SUGAKU EXPOSITIONS
Volume 31, Number 2, December 2018, Pages 117–142
http://dx.doi.org/10.1090/suga/431
Article electronically published on September 19, 2018

MARTINGALES AND FUNCTION SPACES

MASATO KIKUCHI

1. Introduction

The notion of a martingale was introduced by Doob in [18], although the word 'martingale' was not used in [18]. Nine years later, that is, in 1949 Doob himself had used the word 'martingale'. In his book [19] published in 1953, the basic theory of martingales was summarized. During the next 25 years, martingale theory was developed drastically. The contribution of P. A. Meyer and his colleague are particularly noteworthy. For the theory developed by them, see Dellacherie [15], Dellacherie and Meyer [16],[17], and Kazamaki [27], [30].

In recent years, applications of martingale theory to mathematical finance have attracted attention; and researchers interested in martingale theory are now increasing. However, a lot of researchers are not interested in martingale theory itself. In this article, we will focus on analytical aspects of martingale theory; we will give an overview of results on norm convergence of martingales and on norm inequalities for martingales, in a Banach function space.

To avoid cluttering the notation, we will consider only discrete-parameter martingales (with values in $\mathbb{R}$). That is, the set of parameters of each martingale is $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. Almost all results in this article hold true for continuous martingales.

2. Preliminaries

2.1. **Martingales.** Throughout this article, we let $(\Omega, \Sigma, \mathbb{P})$ be a nonatomic probability space, that is, a probability space with no atom. Here, by an *atom*, we mean a set $A \in \Sigma$ with positive measure such that if $A' \in \Sigma$ is a subset of A, then either $\mathbb{P}(A') = 0$ or $\mathbb{P}(A') = \mathbb{P}(A)$. If A is an atom, then every random variable (in other words, Σ-measurable function) is essentially constant on A. In contrast, our assumption that $(\Omega, \Sigma, \mathbb{P})$ is nonatomic guarantees that there exists a random variable with arbitrary distribution. Moreover, our assumption guarantees that for any $t \in [0, 1]$, there exists a set $A \in \Sigma$ such that $\mathbb{P}(A) = t$ (see [12, p. 44]).

Let $x \in L_1(\Omega)$ and let $\mathcal{A}$ be a sub-σ-algebra of Σ. We let $\mathbb{E}[x\,|\mathcal{A}]$ denote the *conditional expectation* of x given $\mathcal{A}$. This means that $\mathbb{E}[x\,|\mathcal{A}]$ is characterized as an essentially unique $\mathcal{A}$-measurable random variable $y \in L_1(\Omega)$ such that

$$\int_A x\,d\mathbb{P} = \int_A y\,d\mathbb{P} \quad \text{for all } A \in \mathcal{A}.$$

Thus $x \mapsto \mathbb{E}[x\,|\mathcal{A}]$ defines a linear projection of $L_1(\Omega)$ onto the linear subspace of $L_1(\Omega)$ consisting of all $\mathcal{A}$-measurable random variables. Moreover, if x is a

This article originally appeared in Japanese in Sūgaku **66** 3 (2014), 249–274.
2010 *Mathematics Subject Classification.* Primary 60G42; Secondary 46E30, 60G46.

nonnegative random variable, then we can define the conditional expectation $\mathbb{E}[x\,|\mathcal{A}]$ by letting $\mathbb{E}[x\,|\mathcal{A}] = \sup_{n\in\mathbb{N}} \mathbb{E}[\min\{x, n\}\,|\mathcal{A}]$.

Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ be a sequence of sub-σ-algebras of Σ. We call $\mathcal{F}$ a *filtration* of Ω if $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{Z}_+$. We let $\mathfrak{F}$ denote the set of all filtrations of Ω. Given $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathfrak{F}$, we say that a stochastic process (that is, a sequence of random variables) $f = (f_n)_{\in\mathbb{Z}_+}$ is *adapted* to $\mathcal{F}$ if each f_n is $\mathcal{F}_n$-measurable. A process $f = (f_n)$ adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)$ is called an $\mathcal{F}$-*martingale* if

$$f_n \in L_1(\Omega) \quad \text{and} \quad \mathbb{E}[f_{n+1}\,|\mathcal{F}_n] = f_n \ \text{ a.s. for all } n \in \mathbb{Z}_+,$$

where "a.s." is the abbreviation for "almost surely". Thus if $f = (f_n)$ is a martingale, then

$$\int_A f_{n+1}\, d\mathbb{P} = \int_A f_n\, d\mathbb{P} \quad \text{for all } n \in \mathbb{Z}_+ \text{ and all } A \in \mathcal{F}_n.$$

Given $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$, we let $\mathcal{M}(\mathcal{F})$ denote the set of all $\mathcal{F}$-martingales, and we let $\mathcal{M} = \bigcup_{\mathcal{F}\in\mathfrak{F}} \mathcal{M}(\mathcal{F})$. We call $f = (f_n) \in \mathcal{M}$ merely a *martingale*. For a martingale $f = (f_n)$, we let Mf denote the *maximal function* of $f = (f_n)$, and we let Sf denote the *square function* of $f = (f_n)$; that is,

$$Mf = \sup_{n\in\mathbb{Z}_+} |f_n| \quad \text{and} \quad Sf = \left[\sum_{n=0}^{\infty} (\Delta_n f)^2\right]^{1/2},$$

where

$$\Delta_0 f = f_0; \qquad \Delta_n f = f_n - f_{n-1}, \quad n \in \mathbb{N}.$$

As is well known, if $f = (f_n) \in \mathcal{M}$ is such that $\sup_{n\in\mathbb{Z}_+} \|f_n\|_{L_1} < \infty$, then $f = (f_n)$ converges a.s. (see [17, p. 24] or [29, p. 65]). Such a martingale $f = (f_n)$ is said to be *bounded* in L_1. More generally, if X is a normed space of random variables and if $\sup_{n\in\mathbb{Z}_+} \|f_n\|_X < \infty$, then $f = (f_n)$ is said to be bounded in X.

A family of random variables $\mathcal{H}$ is said to be *uniformly integrable* if

$$\lim_{\lambda\to\infty} \sup_{x\in\mathcal{H}} \int_{\{\omega\in\Omega:\ |x(\omega)|>\lambda\}} |x|\, d\mathbb{P} = 0.$$

Let $\{x_n\}$ be a sequence of random variables which converges a.s. Then $\{x_n\}$ is uniformly integrable (as a family of random variables) if and only if it converges in $L_1(\Omega)$ with respect to the norm topology. It is clear from the definition that if $\mathcal{H}$ is uniformly integrable, then $\sup_{x\in\mathcal{H}} \|x\|_{L_1} < \infty$. However, the converse is false. On the other hand, if $1 < p \le \infty$ and $\sup_{x\in\mathcal{H}} \|x\|_{L_p} < \infty$, then $\mathcal{H}$ is uniformly integrable. For more details, see, e.g., [16, pp. 33–39].

Let $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$. If $f = (f_n) \in \mathcal{M}(\mathcal{F})$ converges a.s., we let f_∞ denote the almost sure limit of $f = (f_n)$. If $f = (f_n)$ is uniformly integrable, then it is bounded in L_1 and hence converges a.s. In this case, we have $f_n = \mathbb{E}[f_\infty\,|\mathcal{F}_n]$ a.s. for all $n \in \mathbb{Z}_+$, and $f = (f_n)$ converges in L_1. Conversely, if there exists $x \in L_1(\Omega)$ such that $f_n = \mathbb{E}[x\,|\mathcal{F}_n]$ a.s., then $f = (f_n)$ is uniformly integrable.

Other results on martingales, mentioned in this article, can be found in [17], [29], [57], [64], [68], [71], and [72].

2.2. **Banach function spaces.** Throughout this article, we let I be the interval $(0,1]$, and we consider $I=(0,1]$ as the probability space equipped with Lebesgue measure μ. From this point of view, a measurable function ϕ on I should be called a "random variable". However, we sometimes call ϕ a "measurable function" to indicate that ϕ is defined on I. A Banach function space, which is one of the central themes of this article, is a Banach lattice of random variables possessing the Fatou property. In order to give the precise definition, we introduce the following notation.

Let X and Y be Banach spaces of random variables on a (same) probability space. We write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y, that is, $X \subset Y$ and the inclusion map is continuous.

Definition 2.1. A *Banach function space* is a real Banach space X of (equivalence classes of) random variables on a probability space which satisfies the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$.
(B2) If $|x| \le |y|$ a.s. and $y \in X$, then $x \in X$ and $\|x\|_X \le \|y\|_X$.
(B3) If $0 \le x_n \uparrow x$ a.s., $x_n \in X$ for all $n \in \mathbb{N}$, and $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_{n\in\mathbb{N}} \|x_n\|_X$.

We adopt the convention that if x is a random variable which does not belong to X, then $\|x\|_X = \infty$.

For example, Lebesgue spaces L_p, Orlicz spaces L_Φ, and Lorentz spaces $L_{p,q}$ are Banach function spaces. (For details of Orlicz spaces, see [53], [54], [60], or [67]; and for details of Lorentz spaces, see [6] or [21].) Moreover, if $1 < p < \infty$, if w is a random variable such that $\mathbb{E}[\mathrm{w}] = 1$ and $\mathrm{w}^{-1/(p-1)} \in L_1(\Omega)$, and if $\mathbb{P}_\mathrm{w}$ is the probability measure defined by $\mathbb{P}_\mathrm{w}(A) = \int_A \mathrm{w}\, d\mathbb{P}$, then the Lebesgue space $L_p(\mathbb{P}_\mathrm{w})$ with respect to $\mathbb{P}_\mathrm{w}$ is a Banach function space over $(\Omega, \Sigma, \mathbb{P})$. Here the condition that $\mathrm{w}^{-1/(p-1)} \in L_1(\Omega)$ guarantees the embedding $L_p(\mathbb{P}_\mathrm{w}) \hookrightarrow L_1(\Omega)$ (cf. [34, Section 3]). We call $L_p(\mathbb{P}_\mathrm{w})$ the *weighted Lebesgue space*. One can define the *weighted Orlicz space* $L_\Phi(\mathbb{P}_\mathrm{w})$ in the same way.

Let X be a Banach function space over Ω or I. For each random variable x, let

$$\|x\|_{X'} = \sup\left\{\mathbb{E}\big[|xy|\big] : y \in X,\ \|y\|_X \le 1\right\}$$

and define

$$X' = \{x \in L_0 \colon \|x\|_{X'} < \infty\},$$

where L_0 denotes the linear space of random variables which are finite a.s. Then X' is a Banach function space. We call X' the *associate space* of X. For any Banach function space X, the associate space of X' coincides with X, that is, $X'' = X$ and $\|x\|_{X''} = \|x\|_X$ for all $x \in X$ (see [6, Theorem 2.7, p. 10]). This shows that the associate space X' and the dual space X^* does not necessarily coincide. Indeed, we have $(L_\infty)' = L_1$, but L_1 is not the dual space of L_∞ (when the underlying probability space is nonatomic). A necessary and sufficient condition for the associate space X' to coincide with X^* is that X has the absolutely continuous norm (see Definition 3.1 and [6, Corollary 4.3, p. 23]).

Among many Banach function spaces, rearrangement-invariant spaces, which are defined as follows, are essentially important in various studies. In what follows, we write $x \simeq_d y$ if x and y are random variables with the same distribution.

Definition 2.2. A Banach function space X is said to be *rearrangement-invariant* or *r.i.* if X has the property that whenever $x \simeq_d y$ and $y \in X$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

By an r.i. space, we mean a rearrangement-invariant Banach function space.

It is easy to see that Lebesgue spaces and Orlicz spaces are r.i. Moreover, Lorentz spaces are also r.i. On the other hand, the weighted Lebesgue space $L_p(\mathbb{P}_{\mathrm{w}})$ is a Banach function space which is *not* r.i. in general, provided $\mathrm{w}^{-1/(p-1)} \in L_1(\Omega)$. If there exist positive constants a and b such that $a \leq \mathrm{w} \leq b$ a.s., then there exists a norm $|||\cdot|||_{L_p(\mathbb{P}_{\mathrm{w}})}$ on $L_p(\mathbb{P}_{\mathrm{w}})$ which is equivalent to the original norm of $L_p(\mathbb{P}_{\mathrm{w}})$ and with respect to which $L_p(\mathbb{P}_{\mathrm{w}})$ is an r.i. space over $(\Omega, \Sigma, \mathbb{P})$. Conversely, if such a norm $|||\cdot|||_{L_p(\mathbb{P}_{\mathrm{w}})}$ exists, then $a \leq \mathrm{w} \leq b$ a.s. for some positive constants a and b (see [32]; cf. [38, Section 4]).

Given a subset A of Ω or of I, we let $\mathbf{1}_A$ denote the indicator function of A. Let X be an r.i. space. For each $t \in [0,1]$, we choose $A \in \Sigma$ so that $\mathbb{P}(A) = t$ and we let

$$\varphi_X(t) = \|\mathbf{1}_A\|_X .$$

If $A, B \in \Sigma$ and $\mathbb{P}(A) = \mathbb{P}(B)$, then $\mathbf{1}_A$ and $\mathbf{1}_B$ have the same distribution. Thus the value of $\|\mathbf{1}_A\|_X$ depends only on the measure of A. Furthermore, since we are assuming that $(\Omega, \Sigma, \mathbb{P})$ is nonatomic, we can certainly choose $A \in \Sigma$ so that $\mathbb{P}(A) = t$. Thus the function $\varphi_X \colon [0,1] \to [0,\infty)$ is well defined; φ_X is called the *fundamental function* of X. For example, $\varphi_{L_p}(t) = t^{1/p}$ when $1 \leq p < \infty$, and $\varphi_{L_\infty}(t) = \mathbf{1}_{(0,1]}(t)$.

Note that the fundamental function is defined when X is r.i. In Section 7, we will introduce the generalized fundamental functions of a Banach function space.

In order to state some of the results in this article, we will use the Boyd indices of an r.i. space. The Boyd indices of an r.i. space X, which describe the nature of X, were introduced by Boyd [8] to establish an interpolation theorem. To give a precise definition of the Boyd indices, we need some further definitions and notation.

Let x be a random variable on Ω. We define a measurable function $x^*(t)$ on $I = (0,1]$ by

$$x^*(t) = \inf\bigl\{\lambda > 0 \colon P(\omega \in \Omega \colon |x(\omega)| > \lambda) \leq t\bigr\}, \quad t \in I,$$

with the convention that $\inf \emptyset = \infty$. We call x^* the *nonincreasing rearrangement* of x. Note that x^* is a nonincreasing right-continuous function whose distribution is the same as that of $|x|$. By regarding I as a probability space, we can define the nonincreasing rearrangement ϕ^* of a measurable function ϕ on I.

It is known that if X is an r.i. space over Ω, then there exists an r.i. space $\hat{X}$ over Ω such that:

- $x \in X$ if and only if $x^* \in \hat{X}$.
- $\|x\|_X = \|x^*\|_{\hat{X}}$ for all $x \in X$.

Such an r.i. space $\hat{X}$ is unique; we call $\hat{X}$ the *Luxemburg representation* of X. For details, see [6, pp. 62–64]. For example, the Luxemburg representation of $L_p(\Omega)$ is $L_p(I)$.

As mentioned before, we denote by L_0 the linear space of random variables which are finite a.s. We also write $L_0(\Omega)$ (resp., $L_0(I)$) to indicate that it is the space of random variables defined on Ω (resp., I). If Z is a Banach function space, we let $B(Z)$ denote the set of all linear operators T satisfying the following conditions:

- The domain of T contains Z and the range of T is contained in L_0.
- The restriction of T to Z is a bounded linear operator on Z into Z.

Given $T \in B(Z)$, we let $\|T\|_{B(Z)}$ denote the operator norm of the restriction of T to Z.

For each number $s > 0$, we define a linear operator $D_s : L_0(I) \to L_0(I)$ by

$$(D_s\phi)(t) = \begin{cases} \phi(st), & st \in I, \\ 0, & st \notin I, \end{cases} \qquad t \in I.$$

If Z is an r.i. space over I, then $D_s \in B(Z)$ for all $s > 0$ and $\|D_s\|_{B(Z)}$ is less than or equal to $\max\{1, (1/s)\}$. Using these operators, we define the *lower Boyd index* α_Z and the *upper Boyd index* β_Z by

$$\alpha_Z = \sup_{0<s<1} \frac{\log \|D_{1/s}\|_{B(Z)}}{\log s} \quad \text{and} \quad \beta_Z = \inf_{1<s<\infty} \frac{\log \|D_{1/s}\|_{B(Z)}}{\log s},$$

respectively. One can show that

$$\alpha_Z = \lim_{s\to 0+} \frac{\log \|D_{1/s}\|_{B(Z)}}{\log s} \quad \text{and} \quad \beta_Z = \lim_{s\to\infty} \frac{\log \|D_{1/s}\|_{B(Z)}}{\log s}.$$

If X is an r.i. space over Ω, then the Boyd indices of X are defined by

$$\alpha_X = \alpha_{\hat{X}} \quad \text{and} \quad \beta_X = \beta_{\hat{X}},$$

where $\hat{X}$ is the Luxemburg representation of X. For every r.i. space X, we have $0 \le \alpha_X \le \beta_X \le 1$. For example, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1, \infty]$. Thus the Boyd indices are ones which extend the role of the index p of L_p. For details of the Boyd indices, see [6, pp. 146–150]; cf. [6, p. 165]. About the matters which are not mentioned in this section, see [6], [7], [12], [21], [22], [55], [56], [58], and [62].

3. Norm convergence of martingales in a Banach function space

Suppose that $f = (f_n)_{n\in\mathbb{Z}_+}$ is a uniformly integrable martingale. Then, as mentioned before, $f = (f_n)$ converges a.s. and it also converges in L_1. Moreover, if $f_\infty \in L_p$ for $p \in (1, \infty)$, then $f = (f_n)$ converges in L_p.

Let X be a Banach function space over Ω and let $f = (f_n)$ be a uniformly integrable martingale. Under what condition does $f = (f_n)$ converge in X? The results in this section show that norm convergence of martingales is closely connected with the uniform boundedness of conditional expectation operators. We begin with a definition.

Definition 3.1. A Banach function space X over Ω is said to have *absolutely continuous norm* if $\|x\mathbf{1}_{A_n}\|_X \downarrow 0$ for every $x \in X$ and every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets in Σ satisfying $A_n \downarrow \emptyset$ a.s. Here, and in what follows, we write $A_n \downarrow \emptyset$ a.s. to mean that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and $\mathbb{P}(\bigcap_{n=1}^\infty A_n) = 0$.

The first result on norm convergence of martingales is as follows.

Theorem 3.1 ([34])**.** *Let X be a Banach function space over Ω which has absolutely continuous norm, let $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$, and let $\mathbb{E}_n$ denote the conditional expectation operator $\mathbb{E}[\,\cdot\,|\mathcal{F}_n]$ for each $n \in \mathbb{Z}_+$. Then the following are equivalent:*

(i) *If $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is uniformly integrable and if $f_\infty \in X$, then*

$$\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0.$$[1]

(ii) *There exists $n_0 \in \mathbb{Z}_+$ such that $\mathbb{E}_n \in B(X)$ for all $n \geq n_0$, and*

$$\sup_{n\geq n_0} \|\mathbb{E}_n\|_{B(X)} < \infty.$$

An example in [34] shows that, in Theorem 3.1, condition (ii) cannot be replaced by the boundedness of each $\mathbb{E}_n$. Thus the uniform boundedness of $\{\mathbb{E}_n \colon n \in \mathbb{Z}_+\}$ is essential.

Let X be an r.i. space. Then

$$\mathbb{E}[\,\cdot\,|\mathcal{A}] \in B(X) \quad \text{and} \quad \big\|\mathbb{E}[\,\cdot\,|\mathcal{A}]\big\|_{B(X)} = 1$$

for all sub-σ-algebras $\mathcal{A}$ of Σ (see [36, Lemma 2]). It follows that if X has absolutely continuous norm, then (i) of Theorem 3.1 holds. The next theorem asserts that the converse is also true.

Theorem 3.2 ([34])**.** *Let X be an r.i. space over Ω, and let $\mathcal{F} = (\mathcal{F}_n)$ and $\mathbb{E}_n$ be as in Theorem* 3.1*. Then the following are equivalent*:

(i) *If $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is uniformly integrable and if $f_\infty \in X$, then*

$$\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0.$$

(ii) *X has absolutely continuous norm.*

Let us consider the case where X is a Lebesgue space, in Theorem 3.2. Note that if $1 \leq p < \infty$, then L_p has absolutely continuous norm. Hence (i) of Theorem 3.2 holds when $X = L_p$. On the other hand, since L_∞ does not have absolutely continuous norm, (i) of Theorem 3.2 does not hold when $X = L_\infty$. In view of this fact and the fact that $\alpha_{L_p} = \beta_{L_p} = 1/p$, one may guess that a necessary and sufficient condition for an r.i. space X to satisfy (i) would be that $\alpha_X > 0$ or that $\beta_X > 0$. However, this is false. Indeed, when $(1 < p < \infty)$, the Lorentz space $L_{p,\infty}$ is an r.i. space which does not have absolutely continuous norm. Hence the conditions of Theorem 3.2 are not satisfied when $X = L_{p,\infty}$; however, we have $\alpha_{L_{p,\infty}} = \beta_{L_{p,\infty}} = 1/p > 0$ (cf. [33, Remark, p. 90]). Thus, it is impossible to characterize an r.i. space X which satisfies the conditions of Theorem 3.2 by using the Boyd indices.

Now let us consider the case where X is a weighted Lebesgue space, in Theorem 3.2. Let w be a strictly positive random variable on Ω such that $\mathbb{E}[\mathrm{w}] = 1$, and let $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$. We also write $\mathrm{w} = (\mathrm{w}_n)$ for the $\mathcal{F}$-martingale generated by w; that is, $\mathrm{w}_n = \mathbb{E}[\mathrm{w}\,|\mathcal{F}_n]$ a.s. for $n \in \mathbb{Z}_+$. Let $n_0 \in \mathbb{Z}_+$. When $1 < p < \infty$, we say that w satisfies *condition $A_p(\mathcal{F})$* for $n \geq n_0$ if there exists a positive constant K such that

$$(A_p) \qquad \sup_{n\geq n_0} \mathbb{E}\left[\left(\frac{\mathrm{w}_n}{\mathrm{w}}\right)^{1/(p-1)} \,\middle|\, \mathcal{F}_n\right] \leq K \quad \text{a.s.};$$

[1] As mentioned in Definition 2.1, we follow the convention that if x is a random variable which does not belong to X, then $\|x\|_X = \infty$. Hence the condition $\lim_n \|f_n - f_\infty\|_X = 0$ implies that $f_n - f_\infty \in X$ for sufficiently large n.

and we say that w satisfies *condition* $A_1(\mathcal{F})$ for $n \geq n_0$ if there exists a positive constant K such that

$$(A_1) \qquad \sup_{n\geq n_0} \frac{\mathrm{w}_n}{\mathrm{w}} \leq K \quad \text{a.s.}$$

These conditions were introduced by Izumisawa and Kazamaki in [25]. It is known that the martingale w satisfying condition $A_p(\mathcal{F})$ can be represented by using a *BMO*-martingale. Through this fact, the results of Izumisawa and Kazamaki contributed to the later development of the theory of *BMO*-martingales. For details, see Kazamaki [28] and the references therein.

As mentioned in the previous section, we define the probability measure $\mathbb{P}_{\mathrm{w}}$ by $\mathbb{P}_{\mathrm{w}}(A) = \int_A \mathrm{w}\, d\mathbb{P}$. Let us consider the weighted Lebesgue space $L_p(\mathbb{P}_{\mathrm{w}})$. In order to guarantee that $L_p(\mathbb{P}_{\mathrm{w}}) \hookrightarrow L_1$, we assume that $\mathrm{w}^{-1/(p-1)} \in L_1$ when we consider the case where $1 < p < \infty$, and we assume that $\mathrm{w}^{-1} \in L_\infty$ when we consider the case where $p = 1$. As in Theorem 3.1, let $\mathbb{E}_n$ denote the conditional expectation operator $\mathbb{E}[\,\cdot\,|\mathcal{F}_n]$. Then

$$\mathbb{E}_n \in B(L_p(\mathbb{P}_{\mathrm{w}})) \quad \text{and} \quad \sup_{n\geq n_0} \|\mathbb{E}_n\|_{B(L_p(\mathbb{P}_{\mathrm{w}}))} < \infty$$

if and only if w satisfies condition $A_p(\mathcal{F})$ for $n \geq n_0$ (see [20]; cf. [37, Lemma 3]). It is clear that if $1 \leq p < \infty$, then $L_p(\mathbb{P}_{\mathrm{w}})$ has absolutely continuous norm. Hence, with the notation above, we have by Theorem 3.1 the following:

Corollary 3.1 ([34])**.** *Suppose that $1 \leq p < \infty$. Then the following are equivalent*:

(i) *If $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is uniformly integrable and if $f_\infty \in L_p(\mathbb{P}_{\mathrm{w}})$, then*

$$\lim_{n\to\infty} \|f_n - f_\infty\|_{L_p(\mathbb{P}_{\mathrm{w}})} = 0.$$

(ii) *There exists $n_0 \in \mathbb{Z}_+$ such that* w *satisfies condition $A_p(\mathcal{F})$ for $n \geq n_0$.*

Furthermore, we can derive a result on convergence of martingales also in a weighted Orlicz space. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a nondecreasing convex function. If $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$, we call Φ a *Young function.* If, in addition, $\Phi(t) > 0$ for $t > 0$ and $\Phi(t)/t \to 0$ as $t \to 0+$, we call Φ an N-function.

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an N-function, and consider the weighted Orlicz space $L_\Phi(\mathbb{P}_{\mathrm{w}})$. To guarantee that $L_\Phi(\mathbb{P}_{\mathrm{w}}) \hookrightarrow L_1$, we assume that $\Psi(\mathrm{w}^{-1})\mathrm{w} \in L_1$ (see [38, Section 4]), where Ψ is the function defined by

$$\Psi(t) = \sup\{st - \Phi(s)\colon 0 \leq s < \infty\}, \quad t \in [0,\infty).$$

We call Ψ the *complementary function* of Φ. As before, let $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$ and $\mathrm{w}_n = \mathbb{E}[\mathrm{w}\,|\mathcal{F}_n]$, $n \in \mathbb{Z}_+$. We say that w satisfies *condition* $A_\Phi(\mathcal{F})$ for $n \geq n_0$ if there exists a positive constant K such that

$$(A_\Phi) \qquad \mathbb{E}\left[\Psi\left(\frac{\Phi(\lambda)\mathrm{w}_n}{K\mathrm{w}\lambda}\right)\mathrm{w}\,\middle|\,\mathcal{F}_n\right] \leq \Phi(\lambda)\,\mathrm{w}_n \quad \text{a.s.}$$

for all $\lambda > 0$ and all $n \geq n_0$. It is easily checked that if $\Phi(t) = t^p$ $(1 < p < \infty)$, then (A_Φ) coincides with (A_p).

If w satisfies condition $A_\Phi(\mathcal{F})$ for $n \geq n_0$, then $\mathbb{E}_n \in B(L_\Phi(\mathbb{P}_{\mathrm{w}}))$ for all $n \geq n_0$ and

$$\sup_{n\geq n_0} \|\mathbb{E}_n\|_{B(L_\Phi(\mathbb{P}_{\mathrm{w}}))} < \infty$$

(see [39, Theorem 2]). On the other hand, according to [54, p. 88], $L_\Phi(\mathbb{P}_{\mathrm{w}})$ has absolutely continuous norm if and only if Φ satisfies the Δ_2^∞-condition.[2] Here we say that Φ satisfies the Δ_2^∞-*condition* if there exist constants $c > 0$ and $t_0 \geq 0$ such that $\Phi(2t) \leq c\Phi(t)$ for all $t \geq t_0$. With the notation above, we have by Theorem 3.1 the following:

Corollary 3.2. *Suppose that an N-function Φ satisfies the Δ_2^∞-condition and that there exists $n_0 \in \mathbb{Z}_+$ such that* w *satisfies condition $A_\Phi(\mathcal{F})$ for $n \geq n_0$.*

If $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is uniformly integrable and $f_\infty \in L_\Phi(\mathbb{P}_{\mathrm{w}})$, then

$$\lim_{n\to\infty} \|f_n - f_\infty\|_{L_\Phi(\mathbb{P}_{\mathrm{w}})} = 0.$$

All the results mentioned above deal with convergence of uniformly integrable martingales. In general, it is not easy to check the uniform integrability of a process. Instead of doing so, one can often use a method of checking the boundedness of the process in a Banach space. For instance, if $1 < p < \infty$, then a martingale which is bounded in L_p is uniformly integrable and converges in L_p. In contrast, a martingale which is bounded in L_1 converges a.s., but it may not be uniformly integrable. A uniformly integrable martingale is bounded in L_1 and converges in L_1. In view of these facts, it is natural to explore a characterization of a Banach function space X such that every martingale bounded in X converges in X. When X is an r.i. space, we have the following:

Theorem 3.3 (cf. [36])**.** *Let X be an r.i. space over Ω. Then the following are equivalent*:

(i) *If $f = (f_n) \in \mathcal{M}$ is bounded in X, then*

$$\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0.$$

(ii) *X has absolutely continuous norm and $t/\varphi_X(t) \to 0$ as $t \to 0+$.*

(iii) *X has absolutely continuous norm and $X \neq L_1$.*

If X satisfies the conditions above, then every martingale $f = (f_n)$ which is bounded in X is uniformly integrable.

In the theorem above, we notice that if $f = (f_n)$ is bounded in X, then it converges a.s., because such a martingale is bounded in L_1.

The equivalence of (i) and (ii) in Theorem 3.3 was proved by the author in [36]. When the equivalence of (i) and (ii) was proved, the author was not aware that (ii) and (iii) are equivalent. We prove here the equivalence of (ii) and (iii).

First recall that $\varphi_X(t)\varphi_{X'}(t) = t$ for all $t \in [0, 1]$ (see [6, Theorem 5.2, p. 66]), where X' denotes the associate space of X. This implies that $t/\varphi_X(t) \to 0+$ as $t \to 0+$ if and only if $\varphi_{X'}(t) \to 0+$ as $t \to 0+$. Thus it suffices to show that $\varphi_{X'}(t) \to 0+$ as $t \to 0+$ if and only if $X \neq L_1$. Moreover, since the condition $X \neq L_1$ can be rewritten as $X' \neq L_\infty$, it suffices to show that $\varphi_{X'}(t) \to 0+$ as $t \to 0+$ if and only if $X' \neq L_\infty$. Note that if $X' = L_\infty$, then

$$\varphi_{X'}(t) = \mathbf{1}_{(0,1]}(t) \to 1 \quad \text{as } t \to 0+.$$

Thus we need only show that if $X' \neq L_\infty$, then $\varphi_{X'}(t) \to 0+$ as $t \to 0+$. Suppose that $X' \neq L_\infty$; that is, $L_\infty \subsetneq X'$. Let $x \in X' \setminus L_\infty$ be nonnegative, and let $\varepsilon > 0$.

[2]Whether an Orlicz space has absolutely continuous norm does not depend on the underlying probability space.

Choose $\lambda > 0$ so that $\|x\|_{X'}/\lambda < \varepsilon$ and let $\delta = \mathbb{P}(\omega \in \Omega\colon x(\omega) > \lambda)$. Since $x \notin L_\infty$, we have that $\delta > 0$. For $t \in (0, \delta)$, choose $A \in \Sigma$ so that $\mathbb{P}(A) = t$. Then

$$\mathbf{1}_A^* = \mathbf{1}_{(0,t)} \leq \mathbf{1}_{(0,\delta)} = \mathbf{1}^*_{\{\omega\in\Omega\colon x(\omega)>\lambda\}}.$$

It follows that for $t \in (0, \delta)$,

$$\begin{aligned}\varphi_{X'}(t) = \|\mathbf{1}_A\|_{X'} = \|\mathbf{1}_A^*\|_{\hat{X}'} &\leq \|\mathbf{1}^*_{\{\omega\in\Omega\colon x(\omega)>\lambda\}}\|_{\hat{X}'} \\ &= \|\mathbf{1}_{\{\omega\in\Omega\colon x(\omega)>\lambda\}}\|_{X'} \leq \frac{\|x\|_{X'}}{\lambda} < \varepsilon.\end{aligned}$$

This shows that $\varphi_{X'}(t) \to 0$ as $t \to 0+$, as was to be shown.

Using Theorem 3.3, one can prove the following:

Corollary 3.3 ([36]). *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a Young function. Then the following are equivalent*:

(i) *If $f = (f_n) \in \mathcal{M}$ is bounded in L_Φ, then*

$$\lim_{n\to\infty} \|f_n - f_\infty\|_{L_\Phi} = 0.$$

(ii) *Φ satisfies the Δ_2^∞-condition and $\Phi(t)/t \to 0$ as $t \to 0+$.*

Note that if a Banach function space X is reflexive, then X has absolutely continuous norm (see [6, p. 23]) and $X \subsetneq L_1$. This fact, together with Theorem 3.3, gives the following:

Corollary 3.4 ([36]). *Let X be a reflexive r.i. space over Ω. If $f = (f_n) \in \mathcal{M}$ is bounded in X, then*

$$\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0.$$

We now return to annotation of Theorem 3.3. It is easily understood that Theorem 3.3 is derived from Theorem 3.2; the following proposition is key to the proof of Theorem 3.3.

Proposition 3.1 (cf. [36]). *Let X be an r.i. space over Ω. Then the following are equivalent*:

(i) *If $\mathcal{H} \subset X$ and $\sup_{x\in\mathcal{H}} \|x\|_X < \infty$, then $\mathcal{H}$ is uniformly integrable.*
(ii) *If $f = (f_n) \in \mathcal{M}$ is bounded in X, then $f = (f_n)$ is uniformly integrable.*
(iii) *$X \neq L_1$.*

Recall that the theorem of de la Vallée Poussin asserts that if $\Phi(t)/t \to \infty$ as $t \to \infty$ and if $\mathcal{H}$ is a family of random variables which is bounded in L_Φ, then $\mathcal{H}$ is uniformly integrable (see [16, Théorème 22, p. 38]). Proposition 3.1 extends the theorem of de la Vallée Poussin.

4. Summability methods and Tauberian theorems

In this section, we consider summability methods and Tauberian theorems for martingales. Let $\{a_n\}_{n\in\mathbb{Z}_+}$ be a sequence of positive numbers such that

$$A_n := \sum_{k=0}^{n} a_k \to \infty \quad \text{as } n \to \infty.$$

For a process (that is, a sequence of random variables) $\eta = (\eta_n)_{n\in\mathbb{Z}_+}$, the *weighted average* $\sigma(\eta) = (\sigma_n(\eta))_{n\in\mathbb{Z}_+}$ is defined by

$$\sigma_n(\eta) = \frac{1}{A_n}\sum_{k=0}^{n} a_k \eta_k, \quad n \in \mathbb{Z}_+.$$

It is clear that if $\eta = (\eta_n)$ converges to η_∞ in L_p, then $\sigma(\eta) = (\sigma_n(\eta))$ converges to the same limit in L_p. However, $\eta = (\eta_n)$ does not necessarily converge in L_p when $\sigma(\eta) = (\sigma_n(\eta))$ converges in L_p. By using the result of Kazamaki and Tsuchikura [31], we can deduce a Tauberian theorem for this summation method. The main result of [31] asserts that a martingale $f = (f_n)$ is bounded in L_p if and only if $\sigma(f) = (\sigma_n(f))$ is bounded in L_p; that is,

$$\sup_{n\in\mathbb{Z}_+} \|\sigma_n(f)\|_{L_p} < \infty \iff \sup_{n\in\mathbb{Z}_+} \|f_n\|_{L_p} < \infty. \tag{4.1}$$

It follows that if $1 < p < \infty$ and if $\sigma(f) = (\sigma_n(f))$ converges in L_p, then $f = (f_n)$ is bounded in L_p and converges in L_p.

There is an analogous theorem for a Banach function space X (see Theorem 4.2 below). It is a consequence of the following theorem, which extends (4.1).

Theorem 4.1 ([37])**.** *Let X be a Banach function space over Ω, let $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$, and let $\mathbb{E}_n$ denote the conditional expectation operator $\mathbb{E}[\,\cdot\,|\mathcal{F}_n]$ for each $n \in \mathbb{Z}_+$. Suppose that $\mathbb{E}_n \in B(X)$ for all $n \in \mathbb{Z}_+$ and that $C := \sup_{n\in\mathbb{Z}_+} \|\mathbb{E}_n\|_{B(X)} < \infty$. Then*

$$\sup_{n\in\mathbb{Z}_+} \|\sigma_n(f)\|_X \le \sup_{n\in\mathbb{Z}_+} \|f_n\|_X \le C \sup_{n\in\mathbb{Z}_+} \|\sigma_n(f)\|_X \tag{4.2}$$

for all $f = (f_n) \in \mathcal{M}(\mathcal{F})$.

If X is an r.i. space, then $\|\mathbb{E}_n\|_{B(X)} = 1$ for all $n \in \mathbb{Z}_+$, without depending on $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$ (see [36, Lemma 2]). Hence (4.2) can be rewritten as

$$\sup_{n\in\mathbb{Z}_+} \|\sigma_n(f)\|_X = \sup_{n\in\mathbb{Z}_+} \|f_n\|_X .$$

From Theorem 4.1 one can deduce the following:

Theorem 4.2 ([37])**.** *Let X, $\mathcal{F} = (\mathcal{F}_n)$, and $\mathbb{E}_n$ be as in Theorem* 4.1*. Suppose that $\mathbb{E}_n \in B(X)$ for all $n \in \mathbb{Z}_+$ and $\sup_{n\in\mathbb{Z}_+} \|\mathbb{E}_n\|_{B(X)} < \infty$. Then the following conditions on $f = (f_n) \in \mathcal{M}(\mathcal{F})$ are equivalent*:

(i) *$f_n \in X$ for all $n \in \mathbb{Z}_+$, $f_\infty \in X$, and $\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0$.*
(ii) *$\sigma_n(f) \in X$ for all $n \in \mathbb{Z}_+$, $f_\infty \in X$, and $\lim_{n\to\infty} \|\sigma_n(f) - f_\infty\|_X = 0$.*

Combining Theorems 3.1 and 4.2 gives the following:

Theorem 4.3 ([37])**.** *Let X, $\mathcal{F} = (\mathcal{F}_n)$, and $\mathbb{E}_n$ be as in Theorem* 4.1*. Suppose that $\mathbb{E}_n \in B(X)$ for all $n \in \mathbb{Z}_+$ and $\sup_{n\in\mathbb{Z}_+} \|\mathbb{E}_n\|_{B(X)} < \infty$. If X has absolutely continuous norm, then the following conditions on $f = (f_n) \in \mathcal{M}(\mathcal{F})$ are equivalent*:

(i) *$f = (f_n)$ is uniformly integrable and $f_\infty \in X$.*
(ii) *$f_n \in X$ for all $n \in \mathbb{Z}_+$, $f_\infty \in X$, and $\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0$.*
(iii) *$\sigma_n(f) \in X$ for all $n \in \mathbb{Z}_+$, $f_\infty \in X$, and $\lim_{n\to\infty} \|\sigma_n(f) - f_\infty\|_X = 0$.*

We now consider a matrix summability method. Let $(a_{mn})_{m,n=0}^{\infty}$ be an infinite (real) matrix satisfying the following conditions:

- There is a positive constant K such that $\sum_{n=0}^{\infty} |a_{mn}| \leq K$ for all $m \in \mathbb{Z}_+$.
- $\lim_{m\to\infty} a_{mn} = 0$ for each $n \in \mathbb{Z}_+$.
- $\lim_{m\to\infty} \sum_{n=0}^{\infty} a_{mn} = 1$.

Let $\{b_n\}_{n\in\mathbb{Z}_+}$ be a sequence of real numbers such that the series $\sum_{n=0}^{\infty} a_{mn}b_n$ converges for all $m \in \mathbb{Z}_+$. Then we define a sequence $\{Tb_m\}_{m\in\mathbb{Z}_+}$ by letting

$$Tb_m = \sum_{n=0}^{\infty} a_{mn}b_n, \quad m \in \mathbb{Z}_+.$$

If $\{b_n\}$ converges to a real number b, then $\{Tb_m\}$ converges to b (cf. [3], [73, p. 74]). However, $\{b_n\}$ may not converge even if $\{Tb_m\}$ converges.

Fix a matrix $(a_{mn})_{m,n=0}^{\infty}$ satisfying the conditions above. For $f = (f_n) \in \mathcal{M}$, we define a process $Tf = (Tf_m)_{m\in\mathbb{Z}_+}$ by

$$Tf_m = \sum_{n=0}^{\infty} a_{mn}f_n, \quad m \in \mathbb{Z}_+, \tag{4.3}$$

provided the sum on the right-hand side converges a.s. for all $m \in \mathbb{Z}_+$. It is clear that if $f = (f_n) \in \mathcal{M}$ is bounded in L_1, then we can define $Tf = (Tf_m)$. More generally, we have the following:

Theorem 4.4 ([36]). *Let X be an r.i. space over Ω. If $f = (f_n) \in \mathcal{M}$ is bounded in L_1, then the following are equivalent*:

(i) $Tf_m \in X$ *for all* $m \in \mathbb{Z}_+$ *and* $\sup_{m\in\mathbb{Z}_+} \|Tf_m\|_X < \infty$.
(ii) $f_n \in X$ *for all* $n \in \mathbb{Z}_+$ *and* $\sup_{n\in\mathbb{Z}_+} \|f_n\|_X < \infty$.

When these equivalent conditions hold, the series in (4.3) *converges in X.*

Suppose that $Tf = (Tf_m)$ converges in X. Then, by Theorem 4.4, $f = (f_n)$ is bounded in X. It follows from Theorem 3.3 that if X has absolutely continuous norm and $X \neq L_1$, then $f = (f_n)$ converges in X. Thus we have the following:

Theorem 4.5 (cf. [36]). *Let X be an r.i. space over Ω, and let $f = (f_n) \in \mathcal{M}$ be bounded in L_1. If X has absolutely continuous norm and if $X \neq L_1$, then the following are equivalent*:

(i) $Tf_m \in X$ *for all* $m \in \mathbb{Z}_+$, $f_\infty \in X$, *and* $\lim_{m\to\infty} \|Tf_m - f_\infty\|_X = 0$.
(ii) $f_n \in X$ *for all* $n \in \mathbb{Z}_+$, $f_\infty \in X$, *and* $\lim_{n\to\infty} \|f_n - f_\infty\|_X = 0$.

When these equivalent conditions hold, the series in (4.3) *converges in X.*

5. Martingale inequalities in r.i. spaces

In this section, we give an overview of results on martingale inequalities in an r.i. space. For the classical martingale inequality mentioned in this section, see, e.g., [11], [17], [23], and [57].

First we consider the Doob inequality and the Burkholder-Davis-Gundy inequality. Recall that the Doob inequality (for uniformly integrable martingales) can be written as

$$\|Mf\|_{L_p} \leq C_p \|f_\infty\|_{L_p}, \quad 1 < p \leq \infty, \tag{5.1}$$

where C_p is a constant depending only on p. Recall also that the Burkholder-Davis-Gundy inequality (for arbitrary martingales) can be written as

$$c_p \|Mf\|_{L_p} \le \|Sf\|_{L_p} \le C_p \|Mf\|_{L_p}, \quad 1 \le p < \infty, \tag{5.2}$$

where c_p and C_p are constants depending only on p.

A Doob type inequality and a Burkholder-Davis-Gundy type inequality in a general function space were first studied by Antipa [1], Johnson-Schechtman [26], and Novikov [65] (cf. [66]). They proved the following result.

Theorem A ([1], [65]; cf. [66]). *Let Z be an r.i. space over $I = (0, 1]$. Then there is a positive constant C_Z such that for every uniformly integrable martingale $f = (f_n)$ over I,*

$$\|Mf\|_Z \le C_Z \|f_\infty\|_Z,$$

if and only if $\beta_Z < 1$.

Theorem B ([1], [26], [65]; cf. [66]). *Let Z be an r.i. space over $I = (0, 1]$. Then there are positive constants c_Z and C_Z such that for every martingale $f = (f_n)$ over I,*

$$c_Z \|Mf\|_Z \le \|Sf\|_Z \le C_Z \|Mf\|_Z,$$

if and only if $\alpha_Z > 0$.

These theorems are proved by the four authors independently in the three papers [1], [26], and [65]. The proofs in the three papers are different from one another. The author of this article, Kikuchi, gave in [35] yet another proof of Theorems A and B, which is also valid for an r.i. space over Ω. Moreover, the argument in [35] revealed the close connection between the validity of some martingale inequalities and Shimogaki's theorem which concerns the boundedness of the linear operators $\mathcal{P}$ and $\mathcal{Q}$. Here the operators $\mathcal{P}$ and $\mathcal{Q}$ are defined for $\phi \in L_0(I)$ by

$$(\mathcal{P}\phi)(t) = \frac{1}{t}\int_0^t \phi(s)\,ds, \quad t \in I,$$
$$(\mathcal{Q}\phi)(t) = \int_t^1 \frac{\phi(s)}{s}\,ds, \quad t \in I,$$

provided the integrals exist for all $t \in I$. Thus the domain of $\mathcal{P}$ (resp., $\mathcal{Q}$) is the set of all measurable functions which are integrable over $(0, t)$ (resp., $(t, 1)$) for each $t \in I$. These operators are (formal) adjoint of each other.

Shimogaki [69] gave a characterization of an r.i. space X for which $\mathcal{P} \in B(\hat{X})$, and a characterization of an r.i. space X for which $\mathcal{Q} \in B(\hat{X})$. His result is stated, in terms of the Boyd indices, as follows.

Theorem C ([69]). *Let X be an r.i. space over Ω. Then $\mathcal{P} \in B(\hat{X})$ if and only if $\beta_X < 1$, and $\mathcal{Q} \in B(\hat{X})$ if and only if $\alpha_X > 0$.*

Under the setting of this article, the author's result [35][3] can be stated as follows.

Theorem 5.1 (cf. [35]). *Let X be an r.i. space over Ω. Then*:

[3]In [35] continuous parameter martingales were considered. Although the results in [35] hold for discrete parameter martingales, the proofs must be modified.

(i) $\mathcal{P} \in B(\hat{X})$ *if and only if there is a positive constant* C_X *such that*

$$\|Mf\|_X \le C_X \|f_\infty\|_X$$

for every uniformly integrable martingale $f = (f_n)$.

(ii) $\mathcal{Q} \in B(\hat{X})$ *if and only if there are positive constants* c_X *and* C_X *such that*

$$c_X \|Mf\|_X \le \|Sf\|_X \le C_X \|Mf\|_X$$

for every martingale $f = (f_n)$.

Theorem 5.1 extends Theorems A and B. In fact, in the statement, the probability space I is replaced by the general nonatomic probability space Ω, However, this extension is not so important. We emphasize that Theorem 5.1 implies that the combination of Theorems A and B is equivalent to Theorem C. That is, by using Theorem 5.1, one can derive Theorem C from Theorems A and B, and one can derive both of Theorems A and B from Theorem C.

Now we consider martingale inequalities in a Banach function space defined by changing the probability measure. As before, let w be a positive random variable such that $\mathbb{E}[\mathrm{w}] = 1$ and define a probability measure $\mathbb{P}_{\mathrm{w}}$ by $\mathbb{P}_{\mathrm{w}}(A) = \int_A \mathrm{w}\, d\mathbb{P}$. For each $x \in L_0(\Omega)$, we let x^*_{w} denote the nonincreasing rearrangement of x with respect to $\mathbb{P}_{\mathrm{w}}$, that is,

$$x^*_{\mathrm{w}}(t) = \inf\bigl\{\lambda > 0 \colon \mathbb{P}_{\mathrm{w}}(\omega \in \Omega \colon\ |x(\omega)| > \lambda) \le t\bigr\}, \quad t \in I.$$

Let X be an r.i. space over $(\Omega, \Sigma, \mathbb{P})$. For each $x \in L_0$, define $X(\mathbb{P}_{\mathrm{w}})$ by

$$X(\mathbb{P}_{\mathrm{w}}) = \{x \in L_0 \colon x^*_{\mathrm{w}} \in \hat{X}\},$$

and let

$$\|x\|_{X(\mathbb{P}_{\mathrm{w}})} = \|x^*_{\mathrm{w}}\|_{\hat{X}}$$

for $x \in X(\mathbb{P}_{\mathrm{w}})$. Then $X(\mathbb{P}_{\mathrm{w}})$ is an r.i. space over $(\Omega, \Sigma, \mathbb{P}_{\mathrm{w}})$. Recall that the validity of Doob type inequalities in X is equivalent to the boundedness of the operator $\mathcal{P}$ on $\hat{X}$. As for Doob type inequalities in $X(\mathbb{P}_{\mathrm{w}})$, there is a close connection between the validity of such inequalities in $X(\mathbb{P}_{\mathrm{w}})$ and the boundedness of the operators $\mathcal{P}_p$ $(1 \le p < \infty)$ defined by

$$(\mathcal{P}_p\phi)(t) = \frac{1}{t^{1/p}} \int_0^t \phi(s) s^{1/p} \frac{ds}{s}, \qquad t \in I. \tag{5.3}$$

Boyd [8] proved the following theorem on the boundedness of the operators $\mathcal{P}_p$.

Theorem D ([8])**.** *Let* X *be an r.i. space over* Ω*, and let* $1 \le p < \infty$*. Then* $\mathcal{P}_p \in B(\hat{X})$ *if and only if* $\beta_X < 1/p$.

For a proof of the theorem above, see [8] or [6, pp. 150–153]. Using this theorem, we can prove the following:

Theorem 5.2 ([35])**.** *Let* X *be an r.i. space over* Ω*, let* w *and* $X(\mathbb{P}_{\mathrm{w}})$ *be as above, let* $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$*, and let* $1 < p < \infty$.

(i) *Suppose that* w *satisfies condition* $A_p(\mathcal{F})$ *and that* $\beta_X < 1/p$*. Then there is a positive constant* C *such that*

$$\|Mf\|_{X(P_{\mathrm{w}})} \le C \|f_\infty\|_{X(P_{\mathrm{w}})} \tag{5.4}$$

for every uniformly integrable $\mathcal{F}$*-martingale* $f = (f_n)$.

(ii) *Suppose that* w *satisfies condition* $A_p(\mathcal{F})$. *If* (5.4) *holds for every uniformly integrable* $\mathcal{F}$*-martingale* $f = (f_n)$, *then* $\beta_X \le 1/p$.

On the one hand, Theorem 5.2 is derived from Theorem D; and, on the other hand, by using the statement of Theorem 5.2, we can prove Theorem D (see [35, Theorem 7]).

In addition to Doob type inequalities and Burkholder-Davis-Gundy type inequalities, we can obtain a lot of martingale inequalities in an r.i. space. To see this, we need some preliminaries. First, define operators $\mathcal{Q}_p$ $(1 < p \le \infty)$ and $\mathcal{R}_p$ $(1 \le p < \infty)$ by:

$$(\mathcal{Q}_p\phi)(t) = \frac{1}{t^{1/p}} \int_t^1 \phi(s) s^{1/p} \,\frac{ds}{s}, \qquad t \in I;$$
$$(\mathcal{R}_p\phi)(t) = \int_0^1 \frac{\phi(s) s^{1/p}}{t^{1/p} + s^{1/p}} \,\frac{ds}{s}, \qquad t \in I.$$

Of course, we let $1/p = 0$ when $p = \infty$ (and hence, $\mathcal{Q}_\infty = \mathcal{Q}$). It is easily checked that if ϕ is a nonnegative measurable function on I, then

$$\frac{1}{2}\big[(\mathcal{P}_p\phi)(t) + (\mathcal{Q}\phi)(t)\big] \le (\mathcal{R}_p\phi)(t) \le (\mathcal{P}_p\phi)(t) + (\mathcal{Q}\phi)(t)$$

for all $t \in I$. We have the following theorem concerning the boundedness of $\mathcal{Q}_p$ and $\mathcal{R}_p$, which is analogous to Theorem D.

Theorem D′ (cf. [8]). *Let* X *be an r.i. space over* Ω.

(i) *Suppose that* $1 < p \le \infty$. *Then* $\mathcal{Q}_p \in B(\hat{X})$ *if and only if* $1/p < \alpha_X$.

(ii) *Suppose that* $1 \le p < \infty$. *Then* $\mathcal{R}_p \in B(\hat{X})$ *if and only if* $0 < \alpha_X$ *and* $\beta_X < 1/p$.

Let X be an r.i. space over Ω and let $1 \le p < \infty$. We define function spaces $H_p(X)$, $\mathcal{H}_p(X)$, and $K(X)$ by:

$$H_p(X) = \{x \in L_0 \colon \mathcal{P}_p x^* \in \hat{X}\}, \qquad \|x\|_{H_p(X)} = \|\mathcal{P}_p x^*\|_{\hat{X}}\,,$$
$$\mathcal{H}_p(X) = \{x \in L_0 \colon \mathcal{R}_p x^* \in \hat{X}\}, \qquad \|x\|_{\mathcal{H}_p(X)} = \|\mathcal{R}_p x^*\|_{\hat{X}}\,,$$
$$K(X) = \{x \in L_0 \colon \mathcal{Q} x^* \in \hat{X}\}, \qquad \|x\|_{K(X)} = \|\mathcal{Q} x^*\|_{\hat{X}}\,.$$

One can show that $H_p(X)$ is an r.i. space and $H_p(X) \hookrightarrow X$, and that if the function $I \ni t \mapsto -\log t \in [0,\infty)$ belongs to $\hat{X}$, then both $\mathcal{H}_p(X)$ and $K(X)$ are r.i. spaces and $\mathcal{H}_p(X) \hookrightarrow K(X) \hookrightarrow X$ (see [42, Lemma 2.3]). On the other hand, if the function $-\log t$ does not belong to $\hat{X}$, then $\mathcal{H}_p(X) = K(X) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the constant function with value 0. From Theorem D′, one sees that if $\beta_X < 1/p$, then $H_p(X) = X$, and that if $0 < \alpha_X$ and $\beta_X < 1/p$, then $\mathcal{H}_p(X) = X$.

With the notation above, we have the following:

Theorem 5.3 ([42]). *Let* X *be an r.i. space over* Ω, *let* $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$, *let* γ *be a nonnegative random variable, let* $\xi = (\xi_n)$ *be a nondecreasing process* (*that is,* $\xi_n \le \xi_{n+1}$ *a.s. for all* $n \in \mathbb{Z}_+$), *and let* $\xi_\infty = \sup_{n\in\mathbb{Z}_+} \xi_n$.

(i) *If* $1 < p < \infty$ *and if the inequality*

$$\mathbb{E}[(\xi_\infty - \xi_{n-1})^p \,|\, \mathcal{F}_n] \le \mathbb{E}[\gamma^p \,|\, \mathcal{F}_n] \tag{5.5}$$

holds a.s. for all $n \in \mathbb{Z}_+$, *then*

$$\|\xi_\infty\|_X \le \|\xi_\infty\|_{H_1(X)} \le 2\,\|\gamma\|_{\mathcal{H}_p(X)}\,. \tag{5.6}$$

(ii) *If* $p = 1$ *and if* (5.5) *holds for all* $n \in \mathbb{Z}_+$, *then*

$$\|\xi_\infty\|_X \le \|\gamma\|_{K(X)}\,. \tag{5.7}$$

Moreover, if $0 < \alpha_X$ *and* $\beta_X < 1/p$, *then* (5.6) *implies that* $\|\xi_\infty\|_X \le C\,\|\gamma\|_X$, *and if* $0 < \alpha_X$, *then* (5.7) *implies that* $\|\xi_\infty\|_X \le C\,\|\gamma\|_X$.

Various martingale inequalities are derived from Theorem 5.3. Let $1 \le p < \infty$ and let $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$. Given $f = (f_n) \in \mathcal{M}(\mathcal{F})$, we let[4]

$$s^{(p)} f = \left\{ \sum_{n=0}^{\infty} \mathbb{E}\big[|\Delta_n f|^p \,\big|\mathcal{F}_{n-1}\big] \right\}^{1/p},$$

and

$$m^{(p)} f = \sup_{0 \le n \le n' < \infty} \mathbb{E}\big[|f_{n'}|^p \,\big|\mathcal{F}_n\big]^{1/p}.$$

An important case is when $p = 2$; there are many results on $s^{(2)} f$. We call $s^{(2)} f$ the *conditioned square function* of $f = (f_n)$. In this article, we write sf for $s^{(2)} f$.

With the notation above, we have the following:

Theorem 5.4 ([42])**.** *Let* X *be an r.i. space over* Ω.

(i) *If* $2 \le p < \infty$, *then there is a positive constant* C_p, *depending only on* p, *such that*

$$\big\|s^{(p)} f\big\|_X \le C_p\,\|f_\infty\|_{\mathcal{H}_p(X)} \tag{5.8}$$

for every uniformly integrable martingale $f = (f_n)$.

If, in addition, $0 < \alpha_X$ *and* $\beta_X < 1/p$, *then the right-hand side of* (5.8) *can be replaced by* $C_{p,X}\,\|f_\infty\|_X$.

(ii) *If* $1 \le p < q < \infty$, *then there is a positive constant* $C_{p,q}$, *depending only on* p *and* q, *such that*

$$\big\|m^{(q)} f\big\|_X \le C_{p,q}\,\|f_\infty\|_{\mathcal{H}_p(X)} \tag{5.9}$$

for every uniformly integrable martingale $f = (f_n)$.

If, in addition, $0 < \alpha_X$ *and* $\beta_X < 1/p$, *then the right-hand side of* (5.9) *can be replaced by* $C_{p,q,X}\,\|f_\infty\|_X$.

If we define an increasing process $\xi = (\xi_n)$ by $\xi_n = (\sum_{k=0}^{n+1} \mathbb{E}[|\Delta_k f|^p\,|\mathcal{F}_{k-1}])^{1/p}$ and a random variable γ by $\gamma = c_p\,|f_\infty|$ with a suitable constant c_p, we can prove that (5.5) holds. This, together with Theorem 5.3, implies that (5.8) holds for every uniformly integrable martingale $f = (f_n)$. In the same way, we can prove (5.9) by using Theorem 5.3.

For some specific r.i. spaces X, we can give an explicit description of the spaces $H_p(X)$, $\mathcal{H}_p(X)$, and $K(X)$. For example, if $1 \le p < \infty$ and $0 \le a < \infty$, then

$$H_p(L_{p,1}(\log L)_a) = \mathcal{H}_p(L_{p,1}(\log L)_a) = L_{p,1}(\log L)_{a+1}, \tag{5.10}$$ [5]

where $L_{p,q}(\log L)_a$ ($1 \le p < \infty$, $1 \le q < \infty$, $-\infty < a < \infty$) denotes the *Lorentz-Zygmund space* consisting of those $x \in L_0$ for which

$$\|x\|_{L_{p,q}(\log L)_a} := \left\{ \int_0^1 \big[t^{1/p}(1 - \log t)^a x^*(t)\big]^q \,\frac{dt}{t} \right\}^{1/q} < \infty.$$

[4]We adopt the convention that $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ for any $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$.

[5]Equality (5.10) means that the spaces in this equality are the same as a set, and their norms are equivalent.

For details of Lorentz-Zygmund spaces, see [5]. From (5.8), (5.9), and (5.10), it follows that

$$\begin{aligned}\left\|s^{(p)}f\right\|_{L_{p,1}(\log L)_a} &\le C_{p,a}\left\|f_\infty\right\|_{L_{p,1}(\log L)_{a+1}}; \\ \left\|m^{(q)}f\right\|_{L_{p,1}(\log L)_a} &\le C_{p,q,a}\left\|f_\infty\right\|_{L_{p,1}(\log L)_{a+1}}.\end{aligned}$$

There are some other examples. Given $a \in (0,\infty)$ and $x \in L_0$, we let

$$\|x\|_{L_{\exp:a}} = \sup_{t\in I}\frac{1}{t(1-\log t)^{1/a}}\int_0^t x^*(s)\,ds = \sup_{t\in I}\frac{(\mathcal{P}x^*)(t)}{(1-\log t)^{1/a}}$$

and we let $L_{\exp:a}$ denote the set of all $x \in L_0$ such that $\|x\|_{L_{\exp:a}} < \infty$. One can show that $L_{\exp:a}$ is an r.i. space, and that $x \in L_{\exp:a}$ if and only if $\exp(\lambda\,|x|^a) \in L_1$ for some $\lambda > 0$. It is not hard to check that

$$K(L_{\exp:1}) = \mathcal{H}_p(L_{\exp:1}) = L_\infty, \qquad 1 \le p < \infty; \tag{5.11}$$

$$K(L_{\exp:a}) = \mathcal{H}_p(L_{\exp:a}) = L_{\exp:(a/(1-a))}, \qquad 1 \le p < \infty,\ \ 0 < a < 1. \tag{5.12}$$

For details of these examples, see [42]. From (5.11), (5.12), and Theorem 5.4, it follows that

$$\begin{aligned}\left\|s^{(p)}f\right\|_{L_{\exp:a}} &\le C_{p,a}\left\|f_\infty\right\|_{L_{\exp:(a/(1-a))}}, && 1 \le p < \infty,\ \ 0 < a < 1; \\ \left\|m^{(q)}f\right\|_{L_{\exp:1}} &\le C_q\left\|f_\infty\right\|_{L_\infty}, && 1 \le q < \infty.\end{aligned}$$

Moreover, by using the conditional form of the Davis inequality (see [14]) and Theorem 5.3, we have inequalities such as:

$$\begin{aligned}\|Mf\|_{L_{\exp:a}} &\le C\,\|Sf\|_{L_{\exp:(a/(1-a))}}, && 0 < a < 1; \\ \|Sf\|_{L_{\exp:1}} &\le C\,\|Mf\|_{L_\infty}.\end{aligned}$$

6. Martingale inequalities in Banach function spaces

In the previous section, we focused on martingale inequalities in r.i. space. More generally, we consider several inequalities in Banach function spaces in this section. From the results in the previous section, we see that the validity of some martingale inequalities in r.i. spaces does not depend on filtrations. In contrast, martingale inequalities in a Banach function space which is not r.i., such as $L_p(\mathbb{P}_{\mathrm{w}})$, strongly depend on filtrations. Thus, it is reasonable to expect that if a martingale inequality in a Banach function space X holds without depending on filtrations, then X is r.i. In fact, this is true in many cases.

We begin with Doob type inequalities. Let X be a Banach function space over Ω. We say that X *can be renormed so as to be r.i.*, if there exists a norm $|||\cdot|||_X$ on X which is equivalent to the original norm $\|\cdot\|_X$ and with respect to which X is an r.i. space. Let X be such a Banach function space, and suppose that $\beta_X < 1$, where β_X stands for the upper Boyd index of the renormed space X. It then follows from Theorems 5.1 and C that the inequality

$$|||Mf|||_X \le C_X\,|||f_\infty|||_X$$

holds for every uniformly integrable martingale $f = (f_n)$. Since the norms $\|\cdot\|_X$ and $|||\cdot|||_X$ are equivalent, the inequality above can be rewritten as

$$\|Mf\|_X \le C_X\,\|f_\infty\|_X\,. \tag{6.1}$$

The following theorem gives a complete characterization of a Banach function space X for which (6.1) holds for every uniformly integrable martingale $f = (f_n)$.

Theorem 6.1 (cf. [32][6]). *Let X be a Banach function space over Ω. Then the following are equivalent*:

(i) *There is a positive constant C_X, depending only on X, such that* (6.1) *holds for every uniformly integrable martingale $f = (f_n)$.*

(ii) *X can be renormed so as to be r.i. and $\beta_X < 1$.*

In the theorem above, the norm $\||\cdot\||_X$ with respect to which X is an r.i. space is given by

$$\||x\||_X = \sup\left\{\int_0^1 x^*(s)\, y^*(s)\, ds \colon y \in X',\ \|y\|_{X'} \le 1\right\}.$$

For details, see the proof of [45, Lemma 5.1]. To prove Theorem 6.1, one needs to show that if (6.1) holds for every uniformly integrable martingale $f = (f_n)$, then X can be renormed so as to be r.i. To do so, we can use the following result, which was proved in a later study.

Proposition 6.1 ([43]). *Let X be a Banach function space over Ω, and let $\mathbb{S}_+$ be the set of all nonnegative simple random variables on Ω. Then the following are equivalent*:

(i) *There is a positive constant C such that if $x \in X$ and if $\mathcal{A}$ is a sub-σ-algebra of Σ, then*
$$\left\|\mathbb{E}[x\,|\mathcal{A}]\right\|_X \le C\,\|x\|_X\,.$$

(ii) *There is a positive constant C such that if $x, y \in \mathbb{S}_+$, if $x \simeq_d y$, and if $\{\omega \in \Omega\colon x(\omega) > 0\} \cap \{\omega \in \Omega\colon y(\omega) > 0\} = \emptyset$ a.s., then*
$$\|x\|_X \le C\,\|y\|_X\,.$$

(iii) *X can be renormed so as to be r.i.*

Let $\mathcal{A}$ be a sub-σ-algebra of Σ. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$ by

$$\mathcal{F}_n = \begin{cases} \mathcal{A}, & n = 0, \\ \Sigma, & n \ge 1, \end{cases} \tag{6.2}$$

and define $f = (f_n) \in \mathcal{M}(\mathcal{F})$ by $f_n = \mathbb{E}[x\,|\mathcal{F}_n]\,,\ n \in \mathbb{Z}_+$. Then $\mathbb{E}[x\,|\mathcal{A}] \le Mf$ a.s., and $f_\infty = x$ a.s. Applying (6.1) to this $f = (f_n)$, we see that the inequality in (i) of Proposition 6.1 holds. Thus we conclude that if (6.1) holds for every uniformly integrable martingale $f = (f_n)$, then X can be renormed so as to be r.i.

As for Burkholder-Davis-Gundy type inequalities, we have the following analogue of Theorem 6.1.

Theorem 6.2 ([45]). *Let X be a Banach function space over Ω. Then the following are equivalent*:

(i) *There is a positive constant C_X, depending only on X, such that*
$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \le C_X\,\|Sf\|_X \tag{6.3}$$
for every martingale $f = (f_n)$.

[6] In some papers by the author, a Banach function space which can be renormed so as to be r.i. is merely said to be r.i.

(ii) *There is a positive constant C_X, depending only on X, such that*

$$\|Mf\|_X \le C_X \|Sf\|_X$$

for every martingale $f = (f_n)$.

(iii) *X can be renormed so as to be r.i. and $\alpha_X > 0$.*

In order to prove Theorem 6.2, as in the proof of Theorem 6.1, it is essential to show that if (6.3) holds for every martingale $f = (f_n)$, then X can be renormed so as to be r.i. However, the proof of this fact is somewhat complicated. As we have seen above, it is easy to derive (i) of Proposition 6.1 from (6.1). However, the author does not know how to derive (i) of Proposition 6.1 directly from (6.3). Nevertheless, one can derive (ii) of Proposition 6.1 from (6.3). The author found two ways to do so; a considerable effort is forced on both ways (cf. [45]).

We now consider Burkholder type inequalities. In 1966, Burkholder proved in his excellent paper [10] that if $1 < p < \infty$, then there are positive constants k_p and K_p, depending only on p, such that

$$k_p \|f_\infty\|_{L_p} \le \|Sf\|_{L_p} \le K_p \|f_\infty\|_{L_p}$$

for every uniformly integrable martingale $f = (f_n)$.

In view of Theorems 6.1 and 6.2, and the result mentioned above, the following theorem is just the expected one.

Theorem 6.3 ([38])**.** *Let X be a Banach function space over Ω. Then the following are equivalent*:

(i) *There are positive constants c_X and C_X, depending only on X, such that*

$$c_X \|f_\infty\|_X \le \|Sf\|_X \le C_X \|f_\infty\|_X \tag{6.4}$$

for every uniformly integrable martingale $f = (f_n)$.

(ii) *X can be renormed so as to be r.i., and $0 < \alpha_X$ and $\beta_X < 1$.*

Using Theorems 6.1 and 6.2, one easily sees that (ii) implies (i). Moreover, one easily sees also that if (i) holds, then X can be renormed so as to be r.i. However, a somewhat complicated calculation is needed to show that if X can be renormed so as to be r.i. and if (i) holds, then $0 < \alpha_X$ and $\beta_X < 1$. For details, see [38].

As is well known, various martingale inequalities have been studied in a lot of papers, and some of them are very important in martingale theory. Most of martingale inequalities are inequalities for the L_p-norms of functionals defined on spaces of martingales. Of course, it is an important topic to seek a characterization of a Banach function space X such that a known martingale inequality in L_p holds with L_p replaced by X. For more results on this topic, see [2], [40], [41], [43], [44], [46]–[49], [51], and [52]. We give here an overview of results in [46], which are results concerning some inequalities for the Doob decompositions of submartingales.

Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathfrak{F}$. Recall that a process $h = (h_n)_{n\in\mathbb{Z}_+}$ is said to be *$\mathcal{F}$-predictable* if each h_n is $\mathcal{F}_{n-1}$-measurable.[7] We let $\wp(\mathcal{F})$ denote the set of all $\mathcal{F}$-predictable processes $h = (h_n)_{n\in\mathbb{Z}_+}$ such that $h_0 = 0$ a.s.

Let $\eta = (\eta_n)$ be an $\mathcal{F}$-submartingale; that is, each η_n is integrable $\mathcal{F}_n$-measurable random variable and $\mathbb{E}[\eta_{n+1}|\mathcal{F}_n] \ge \eta_n$ a.s. for all $n \in \mathbb{Z}_+$. It is well known that there exist $g = (g_n) \in \mathcal{M}(\mathcal{F})$ and $h = (h_n) \in \wp(\mathcal{F})$ such that

$$\eta_n = g_n + h_n \quad \text{a.s. for all } n \in \mathbb{Z}_+.$$

[7] Recall our convention that $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$ for any $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$.

Such $g = (g_n)$ and $h = (h_n)$ are unique. It is easily seen that $h = (h_n)$ is a nondecreasing process. The above expression of $\eta = (\eta_n)$ is called the *Doob decomposition* of $\eta = (\eta_n)$. In what follows, whenever we consider the Doob decomposition of a submartingale, we will let $g = (g_n)$ be a martingale and let $h = (h_n)$ be a predictable process such that $h_0 = 0$ a.s.

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a Young function and let $f = (f_n) \in \mathcal{M}(\mathcal{F})$. If the composite random variables $\Phi(|f_n|) \equiv \Phi \circ |f_n|$ are integrable, then the process $\Phi(|f|) := (\Phi(|f_n|))$ is an $\mathcal{F}$-submartingale and hence it is decomposed as

$$\Phi(|f_n|) = g_n + h_n \quad \text{a.s. for all } n \in \mathbb{Z}_+.$$

Furthermore, from [64, Proposition VIII-1-4] one can deduce that if $1 \le p < \infty$ and if $\Phi(Mf) \equiv \Phi \circ Mf \in L_p$, then $h_\infty = \sup_n h_n \in L_p$ and

$$\|h_\infty\|_{L_p} \le p \, \|\Phi(Mf)\|_{L_p}.$$

The following theorem gives a characterization of a Banach function space X such that the inequality above holds with L_p replaced by X.

Theorem 6.4 ([46])**.** *Let X be a Banach function space over Ω, and let Φ be as above. Given $\mathcal{F} = (\mathcal{F}_n) \in \mathfrak{F}$, let $\mathcal{S}(\mathcal{F})$ denote the set of all nonnegative $\mathcal{F}$-submartingales. Then the following are equivalent*:

(i) *There is a positive constant C such that if $\eta = (\eta_n) \in \mathcal{S}(\mathcal{F})$ is decomposed as $\eta_n = g_n + h_n$ a.s. for all $n \in \mathbb{Z}_+$, then*

$$\|h_\infty\|_X \le C \sup_{n \in \mathbb{Z}_+} \|\eta_n\|_X.$$

(ii) *There is a positive constant C such that if $\eta = (\eta_n) \in \mathcal{S}(\mathcal{F})$ is decomposed as $\eta_n = g_n + h_n$ a.s. for all $n \in \mathbb{Z}_+$, then*

$$\sup_{n \in \mathbb{Z}_+} \|g_n\|_X \le C \sup_{n \in \mathbb{Z}_+} \|\eta_n\|_X.$$

(iii) *There is a positive constant C such that if $f = (f_n) \in \mathcal{M}(\mathcal{F})$, if $\Phi(|f_n|)$ is integrable for all $n \in \mathbb{Z}_+$, and if $\Phi(|f|) = (\Phi(|f_n|))$ is decomposed as $\Phi(|f_n|) = g_n + h_n$ a.s. for all $n \in \mathbb{Z}_+$, then*

$$\|h_\infty\|_X \le C \, \|\Phi(Mf)\|_X.$$

(iv) *There is a positive constant C such that if $f = (f_n) \in \mathcal{M}(\mathcal{F})$, if $\Phi(|f_n|)$ is integrable for all $n \in \mathbb{Z}_+$, and if $\Phi(|f|) = (\Phi(|f_n|))$ is decomposed as $\Phi(|f_n|) = g_n + h_n$ a.s. for all $n \in \mathbb{Z}_+$, then*

$$\sup_{n \in \mathbb{Z}_+} \|g_n\|_X \le C \, \|\Phi(Mf)\|_X.$$

(v) *X can be renormed so as to be r.i. and $\alpha_X > 0$.*

When these equivalent conditions hold, the constant C in each of (i)–(iv) *depends only on X.*

Is it possible to find a characterization of a Banach function space X such that the inequality in (iii) (or (iv)) holds with Mf replaced by Sf? Unfortunately the author does not know a complete characterization of such a Banach function space. However, under the assumption that Φ satisfies the Δ_2-condition, the following theorem gives a characterization of such a Banach function space.

Theorem 6.5 ([46])**.** *Let X and Φ be as in Theorem* 6.4.

(i) *Suppose that at least one of the following conditions are satisfied:*
   (a) *There is a positive constant* C *such that if* $f = (f_n) \in \mathcal{M}(\mathcal{F})$, *if* $\Phi(|f_n|)$ *is integrable for all* $n \in \mathbb{Z}_+$, *and if* $\Phi(|f|) = (\Phi(|f_n|))$ *is decomposed as* $\Phi(|f_n|) = g_n + h_n$ *a.s., then*

$$\|h_\infty\|_X \leq C \, \|\Phi(Sf)\|_X \, .$$

   (b) *There is a positive constant* C *such that if* $f = (f_n) \in \mathcal{M}(\mathcal{F})$, *if* $\Phi(|f_n|)$ *is integrable for all* $n \in \mathbb{Z}_+$, *and if* $\Phi(|f|) = (\Phi(|f_n|))$ *is decomposed as* $\Phi(|f_n|) = g_n + h_n$ *a.s., then*

$$\sup_{n \in \mathbb{Z}_+} \|g_n\|_X \leq C \, \|\Phi(Sf)\|_X \, .$$

   Then X *can be renormed so as to be r.i. and* $\alpha_X > 0$.

(ii) *Suppose that* Φ *satisfies the* Δ_2*-condition; that is, there is a positive constant* c *such that* $\Phi(2t) \leq c\Phi(t)$ *for all* $t \geq 0$. *If* X *is renormed so as to be r.i. and* $\alpha_X > 0$, *then both* (a) *and* (b) *hold with a constant* C *depending only on* X *and* Φ.

In order to prove Theorem 6.4 and (i) of Theorem 6.5, one can use the methods of [38] and [45] (though some effort is forced). On the other hand, in order to prove (ii) of Theorem 6.5, one needs the following:

Theorem 6.6 ([46])**.** *Let* X *be an i.r. space over* Ω *such that* $\alpha_X > 0$, *and suppose that a Young function* $\Phi \colon [0, \infty) \to [0, \infty)$ *satisfies the* Δ_2*-condition. Then there are positive constants* $c_{X,\Phi}$ *and* $C_{X,\Phi}$ *such that*

$$c_{X,\Phi} \, \|\Phi(Sf)\|_X \leq \|\Phi(Mf)\|_X \leq C_X \, \|\Phi(Sf)\|_X$$

for every martingale $f = (f_n)$.

Of course, Theorem 6.6 is an extension of Theorem B (Burkholder-Davis-Gundy type inequality). According to [59, Theorems 3.2 and 4.2] (cf. [9]), a Young function Φ satisfies the Δ_2^∞-condition if and only if $\alpha_{L_\Phi} > 0$. Combining this fact with Theorem B, one may think that the proof of Theorem 6.6 is easy. However it is not so easy to prove Theorem 6.6. In fact, to prove it, one needs to utilize several theorems whose proofs are not so easy, such as an interpolation theorem of Astashkin and Maligranda [4] and a theorem of Cwikel [13] concerning a property of interpolation spaces.

We close this section with an application of Theorems 6.4 and 6.5. Suppose that $f = (f_n) \in \mathcal{M}(\mathcal{F})$ is bounded in L_2, and define a process $h = (h_n)$ by

$$h_0 = 0, \qquad h_n = \sum_{k=1}^{n} \mathbb{E}\big[|\Delta_k f|^2 \,\big|\mathcal{F}_{k-1}\big] \, , \quad n \in \mathbb{N}.$$

Then $h = (h_n) \in \wp(\mathcal{F})$, and the process $g = (g_n)$ defined by $g_n = f_n^2 - h_n$, $n \in \mathbb{Z}_+$, is an $\mathcal{F}$-martingale. Thus the submartingale $f^2 = (f_n^2)$ is decomposed as

$$f_n^2 = g_n + h_n \quad \text{a.s. for all } n \in \mathbb{Z}_+.$$

Since $h_\infty = (sf)^2 - \mathbb{E}\big[f_0^2\big]$, we have the following theorem by using Theorems 6.4 and 6.5.

Theorem 6.7 ([46])**.** *Let* X *be a Banach function space over* Ω. *Then the following are equivalent*:

(i) *There is a positive constant C such that*

$$\left\|(sf)^2\right\|_X \le C \sup_{n\in\mathbb{Z}_+} \left\|f_n^2\right\|_X$$

for every martingale $f=(f_n)$ which is bounded in L_2.

(ii) *There is a positive constant C such that*

$$\left\|(sf)^2\right\|_X \le C \left\|(Mf)^2\right\|_X$$

for every martingale $f=(f_n)$ which is bounded in L_2.

(iii) *There is a positive constant C such that*

$$\left\|(sf)^2\right\|_X \le C \left\|(Sf)^2\right\|_X$$

for every martingale $f=(f_n)$ which is bounded in L_2.

(iv) *X can be renormed so as to be r.i. and $\alpha_X > 0$.*

When these equivalent conditions hold, the constant C in each of (i)–(iii) *depends only on X.*

7. Weak type inequality in Banach function spaces

In this section, we give an overview of some weak type inequalities in Banach function spaces. Recall that when $1 < p \le \infty$ the Doob inequality (5.1) is valid, and that when $p=1$ it is not valid. On the other hand, the weak type inequality

$$\sup_{\lambda>0} \lambda\mathbb{P}\big(\omega\in\Omega\colon (Mf)(\omega) > \lambda\big)^{1/p} \le \|f_\infty\|_{L_p} \tag{7.1}$$

holds for every uniformly integrable martingale $f=(f_n)$, not only when $1<p\le\infty$ but also when $p=1$. Our interest here is in extending this inequality.

We begin with introducing a notion of a weak space. Let X be a Banach function space over Ω. For each $x\in L_0$, let

$$\|x\|_{\text{w-}X} = \sup_{\lambda>0} \lambda \left\|\mathbf{1}_{\{\omega\in\Omega\colon |x(\omega)|>\lambda\}}\right\|_X,$$

and define

$$\text{w-}X = \{x\in L_0 : \|x\|_{\text{w-}X} < \infty\}.$$

For example, w-L_p coincides with $L_{p,\infty}$ as a set (see [70, Lemma 3.8, p. 191]). It is clear from the definition of w-X that if $x\in X$, then $x\in$ w-X and $\|x\|_{\text{w-}X} \le \|x\|_X$. It is also clear that $\|x\|_{\text{w-}X} = 0$ if and only if $x=0$ a.s., and that if $x\in$ w-X and $\alpha\in\mathbb{R}$, then $\|\alpha x\|_{\text{w-}X} = |\alpha|\,\|x\|_{\text{w-}X}$. Although the functional $\|\cdot\|_{\text{w-}X}$ does not satisfy the triangle inequality, we have

$$\|x+y\|_{\text{w-}X} \le 2(\|x\|_{\text{w-}X} + \|y\|_{\text{w-}X})$$

for all x, $y\in$ w-X. That is, $\|\cdot\|_{\text{w-}X}$ is a quasi-norm; and hence there is a metric d on w-X such that $d(x,x_n)\to 0$ as $n\to\infty$ if and only if $\|x-x_n\|_{\text{w-}X}\to 0$ as $n\to\infty$ (see [7, p. 59]). It is straightforward to check that w-X is complete with respect to d. Thus w-X is a quasi-Banach space. In fact, w-X is a *quasi-Banach function space*; that is, $L_\infty \hookrightarrow$ w-$X \hookrightarrow L_0$, and w-X satisfies (B2) and (B3). Moreover, w-X is a maximal quasi-Banach function space in the sense of [24].

Note that the weak type inequality (7.1) can be rewritten as

$$\|Mf\|_{\text{w-}L_p} \le \|f_\infty\|_{L_p}.$$

Our aim is to give a characterization of a Banach function space X such that the analogous inequality

$$\|Mf\|_{\text{w-}X} \le C\,\|f_\infty\|_X \tag{7.2}$$

holds for every uniformly integrable martingale $f = (f_n)$. We begin with extending the notion of fundamental function of an r.i. space. For each $t \in [0,1]$, let

$$\Sigma(t) = \{A \in \Sigma : \mathbb{P}(A) = t\}$$

and define a function $\overline{\varphi}_X : [0,1] \to [0,\infty)$ by

$$\overline{\varphi}_X(t) = \sup\bigl\{\|1_A\|_X : A \in \Sigma(t)\bigr\}, \quad t \in [0,1].$$

It is then clear that if X is r.i., then $\overline{\varphi}_X(t) = \varphi_X(t)$. One can easily check that $t \le \overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t)$ even when X is not r.i. Furthermore, according to [50, Lemma 1], $\overline{\varphi}_X(t)$ is a *quasi-concave* function; that is,

- $\overline{\varphi}_X(t) = 0$ if and only if $t = 0$.
- $\overline{\varphi}_X(t)$ is nondecreasing on $[0,1]$.
- $\overline{\varphi}_X(t)/t$ is nonincreasing on $(0,1]$.

It follows that the set of all $x \in L_0$ such that

$$\|x\|_{M(\overline{\varphi}_X)} = \sup_{t\in I} \frac{\overline{\varphi}_X(t)}{t} \int_0^t x^*(s)\,ds < \infty$$

forms an r.i. space over Ω (see [6, p. 69]); this r.i. space is denoted by $M(\overline{\varphi}_X)$.

By using the following theorem, we can deduce a characterization of a Banach function space X such that (7.2) holds for every uniformly integrable martingale $f = (f_n)$. In what follows, we say that a linear operator T is an L_1–L_∞*-contraction* if $T \in B(L_1) \cap B(L_\infty)$ and $\max\{\|T\|_{B(L_1)}, \|T\|_{B(L_\infty)}\} \le 1$.

Theorem 7.1 ([50])**.** *Let X be a Banach function space. The following are equivalent*:

(i) *There is a positive constant C such that for all L_1–L_∞-contractions T and all $x \in X$,*

$$\|Tx\|_{\text{w-}X} \le C\,\|x\|_X .$$

(ii) *There is a positive constant C such that for all sub-σ-algebras $\mathcal{A}$ and all $x \in X$,*

$$\bigl\|\mathbb{E}[x\,|\mathcal{A}]\bigr\|_{\text{w-}X} \le C\,\|x\|_X .$$

(iii) *There is a positive constant C such that for all $t \in [0,1]$,*

$$\overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t) \le Ct.$$

(iv) *$X \hookrightarrow M(\overline{\varphi}_X)$, that is, there is a positive constant C such that for all $x \in X$,*

$$\|x\|_{M(\overline{\varphi}_X)} \le C\,\|x\|_X .$$

When these equivalent conditions hold, w-X *coincides with the quasi-Banach space $M^*(\overline{\varphi}_X)$ consisting of all $x \in L_0$ such that*

$$\|x\|_{M^*(\overline{\varphi}_X)} := \sup_{t\in I}\bigl[\,\overline{\varphi}_X(t)\,x^*(t)\bigr] < \infty,$$

and the quasi-norms of w-X *and $M^*(\overline{\varphi}_X)$ are equivalent.*

Suppose that condition (ii) of Theorem 7.1 holds, and that $f = (f_n)$ is a uniformly integrable martingale. Let $\lambda > 0$ and define a stopping time τ by

$$\tau(\omega) = \min\{n \in \mathbb{Z}_+ : |f_n(\omega)| > \lambda\},$$

with the convention that $\min \emptyset = \infty$. Then

$$\{\omega \in \Omega\colon (Mf)(\omega) > \lambda\} = \{\omega \in \Omega\colon \tau(\omega) < \infty\} \in \mathcal{F}_\tau$$

and on this set, $\mathbb{E}[|f_\infty|\,|\mathcal{F}_\tau] \geq |f_\tau| > \lambda$ a.s. Since $\|\mathbf{1}_A\|_{\text{w-}X} = \|\mathbf{1}_A\|_X$ for all $A \in \Sigma$, (ii) of Theorem 7.1 implies that

$$\lambda \left\|\mathbf{1}_{\{\omega\colon (Mf)(\omega)>\lambda\}}\right\|_X = \left\|\lambda \mathbf{1}_{\{\omega\colon \tau(\omega)<\infty\}}\right\|_{\text{w-}X} \leq \left\|\mathbb{E}\big[|f_\infty|\big|\mathcal{F}_\tau\big]\right\|_{\text{w-}X} \leq C\,\|f_\infty\|_X\,.$$

Since $\lambda > 0$ is arbitrary, (7.2) follows.

Conversely, suppose that (7.2) holds for every uniformly integrable martingale $f = (f_n)$. Let $\mathcal{A}$ be a sub-σ-algebra and let $x \in X$. Define $\mathcal{F} = (F_n) \in \mathfrak{F}$ by (6.2) and let $f = (f_n)$ be the martingale defined by $f_n = \mathbb{E}[x\,|\mathcal{F}_n]$, $n \in \mathbb{Z}_+$. Then, $\mathbb{E}[x\,|\mathcal{A}] \leq Mf$ a.s., and hence by (7.2),

$$\left\|\mathbb{E}[x\,|\mathcal{A}]\right\|_{\text{w-}X} \leq \|Mf\|_{\text{w-}X} \leq C\,\|f_\infty\|_X = C\,\|x\|_X\,,$$

that is, (ii) of Theorem 7.1 holds. Thus we have the following:

Theorem 7.2 ([50])**.** *Let X be a Banach function space over Ω. Then inequality* (7.2) *holds for every uniformly integrable martingale $f = (f_n)$ if and only if equivalent conditions* (i)–(iv) *in Theorem* 7.1 *hold.*

If X is r.i., then $\overline{\varphi}_X(t)\,\overline{\varphi}_{X'}(t) = t$ for all $t \in [0,1]$, and hence (7.2) holds for every uniformly integrable martingale $f = (f_n)$. However, an example in [50, Section 5] shows that X may not be renormed so as to be r.i., even if (7.2) holds for every uniformly integrable martingale $f = (f_n)$.

On the other hand, as the following theorem shows, when X is a weighted Orlicz space (or weighted Lebesgue space), (7.2) holds for every uniformly integrable martingale $f = (f_n)$ if and only if X can be renormed so as to be r.i.

Let w be a strictly positive random variable such that $\mathbb{E}[\mathrm{w}] = 1$ and let $\mathbb{P}_\mathrm{w}$ be the probability measure defined by $\mathbb{P}_\mathrm{w}(A) = \int_A \mathrm{w}\,d\mathbb{P}$. With this notation, we have the following:

Theorem 7.3 ([50])**.** *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a Young function satisfying the Δ_2^∞-condition. Then the following are equivalent*:

(i) *There is a positive constant C such that for all sub-σ-algebras $\mathcal{A}$ and all $x \in L_\Phi(\mathbb{P}_\mathrm{w})$,*

$$\left\|\mathbb{E}[x\,|\mathcal{A}]\right\|_{\text{w-}L_\Phi(\mathbb{P}_\mathrm{w})} \leq C\,\|x\|_{L_\Phi(\mathbb{P}_\mathrm{w})}\,.$$

(ii) *There are positive constants a, b such that $a \leq \mathrm{w} \leq b$ a.s.*

(iii) *$L_\Phi(\mathbb{P}_\mathrm{w})$ is a Banach function space over $(\Omega, \Sigma, \mathbb{P})$ and it can be renormed so as to be r.i. with respect to $\mathbb{P}$.*

In the theorem above, the assumption that Φ satisfies the Δ_2^∞-condition is essential. See [50, Section 5] for details.

As we have seen above, martingale theory is useful for analyzing the structure of Banach function spaces. The author is glad if the results in this article give a new understanding of the relation between martingale theory and the structure of Banach function spaces.

8. Postscript

Although martingales are indispensable tools in probability theory, a lot of researchers of probability theory are not interested in martingale theory itself. Especially in Japan, very few researchers have been studying martingale theory itself. Recently, however, some researchers of harmonic analysis in Japan came to pay attention to martingale theory. In fact, studies of martingales from a viewpoint of harmonic analysis have already begun. For instance, Miyamoto-Nakai-Sadasue [61] and Nakai-Sadasue [63] are such studies. The author expects the progress of further new studies.

Acknowledgment

The author would like to thank the referees for some helpful comments.

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Department of Mathematics, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan

Email address: `kikuchi@sci.u-toyama.ac.jp`