to probability theory and diffusion theory. These estimates were proved by the spectral method based on the Bloch representation of the fundamental solution of the parabolic equation. Actually, a pointwise estimate and an integral estimate were proved for the fundamental solution. The latter can be regarded as a kernel of the operator exponential corresponding to the nonstationary diffusion equation. As an immediate consequence of estimates for the fundamental solution, an error estimate of homogenization in the L^∞-norm was derived with a constant on the right-hand side, allowing one to reformulate the result as an estimate in the

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operator ($L^\infty \to L^\infty$)-norm for the difference of the semigroups corresponding to the original and homogenized problems. Moreover, in [5] it was shown that the estimates for the fundamental solution established in [4] naturally imply not only an L^∞-estimate of homogenization but also similar estimates in L^s-norm for any $s \ge 1$ with a universal constant on its right-hand side. More precisely, these estimates are of the form

$$\|u^\varepsilon(\,\cdot\,,t) - u^0(\,\cdot\,,t)\|_{L^s(\mathbb{R}^d)} \le c_0 \frac{\varepsilon}{\sqrt{t}} \|f\|_{L^s(\mathbb{R}^d)}, \quad 1 \le s \le \infty. \tag{1.1}$$

Here, $u^\varepsilon(x,t)$ is a solution of the original strongly nonhomogeneous Cauchy problem in the half-space $\{(x,t): \ x \in \mathbb{R}^d, t \ge 0\}$, i.e.,

$$\partial_t u^\varepsilon + A_\varepsilon u^\varepsilon = 0 \ \text{ for } t > 0, \quad u^\varepsilon(x,0) = f(x),$$

where $A_\varepsilon = -\operatorname{div}(a_\varepsilon(x)\nabla)$ is a diffusion operator with an ε-periodic diffusion matrix $a_\varepsilon(x) = a(x/\varepsilon)$, $0 < \varepsilon \le 1$, and $a(y)$ is a measurable symmetric 1-periodic matrix such that

$$\nu\mathbf{1} \le a \le 1/\nu\mathbf{1}, \quad \nu > 0,\ \mathbf{1} \ \text{is the identity matrix;}$$

$u^0(x,t)$ is a solution of the homogenized Cauchy problem

$$\partial_t u^0 + A_0 u^0 = 0 \ \text{ for } \ t > 0, \quad u^0(x,0) = f(x),$$

where $A_0 = -\operatorname{div}(a^0\nabla)$ is a diffusion operator with a constant matrix $a^0 > 0$ called the effective diffusion matrix and calculated via a well-known procedure; the constant c_0 on the right-hand side of (1.1) depends only on the dimension d and the ellipticity constant ν of the matrix $a(y)$.

The solutions of the original and the homogenized problems can be represented with the help of operator exponentials, namely,

$$u^\varepsilon(x,t) = e^{-tA_\varepsilon} f(x), \quad u^0(x,t) = e^{-tA_0} f(x).$$

Then taking, e.g., $s = 2$, from (1.1) we obtain an estimate in the operator L^2-norm for the difference of exponentials

$$\|e^{-tA_\varepsilon} - e^{-tA_0}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le c_0 \frac{\varepsilon}{\sqrt{t}}. \tag{1.2}$$

Now, representing the resolvent as the Laplace transform of the semigroup, we can deduce an estimate for the difference of resolvents

$$\|(A_\varepsilon + 1)^{-1} - (A_0 + 1)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le c\varepsilon, \quad c = \operatorname{const}(d,\nu), \tag{1.3}$$

(see [5] for the details).

The interest to operator-type estimates of homogenization arose again after the paper [6] by M. Sh. Birman and T. A. Suslina, which served as a new impetus to the spectral branch of homogenization theory. Estimates like (1.2), (1.3) were proved in L^2-norms for a broad class of selfadjoint matrix elliptic operators with the help of an operator-theoretic approach suggested by the authors of [6].

Over the past decade and a half, many interesting results on operator-type homogenization estimates were obtained by efforts of numerous mathematicians, and various approaches were elaborated for this. An overview of these results was given in [5].

From the methodological point of view, here we continue the studies going back to [7,8] and demonstrate the application of the so-called *shift method*, first suggested by Zhikov in [7], for deriving operator-type estimates in the homogenization problems under consideration in the present paper. These are second order nonselfadjoint elliptic equations with unbounded coefficients in lower order terms. All the coefficients of equations oscillate rapidly with a period $\varepsilon \to 0$. In this context, the unbounded coefficients in lower order terms satisfy certain integrability conditions over the unit cell of periodicity, with exponents determined by the space dimension. These integrability exponents are dictated by embedding theorems and they are precisely the same as those arising in existence conditions in the classical theory of elliptic equations with unbounded coefficients (see [9, Chapter III]).

To some extent, homogenization of equations with unbounded coefficients was touched upon earlier in the paper [10], where equations with degenerate weights, including those that may grow to infinity, were studied. The conditions under which the homogenization estimates were proved in [10] agree with those formulated in the present paper.

Operator-type homogenization estimates for second order selfadjoint elliptic equations (in particular, vector equations) with unbounded coefficients in lower order terms were studied in [11, 12], where the authors applied an approach based on the ideas of [6]. In the items overlapping with [11, 12], here we obtain similar results but under slightly weaker assumptions.

§2. Setting of the problem. The main result

2.1. Elliptic equation with unbounded coefficients in lower order terms. In the entire space $\mathbb{R}^d$, $d \geq 2$, we consider a divergent-type second order elliptic equation

$$\begin{gathered} u^\varepsilon \in H^1(\mathbb{R}^d), \quad A_\varepsilon u^\varepsilon + \lambda u^\varepsilon = f, \quad f \in L^2(\mathbb{R}^d), \\ A_\varepsilon u^\varepsilon = -\operatorname{div}(a_\varepsilon \nabla u^\varepsilon + \alpha_\varepsilon u^\varepsilon) + \beta_\varepsilon \cdot \nabla u^\varepsilon + \gamma_\varepsilon u^\varepsilon \end{gathered} \tag{2.1}$$

with a small parameter $\varepsilon \in (0,1)$. The coefficients of the equation are ε-periodic and, thus, oscillate rapidly as $\varepsilon \to 0$. For example, the matrix $a_\varepsilon = a_\varepsilon(x)$ of leading coefficients is obtained as follows: $a_\varepsilon(x) = a(\frac{x}{\varepsilon})$, where $a(y)$ is a measurable 1-periodic matrix with real entries (a is not necessarily symmetric). The periodicity cell is the unit cube $Y = \big[-\frac{1}{2}, \frac{1}{2}\big)^d$. Similarly, via 1-periodic functions $\alpha(y)$, $\beta(y)$, $\gamma(y)$ we obtain the coefficients in the lower order terms of equation (2.1).

Throughout the paper, the function spaces (e.g., $L^2(\mathbb{R}^d)$, $H^1(\mathbb{R}^d)$) are assumed to consist of real-valued functions. The coefficients of equation (2.1) are also real-valued.

We assume that the matrix $a(y)$ satisfies the following ellipticity and boundedness conditions:

$$\mu|\xi|^2 \leq a(y)\xi \cdot \xi, \quad a(y)\xi \cdot \eta \leq \mu^{-1}|\xi|\,|\eta|, \quad \xi, \eta \in \mathbb{R}^d, \quad \mu > 0. \tag{2.2}$$

As for the 1-periodic vector-valued functions $\alpha(y)$, $\beta(y)$ and the scalar function $\gamma(y)$, they can be unbounded but obey the following integrability condition:

$$\text{the norms } \|\alpha\|_{L^{2p}(Y)^d},\ \|\beta\|_{L^{2p}(Y)^d},\ \|\gamma\|_{L^p(Y)} \text{ are finite,} \tag{2.3}$$

where

$$p = \frac{d}{2} \text{ if } d > 2, \quad p > 1 \text{ if } d = 2. \tag{2.4}$$

By a solution of equation (2.1) we mean a function u^ε satisfying the integral identity

$$\begin{aligned} \int_{\mathbb{R}^d} (a_\varepsilon \nabla u^\varepsilon + \alpha_\varepsilon u^\varepsilon) \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^d} \beta_\varepsilon \cdot \nabla u^\varepsilon \varphi dx + \int_{\mathbb{R}^d} \gamma_\varepsilon u^\varepsilon \varphi \, dx + \lambda \int_{\mathbb{R}^d} u^\varepsilon \varphi \, dx \\ = \int_{\mathbb{R}^d} f \varphi \, dx \quad \text{for all} \quad \varphi \in H^1(\mathbb{R}^d). \end{aligned} \tag{2.5}$$

To justify the well-posedness of this setting, we need to study the bilinear form on the left-hand side of (2.5):

$$\begin{aligned} B_\varepsilon(u,v) = \int_{\mathbb{R}^d} (a_\varepsilon \nabla u + \alpha_\varepsilon u) \cdot \nabla v \, dx + \int_{\mathbb{R}^d} \beta_\varepsilon \cdot \nabla u v \, dx + \int_{\mathbb{R}^d} \gamma_\varepsilon u v \, dx + \lambda \int_{\mathbb{R}^d} u v \, dx, \\ u, v \in H^1(\mathbb{R}^d). \end{aligned} \tag{2.6}$$

We show that all integrals in (2.6) are finite. For this, the following assertion from [10] is helpful.

Lemma 2.1. *Let $\rho_\varepsilon(x) = \rho\big(\frac{x}{\varepsilon}\big)$, where $\rho \geq 0$, $\rho \in L^p_{\rm per}(Y)$, and let the exponent p be the same as in* (2.4)*. Then*

$$(2.7)\qquad (\rho_\varepsilon u, u) \leq c_0(\|u\|^2 + \varepsilon^2\|\nabla u\|^2), \quad c_0 = \mathrm{const}(d, \|\rho\|_{L^p(Y)}).$$

In the dimension $d = 2$ the constant c_0 depends also on the exponent p itself.

Here and in what follows, we use the simplified notation for the inner product and the norm in $L^2(\mathbb{R}^d)$:

$$(\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}.$$

We often make no difference in notation for spaces of scalar and vector functions.

To make our exposition full, we give the proof of Lemma 2.1 at the end of the paper (see Subsection 6.3). But now we state the following claim.

Corollary to Lemma 2.1. *For the terms in* (2.6)*, we have*:

i) $(\gamma_\varepsilon u, v) \leq c_1(\|u\| + \varepsilon\|\nabla u\|)(\|v\| + \varepsilon\|\nabla v\|), \quad c_1 = \mathrm{const}(d, \|\gamma\|_{L^p(Y)})$;

ii) $(\alpha_\varepsilon u, \nabla v) \leq c_2(\|u\| + \varepsilon\|\nabla u\|)\|\nabla v\|, \quad c_2 = \mathrm{const}(d, \|\alpha\|_{L^{2p}(Y)^d})$;

iii) $(\beta_\varepsilon \cdot \nabla u, v) \leq c_3(\|v\| + \varepsilon\|\nabla v\|)\|\nabla u\|, \quad c_3 = \mathrm{const}(d, \|\beta\|_{L^{2p}(Y)^d})$.

In the dimension $d = 2$, the constants in these estimates depend also on the exponent p itself.

Indeed, since $(\gamma_\varepsilon u, v) \leq (\rho_\varepsilon u, u)^{\frac{1}{2}}(\rho_\varepsilon v, v)^{\frac{1}{2}}$, $\rho_\varepsilon = |\gamma_\varepsilon|$, statement *i)* follows immediately from (2.7) and the numerical inequality $(a^2 + b^2)^{\frac{1}{2}} \leq a + b$ valid for any a, $b \geq 0$.

Assertions ii) and iii) are proved similarly. For example, in the case of ii) we write the inequality

$$(\alpha_\varepsilon u, \nabla v) \leq \|\alpha_\varepsilon u\|\,\|\nabla v\| = (|\alpha_\varepsilon|^2 u, u)^{\frac{1}{2}}\|\nabla v\|$$

and apply estimate (2.7) to the form $(|\alpha_\varepsilon|^2 u, u)$ setting $\rho_\varepsilon = |\alpha_\varepsilon|^2$.

Assertions i)–iii) combined with condition (2.2) ensure the boundedness property of the form (2.6) for all ε and λ and the coercivity property for relevant ε and λ. We prove here only less evident coercivity property, which means that

$$(2.8)\qquad \text{there exists } \mu_0 > 0 \text{ with } B_\varepsilon(u, u) \geq \mu_0\|u\|^2_{H^1(\mathbb{R}^d)} \quad \text{for all} \quad u \in H^1(\mathbb{R}^d)$$

for sufficiently large $\lambda \geq \lambda_0 > 0$ and sufficiently small $\varepsilon \leq \varepsilon_0$. The values λ_0, ε_0, and μ_0 depend on the dimension d, the ellipticity constant μ, the norms in (2.3), and also the exponents of these norms in the dimension $d = 2$, which is seen from what follows. Indeed, we start with the inequality

$$B_\varepsilon(u, u) \geq \mu\|\nabla u\|^2 + \lambda\|u\|^2 - |(\gamma_\varepsilon u, u)| - |(\alpha_\varepsilon u, \nabla u)| - |(\beta_\varepsilon \cdot \nabla u, u)|,$$

where, in particular, the ellipticity property (2.2) is taken into account. Then we apply estimates i)–iii) to each form an the right-hand side. Hence, using the inequality $ab \leq \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$ for any $\delta > 0$, we successively deduce the inequalities

$$\begin{aligned}
B_\varepsilon(u, u) &\geq \mu\|\nabla u\|^2 + \lambda\|u\|^2 - 2c_1(\|u\|^2 + \varepsilon^2\|\nabla u\|^2) \\
&\qquad\qquad - c_2(\|u\| + \varepsilon\|\nabla u\|)\|\nabla u\| - c_3(\|u\| + \varepsilon\|\nabla u\|)\|\nabla u\| \\
&= (\mu - 2c_1\varepsilon^2 - c_2\varepsilon - c_3\varepsilon)\|\nabla u\|^2 + (\lambda - 2c_1)\|u\|^2 - (c_2 + c_3)\|u\|\|\nabla u\| \\
&\geq \Big(\mu - 2c_1\varepsilon^2 - c_2\varepsilon - c_3\varepsilon - \frac{\delta}{2}\Big)\|\nabla u\|^2 + \Big(\lambda - 2c_1 - \frac{1}{2\delta}(c_2 + c_3)^2\Big)\|u\|^2 \\
&\geq \frac{1}{4}\mu\|\nabla u\|^2 + \|u\|^2,
\end{aligned}$$

whenever $\delta = \mu$, ε is so small that $2c_1\varepsilon^2 + (c_2 + c_3)\varepsilon < \frac{\mu}{4}$, and λ is so large that $\lambda - 2c_1 - \frac{1}{2\mu}(c_2 + c_3)^2 > 1$. It remains to set $\mu_0 = \min\{\frac{1}{4}\mu, 1\}$. The value $\lambda_0 > 0$ will be finally selected later while considering the homogenized equation.

Using the Lax–Milgram lemma combined with the boundedness and coercivity properties of the form (2.6), we infer the existence and uniqueness of a solution of (2.1). Letting $\varphi = u^\varepsilon$ in (2.5), we obtain the energy identity

$$B_\varepsilon(u^\varepsilon, u^\varepsilon) = (f, u^\varepsilon),$$

whence by (2.8) we get the energy estimate

$$\|u^\varepsilon\|_{H^1(\mathbb{R}^d)} \leq \frac{1}{\mu_0}\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.9}$$

2.2. Homogenized equation. With problem (2.1) we associate the homogenized problem

$$\begin{gathered} u \in H^1(\mathbb{R}^d), \quad A_0 u + \lambda u = f, \\ A_0 u = -\operatorname{div}(a^0\nabla u + \alpha^0 u) + \beta^0 \cdot \nabla u + \gamma^0 u, \end{gathered} \tag{2.10}$$

where a^0 is a constant positive definite matrix and the coefficients in the lower order terms are also constant. Formulas for finding a^0, α^0, β^0, γ^0 are given below (see (2.16) and (2.17)).

A solution of (2.10) is understood in the sense of the integral identity

$$B_0(u, \varphi) = (f, \varphi), \quad \varphi \in H^1(\mathbb{R}^d),$$

where we have a bilinear form similar to (2.6) but with constant coefficients, namely

$$B_0(u, v) = \int_{\mathbb{R}^d} (a^0\nabla u + \alpha^0 u) \cdot \nabla v\, dx + \int_{\mathbb{R}^d} \beta^0 \cdot \nabla u v\, dx + \int_{\mathbb{R}^d} \gamma^0 u v\, dx + \lambda \int_{\mathbb{R}^d} u v\, dx.$$

There exists a unique solution of (2.10) for sufficiently large λ, thanks to the Lax–Milgram lemma. Since the coefficients in the homogenized equation are constant, along with the energy estimate (2.9) for the solution u, we have the elliptic estimate

$$\|u\|_{H^2(\mathbb{R}^d)} \leq c\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.11}$$

In the sequel, problems (2.1) and (2.10) are considered for suitable λ and ε, $\lambda \geq \lambda_0$ and $\varepsilon \leq \varepsilon_0$, where the choice of λ_0 and ε_0 guarantees than the operators A_ε, A are coercive; consequently, the two problems are well posed. This specification of the parameters will be often omitted in what follows.

Our aim is to prove the following claim.

Theorem 2.2. *For the difference of solutions of problems* (2.1) *and* (2.10), *we have the estimate*

$$\|u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \tag{2.12}$$

where $\lambda \geq \lambda_0$ and $\varepsilon \leq \varepsilon_0$. The constant C depends on the dimension d, the ellipticity constant μ, the parameter λ, and the norms $\|\alpha\|_{L^{2p}(Y)^d}$, $\|\beta\|_{L^{2p}(Y)^d}$, $\|\gamma\|_{L^p(Y)}$. In dimension $d = 2$, the constant C depends on the exponent p itself.

Inequality (2.12) implies an estimate in the operator $(L^2 \to L^2)$-norm for the difference between the resolvents of the original and homogenized operators:

$$\|(A_\varepsilon + \lambda)^{-1} - (A_0 + \lambda)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C\varepsilon. \tag{2.13}$$

The resolvent $(A_\varepsilon + \lambda)^{-1}$ can be regarded as an operator acting from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$ and, in the corresponding operator $(L^2{\to}H^1)$-norm, its approximation is the sum $(A_0 + \lambda)^{-1} + \varepsilon K_\varepsilon$, where the correcting operator $\varepsilon K_\varepsilon$ is defined in (5.12). This assertion is formulated more presisely in Theorem 5.1.

2.3. Cell problems and homogenized equation coefficients. We consider the periodic problems

$$N_j \in H^1_{\mathrm{per}}(Y), \quad \operatorname{div}_y[a(y)(e^j + \nabla_y N_j(y))] = 0, \quad \langle N_j \rangle = 0, \quad j = 1, \dots, d, \tag{2.14}$$

where $e^1, \dots, e^d$ is the canonical basis in $\mathbb{R}^d$; and similarly,

$$N_0 \in H^1_{\mathrm{per}}(Y), \quad \operatorname{div}_y[a(y)\nabla N_0(y) + \alpha(y)] = 0, \quad \langle N_0 \rangle = 0. \tag{2.15}$$

where

$$\langle \cdot \rangle = \int_Y \cdot \, dy$$

denotes the mean value over the cell.

Solutions of cell problems are understood in the sense of integral identities on smooth periodic functions. As an example, for problem (2.14) we have the identity

$$\int_Y [a(y)(e^j + \nabla_y N_j(y))] \cdot \nabla\varphi(y)\, dy = 0 \quad \text{for all} \quad \varphi \in C^\infty_{\mathrm{per}}(Y),$$

which means that $a(y)(e^j + \nabla_y N_j(y))$ is a solenoidal vector of class $L^2_{\mathrm{per}}(Y)^d$. This will be used later.

Each of problems (2.14) and (2.15) has a unique solution by the Lax–Milgrame lemma. In this regard, for equation (2.15) it should be noted that $\alpha \in L^2_{\mathrm{per}}(Y)^d$ due to (2.3) and (2.4) and, thus, this equation can be written in the form $\operatorname{div}_y[(a(y)\nabla N_0(y)] = -\operatorname{div}_y \alpha(y)$, where the function on the right-hand side belongs to the dual space $(H^1_{\mathrm{per}}(Y))^*$.

Using the solutions of the cell problems (2.14) and (2.15), we introduce the coefficients of the homogenized equation (2.10). The homogenized matrix a^0 is defined by the formulas

$$a^0 e^j = \langle a(e^j + \nabla_y N_j \rangle, \quad j = 1, \dots, d; \tag{2.16}$$

the other coefficients are the following mean values:

$$\alpha^0 = \langle a\nabla_y N_0 + \alpha \rangle, \quad \beta^0 = \langle (\mathbf{1} + \nabla_y N)^T \beta \rangle, \quad \gamma^0 = \langle \beta \cdot \nabla_y N_0 + \gamma \rangle, \tag{2.17}$$

where the vector $N = \{N_1, N_2, \dots, N_d\}$ consists of solutions to problem (2.14), $\mathbf{1}$ is the unit matrix, the matrix ∇N has the entries $\{\nabla N\}_{ij} = \frac{\partial N_j}{\partial y_i}$, and by M^T we denote the transpose of M.

It is well known (see [1–3]) that the homogenized matrix a^0 is positive definite.

2.4. Some remarks. Operator-type estimates (2.13) and (5.12) will follow from the analysis performed in §§3–5. Possibly, our main result is the so-called estimate “integrated over the shift parameter” in Theorem 4.2. Namely, this “integrated” estimate implies desired L^2- and H^1-estimates of homogenization as simple corollaries. The specifics of the scalar case are used nowhere in the proof of Theorem 4.2. Here we mean the maximum principle and its consequences. Thereby, the results implied by Theorem 4.2 are also valid for vector equations. The scalar case specifics are taken into account only in §6, where, under slightly stronger requirements on the coefficients in lower order terms, we prove the H^1-estimate with the approximation that is usual for classical homogenization theory.

Remark 1. Since the coefficients of the operator A_0 are constant, it is possible to simplify the homogenized equation (2.10) rewriting it with a symmetric matrix in the principal part and with only one drift vector, namely

$$A_0 u = -\sum_{i,j} (a^0)^s_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + (\alpha^0 + \beta^0) \cdot \nabla u + (\gamma^0 + \lambda) u,$$

where $(a^0)^s$ is the symmetric part of a^0. But here we do not do this deliberately in order to preserve the original equation structure and to employ this similarity in the proof.

Remark 2. In [13] and [5], homogenization operator-type estimates were proved for divergent nonselfadjoint elliptic equations of arbitrary even order and with lower order terms, but with coefficients that are all bounded.

Remark 3. In [11] and [12], homogenization operator-type estimates were studied for selfadjoint elliptic vector equations with unbounded coefficients under assumptions stronger than in (2.3), (2.4).

Remark 4. One may consider a diffusion operator with lower order terms coefficients that are unbounded in another sense. This is the operator with a drift growing to infinity as $\varepsilon \to 0$,

$$A_\varepsilon = -\mathrm{div}(a_\varepsilon \nabla) + \varepsilon^{-1}\beta_\varepsilon \cdot \nabla, \tag{2.18}$$

where the ε-periodic coefficients of A_ε are bounded for each fixed ε. This requires quite a different homogenization procedure where the drift affects the effective diffusion matrix, and the homogenized equation (though with constant coefficients) still preserves a drift vector unboundedly growing as $\varepsilon \to 0$.

In the situation of equation (2.1), we have drift vectors that are pointwise unbounded but have some bounded integral mean value. Then formula (2.16) shows that the drift in (2.1) does not affect the effective diffusion matrix a^0. On the other hand, the effective drift vectors α^0 and β^0 (see (2.17)) depend on the original diffusion matrix $a(y)$ directly (in case of α^0), or indirectly via the solutions N_j of (2.14) (in the case of β^0).

For a diffusion equation with the operator (2.18), homogenization operator-type estimates were obtained in [14] by using the spectral method based on the Bloch representation of nonselfadjoint operators.

Remark 5. What concerns second order elliptic equations with unbounded coefficients in the principal part, homogenization operator-type estimates were established in the following case: the original diffusion matrix splits into the sum $a(y) = a^s(y) + b(y)$ of symmetric and skew-symmetric parts so that the symmetric matrix $a^s(y)$ satisfies a condition of the type (2.2) and the skew-symmetric matrix $b(y)$ has entries in BMO (the space of functions with bounded mean oscillation). The details can be found in [5].

Added in proof. When this paper had already been submitted and was in peer review, the paper [15] appeared, to which our attention was drawn by the reviewer. This paper concerns homogenization of a nonselfadjoint elliptic operator in a domain that is an infinite cylinder whose section is an n-dimensional torus. The coefficients of the operator are periodic and oscillate rapidly as $\varepsilon \to 0$ in the variables of the cylinder's "ruling". Homogenization is only taken over these variables. In the case where $n = 0$ (i.e., the "0-dimensional torus" is the section of the cylinder), the problem of "partial homogenization", treated in [15], includes our problem of homogenization in the entire space $\mathbb{R}^d$. Thus, the objects under consideration in our paper and in [15] partly intersect. So, it is natural to expect partial overlapping of conditions on coefficients under which homogenization results are proved. In [15], the coefficients in lower order terms are assumed to be multipliers between Sobolev spaces. In particular, they may be elements of Lebesgue spaces with appropriate exponents. In this case, we manage to cope with the less restrictive conditions (2.3) and (2.4) than those in [15].

§3. Discrepancy of the first approximation in the equation

We try to construct an approximation to the solution of problem (2.1) in the H^1-norm, shortly called a first approximation. Following the classical homogenization theory (see [1–3]), in we seek it the form

$$v^\varepsilon(x) = u(x) + \varepsilon N_j(y)\frac{\partial u(x)}{\partial x_j} + \varepsilon N_0(y)u(x), \quad y = \frac{x}{\varepsilon}, \tag{3.1}$$

where the N_j $(j = 0, 1, \dots, d)$ are solutions of cell problems and u is a solution of the homogenized problem. (Here and in what follows, summation over repeated indices from 1 to d is assumed if it is not stipulated otherwise.) The additional terms of order of $O(\varepsilon)$ to the zero approximation u in formulas of the type (3.1) are usually called a corrector.

Under our assumptions about regularity of the data in the original problem (we mean the right-hand side function f, the coefficients introduced via the matrix a, and the functions α, β, and γ), the function v^ε may fail to belong to $H^1(\mathbb{R}^d)$. To make our actions consistent, we assume initially that $f \in C_0^\infty(\mathbb{R}^d)$. Then the solution u of the homogenized problem is smooth and decays at infinity sufficiently rapidly, so that v^ε belongs to $H^1(\mathbb{R}^d)$.

For the first approximation v^ε, we calculate its gradient and flow:

$$\begin{aligned}
\nabla v^\varepsilon(x) &= \nabla u(x) + \nabla_y N_j(y)\frac{\partial u(x)}{\partial x_j} + \nabla_y N_0(y)u(x) \\
&\qquad + \varepsilon N_j(y)\nabla\left(\frac{\partial u(x)}{\partial x_j}\right) + \varepsilon N_0(y)\nabla u(x) \\
&= (e^j + \nabla_y N_j(y))\frac{\partial u(x)}{\partial x_j} + \nabla_y N_0(y)u(x) + \varepsilon\nabla^2 u(x)N(y) + \varepsilon N_0(y)\nabla u(x), \\
a(y)&\nabla v^\varepsilon(x) + \alpha(y)v^\varepsilon(x) \\
&= a(y)(e^j + \nabla_y N_j(y))\frac{\partial u(x)}{\partial x_j} + [a(y)\nabla_y N_0(y) + \alpha(y)]u(x)\varepsilon a(y)\nabla^2 u(x)N(y) \\
&\qquad + \varepsilon a(y)N_0(y)\nabla u(x) + \varepsilon\alpha(y)(N(y)\cdot\nabla u(x)) + \varepsilon\alpha(y)N_0(y)u(x), \ \ y = \frac{x}{\varepsilon}.
\end{aligned} \tag{3.2}$$

Compare this flow with the flow $a^0\nabla u + \alpha^0 u$ of the homogenized equation. For their difference we have

$$\begin{aligned}
R_\varepsilon &\equiv a(y)\nabla v^\varepsilon(x) + \alpha(y)v^\varepsilon(x) - a^0\nabla u(x) - \alpha^0 u(x) \\
&= \left[a(y)(e^j + \nabla_y N_j(y)) - a^0 e^j\right]\frac{\partial u(x)}{\partial x_j} + [a(y)\nabla_y N_0(y) + \alpha(y) - \alpha^0]u(x) \\
&\qquad + \varepsilon a(y)\nabla^2 u(x)N(y) + \varepsilon a(y)N_0(y)\nabla u(x) \\
&\qquad + \varepsilon\alpha(y)(N(y)\cdot\nabla u(x)) + \varepsilon\alpha(y)N_0(y)u(x) \\
&= g^j(y)\frac{\partial u(x)}{\partial x_j} + g^0(y)u(x) + \varepsilon a(y)\nabla^2 u(x)N(y) + \varepsilon a(y)N_0(y)\nabla u(x) \\
&\qquad + \varepsilon\alpha(y)(N(y)\cdot\nabla u(x)) + \varepsilon\alpha(y)N_0(y)u(x), \ \ y = \frac{x}{\varepsilon},
\end{aligned} \tag{3.3}$$

where the vectors in square brackets are

$$g^j(y) = a(y)(e^j + \nabla_y N_j(y)) - a^0 e^j, \quad g^0(y) = a(y)\nabla_y N_0(y) + \alpha(y) - \alpha^0. \tag{3.4}$$

These are periodic solenoidal vectors from $L^2_{\mathrm{per}}(Y)^d$ due to equations (2.14) and (2.15), and moreover, they have zero mean value (see the definitions of a^0 and α^0 in (2.16), (2.17)). It is known that such vectors can be represented in terms of a matrix potential (see [3, Chapter I, §1]).

Lemma 3.1. *Let $t \in L^2_{\mathrm{per}}(Y)^d$ be a solenoidal vector with zero mean value*: $\mathrm{div}_y t(y) = 0$, $\langle t \rangle = 0$. *Then there exists a skew-symmetric matrix $T \in H^1_{\mathrm{per}}(Y)^{d\times d}$, $T = \{T_{ik}\}$, $T_{ik} = -T_{ki}$, such that*

$$
\begin{gathered}
t(y) = \mathrm{div}_y T(y), \\
\|T\|_{H^1_{\mathrm{per}}(Y)^{d\times d}} \le c\|t\|_{L^2_{\mathrm{per}}(Y)^d}.
\end{gathered} \tag{3.5}
$$

Hence, writing the vectors (3.4) in terms of the matrix potential gives

$$
g^j(y) = \mathrm{div}_y G^j(y), \quad G^j \in H^1_{\mathrm{per}}(Y)^{d\times d}, \quad j = 0, 1, \dots, d, \tag{3.6}
$$

and, as a consequence (without summation over the repeated index j),

$$
\begin{gathered}
g^j(y)\frac{\partial u(x)}{\partial x_j} = \varepsilon \mathrm{div}\left(G^j(y)\frac{\partial u(x)}{\partial x_j}\right) - \varepsilon G^j(y)\nabla\frac{\partial u(x)}{\partial x_j}, \quad j = 1, \dots, d, \\
g^0(y)u(x) = \varepsilon\mathrm{div}(G^0(y)u(x)) - \varepsilon G^0(y)\nabla u(x), \quad y = \frac{x}{\varepsilon}.
\end{gathered} \tag{3.7}
$$

We claim that the first terms on the right-hand sides are solenoidal vectors. Indeed,

$$
\int_{\mathbb{R}^d} \mathrm{div}\left(G^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u(x)}{\partial x_j}\right)\cdot\nabla\varphi(x)\,dx = -\int_{\mathbb{R}^d}\frac{\partial u(x)}{\partial x_j}G^j\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2\varphi(x)\,dx = 0
$$

and

$$
\int_{\mathbb{R}^d} \mathrm{div}\left(G^0\left(\frac{x}{\varepsilon}\right)u(x)\right)\cdot\nabla\varphi(x)\,dx = -\int_{\mathbb{R}^d} u(x)G^0\left(\frac{x}{\varepsilon}\right)\cdot\nabla^2\varphi(x)\,dx = 0
$$

if $\varphi \in C_0^\infty(\mathbb{R}^d)$, because G^j and G^0 are skew-symmetric and $\nabla^2\varphi$ is symmetric.

Representations (3.7) allow us to obtain an expression with a factor ε for $\mathrm{div}R_\varepsilon$, namely

$$
\begin{aligned}
\mathrm{div}R_\varepsilon \overset{(3.3)}{=} \varepsilon\mathrm{div}\Big[&a(y)\nabla^2u(x)N(y) + a(y)N_0(y)\nabla u(x) + \alpha(y)(N(y)\cdot\nabla u(x)) \\
&+ \alpha(y)N_0(y)u(x) - G^0(y)\nabla u(x) - G^j(y)\nabla\frac{\partial u(x)}{\partial x_j}\Big], \quad y = \frac{x}{\varepsilon}.
\end{aligned} \tag{3.8}
$$

Now, for the first approximation v^ε we take the so-called scalar flow

$$
\begin{aligned}
&\beta(y)\cdot\nabla v^\varepsilon(x) + \gamma(y)v^\varepsilon(x) \\
&\quad = \beta(y)\cdot(e^j + \nabla_y N_j(y))\frac{\partial u(x)}{\partial x_j} + \big[\beta(y)\cdot\nabla_y N_0(y) + \gamma(y)\big]u(x) + \varepsilon\beta(y)\cdot(\nabla^2u(x)N(y)) \\
&\qquad + \varepsilon\beta(y)\cdot\nabla u(x)N_0(y) + \varepsilon\gamma(y)N(y)\cdot\nabla u(x) + \varepsilon\gamma(y)N_0(y)u(x), \quad y = \frac{x}{\varepsilon},
\end{aligned}
$$

and compare it with the similar scalar flow $\beta^0\cdot\nabla u(x) + \gamma^0 u(x)$ for the homogenized equation. The difference of these flows can be written as

$$
\begin{aligned}
r_\varepsilon &\equiv \beta(y)\cdot\nabla v^\varepsilon(x) + \gamma(y)v^\varepsilon(x) - \beta^0\cdot\nabla u(x) - \gamma^0 u(x) \\
&= \big[\beta(y)\cdot(e^j + \nabla_y N_j(y)) - \beta_j^0\big]\frac{\partial u(x)}{\partial x_j} + [\beta(y)\cdot\nabla_y N_0(y) + \gamma(y) - \gamma^0]u(x) \\
&\quad + \varepsilon\beta(y)\cdot(\nabla^2u(x)N(y)) + \varepsilon\beta(y)\cdot\nabla u(x)N_0(y) + \varepsilon\gamma(y)N(y)\cdot\nabla u(x) \\
&\quad + \varepsilon\gamma(y)N_0(y)u(x), \quad y = \frac{x}{\varepsilon}.
\end{aligned} \tag{3.9}
$$

Consider the functions

$$
\begin{gathered}
s_j(y) = \beta(y)\cdot(e^j + \nabla_y N_j(y)) - \beta_j^0, \quad j = 1, \dots, d, \\
s_0(y) = \beta(y)\cdot\nabla_y N_0(y) + \gamma(y) - \gamma^0
\end{gathered} \tag{3.10}
$$

standing in the square brackets in (3.9). They have zero mean value by the definition of the constants β_j^0 and γ^0 (see (2.17)). Moreover, for all j we have

$$s_j \in L_{\rm per}^{\frac{2p}{p+1}}(Y). \tag{3.11}$$

Indeed, since $\beta \in L^{2p}_{\rm per}(Y)^d$ and $\nabla_y N_j \in L^2_{\rm per}(Y)^d$, it follows that

$$\int_Y |\nabla_y N_j(y)\cdot \beta(y)|^{\frac{2p}{p+1}}\,dy \le \left(\int_Y |\nabla_y N_j(y)|^2 dy\right)^{\frac{p}{p+1}} \left(\int_Y |\beta(y)|^{2p}\,dy\right)^{\frac{1}{p+1}} < \infty$$

by the Hölder inequality, whence $\nabla_y N_j \cdot \beta \in L_{\rm per}^{\frac{2p}{p+1}}(Y)$. Similarly, $\beta \in L_{\rm per}^{\frac{2p}{p+1}}(Y)^d$. Hence, by the definition $(3.10)_1$ relation (3.11) is true for all s_j, $j \ge 1$. As for s_0, we can use similar arguments, observing that $\gamma \in L^p_{\rm per}(Y)$ and $L^p_{\rm per}(Y) \subset L_{\rm per}^{\frac{2p}{p+1}}(Y)$. The last inclusion follows from the fact that $p > \frac{2p}{p+1}$ valid for $p > 1$, which is ensured by (2.4).

The proof of the following assertion is given at the end of §4.

Lemma 3.2. *There exist vectors $S^j \in L^2_{\rm per}(Y)^d$, $j = 0, 1, \dots, d$, such that $s_j(y) = {\rm div}_y S^j(y)$, where*

$$\|S^j\|_{L^2_{\rm per}(Y)^d} \le c\|s_j\|_{L_{\rm per}^{\frac{2p}{p+1}}(Y)}, \quad c = {\rm const}(d). \tag{3.12}$$

Using the formulas from Lemma 3.2, we have

$$\begin{aligned} s_j(y)\frac{\partial u(x)}{\partial x_j} &= \varepsilon{\rm div}\left(S^j(y)\frac{\partial u(x)}{\partial x_j}\right) - \varepsilon S^j(y)\cdot\nabla\left(\frac{\partial u(x)}{\partial x_j}\right), \\ s_0(y)u(x) &= \varepsilon{\rm div}\left(S^0(y)u(x)\right) - \varepsilon S^0(y)\cdot\nabla u(x), \quad y = \frac{x}{\varepsilon}. \end{aligned}$$

Then the difference r_ε (see (3.9), (3.10)) can be written in the form

$$\begin{aligned} r_\varepsilon = \varepsilon{\rm div}&\left(S^j(y)\frac{\partial u(x)}{\partial x_j} + S^0(y)u(x)\right) \\ &+ \varepsilon\beta(y)\cdot(\nabla^2 u(x)N(y)) + \varepsilon\beta(y)\cdot\nabla u(x)N_0(y) + \varepsilon\gamma(y)N(y)\cdot\nabla u(x) \\ &+ \varepsilon\gamma(y)N_0(y)u(x) - \varepsilon S^j(y)\cdot\nabla\left(\frac{\partial u(x)}{\partial x_j}\right) - \varepsilon S^0(y)\cdot\nabla u(x), \quad y = \frac{x}{\varepsilon}. \end{aligned} \tag{3.13}$$

The equations for u^ε and u, and also the representations for differences of flows yield the identity

$$\begin{aligned} A_\varepsilon(v^\varepsilon - u^\varepsilon) + \lambda(v^\varepsilon - u^\varepsilon) &= A_\varepsilon v^\varepsilon - A_\varepsilon u^\varepsilon + \lambda v^\varepsilon - \lambda u^\varepsilon = A_\varepsilon v^\varepsilon + \lambda v^\varepsilon - f \\ &= A_\varepsilon v^\varepsilon + \lambda v^\varepsilon - A_0 u - \lambda u = -{\rm div}R_\varepsilon + r_\varepsilon + \lambda(v^\varepsilon - u) \\ &= -{\rm div}R_\varepsilon + r_\varepsilon + \varepsilon\lambda N(y)\cdot\nabla u(x) + \varepsilon\lambda N_0(y)u(x), \end{aligned} \tag{3.14}$$

where ${\rm div}R_\varepsilon$ and r_ε are given in (3.8) and (3.13). In other words, the function $w^\varepsilon = v^\varepsilon - u^\varepsilon$ solves the equation

$$A_\varepsilon w^\varepsilon + \lambda w^\varepsilon = f_\varepsilon + {\rm div}F_\varepsilon, \tag{3.15}$$

where the functions f_ε, F_ε are expressed in terms of the solutions of auxiliary cell problems and the homogenized problem in accordance with (3.8), (3.13), and (3.14). Namely,

$$F_\varepsilon(x) = -R_\varepsilon(x) + \varepsilon S^j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u(x)}{\partial x_j} + \varepsilon S^0\Big(\frac{x}{\varepsilon}\Big)u(x), \tag{3.16}$$

where

$$R_\varepsilon(x) \overset{(3.8)}{=} \varepsilon\Big[a(y)\nabla^2 u(x)N(y) + a(y)N_0(y)\nabla u(x) + \alpha(y)(N(y)\cdot\nabla u(x)) + \alpha(y)N_0(y)u(x) - G^0(y)\nabla u(x) - G^j(y)\nabla\frac{\partial u(x)}{\partial x_j}\Big], \quad y = \frac{x}{\varepsilon},$$

and

$$f_\varepsilon(x) \overset{(3.13),(3.14)}{=} \varepsilon\Big[\beta(y)\cdot(\nabla^2 u(x)N(y)) + \beta(y)\cdot\nabla u(x)N_0(y) + \gamma(y)N(y)\cdot\nabla u(x) + \gamma(y)N_0(y)u(x) - S^j(y)\cdot\nabla\frac{\partial u(x)}{\partial x_j} - S^0(y)\cdot\nabla u(x) + \lambda N(y)\cdot\nabla u(x) + \lambda N_0(y)u(x)\Big]\Big|_{y=\frac{x}{\varepsilon}}. \tag{3.17}$$

Equation (3.15) is of a more general form than the original equation (2.1). Nevertheless, a counterpart of the energy estimate (2.9) is valid for equation (3.15), namely,

$$\|w^\varepsilon\|_{H^1(\mathbb{R}^d)} \le c_0\big(\|f_\varepsilon\|_{L^2(\mathbb{R}^d)} + \|F_\varepsilon\|_{L^2(\mathbb{R}^d)}\big). \tag{3.18}$$

Taking the structure of the functions f_ε and F_ε into account, from (18) we deduce the estimate

$$\|v^\varepsilon - u^\varepsilon\|^2_{H^1(\mathbb{R}^d)} \le c_0\varepsilon^2\sum_i\int_{\mathbb{R}^d}\left|b_i\left(\frac{x}{\varepsilon}\right)\right|^2|\Phi_i(x)|^2\,dx. \tag{3.19}$$

Here $\Phi_i(x)$ stands for the function $u(x)$ or its gradients $\nabla u(x)$, $\nabla^2 u(x)$, so that

$$\|\Phi_i\|_{L^2(\mathbb{R}^d)} \le c\|f\|_{L^2(\mathbb{R}^d)} \tag{3.20}$$

by the elliptic estimate (2.11); the factors $b_i(y)$ are built out of

$$G^j(y),\ S^j(y),\ N_j(y), \tag{3.21}$$

$$N_j(y)\alpha(y),\ N_j(y)\beta(y),\ \gamma(y)N_j(y). \tag{3.22}$$

In the general case, the factors $b_i(y)$ are not in $L^\infty(Y)$. Therefore, we cannot exclude them from the right-hand side integrals in (3.19) and then, due to (3.20), arrive at the estimate

$$\|u^\varepsilon - v^\varepsilon\|_{H^1(\mathbb{R}^d)} \le C\varepsilon\|f\|_{L^2(\mathbb{R}^d)} \tag{3.23}$$

with a constant depending only on the quantities listed in Theorem 2.2. In the sequel, we explain how to overcome this difficulty by changing slightly the first approximation.

§4. The integrated estimate

4.1. Shift in coefficients. Shifted first approximations. Consider a family of problems with shift in coefficients, namely

$$\begin{gathered} -\operatorname{div}[a(y+\omega)\nabla u^\varepsilon(x,\omega) + \alpha(y+\omega)u^\varepsilon(x,\omega)] \\ + \beta(y+\omega)\cdot\nabla u^\varepsilon(x,\omega) + \gamma(y+\omega)u^\varepsilon(x,\omega) + \lambda u^\varepsilon(x,\omega) = f(x), \\ y = \frac{x}{\varepsilon}, \quad \omega\in Y. \end{gathered} \tag{4.1}$$

Here, we have the same right-hand side function $f(x)$ as in (2.1). Evidently, equation (2.1) is presented in this family when $\omega = 0$. The solutions of the auxiliary cell problems corresponding to (4.1) are $N_j(y+\omega)$ and $N_0(y+\omega)$, i.e., they are obtained by appropriate shifting from solutions of (2.14), (2.15). Consequently, the homogenized matrix and the homogenized equation do not depend on the parameter ω, and in accordance with (3.1) the corresponding first approximation to the solution of (4.1) will be of the form

$$v^\varepsilon(x,\omega) = u(x) + \varepsilon N_j(y+\omega)\frac{\partial u(x)}{\partial x_j} + \varepsilon N_0(y+\omega)u(x), \quad y = \frac{x}{\varepsilon}. \tag{4.2}$$

We have seen above that the difference $w^\varepsilon(x,\omega) = v^\varepsilon(x,\omega) - u^\varepsilon(x,\omega)$ satisfies the estimate of the type (3.19), namely,

$$\|w^\varepsilon(\,\cdot\,,\omega)\|^2_{H^1(\mathbb{R}^d)} \le c_0\varepsilon^2 \sum_i \int_{\mathbb{R}^d} \left| b_i\left(\frac{x}{\varepsilon}+\omega\right)\right|^2 |\Phi_i(x)|^2\,dx.$$

After integrating this inequality with respect to $\omega \in Y$, we eliminate the functions $|b_i(y+\omega)|^2$ from the right-hand side integrals, replacing them by the mean values over the cell Y:

$$\begin{aligned}\int_Y \|w^\varepsilon(\,\cdot\,,\omega)\|^2_{H^1(\mathbb{R}^d)}\,d\omega &\le c_0\varepsilon^2 \sum_i \int_{\mathbb{R}^d} |\Phi_i(x)|^2 \int_Y \left| b_i\left(\frac{x}{\varepsilon}+\omega\right)\right|^2\,d\omega\,dx \\ &= c_0\varepsilon^2 \sum_i \langle |b_i|^2\rangle \int_{\mathbb{R}^d} |\Phi_i(x)|^2\,dx \\ &\le c_0\varepsilon^2 \|u\|^2_{H^2(\mathbb{R}^d)} \sum_i \langle |b_i|^2\rangle \overset{(2.11)}{\le} c\varepsilon^2 \|f\|^2_{L^2(\mathbb{R}^d)} \sum_i \langle |b_i|^2\rangle.\end{aligned} \tag{4.3}$$

This is possible provided that

$$b_i \in L^2_{\mathrm{per}}(Y).$$

for all i. Let us find out whether this integrability property is valid for all functions listed in (3.21), (3.22). There may only be doubts about functions on the list (3.22), because they are products of $N_j \in H^1_{\mathrm{per}}(Y)$ by one of the unbounded multipliers $\alpha(y)$, $\beta(y)$, and $\gamma(y)$. Note that the functions listed in (3.21) belong to $L^2_{\mathrm{per}}(Y)$ by their choice (see (2.14), (2.15), (3.6) and Lemma 3.2).

First, we verify that $N_j\alpha, N_j\beta \in L^2_{\mathrm{per}}(Y)^d$. Indeed, by Hölder's inequality we have

$$\begin{aligned}\int_Y |N_j(y)\alpha(y)|^2\,dy &\le \left(\int_Y |\alpha^2(y)|^p\,dy\right)^{\frac{1}{p}} \left(\int_Y |N_j(y)|^{\frac{2p}{p-1}}\,dy\right)^{\frac{p-1}{p}} \\ &\le c\left(\int_Y |N_j(y)|^{\frac{2p}{p-1}}\,dy\right)^{\frac{p-1}{p}} \le C\int_Y |\nabla N_j(y)|^2\,dy < \infty,\end{aligned} \tag{4.4}$$

where we have also used condition (2.3) for α and the Sobolev inequality (see, e.g., [9, Chapter II, §2]) for $N_j \in H^1(Y)$. Inequalities (4.4) hold true in any dimension $d \ge 2$. In the case where $d > 2$, we have $p = \frac{d}{2}$ (see (2.4)), so that $\frac{2p}{p-1} = \frac{2d}{d-2}$ coincides with the Sobolev exponent. For $d = 2$, we have the exponent $p > 1$, and the Sobolev embedding theorem yields $\|N_j\|_{L^q(Y)} \le c\|\nabla N_j\|_{L^2(Y)}$ for any $q > 1$, in particular, for $q = \frac{2p}{p-1}$.

The second function on the list (3.22) also belongs to $L^2_{\mathrm{per}}(Y)$ and this is ensured by the property $\beta \in L^{2p}_{\mathrm{per}}(Y)^d$. What concerns the last function in (3.22), the above arguments do not work. The reason is that, in accordance with (2.3), the periodic multiplier γ possesses a weaker integrability property compared to α and β. In the general case, the embedding theorem shows that $N_j\gamma \in L^{\frac{2d}{d+2}}_{\mathrm{per}}(Y)$ if $d > 2$ (because $N_j \in L^{\frac{2d}{d-2}}_{\mathrm{per}}(Y)$, $\gamma \in L^{\frac{d}{2}}_{\mathrm{per}}(Y)$, and $\frac{d-2}{2d} + \frac{2}{d} = \frac{d+2}{2d}$) and the exponent $\frac{2d}{d+2}$ does not attain the value 2. The same problem arises in dimension $d = 2$.

The problem of inadequate integrability of the functions $\sigma(y) = N_j(y)\gamma(y)$ on the cell Y is overcome in the following way. In (3.17), these periodic functions are multiplied by $z(x)$, where $z(x)$ is either $u(x)$ or $\nabla u(x)$, so that $z \in H^1(\mathbb{R}^d)$ in any case. We can transform this product by using the possibility of additional differentiation of $z(x)$. New periodic multipliers that emerge via this transformation will be integrable with appropriate exponents. This transformation procedure is described in the following lemma, which will be proved later.

Lemma 4.1. *Suppose $z \in H^1(\mathbb{R}^d)$ and $\sigma \in L^q_{\mathrm{per}}(Y)$, $q = \frac{2d}{d+2}$. Then*

$$\begin{aligned} &\sigma(y) = \langle\sigma\rangle + \operatorname{div}_y \rho(y), \quad \rho \in L^2_{\mathrm{per}}(Y)^d, \\ &\sigma\Big(\frac{x}{\varepsilon}\Big) z(x) = \langle\sigma\rangle z(x) + \varepsilon\Big[\operatorname{div}\Big(\rho\Big(\frac{x}{\varepsilon}\Big) z(x)\Big) - \rho\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla z(x)\Big]. \end{aligned} \tag{4.5}$$

Moreover,

$$\|\rho\|_{L^2_{\mathrm{per}}(Y)^d} \le c\|\sigma\|_{L^q_{\mathrm{per}}(Y)}, \quad c = \mathrm{const}(d). \tag{4.6}$$

We apply (4.5) to the terms in (3.17) that contain $N_j(y)\gamma(y)$, namely,

$$\begin{aligned} \varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\gamma\Big(\frac{x}{\varepsilon}\Big) z(x) = \varepsilon\langle N_j\gamma\rangle z(x) - \varepsilon^2\rho_j\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla z(x) + \varepsilon^2 \operatorname{div}\Big(\rho_j\Big(\frac{x}{\varepsilon}\Big) z(x)\Big), \\ \rho_j \in L^2_{\mathrm{per}}(Y)^d. \end{aligned} \tag{4.7}$$

Here, the expressions with the operator div occurring inf_ε will go to the second component $\operatorname{div} F_\varepsilon$ of the right-hand side in (3.15).

Thus, we obtain (3.15) with the transformed right-hand side $f_\varepsilon + \operatorname{div} F_\varepsilon$, the structure of which dictates estimate (3.18) in the form of (3.19) with a new set of functions $b_i(y)$. Now, the function γN_j does not occur on the list (3.22). Instead of this, the list (3.21) will be supplemented with $\langle N_j\gamma\rangle$ and ρ_j (see (4.7)).

As a result, we obtain estimate (4.3) in which $b_i \in L^2_{\mathrm{per}}(Y)$ for all i. In short this estimate can be written as

$$\int_Y \|v^\varepsilon(\,\cdot\,,\omega) - u^\varepsilon(\,\cdot\,,\omega)\|^2_{H^1(\mathbb{R}^d)}\, d\omega \le C\varepsilon^2\|f\|^2_{L^2(\mathbb{R}^d)}, \tag{4.8}$$

where the constant C depends on the quantities listed in Theorem 2.2.

4.2. Estimate averaged over the shift parameter. Now we investigate the possibility of replacing the function $u^\varepsilon(x,\omega)$ by $u^\varepsilon(x+\varepsilon\omega)$ in (4.8). Note that $u^\varepsilon(x+\varepsilon\omega)$ is a solution of the equation

$$\begin{aligned} - \operatorname{div}\left[a(y+\omega)\nabla u^\varepsilon(x+\varepsilon\omega) + \alpha(y+\omega)u^\varepsilon(x+\varepsilon\omega)\right] + \beta(y+\omega)\cdot\nabla u^\varepsilon(x+\varepsilon\omega) \\ + \gamma(y+\omega)u^\varepsilon(x+\varepsilon\omega) + \lambda u^\varepsilon(x+\varepsilon\omega) = f(x+\varepsilon\omega), \quad y = \frac{x}{\varepsilon}. \end{aligned} \tag{4.9}$$

From (4.9) and (4.1) it is seen that $u^\varepsilon(x+\varepsilon\omega)$ and $u^\varepsilon(x,\omega)$ satisfy one and the same equation but with the different right-hand side functions $f(x+\varepsilon\omega)$ and $f(x)$, respectively. To compare these right-hand sides, we employ the following inequality (its proof can be found in [5, 7, 8]):

$$\|f(\,\cdot\,+\varepsilon\omega) - f(\,\cdot\,)\|_{H^{-1}(\mathbb{R}^d)} \le \varepsilon|\omega|c\|f\|_{L^2(\mathbb{R}^d)}, \quad \omega\in Y, \quad c = \mathrm{const}(d),$$

whenever $f \in L^2(\mathbb{R}^d)$.

So, the difference $z^\varepsilon_\omega(x) = u^\varepsilon(x,\omega) - u^\varepsilon(x+\varepsilon\omega)$ satisfies the equation

$$z^\varepsilon_\omega \in H^1(\mathbb{R}^d), \quad A_\varepsilon z^\varepsilon_\omega + \lambda z^\varepsilon_\omega = F^\varepsilon_\omega \in H^{-1}(\mathbb{R}^d)$$

with the right-hand side $F^\varepsilon_\omega(x) = f(x+\varepsilon\omega) - f(x)$. We have the energy estimate

$$\|z^\varepsilon_\omega\|_{H^1(\mathbb{R}^d)} \le C\|F^\varepsilon_\omega\|_{H^{-1}(\mathbb{R}^d)}.$$

Hence,

$$\int_{\mathbb{R}^d} \left(|u^\varepsilon(x,\omega) - u^\varepsilon(x+\varepsilon\omega)|^2 + |\nabla u^\varepsilon(x,\omega) - \nabla u^\varepsilon(x+\varepsilon\omega)|^2\right) dx \le C\varepsilon^2\int_{\mathbb{R}^d}|f(x)|^2\, dx$$

for all $\omega \in Y$. This allows us to replace $u^\varepsilon(x,\omega)$ with $u^\varepsilon(x+\varepsilon\omega)$ in (4.8). As a result, the averaged (over the shifting parameter $\omega \in Y$) estimate is proved.

Theorem 4.2. *Assume that $u^\varepsilon(x)$ is a solution of* (2.1) *and $v^\varepsilon(x,\omega)$ is defined in* (4.2). *Then under the conditions $\varepsilon \le \varepsilon_0$ and $\lambda \ge \lambda_0$ we have the estimate*

$$\int_Y \int_{\mathbb{R}^d} \left(|u^\varepsilon(x+\varepsilon\omega) - v^\varepsilon(x,\omega)|^2 + |\nabla u^\varepsilon(x+\varepsilon\omega) - \nabla v^\varepsilon(x,\omega)|^2\right) d\omega\, dx \\ \le C\varepsilon^2 \int_{\mathbb{R}^d} |f(x)|^2\, dx. \tag{4.10}$$

The constant C depends on the dimension d, the ellipticity constant μ, the parameter λ, and the norms $\|\alpha\|_{L^{2p}_{\mathrm{per}}(Y)^d}$, $\|\beta\|_{L^{2p}_{\mathrm{per}}(Y)^d}$, and $\|\gamma\|_{L^{p}_{\mathrm{per}}(Y)}$ with the same stipulation in the case of $d=2$ as in Theorem 2.2.

4.3. Proof of auxiliary assertions. Here we prove Lemmas 3.2 and 4.1. We use the following fact: if $s \in L^q_{\mathrm{per}}(Y)$, $q>1$, and $\langle s\rangle = 0$, then there exists a vector $S \in W^{1,q}_{\mathrm{per}}(Y)^d$ such that

$$\mathrm{div}_y S(y) = s(y).$$

To prove this, it suffices to take a solution of the periodic problem

$$\Delta_y U = s, \quad U \in W^{2,q}_{\mathrm{per}}(Y)$$

(which exists by the elliptic theory) and to put $S = \nabla_y U$. Thus, from $s \in L^{\frac{2p}{p+1}}_{\mathrm{per}}(Y)$ it follows that $S \in W^{1,\frac{2p}{p+1}}_{\mathrm{per}}(Y)^d$, whence $S \in L^2_{\mathrm{per}}(Y)^d$ by the embedding theorem, as required. Lemma 3.2 is proved.

The above arguments prove also the first representation in (4.5) together with estimate (4.6). The second representation in (4.5) is an immediate consequence of the first. This proves Lemma 4.1.

§5. Corollaries to the integrated estimate

We deduce consequences of Theorem 4.2.

1. First, from the integrated estimate (4.10), we deduce the L^2-estimate (2.12). Indeed, discarding the term with gradients in (4.10) and changing the order of integration, we obtain

$$\int_{\mathbb{R}^d} \int_Y |u^\varepsilon(x+\varepsilon\omega) - v^\varepsilon(x,\omega)|^2\, d\omega\, dx \le C\varepsilon^2 \int_{\mathbb{R}^d} |f(x)|^2\, dx,$$

and then applying the Cauchy–Schwarz inequality in the inner integral with respect to ω, finally we get

$$\int_{\mathbb{R}^d} \left| \int_Y u^\varepsilon(x+\varepsilon\omega) d\omega - u(x)\right|^2 dx \le C\varepsilon^2 \int_{\mathbb{R}^d} |f(x)|^2\, dx, \tag{5.1}$$

where we have taken into account that

$$\int_Y v^\varepsilon(x,\omega)\, d\omega = u(x) + \varepsilon\left(\int_Y N_j\Big(\frac{x}{\varepsilon}+\omega\Big)\, d\omega\right)\frac{\partial u(x)}{\partial x_j} \\ + \varepsilon\left(\int_Y N_0(\frac{x}{\varepsilon}+\omega)\, d\omega\right) u(x) = u(x), \tag{5.2}$$

because $\langle N_j\rangle = 0$ for all $j = 0,1,\dots,d$.

Note that $\int_Y u^\varepsilon(x+\varepsilon\omega)\, d\omega$ is the Steklov averaging (called also Steklov smoothing) of the original solution $u^\varepsilon(x)$. The following property of the Steklov smoothing

$$(S^\varepsilon \varphi)(x) = \int_Y \varphi(x+\varepsilon\omega)\, d\omega$$

is well known (see [5, 7, 8]):

$$\|S^\varepsilon\varphi - \varphi\|_{L^2(\mathbb{R}^d)} \le c_0\varepsilon\|\nabla\varphi\|_{L^2(\mathbb{R}^d)}, \quad c_0 = \mathrm{const}(d). \tag{5.3}$$

Using (5.3), we write

$$\left\| \int_Y u^\varepsilon(\,\cdot\, + \varepsilon\omega)\,d\omega - u^\varepsilon(\,\cdot\,)\right\|_{L^2(\mathbb{R}^d)} \le c_0\varepsilon\|\nabla u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le c\varepsilon\|f\|_{L^2(\mathbb{R}^d)}, \tag{5.4}$$

where the energy inequality (2.9) is employed at the last step.

From (5.1), (5.4), and the triangle inequality

$$\|u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} \le \left\|u^\varepsilon(\,\cdot\,) - \int_Y u^\varepsilon(\,\cdot\, + \varepsilon\omega)\,d\omega\right\|_{L^2(\mathbb{R}^d)} + \left\|\int_Y u^\varepsilon(\,\cdot\, + \varepsilon\omega)\,d\omega - u(\,\cdot\,)\right\|_{L^2(\mathbb{R}^d)}$$

we obtain (2.12).

2. Now we do not discard the gradients in (4.10). Changing the order of integration and applying the Cauchy–Schwarz inequality in the inner integral, we find

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\int_Y (u^\varepsilon(x+\varepsilon\omega) - v^\varepsilon(x,\omega))\,d\omega\right|^2 dx + \int_{\mathbb{R}^d}\left|\nabla\int_Y (u^\varepsilon(x+\varepsilon\omega) - v^\varepsilon(x,\omega))\,d\omega\right|^2 dx \\ \le C\varepsilon^2\int_{\mathbb{R}^d}|f(x)|^2\,dx.\end{aligned} \tag{5.5}$$

Observing that (see (5.2))

$$\int_Y v^\varepsilon(x,\omega)\,d\omega = u(x), \quad \int_Y \nabla v^\varepsilon(x,\omega)\,d\omega = \nabla u(x)$$

and that

$$\int_Y u^\varepsilon(x+\varepsilon\omega)\,d\omega = (S^\varepsilon u^\varepsilon)(x)$$

is the Steklov smoothing of the function $u^\varepsilon(x)$, we see that estimate (5.5) can be rewritten as

$$\|S^\varepsilon u^\varepsilon - u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla(S^\varepsilon u^\varepsilon - u)\|^2_{L^2(\mathbb{R}^d)} \le C\varepsilon^2\|f\|^2_{L^2(\mathbb{R}^d)},$$

i.e.,

$$\|S^\varepsilon u^\varepsilon - u\|_{H^1(\mathbb{R}^d)} \le c\varepsilon\|f\|_{L^2(\mathbb{R}^d)} \tag{5.6}$$

with a constant on the right-hand side of the same type as in (2.12). An interesting property of this H^1-estimate should be mentioned: it does not involve any corrector.

3. Estimate (4.10) can be transformed somewhat differently in order to carry the smoothing operator from $u^\varepsilon(x)$ over to the shifted first approximation $v^\varepsilon(x,\omega) \equiv v^\varepsilon_\omega(x)$. The change of variable $x \to x + \varepsilon\omega$ yields

$$\int_{\mathbb{R}^d}\left(\int_Y |u^\varepsilon(x) - v^\varepsilon_\omega(x-\varepsilon\omega)|^2\,d\omega + \int_Y |\nabla u^\varepsilon(x) - \nabla v^\varepsilon_\omega(x-\varepsilon\omega)|^2\,d\omega\right) dx \le c\varepsilon^2\int_{\mathbb{R}^d} f^2(x)\,dx,$$

whence, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^d}\left[u^\varepsilon(x) - \int_Y v^\varepsilon_\omega(x-\varepsilon\omega)\,d\omega\right]^2 dx + \int_{\mathbb{R}^d}\left[\nabla u^\varepsilon(x) - \int_Y \nabla v^\varepsilon_\omega(x-\varepsilon\omega)\,d\omega\right]^2 dx \\ \le c\varepsilon^2\int_{\mathbb{R}^d} f^2(x)\,dx.\end{aligned} \tag{5.7}$$

Observe that

$$
\begin{aligned}
&v^\varepsilon_\omega(x-\varepsilon\omega)=u(x-\varepsilon\omega)+\varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u}{\partial x_j}(x-\varepsilon\omega)+\varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)u(x-\varepsilon\omega),\\
&\int_Y v^\varepsilon_\omega(x-\varepsilon\omega)\,d\omega\\
&\qquad=\int_Y u(x-\varepsilon\omega)\,d\omega+\varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\int_Y\frac{\partial u}{\partial x_j}(x-\varepsilon\omega)\,d\omega+\varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)\int_Y u(x-\varepsilon\omega)\,d\omega\\
&\qquad=S^\varepsilon u+\varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}S^\varepsilon u+\varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)S^\varepsilon u,
\end{aligned}
$$

where the Steklov smoothing $S^\varepsilon u$ arises in a natural way. Therefore, (5.7) means that

$$
\begin{aligned}
&\Big\|u^\varepsilon(\,\cdot\,)-(S^\varepsilon u)(\,\cdot\,)-\varepsilon N_j\Big(\frac{\cdot}{\varepsilon}\Big)\frac{\partial}{\partial x_j}(S^\varepsilon u)(\,\cdot\,)-\varepsilon N_0\Big(\frac{\cdot}{\varepsilon}\Big)(S^\varepsilon u)(\,\cdot\,)\Big\|_{H^1(\mathbb{R}^d)}\\
&\qquad\qquad\leq c\varepsilon\|f\|_{L^2(\mathbb{R}^d)}.
\end{aligned}
\tag{5.8}
$$

The expression

$$
\widetilde{v}^\varepsilon(x)=S^\varepsilon u(x)+\varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}S^\varepsilon u(x)+\varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)S^\varepsilon u(x)
\tag{5.9}
$$

is called the smoothed first approximation. We write (5.8) briefly as

$$
\|u^\varepsilon-\widetilde{v}^\varepsilon\|_{H^1(\mathbb{R}^d)}\leq c\varepsilon\|f\|_{L^2(\mathbb{R}^d)}.
$$

Estimate (5.8) can be simplified. Indeed,

$$
\|S^\varepsilon u-u\|_{H^1(\mathbb{R}^d)}\leq c_1\varepsilon(\|\nabla u\|_{L^2(\mathbb{R}^d)}+\|\nabla^2 u\|_{L^2(\mathbb{R}^d)})\leq c\varepsilon\|f\|_{L^2(\mathbb{R}^d)}
$$

by the properties of the smoothing operator, in view of the elliptic estimate for u. In other words, the function

$$
\widehat{v}^\varepsilon(x)=u(x)+\varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial x_j}S^\varepsilon u(x)+\varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)S^\varepsilon u(x)
\tag{5.10}
$$

can also be taken as an H^1-approximation. It is called the first approximation with smoothed corrector. The following estimate holds true:

$$
\Big\|u^\varepsilon(\,\cdot\,)-u(\,\cdot\,)-\varepsilon N_j\Big(\frac{\cdot}{\varepsilon}\Big)\frac{\partial}{\partial x_j}(S^\varepsilon u)(\,\cdot\,)-\varepsilon N_0\Big(\frac{\cdot}{\varepsilon}\Big)(S^\varepsilon u)(\,\cdot\,)\Big\|_{H^1(\mathbb{R}^d)}\leq c\varepsilon\|f\|_{L^2(\mathbb{R}^d)},
$$

or in short,

$$
\|u^\varepsilon-\widehat{v}^\varepsilon\|_{H^1(\mathbb{R}^d)}\leq c\varepsilon\|f\|_{L^2(\mathbb{R}^d)}.
\tag{5.11}
$$

Now we summarize our results about approximations in the H^1-norm.

Theorem 5.1. *Let $u^\varepsilon(x)$ be a solution of problem* (2.1), *and let $\widehat{v}^\varepsilon(x)$ be a first approximation with smoothed corrector* (*see* (5.10)). *Then under the conditions $\varepsilon\leq\varepsilon_0$ and $\lambda\geq\lambda_0$, inequality* (5.11) *holds true with a constant of the same type as in* (2.12).

Moreover, the solutions of the original and the homogenized problems are close in the H^1-norm in the sense of estimate (5.6).

Relation (5.11) shows that we have proved an estimate in the operator $(L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d))$-norm for the resolvent $(A_\varepsilon+\lambda)^{-1}$ of the original operator and of its approximation. Namely,

$$
\begin{aligned}
&\|(A_\varepsilon+\lambda)^{-1}-(A_0+\lambda)^{-1}-\varepsilon K_\varepsilon\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leq C\varepsilon,\\
&K_\varepsilon f=N\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla S^\varepsilon(A_0+\lambda)^{-1}f+N_0\Big(\frac{x}{\varepsilon}\Big)S^\varepsilon(A_0+\lambda)^{-1}f.
\end{aligned}
\tag{5.12}
$$

§6. Error estimate for the usual first approximation

6.1. The question arises as to whether it is possible to get rid of the smoothing operator in (5.11) and, thus, proceed to an H^1-estimate with the usual first approximation (3.1), possibly after some strengthening of the original conditions on the data of the problem. The affirmative answer is given below.

Theorem 6.1. *In dimension $d \geq 3$, assume the following conditions on the coefficients in the lower order terms of equation* (2.1):

$$\alpha \in L^s_{\mathrm{per}}(Y)^d \ \textit{for } s > d, \quad \beta \in L^d_{\mathrm{per}}(Y)^d, \quad \gamma \in L^{\frac{d}{2}}_{\mathrm{per}}(Y)^d. \tag{6.1}$$

Then the function $v^\varepsilon(x)$ defined in (3.1) *approximates the solution $u^\varepsilon(x)$ of problem* (2.1) *with estimate* (3.23)*, where the constant C depends on d, the ellipticity constant μ, the parameter λ, and the norms $\|\alpha\|_{L^s(Y)^d}$, $\|\beta\|_{L^d(Y)^d}$, $\|\gamma\|_{L^{\frac{d}{2}}(Y)}$. In dimension $d = 2$, under the same conditions* (2.3), (2.4) *as before, estimate* (3.23) *holds true with a constant of the same type as in Theorem* 2.2.

First, we claim that under condition (6.1) the function $v^\varepsilon(x)$ belongs to $H^1(\mathbb{R}^d)$. Indeed, the generalized maximum principle (see [16, Chapter II, Appendix B]) can be applied to the cell problems (2.14) and (2.15), because in both cases we have a scalar equation of the type

$$\mathrm{div}_y[(a(y)\nabla N(y)] = \mathrm{div}_y F(y)$$

with a function $F \in L^s(Y)$, $s > d$. For example, on the right-hand side of (2.14) we have the function $-\mathrm{div}_y[(a(y)e^j]$ and the desired integrability property is ensured merely by the boundedness condition (2.2). As for (2.15), the right-hand side function α is of class $L^s_{\mathrm{per}}(Y)^d$, $s > d$, by assumptions. Therefore,

$$\|N_j\|_{L^\infty} \leq c, \quad j = 0, 1, \dots, d, \tag{6.2}$$

and, clearly, all the terms in $v^\varepsilon(x)$ and $\nabla v^\varepsilon(x)$ (see (3.1) and (3.2)) containing the multiplier N_j belong to $L^2(\mathbb{R}^d)$. As an example, by (2.11) and (6.2) we obtain

$$\begin{aligned}&\|\nabla^2 u(\,\cdot\,)N(\,\cdot\,/\varepsilon)\|_{L^2(\mathbb{R}^d)^d} \leq C, \quad \|N_0(\,\cdot\,/\varepsilon)\nabla u(\,\cdot\,)\|_{L^2(\mathbb{R}^d)^d} \leq C,\\ &\|N(\,\cdot\,/\varepsilon)\cdot\nabla u(\,\cdot\,)\|_{L^2(\mathbb{R}^d)} \leq C.\end{aligned}$$

The terms in $\nabla v^\varepsilon(x)$ that involve the multiplier ∇N_j also belong to $L^2(\mathbb{R}^d)$ thanks to the following property of this multiplier.

Lemma 6.2. i) *Let N_0 be a solution of problem* (2.15) *under the condition $\alpha \in L^s_{\mathrm{per}}(Y)^d$, $s > d$. Then for sufficiently small ε, $\varepsilon \leq \varepsilon_0$, the gradient $\nabla N_0(y)|_{y=x/\varepsilon}$ is a multiplier from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)^d$. Moreover, we have the estimate*

$$\int_{\mathbb{R}^d} \left|(\nabla_y N_0)\Big(\frac{x}{\varepsilon}\Big)w(x)\right|^2 dx \leq C\int_{\mathbb{R}^d}(|w(x)|^2 + \varepsilon^2|\nabla w(x)|^2)\,dx \ \ \textit{for all } w \in H^1(\mathbb{R}^d), \tag{6.3}$$

where the constant C depends on the dimension d, the ellipticity constant μ, and the norm $\|\alpha\|_{L^s_{\mathrm{per}}(Y)^d}$.

ii) *A similar multiplier property is valid for the gradient $\nabla N_j(y)|_{y=x/\varepsilon}$ of the solution of* (2.14)*, $j = 1, \dots, d$. We have*

$$\int_{\mathbb{R}^d} \left|(\nabla_y N_j)\Big(\frac{x}{\varepsilon}\Big)w(x)\right|^2 dx \leq C\int_{\mathbb{R}^d}(|w(x)|^2 + \varepsilon^2|\nabla w(x)|^2)\,dx \ \ \textit{for all } w \in H^1(\mathbb{R}^d), \tag{6.4}$$

where the constant C depends on the dimension d and the ellipticity constant μ.

Proof. Here we consider only the more complicated assertion i). Assertion ii) of this lemma was proved in [17] and [8] and was used for the first time to study the first approximation in [8].

We begin the proof with an equivalent formulation of the integral identity for the cell problem. By the definition of a periodic solenoidal vector $b \in L^2_{\mathrm{per}}(Y)^d$ such that $\operatorname{div}_y b = 0$, we have the following integral identity on periodic smooth functions:

$$\int_Y b(y) \cdot \nabla\varphi(y)\, dy = 0 \quad \text{for all} \quad \varphi \in C^\infty_{\mathrm{per}}(\square).$$

Therefore, the next integral identity on finitary functions is valid:

$$\int_{\mathbb{R}^d} b(x/\varepsilon) \cdot \nabla\psi\, dx = 0 \quad \text{for all} \quad \psi \in C_0^\infty(\mathbb{R}^d),$$

where, by closure, the test functions may be chosen in $H^1(\mathbb{R}^d)$.

Now suppose that $b(y) = a(y)\nabla N(y) + \alpha(y)$, where $N = N_0$ is a solution of (2.15), and plug the test function

$$\psi(x) = N(x/\varepsilon)|w(x)|^2, \quad w \in C_0^\infty(\mathbb{R}^d),$$

in the last integral identity. Using systematically the notation $b_\varepsilon(x) = b(x/\varepsilon)$ for ε-periodic functions, we obtain

$$\begin{aligned} 0 = \varepsilon \int_{\mathbb{R}^d} b_\varepsilon \cdot \nabla\psi\, dx \iff J &\equiv \int_{\mathbb{R}^d} a_\varepsilon (\nabla N)_\varepsilon w \cdot (\nabla N)_\varepsilon w\, dx \\ &= -2\varepsilon \int_{\mathbb{R}^d} a_\varepsilon (\nabla N)_\varepsilon w \cdot N_\varepsilon \nabla w\, dx - \int_{\mathbb{R}^d} \alpha_\varepsilon w \cdot (\nabla N)_\varepsilon w\, dx \\ &\qquad - 2\varepsilon \int_{\mathbb{R}^d} \alpha_\varepsilon w \cdot N_\varepsilon \nabla w\, dx. \end{aligned} \tag{6.5}$$

The quadratic form J in (6.5) is estimated from below with the help of the ellipticity condition (2.2), namely,

$$\mu \|(\nabla N)_\varepsilon w\|^2_{L^2(\mathbb{R}^d)} \le J.$$

Now, we are going to estimate from above all the three components of J. For this, observe that $\|N\|_{L^\infty} \le c$, as it has been explained before. We have

$$\begin{aligned} &2\varepsilon \int_{\mathbb{R}^d} a_\varepsilon (\nabla N)_\varepsilon w \cdot N_\varepsilon \nabla w\, dx \\ &\qquad \le \delta \int_{\mathbb{R}^d} a_\varepsilon (\nabla N)_\varepsilon w \cdot (\nabla N)_\varepsilon w\, dx + C_\delta \varepsilon^2 \int_{\mathbb{R}^d} a_\varepsilon N_\varepsilon \nabla w \cdot N_\varepsilon \nabla w\, dx \\ &\qquad \overset{(6.5),(2.2)}{\le} \delta J + \mu^{-1} C_\delta \|N\|^2_{L^\infty} \|\varepsilon \nabla w\|^2_{L^2(\mathbb{R}^d)}, \\ &\int_{\mathbb{R}^d} \alpha_\varepsilon w \cdot (\nabla N)_\varepsilon w\, dx \le \delta \|(\nabla N)_\varepsilon w\|^2_{L^2(\mathbb{R}^d)} + C_\delta \int_{\mathbb{R}^d} |\alpha_\varepsilon|^2 |w|^2\, dx, \\ &2\varepsilon \int_{\mathbb{R}^d} \alpha_\varepsilon w \cdot N_\varepsilon \nabla w\, dx \le \|N\|_{L^\infty} \Big(\|\varepsilon \nabla w\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\alpha_\varepsilon|^2 |w|^2\, dx \Big), \end{aligned}$$

where $\delta > 0$ can be arbitrarily small. We also recall that, by Lemma 2.1,

$$\int_{\mathbb{R}^d} |\alpha_\varepsilon|^2 |w|^2\, dx \le c_0 \big(\|w\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^2 \|\nabla w\|^2_{L^2(\mathbb{R}^d)} \big).$$

The above inequalities show that estimate (6.3) is true for $w \in C_0^\infty(\mathbb{R}^d)$ and, by closure, it is valid for any $w \in H^1(\mathbb{R}^d)$. Thus, the proof is complete.

6.2. We proceed to the proof of the estimate

$$\|u^\varepsilon - v^\varepsilon\|_{H^1(\mathbb{R}^d)} \le C\varepsilon \|f\|_{L^2(\mathbb{R}^d)}.$$

For this, we are going to replace in the H^1-estimate (5.11) the smoothed approximation by the usual first approximation.

To be it more demonstrative, we write estimate (5.11) in a detailed version, using the simplified notation $z^\varepsilon = S^\varepsilon u$ for the Steklov smoothing.

Namely,

$$\begin{aligned} &\int_{\mathbb{R}^d} \Big| u^\varepsilon(x) - u(x) - \varepsilon N\Big(\frac{x}{\varepsilon}\Big)\cdot \nabla z^\varepsilon(x) - \varepsilon N_0\Big(\frac{x}{\varepsilon}\Big) z^\varepsilon(x)\Big|^2\,dx \\ &\quad + \int_{\mathbb{R}^d} \Big|\nabla u^\varepsilon(x) - \nabla\Big(u(x) + \varepsilon N\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla z^\varepsilon(x) + \varepsilon N_0\Big(\frac{x}{\varepsilon}\Big) z^\varepsilon(x)\Big)\Big|^2\,dx \\ &\qquad \le c\varepsilon^2 \int_{\mathbb{R}^d} f^2\,dx. \end{aligned} \tag{6.6}$$

By (6.2) and the simplest property of smoothing

$$\|S^\varepsilon u\|_{L^2(\mathbb{R}^d)} \le \|u\|_{L^2(\mathbb{R}^d)}, \tag{6.7}$$

estimate (6.6) survives if z^ε is replaced with u in the first integral. The smoothed corrector in the second integral requires a subtler treatment. As for the expression (without summation over j)

$$\varepsilon\nabla\Big(N_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial z^\varepsilon(x)}{\partial x_j}\Big) = \varepsilon N_j\Big(\frac{x}{\varepsilon}\Big)\nabla\frac{\partial z^\varepsilon(x)}{\partial x_j} + (\nabla_y N_j)\Big(\frac{x}{\varepsilon}\Big)\frac{\partial z^\varepsilon(x)}{\partial x_j}, \quad j = 1,\dots,d, \tag{6.8}$$

there is no problem with the first term thanks to the above-mentioned property of smoothing, the boundedness of N^j, and the elliptic estimate for u. In the second term of (6.8) smoothing can be omitted, because

$$\begin{aligned} \Big\|\nabla_y N_j\Big(\frac{\partial z^\varepsilon}{\partial x_j} - \frac{\partial u}{\partial x_j}\Big)\Big\|_{L^2} &\le C\Big(\varepsilon\Big\|\nabla\Big(\frac{\partial z^\varepsilon}{\partial x_j} - \frac{\partial u}{\partial x_j}\Big)\Big\|_{L^2} + \Big\|\frac{\partial z^\varepsilon}{\partial x_j} - \frac{\partial u}{\partial x_j}\Big\|_{L^2}\Big) \\ &\le C\Big(\varepsilon\Big\|\nabla\frac{\partial z^\varepsilon}{\partial x_j}\Big\|_{L^2} + \varepsilon\Big\|\nabla\frac{\partial u}{\partial x_j}\Big\|_{L^2} + c_0\varepsilon\Big\|\nabla\frac{\partial u}{\partial x_j}\Big\|_{L^2}\Big) \\ &\le C_1\varepsilon\|\nabla^2 u\|_{L^2} \le c\varepsilon\|f\|_{L^2} \end{aligned} \tag{6.9}$$

by (6.4), the smoothing properties (6.7), (5.3), and the elliptic estimate for u.

The gradient of the second summand in the smoothed corrector is studied similarly because it has the form

$$\nabla\Big(\varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)z^\varepsilon(x)\Big) = \varepsilon N_0\Big(\frac{x}{\varepsilon}\Big)z^\varepsilon(x) + (\nabla N_0)\Big(\frac{x}{\varepsilon}\Big)z^\varepsilon(x).$$

Here, we use the boundedness of N_0, estimate (6.3) and again various properties of the Steklov smoothing.

Thus, we have verified that the Steklov smoothing can be omitted everywhere in (6.6) (in other words, we may replace everywhere $z^\varepsilon = S^\varepsilon u$ by the function u itself), arriving at the desired H^1-estimate with the usual first approximation v^ε. This completes the proof of Theorem 6.1. □

6.3. Here we present the proof of Lemma 2.1, which plays a key role in our considerations. For definiteness, let $d > 2$. By homothety arguments, it suffices to consider only the case where $\varepsilon = 1$. This implies that the weight $\rho(x)$ is 1-periodic.

Split $\mathbb{R}^d$ into unit cubes. In each unit cube Y we have the estimate

$$\frac{1}{2}\int_Y u^2\rho\,dx \le \int_Y (u - \langle u\rangle)^2\rho\,dx + \int_Y \langle u\rangle^2\rho\,dx, \quad \text{where } \langle u\rangle = \int_Y u\,dx, \quad \langle u\rangle^2 \le \langle u^2\rangle.$$

Applying the Hölder and Sobolev inequalities, we get

$$\int_Y (u-\langle u\rangle)^2\rho\,dx \le \Big(\int_Y \rho^{\frac{d}{2}}\,dx\Big)^{\frac{2}{d}}\Big(\int_Y (u-\langle u\rangle)^{\frac{2d}{d-2}}\,dx\Big)^{\frac{d-2}{d}} \le c_S\|\rho\|_{L^{\frac{d}{2}}(Y)}\int_Y |\nabla u|^2\,dx,$$

whence

$$\int_Y u^2\rho\,dx \le 2\langle\rho\rangle\int_Y u^2\,dx + 2c_S\|\rho\|_{L^{\frac{d}{2}}(Y)}\int_Y|\nabla u|^2\,dx \le c_0\|u\|^2_{H^1(Y)},$$
$$c_0 = \mathrm{const}\,\big(d, \|\rho\|_{L^{\frac{d}{2}}(Y)}\big).$$

Summing up this estimate over all cubes of the splitting yields inequality (2.7) for $\varepsilon = 1$. This completes the proof of Lemma 2.1.

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