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# OPERATOR THEORY IN THE COMPLEX GINZBURG-LANDAU EQUATION

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ABSTRACT. Strong well-posedness for the complex Ginzburg-Landau equation

$$\partial u/\partial t + (\lambda + i\,\alpha)(-\Delta)u + (\kappa + i\,\beta)|u|^{q-2}u - \gamma u = 0$$

is discussed from the viewpoint of operator theory. It is concluded that the solution operators from $L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$, into itself form a semigroup of quasi-contractions when $\kappa^{-1}|\beta| \le c_q^{-1} := 2\sqrt{q-1}/|q-2|$ (without any upper bound on $q \ge 2$), and a non-contraction semigroup of Lipschitz operators when $\kappa^{-1}|\beta| > c_q^{-1}$ $(2 \le q \le 2 + 4/N)$. The assertion is proved by energy methods based on monotonicity methods. Also compactness methods are valid for the Cauchy problem when the initial value belongs to $H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ in addition to the restriction $(\alpha/\lambda, \beta/\kappa) \in CGL(c_q^{-1})$; in this case solutions are unique under sub-critical condition: $q \in [2, 2 + 4/(N-2)_+]$.

## 1. PREFACE

The complex Ginzburg-Landau equation is one of non-linear partial differential equations of parabolic type (in what follows, it is abbreviated as a CGL equation). The theme of this article is the Cauchy problem and initial-boundary value problem for a CGL equation:

$$(\mathbf{CGL})_\Omega \quad \begin{cases} \dfrac{\partial u}{\partial t} + (\lambda + i\,\alpha)(-\Delta)u + (\kappa + i\,\beta)|u|^{q-2}u - \gamma u = 0 & \text{on } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Here $\Omega$ denotes a (bounded or unbounded) domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$. The boundary $\partial\Omega$ is assumed to be reasonable in the sense that the Laplacian $-\Delta$ in the equation together with its boundary condition induces a selfadjoint realization $S$ in $L^2(\Omega)$, with domain $D(S) = H_0^1(\Omega) \cap H^2(\Omega)$ (cf. Miyajima [36, Remark 11.17]). Besides, since $i = \sqrt{-1}$, we have to notice two complex coefficients in front of the spatial-derivative and non-linear terms of the CGL equation. This is reflected in the letter "C" in the abbreviated symbol of the equation (for "GL" which comes from the two founders' names of the theory of superconductivity; cf. Jimbo-Morita [19]). Therefore, the unknown $u$ becomes a complex-valued function of $(x,t) \in \Omega \times [0,\infty)$. As for the real-parts of those coefficients we assume that $\lambda$, $\kappa \in \mathbb{R}_+ := (0,\infty)$. The rest of the parameters are also constants such that $\alpha$, $\beta$, $\gamma \in \mathbb{R}$, and $q \in [2,\infty)$. First, since $\lambda > 0$, the CGL equation is regarded as one of parabolic type, however, the available tools are restricted because it is usually dealt with in complex $L^2$-space

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(for an aspect of complex-valuedness see Cazenave-Dickstein-Weissler [10]). Second, since $\kappa > 0$, it is expected that such a non-linearity yields global solvability of problem $(\mathbf{CGL})_\Omega$. That is, the above-mentioned assumption simplifies the problem. Nevertheless, the unrestricted exponent $q$ of the non-linear term and large values of $|\beta|$ are not so simply dealt with.

The purpose of this article is to give a survey focused on an operator-theoretic approach to the **strong well-posedness** of $(\mathbf{CGL})_\Omega$ as one of initial value problems for (semilinear) evolution equations in complex $L^2$-space. Because of the complex coefficient in front of the non-linear term we have to deal with not only semigroups of (quasi-)contractions but also non-contraction semigroups of (locally) Lipschitz operators as solution operators mapping $L^2$ into itself. Unfortunately, the generation theorem for such a class of non-contraction semigroups, corresponding to the Hille-Yosida theorem, is not yet known. Accordingly, we employ so-called energy methods based on monotonicity (accretivity) methods instead of the unfinished generation theorem to prove the strong well-posedness of $(\mathbf{CGL})_\Omega$ in $L^2$. Sometimes compactness methods are also useful, but an additional restriction is imposed on the initial values. It is possible to transform $(\mathbf{CGL})_\Omega$ into an integral equation by using the semigroup $\{e^{(\lambda+i\alpha)t\Delta}; t \geq 0\}$; however, here we do not employ such a device though it is very popular in the general theory of semilinear evolution equations (this is the main theme of a book by Ogawa [37] published recently). The substitutive approximate problem is constructed by replacing the $m$-sectorial operator $u \mapsto |u|^{q-2}u$ in $L^2$ induced from the non-linear term of the CGL equation with the family of its Yosida approximations. That is, accretivity methods are essentially methods of Yosida approximation which ensure the global solvability of the approximate problem.

Physicists say that the CGL equation has a meaning of an **amplitude equation** derived by "contraction" from fundamental systems of equations describing physical phenomena. Y. Kuramoto reported in [25] his experience around 1974 to derive the CGL equation as a contraction of a certain system of reaction-diffusion equations. At that time it was the dawn of non-linear science so that the name CGL equation had not yet been fixed. That is, CGL equation is about forty years old. Incidentally, it was around 1995 when the name complex Ginzburg-Landau equation was fixed in mathematical papers. At that time the author started his study of the well-posedness. In fact, the name Ginzburg-Landau equation without "complex" was used in Temam [60] and Yang [64] (cf. also Unai-Okazawa [62]), while "complex" was added to the titles in Ginibre-Velo [14], [15] and Levermore-Oliver [26], [27].

Before discussing $(\mathbf{CGL})_\Omega$ there is a simpler problem worth mentioning:

$$(\mathbf{NCGL})_\Omega \qquad \begin{cases} \dfrac{\partial u}{\partial t} - i\Delta u + |u|^{q-2}u = 0 & \text{on } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

because the same accretivity methods apply (the study of its strong solvability goes back to Pecher-von Wahl [54] and Shigeta [57]). From an operator-theoretic viewpoint $(\mathbf{NCGL})_\Omega$ plays the role of a model problem to explain the validity of the non-linear Hille-Yosida theorem in a complex Hilbert space. It took the keen eyes of Ghoussoub [12] to call $(\mathbf{NCGL})_\Omega$ the non-diffusive complex Ginzburg-Landau equation though the linear terms of the equation in $(\mathbf{NCGL})_\Omega$ are the same as those in the non-linear Schrödinger equation which was reflected in the titles in [54] and [57]. Here "nondiffusive" means that there is no smoothing effect of solution operators on initial values. But its parabolic regularization is possible if $u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \cap L^{2(q-1)}(\Omega)$. In fact, a solution $u(\cdot)$ to $(\mathbf{NCGL})_\Omega$ is given as the limit of a sequence $\{u_n(\cdot)\}$ of solutions to $(\mathbf{CGL})_\Omega$ with $\lambda = n^{-1}$ $(n \in \mathbb{N})$, $\alpha = 1 = \kappa$, and $\beta = 0 = \gamma$: $\|u(t) - u_n(t)\|_{L^2} \le (t/2n)^{1/2}\|\nabla u_0\|_{L^2}$ (cf. [45, Theorem 1.3]).

Besides, since we have assumed that $\lambda,\ \kappa \in \mathbf{R}_+$ in $(\mathbf{CGL})_\Omega$, the following problem for the non-linear Schrödinger equation:

$$(\mathbf{NLS})_\Omega \qquad \begin{cases} \dfrac{\partial u}{\partial t} - i\Delta u + i|u|^{q-2}u = 0 & \text{on } \ \Omega \times (0,\infty), \\ u = 0 & \text{on } \ \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

is not a particular case of $(\mathbf{CGL})_\Omega$, however, the transition from $(\mathbf{CGL})_\Omega$ to $(\mathbf{NLS})_\Omega$ is quite challenging as a singular perturbation problem:

$$(\mathbf{NLS})_\Omega = \lim_{\substack{\lambda\downarrow 0,\ \kappa\downarrow 0 \\ \gamma\to 0}} (\mathbf{CGL})_\Omega. \tag{1.1}$$

In particular, as for the convergence in (1.1) where $\Omega = \mathbb{R}^N$, there was rapid progress of the study in the early 2000s (cf. Machihara-Nakamura [29], Ogawa-Yokota [38]). In this article we do not pay attention to the detailed properties of solutions of CGL or NCGL-like equations such as (1.1) and asymptotic behaviors (cf. Hayashi-Kaikina-Naumkin [17], Kita-Shimomura [21]). The literature in those areas will be found in the bibliography in [17], [21], [29], [38].

The study of the existence of solutions to $(\mathbf{CGL})_\Omega$ started in [60], [64] at the end of the 1980s and then through [14], [15], [26], and [27] in the middle of the 1990s it arrives at Okazawa-Yokota [47] in which the smoothing effect was first shown (under unsatisfactory restriction $\kappa^{-1}|\beta| \le 2\sqrt{q-1}/(q-2)$). At that moment it was recognized that the solution operators to $(\mathbf{CGL})_\Omega$ formed quasi-contraction semigroups. After Okazawa-Yokota [48] a more general class of (locally) Lipschitz semigroups entered the stage. The increase of the argument (or the ratio $|\beta|/\kappa$) of coefficient $\kappa + i\beta$ results in the change of classes of semigroups from contractive ones to non-contractive. This happens also in the case of $(C_0)$ semigroups of linear operators (see Metafune-Okazawa-Sobajima-Yokota [34]). The progress motivated the group of Y. Kobayashi, N. Tanaka and T. Matsumoto to publish a series of research papers on $(\mathbf{CGL})_\Omega$ (cf. [22], [31]–[33]); in particular, they dealt with $L^p$-theory $(p \ne 2)$ in [32], [33]. Before noticing the applicability to $(\mathbf{CGL})_\Omega$, they aimed for the construction of the general theory of (locally) Lipschitz semigroups (cf. [23], [24]) in connection with hyperbolic systems of conservation laws. Besides, it is pointed out by Ghoussoub [13] that one of general theories in the calculus of variations can also be applied to $(\mathbf{CGL})_\Omega$. In spite of their contribution fairly decisive results on $(\mathbf{CGL})_\Omega$ seem to be derived by employing the notion of subdifferential operators in a complex Hilbert space. All results in this direction are put together in Okazawa-Yokota [49] and Clément-Okazawa-Sobajima-Yokota [11]. The present article is a reconstruction of the related results around [49], [11] in a form to extract the essence.

The contents of this article are stated as follows. Section 2 is concerned with $(\mathbf{NCGL})_\Omega$. Theorem 3.1 (**I**), (**II**) in Section 3 are the main theorem for $(\mathbf{CGL})_\Omega$. In

Section 3 the proof of Theorem 3.1 (**I**) is outlined in a form to derive the smoothing effect on initial values in addition to the reasoning based on what is given in Section 2. In Section 4 an abstract theorem is introduced to prove Theorem 3.1 (**II**). In Section 5 compactness methods are applied to $(\mathbf{CGL})_{\mathbb{R}^N}$. Several remarks are in order in Section 6.

## 2. Non-diffusive complex Ginzburg-Landau equation

We solve problem $(\mathbf{NCGL})_\Omega$ by regarding the unknown $u(x,t)$ as a function $u(t)$ defined on the time inverval $[0,\infty)$ with values in the complex Hilbert space $X := L^2 = L^2(\Omega)$; note that $X$ is a space of functions depending only on $x$. When we write just $u(t)$, we mean that $u(t) \in X$, that is, the function $x \mapsto u(x,t)$. In this way $(\mathbf{NCGL})_\Omega$ is formulated as one of the abstract Cauchy problems (ACP):

$$(\mathbf{ACP})_{00} \qquad \begin{cases} \dfrac{du}{dt} + (iS + B)u(t) = 0 \quad \text{a.e. on } (0,\infty), \\ u(0) = u_0. \end{cases}$$

Here the operators $S$ and $B$ with domains and ranges in $X = L^2$ are defined as follows:

$$\begin{cases} Su := -\Delta u, & D(S) := H_0^1(\Omega) \cap H^2(\Omega), \\ Bu := |u|^{q-2}u, & D(B) := L^{2(q-1)}(\Omega) \cap L^2(\Omega). \end{cases} \tag{2.1}$$

Of great importance are the non-negative selfadjointness of $S$ and $m$-sectoriality of $B$ (the term "sectorial" is, for example, employed in a book by Goldstein [16]). The solvability of $(\mathbf{NCGL})_\Omega$ is reduced to the decision of the $m$-accretivity of $iS + B$ in $(\mathbf{ACP})_{00}$. In fact, the non-linear Hille-Yosida theorem is applied to such kind of (ACP); the theorem in Hilbert space is sometimes called the Komura-Kato theorem (see Showalter [58, Section IV.3]) as a particular case of the Crandall-Liggett theorem in general Banach space (cf. Brezis [4], Miyadera [35], Barbu [3]). Thus we are led to the next

**Theorem 2.1** (cf. [45])**.** *Let $q \ge 2$. Then for every $v_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ there exists a unique strong solution $v(\cdot) \in C([0,\infty); L^2(\Omega))$ to $(\mathbf{NCGL})_\Omega$ belonging to the following class*:

$$v(\cdot) \in C^{0,1}([0,\infty); L^2(\Omega)) \cap C^{0,1/2}([0,\infty); H_0^1(\Omega)) \cap C^{0,1/q}([0,\infty); L^q(\Omega)),$$
$$(d/dt)v(\cdot),\ \Delta v(\cdot),\ |v|^{q-2}v(\cdot) \in L^\infty(0,\infty; L^2(\Omega)).$$

*Define the family $\{U(t); t \ge 0\}$ of solution operators by $U(t)v_0 := v(t)$. Then*

$$\|U(t)v_0 - U(t)w_0\|_{L^2} \le \|v_0 - w_0\|_{L^2}, \quad v_0,\ w_0 \in H^2 \cap H_0^1 \cap L^{2(q-1)}. \tag{2.2}$$

*Consequently, $\{U(t);\ t \ge 0\}$ can be extended to a contraction semigroup on $L^2(\Omega)$.*

The symbols of function spaces in Theorem 2.1 are standard (see, e.g., [5], [36]). Simply speaking, $f \in L^p = L^p(\Omega)$ maens that $|f|^p$ is integrable over $\Omega$, while $g \in H^k = H^k(\Omega)$ $(k = 1, 2)$ maens that $g$ is a function over $\Omega$ whose (distributional) $j$-th derivatives belong to $L^2$ $(0 \le j \le k)$. Since $H_0^1 = H_0^1(\Omega)$ is nothing but the closure of $C_0^\infty(\Omega) (\subset H^1(\Omega))$ with respect to the norm of $H^1(\Omega)$, $u \in H_0^1(\Omega)$ implies that the boundary condition, $u = 0$ on $\partial\Omega$, in $(\mathbf{CGL})_\Omega$ is reflected in the definition of domain $D(S)$.

*Remark* 2.2 (Separation property of $iS+B$). Put $q_N := (N+2)/(N-2)$ for $N \geq 3$ (cf. [54]). Then the characterization of the domain $D(iS+B)$ is divided into two cases depending on the exponent $r := q-1$ of non-linearity:

$$D(iS+B) = \begin{cases} D(S) \subset D(B), & r \leq q_N, \\ D(S) \cap D(B), & r > q_N. \end{cases}$$

That is, when $r \leq q_N$, $iS+B$ is regarded as a linear operator $iS$ with relatively bounded non-linear perturbation $B$ (cf. [57]). But both cases are unified as

$$\|Su\|_{L^2}^2 + \|Bu\|_{L^2}^2 \leq r\|(iS+B)u\|_{L^2}^2, \quad u \in D(S) \cap D(B), \tag{2.3}$$

which may be referred to as the **separation property** of $iS+B$; this term is borrowed from that in the domain characterization of Schrödinger operators (see Section 5).

First it will be desirable to make sure of the $m$-accretivity of $B$ in (2.1). Namely, the accretivity of $B$ and its maximality are respectively described as $\mathrm{Re}\,(Bu-Bv, u-v)_{L^2} \geq 0$ ($u,\, v \in D(B)$) and the range condition: $R(1+B) = L^2$ ($\Leftrightarrow R(1+\varepsilon B) = L^2$ ($\varepsilon > 0$)). This concludes that $1+\varepsilon B$ is bijective. In fact, the accretivity of $B$ implies its injectivity: $\|u-v\|_{L^2}^2 \leq \mathrm{Re}\,((1+\varepsilon B)u-(1+\varepsilon B)v, u-v)_{L^2}$ ($\varepsilon > 0$). More precisely, it should be emphasized that the accretivity of $B$ is replaced with its sectoriality, that is, there is a constant $c \geq 0$ such that

$$|\mathrm{Im}\,(Bu-Bv, u-v)_{L^2}| \leq c\,\mathrm{Re}\,(Bu-Bv, u-v)_{L^2}, \quad u,\, v \in D(B),$$

where $c$ is exactly given by (2.9) below. Actually, the key word in Proof of Theorem 2.1 is the sectoriality or **sectorial-valuedness** the notion of which was first introduced by Kato [20, Section V.3.10] for linear operators and then extended to non-linear operators in [62] and [47] (for the second example see Remark 2.7 below).

As a beginning of **Proof of Theorem 2.1** we now introduce the realization of the Dirichlet-Laplace operator in $L^q = L^q(\Omega)$:

$$A_q := -\Delta, \quad D(A_q) := W_0^{1,q}(\Omega) \cap W^{2,q}(\Omega), \quad 1 < q < \infty.$$

Then $A_q$ becomes $m$-sectorial in the sense of $L^q$ (Ouhabaz [52, Theorem 3.9]):

$$\frac{|\mathrm{Im}\,\langle A_q u, F_q(u)\rangle_{L^q,L^{q'}}|}{\mathrm{Re}\,\langle A_q u, F_q(u)\rangle_{L^q,L^{q'}}} \leq c_q := \frac{|q-2|}{2\sqrt{q-1}}, \quad 0 \neq u \in D(A_q). \tag{2.4}$$

Here we denote by $F_q$ the duality mapping from $L^q$ to $L^{q'}$: $F_q(u) := |u|^{q-2}u$ ($1 < q < \infty$), where $q'$ is the Hölder conjugate of $q$. The constant $c_q$ in (2.4) for the Laplacian was found by Henry [18, p.32], but the property (2.4) is also shared by elliptic operators in divergence form (cf. Okazawa [41]). See also Bakry [2], Liskevich-Perelmuter [28] for the class of $(C_0)$ semigroups whose negative infinitesimal generators satisfy (2.4). In this connection we consider the family $\{A_{q,\varepsilon};\, \varepsilon > 0\}$ of Yosida approximations of $A_q$

$$A_{q,\varepsilon} := A_q(1+\varepsilon A_q)^{-1} = \varepsilon^{-1}[1-(1+\varepsilon A_q)^{-1}] \in B(L^q), \quad \varepsilon > 0. \tag{2.5}$$

Here $B(L^q)$ denotes the set of all bounded linear operators on $L^q$. The term "approximation" reflects the fact that $\|A_q u - A_{q,\varepsilon}u\| = \|[1-(1+\varepsilon A_q)^{-1}]A_q u\| \to 0$ ($\varepsilon \downarrow 0$) for every $u \in D(A_q)$. Then the sectoriality, the key in this section, is summarized as follows:

**Lemma 2.3** (Sectoriality 1)**.** *Let* $A_{q,\varepsilon}$ *and* $F_q$ *be as defined above. Then for* $v$, $w \in L^q$,

$$|\mathrm{Im}\,\langle v-w, F_q(v)-F_q(w)\rangle_{L^q,L^{q}{}'}| \le c_q \mathrm{Re}\,\langle v-w, F_q(v)-F_q(w)\rangle_{L^q,L^{q}{}'}, \tag{2.6}$$

$$|\mathrm{Im}\,\langle A_{q,\varepsilon}v, F_q(v)\rangle_{L^q,L^{q'}}| \le c_q \mathrm{Re}\,\langle A_{q,\varepsilon}v, F_q(v)\rangle_{L^q,L^{q'}}. \tag{2.7}$$

In fact, let $X$ be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Then we have

$$\frac{\left|\mathrm{Im}(\|z\|^{q-2}z-\|w\|^{q-2}w, z-w)\right|}{\mathrm{Re}(\|z\|^{q-2}z-\|w\|^{q-2}w, z-w)} \le c_q, \quad z,\ w\in X \ \ (z\ne w),\ 1<q<\infty. \tag{2.8}$$

This inequality was first established by Liskevich-Perelmuter [28] when $X=\mathbb{C}$ and $\|\cdot\| = |\cdot|$, and then extended to the general case in [47, Lemma 2.1] with a fairly simplified proof. Now (2.6) is derived by a computation based on (2.8) with $X=\mathbb{C}$. Next, to prove (2.7) it suffices to note that $\mathrm{Im}\,\langle A_{q,\varepsilon}v, F_q(v)\rangle_{L^q,L^{q'}}$ can be decomposed as

$$\begin{aligned}&\mathrm{Im}\,\langle A_{q,\varepsilon}v, F_q(v)\rangle_{L^q,L^{q'}}\\ &= \varepsilon^{-1}\mathrm{Im}\,\langle v-(1+\varepsilon A_q)^{-1}v, F_q(v)-F_q((1+\varepsilon A_q)^{-1}v)\rangle_{L^q,L^{q'}}\\ &\quad + \mathrm{Im}\,\langle A_q(1+\varepsilon A_q)^{-1}v, F_q((1+\varepsilon A_q)^{-1}v)\rangle_{L^q,L^{q'}}\end{aligned}$$

by virtue of (2.5) and the definition of $F_q$. The first term on the right-hand side is estimated by (2.6) with $w := (1+\varepsilon A_q)^{-1}v \in L^q$, while the second term on the right-hand side is estimated by (2.4) with $u := (1+\varepsilon A_q)^{-1}v \in D(A_q)$. This finishes the proof.

**Lemma 2.4** (Sectoriality 2)**.** *Let* $S$ *and* $B$ *be as defined in* (2.1)*. Let* $\{S_\varepsilon;\, \varepsilon > 0\} = \{A_{2,\varepsilon};\, \varepsilon>0\}$ *denote the family of Yosida approximations of* $S = A_2$*. Then the inequalities in* Lemma 2.3 *are respectively translated into those in* $L^2$:

$$|\mathrm{Im}\,(Bu-Bv, u-v)_{L^2}| \le c_q\mathrm{Re}\,(Bu-Bv,u-v)_{L^2}, \quad u,\ v\in D(B), \tag{2.9}$$

$$|\mathrm{Im}\,(S_\varepsilon v, Bv)_{L^2}| \le c_q\mathrm{Re}\,(S_\varepsilon v, Bv)_{L^2}, \quad v\in D(B),\ \ \varepsilon>0. \tag{2.10}$$

*Here* $c_q$ *is exactly the* **sectoriality constant** *of* $B$ *and its Yosida approximation*:

$$|\mathrm{Im}\,(B_\varepsilon u-B_\varepsilon v, u-v)_{L^2}| \le c_q\mathrm{Re}\,(B_\varepsilon u-B_\varepsilon v,u-v)_{L^2}, \quad u,\ v\in D(B); \tag{2.11}$$

*note that* $B_\varepsilon := \varepsilon^{-1}[1-(1+\varepsilon B)^{-1}]$ $(\varepsilon>0)$ *is well defined also for a non-linear* $m$*-sectorial operator* $B$.

In fact, when $u$, $v\in D(B)$, we have the equality

$$(Bu-Bv,u-v)_{L^2} = \overline{\langle u-v, F_q(u)-F_q(v)\rangle_{L^q,L^{q}{}'}} \quad (q\ge 2).$$

Since the Cauchy-Schwarz inequality applies to give

$$D(B) = L^2\cap L^{2(q-1)} \subset L^2\cap L^q \quad (q\ge 2), \tag{2.12}$$

it suffices to note that $Bu = F_q(u) \in L^2\cap L^{q}{}'$ when $u\in D(B)$. In this way we can show that $B$ has the sectoriality (2.9) in $L^2$ which is a property very close to the non-negative selfadjointness (when $q=2$). Because the description on the sectoriality is already so long, the readers are referred to [45, Lemma 3.1] for the maximality.

In the same way we can prove (2.10). In fact, it is easy to see that

$$(S_\varepsilon v, Bv)_{L^2} = (A_{2,\varepsilon}v, Bv)_{L^2} = \langle A_{q,\varepsilon}v, F_q(v)\rangle_{L^q,L^{q'}} \quad \forall\, v\in D(B).$$

Here (2.12) and (2.5) are again useful. On the one hand, we have

$$(1+\varepsilon S)^{-1}v = (1+\varepsilon A_q)^{-1}v \;\Rightarrow\; S_\varepsilon v = A_{q,\varepsilon}v \in L^2\cap L^q \;\;\forall\, v\in D(B)\subset L^2\cap L^q.$$

On the other hand, we have $Bv = |v|^{q-2}v = F_q(v) \in L^2\cap L^{q'}$ for $v\in D(B)$.

The inequality (2.10) is connected with the maximality of $iS+B$ through the separation property of $iS_\varepsilon + B$ which is now stated as a corollary. Incidentally, it seems to be natural to regard $iS+B$ as the sum of mutually even operators.

**Corollary 2.5** ([45, Proof of Theorem 2.3])**.** *Let $S$ and $B$ be as defined in* (2.1). *Then*

$$\|S_\varepsilon v\|_{L^2}^2 + \|Bv\|_{L^2}^2 \le (q-1)\|(iS_\varepsilon + B)v\|_{L^2}^2, \quad v\in D(B),\;\; \varepsilon>0. \tag{2.13}$$

*Thus one can get* (2.3) *by letting $\varepsilon\downarrow 0$ in* (2.13) *with $v=u\in D(S)\cap D(B)$.*

In this way we have finished the preparation to show the $m$-accretivity of $iS+B$. Since $S$ is non-negative selfadjoint, the accretivity of $iS+B$ follows easily from (2.9):

$$\mathrm{Re}\,((iS+B)u-(iS+B)v, u-v)_{L^2} = \mathrm{Re}\,(Bu-Bv,u-v)_{L^2}\ge 0.$$

It remains to prove its maximality: $R(1+iS+B)=L^2$. To this end we consider a family $\{u_\varepsilon;\, \varepsilon>0\}$ of solutions of approximate equations

$$u_\varepsilon + iS_\varepsilon u_\varepsilon + Bu_\varepsilon = f\in L^2 \;\Big(\Leftrightarrow\; R(1+iS_\varepsilon+B)=L^2\Big), \quad \varepsilon>0. \tag{2.14}$$

For the solvability of (2.14) we can simply modify the reasoning in the linear case as given by Brezis [5, Proposition VII.1]. Besides, $f\in R(1+iS+B)$ if and only if $\|S_\varepsilon u_\varepsilon\|_{L^2}$ is bounded as $\varepsilon\downarrow 0$ (see Brezis-Crandall-Pazy [6]). (2.13) is essential here. In fact, rewriting (2.14) as $(iS_\varepsilon+B)u_\varepsilon = f-u_\varepsilon$, we see from (2.13) with $v:=u_\varepsilon$ that

$$\|S_\varepsilon u_\varepsilon\|_{L^2}\le \sqrt{q-1}\,\|f-u_\varepsilon\|_{L^2}.$$

Therefore, the boundedness of $\|S_\varepsilon u_\varepsilon\|_{L^2}$ is replaced with that of $\|u_\varepsilon\|_{L^2}$ which is guaranteed by the fact that $D(S)\cap D(B)\neq\emptyset$. That is, it is revealed in [45] that the solvability of $(\mathbf{NCGL})_\Omega$ is closely connected with sectoriality of $-\Delta$ in $L^q$ (see (2.4)). Once the structure of the proof is clear, it is possible to modify the abstract theory in a form to be applied to the pair of $S=-\Delta$ and $Bu=|u|^{q-2}u$, respectively, replaced with an elliptic operator in divergence form and non-linearity of non-power type (see Section 6.2).

*Remark* 2.6 (The relation between $S_\lambda$ and $B_\varepsilon$)*.* To deal with $(\mathbf{CGL})_\Omega$ in the subsequent sections we shall employ conditions of the form

$$|\mathrm{Im}\,(Su,B_\varepsilon u)_{L^2}|\le c_q\mathrm{Re}\,(Su,B_\varepsilon u)_{L^2}, \quad u\in D(S),\;\; \varepsilon>0 \tag{2.15}$$

instead of (2.10). Here we want to show that (2.10) is equivalent to (2.15). To this end, it suffices to show that (2.10) implies

$$|\mathrm{Im}\,(S_\lambda w,B_\varepsilon w)_{L^2}|\le c_q\mathrm{Re}\,(S_\lambda w,B_\varepsilon w)_{L^2}, \quad w\in L^2(\Omega),\;\; \lambda>0,\;\; \varepsilon>0. \tag{2.16}$$

Here $\{S_\lambda;\, \lambda>0\}$ denotes the family of Yosida approximations of $S$. In fact, we have (2.15) by letting $\lambda\downarrow 0$ in (2.16) with $w=u\in D(S)$. It is an advantage that there is nothing to care about the domains of $S$ and $B$ in (2.16). Now let $w\in L^2(\Omega)$. Then, since $B_\varepsilon = B(1+\varepsilon B)^{-1}$ $(\varepsilon>0)$, $\mathrm{Im}\,(S_\lambda w,B_\varepsilon w)_{L^2}$ is decomposed as

$$\begin{aligned}\mathrm{Im}\,(S_\lambda w,B_\varepsilon w)_{L^2} &= \varepsilon^{-1}\mathrm{Im}\,(S_\lambda w - S_\lambda(1+\varepsilon B)^{-1}w, w-(1+\varepsilon B)^{-1}w)_{L^2}\\ &\quad + \mathrm{Im}\,(S_\lambda(1+\varepsilon B)^{-1}w, B(1+\varepsilon B)^{-1}w)_{L^2}.\end{aligned}$$

The first term on the right-hand side vanishes by virtue of the selfadjointness of $S_\lambda$. Applying (2.10) to the second term on the right-hand side and then using the non-negativity of $S_\lambda$, we can obtain (2.15):

$$\begin{aligned}|\mathrm{Im}\,(S_\lambda w, B_\varepsilon w)_{L^2}| &\le c_q \mathrm{Re}\,(S_\lambda(1+\varepsilon B)^{-1}w, B(1+\varepsilon B)^{-1}w)_{L^2}\\ &\quad + \frac{c_q}{\varepsilon}\,\mathrm{Re}\,(S_\lambda w - S_\lambda(1+\varepsilon B)^{-1}w, w-(1+\varepsilon B)^{-1}w)_{L^2}\\ &= c_q\mathrm{Re}\,(S_\lambda(1+\varepsilon B)^{-1}w, B_\varepsilon w)_{L^2}\\ &\quad + c_q \mathrm{Re}\,(S_\lambda w - S_\lambda(1+\varepsilon B)^{-1}w, B_\varepsilon w)_{L^2}\\ &= c_q\mathrm{Re}\,(S_\lambda w, B_\varepsilon w)_{L^2}.\end{aligned}$$

Conversely, we can employ (2.11) to come back to (2.16) from (2.15).

*Remark* 2.7 (cf. [47]). Another example of a non-linear $m$-sectorial operator is given by the $p$-Laplacian $-\Delta_p$ ($p \in [2,\infty)$), with Dirichlet boundary condition, in $L^2(\Omega)$, where $\Omega$ is a bounded domain. In fact, integration by parts yields

$$((-\Delta_p)u - (-\Delta_p)v, u-v)_{L^2} = (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v, \nabla u - \nabla v)_{L^2}.$$

Therefore, it follows from (2.8) with $X = \mathbb{C}^N$ and $q = p$ that for $u$, $v \in D(\Delta_p)$,

$$|\mathrm{Im}\,((-\Delta_p)u - (-\Delta_p)v, u-v)_{L^2}| \le c_p\,\mathrm{Re}\,((-\Delta_p)u - (-\Delta_p)v, u-v)_{L^2};$$

note that $-\Delta_p$ is realized as a sub-differential of convex function $\varphi_p(u) := \frac{1}{p}\|\nabla u\|_{L^p}^p$ with effective domain $D(\varphi_p) := W_0^{1,p}(\Omega)$ (for the definition of sub-differentials see Section 4).

## 3. Complex Ginzburg-Landau equation (1) accretivity methods

The result on $(\mathbf{CGL})_\Omega$ with $u_0 \in L^2$ is stated in [47] and [49]. The role of the sectoriality constant $c_q$ in these works is stated as follows. The mapping $u \mapsto (\kappa + i\beta)|u|^{q-2}u$ defines an accretive operator when (**I**) $\kappa^{-1}|\beta| \le c_q^{-1}$, but it does not when (**II**) $\kappa^{-1}|\beta| > c_q^{-1}$. This leads to the decision of the classes of semigroups to which solution operators belong. Moreover, there is a big difference between these two cases. When $\kappa^{-1}|\beta| \le c_q^{-1}$, there is no upper restriction on the exponent $q-1$ of non-linear term, but when $\kappa^{-1}|\beta| > c_q^{-1}$, it is restricted by a constant depending on the spatial dimension: $1 \le q-1 \le 1 + 4/N$. Thus the result is devided into these two cases.

**Theorem 3.1.** *Let* $c_q := \dfrac{q-2}{2\sqrt{q-1}}$ $(q \ge 2)$ *in addition to* $N \in \mathbb{N}$, $\lambda$, $\kappa \in \mathbb{R}_+$ *and* $\alpha$, $\beta$, $\gamma \in \mathbb{R}$.
(**I**) (Accretive non-linearity). *Impose restriction* $\kappa^{-1}|\beta| \in [0, c_q^{-1}]$ *on the coefficient* $\kappa + i\beta$. *Then for every* $u_0 \in L^2(\Omega)$ *there exists a unique strong solution* $u(\cdot) \in C([0,\infty); L^2(\Omega))$ *to* $(\mathbf{CGL})_\Omega$ *satisfying the following*:

(3.1) $$u(\cdot) \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Omega)) \cap C^{0,1/2}_{\mathrm{loc}}(\mathbb{R}_+; H^1_0(\Omega)) \cap C^{0,1/q}_{\mathrm{loc}}(\mathbb{R}_+; L^q(\Omega)),$$

(3.2) $$(d/dt)u(\cdot),\ \Delta u(\cdot),\ |u|^{q-2}u(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Omega)),$$

(3.3) $$\|u(t)\|_{L^2} \le e^{\gamma t}\|u_0\|_{L^2} \quad \forall\, t \ge 0.$$

*Set* $U(t)u_0 := u(t)$. *Then* $\{U(t);\, t \ge 0\}$ *forms a quasi-contraction semigroup on* $L^2(\Omega)$:

(3.4) $$\|U(t)u_0 - U(t)v_0\|_{L^2} \le e^{\gamma t}\|u_0 - v_0\|_{L^2}, \quad u_0,\ v_0 \in L^2(\Omega).$$

(**II**) (General non-linearity). *Impose restriction* $\kappa^{-1}|\beta| \in (c_q^{-1}, \infty)$ *on the coefficient* $\kappa + i\beta$, *where the exponent* $q$ *of non-linearity is assumed to satisfy*

$$2 \le q \le 2 + \frac{4}{N}. \tag{3.5}$$

*Then for every* $u_0 \in L^2(\Omega)$ *there exists a unique strong solution* $u(\cdot) \in C([0,\infty); L^2)$ *to* $(\mathbf{CGL})_\Omega$ *satisfying the following*:

$$u(\cdot) \in C^{0,1/2}_{\rm loc}(\mathbb{R}_+; L^2(\Omega)) \cap C(\mathbb{R}_+; H^1_0(\Omega)), \tag{3.6}$$

$$(d/dt)u(\cdot),\ \Delta u(\cdot),\ |u|^{q-2}u \in L^2_{\rm loc}(\mathbb{R}_+; L^2(\Omega)), \tag{3.7}$$

$$\|u(t)\|_{L^2} \le e^{\gamma t}\|u_0\|_{L^2} \quad \forall\, t \ge 0. \tag{3.8}$$

*Set* $U(t)u_0 := u(t)$. *Then* $\{U(t);\, t \ge 0\}$ *forms a locally* Lipschitz *semigroup on* $L^2(\Omega)$:

$$\|U(t)u_0 - U(t)v_0\|_{L^2} \le e^{K(t, \|u_0\| \vee \|v_0\|)}\|u_0 - v_0\|_{L^2}, \quad u_0,\ v_0 \in L^2(\Omega), \tag{3.9}$$

*where* $K(t,M) := K_1 t + K_2 e^{2\gamma_+ + t}M^2$ *and* $\gamma_+ := \max\{0, \gamma\}$. *Furthermore,* $K_1$ *and* $K_2$ *are positive constants depending only on* $\lambda$, $\kappa$, $\beta$, $\gamma$, $q$, *and* $N$.

*Remark* 3.2 ($|\beta|$-dependence in case (**II**)). Two constants in $K(t,M)$ are given as

$$K_1 := \gamma + (1-\theta)(|\beta| - c_q^{-1}\kappa)C_1, \quad K_2 := \frac{\theta}{2\,q\,\kappa}(|\beta| - c_q^{-1}\kappa)C_1.$$

Then since $K_1 \to \gamma$ and $K_2 \to 0$ as $|\beta| \downarrow c_q^{-1}\kappa$, it follows that $K(t,M) \to \gamma\, t$ ($|\beta| \downarrow c_q^{-1}\kappa$) and hence (3.4) is the limiting case of (3.9). Here we note further that

$$\theta = f_N(q) := \frac{2(q-2)}{2q - N(q-2)} \in [0,1] \ \Leftrightarrow\ q \in \Big[2, 2 + \frac{4}{N}\Big]. \tag{3.10}$$

In fact, we see that

$$0 = f_N(2) \le f_N(q) \le f_N(2 + 4/N) = 1$$

(but $C_1 > 0$ is not so explicit).

*Remark* 3.3 (Smoothing effect). Define $S$ and $B$ as in (2.1). Then these operators are also useful in the proof of Theorem 3.1. Since $L^2 = \overline{D(S) \cap D(B)}$, the closure of $D(S) \cap D(B)$, (3.2) and (3.7) are described as

$$U(t)[\overline{D(S) \cap D(B)}] \subset D(S) \cap D(B) \quad \text{a.a. } t > 0. \tag{3.11}$$

This phenomenon is called the **smoothing effect** of $\{U(t)\}$ on the initial values which is peculiar to parabolic type evolution equations. In particular, if $U(t)$ maps $D(S) \cap D(B)$ into itself for $t > 0$ (cf. [4], [35], [58], [3]), then (3.11) holds for every $t > 0$.

Proof of Theorem 3.1 (**I**) is divided into two parts. The first part is concerned with initial values in $H^2 \cap H^1_0 \cap L^{2(q-1)}$. Then based on the first part the reasoning is extended to initial values in $L^2$. Proof of (**II**) will be outlined in the next section. Proof of (**I**) is summarized as follows. Under the restriction $\kappa^{-1}|\beta| \le c_q^{-1}$ a quasi-contraction semigroup $\{U(t)\}$ on $L^2$ is constructed. If $v_0 \in H^2 \cap H^1_0 \cap L^{2(q-1)}$, then $v(t) := U(t)v_0$ is a unique strong solution to $(\mathbf{CGL})_\Omega$. But we need the notion of subdifferential operators to show that for $u_0 \in L^2$, $u(t) := U(t)u_0$ is really a unique strong solution to $(\mathbf{CGL})_\Omega$.

*Proof of Theorem* 3.1 (**I**) (outline).

*Step* 1 (*Initial values in* $D(S) \cap D(B)$). First we describe the existence proof for the initial value $v_0 \in H^2 \cap H_0^1 \cap L^{2(q-1)}$ compared with that of Theorem 2.1.

$(\mathbf{ACP})_{00}$ in Section 2 is now replaced with

$$(\mathbf{ACP})_0 \quad \begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)Su(t) + (\kappa + i\beta)Bu(t) - \gamma u = 0 \quad \text{a.e. on } (0, \infty), \\ u(0) = v_0. \end{cases}$$

That is, $iS$ and $B$ in $(\mathbf{ACP})_{00}$ are replaced with $(\lambda + i\alpha)S$ and $(\kappa + i\beta)B$ as in $(\mathbf{ACP})_0$. On the one hand, $(\lambda+i\alpha)S$ keeps the same $m$-accretivity as $iS$ by virtue of non-negative selfadjointness of $S$. On the other hand, the accretivity of $(\kappa+i\beta)B$ is kept by (2.9) under condition $|\beta| \le c_q^{-1}\kappa$:

$$\begin{aligned} &\operatorname{Re}\{(\kappa + i\beta)(Bu - Bv, u - v)\} \\ &\ge \kappa \operatorname{Re}(Bu - Bv, u - v) - |\beta| \cdot |\operatorname{Im}(Bu - Bv, u - v)| \\ &\ge (c_q^{-1}\kappa - |\beta|)|\operatorname{Im}(Bu - Bv, u - v)| \ge 0. \end{aligned}$$

The inequality (2.13) of the separation property (implying the maximality of $iS+B$) is now replaced with

$$\lambda\|Sv\|_{L^2} \le \|(\lambda + i\alpha)Sv + (\kappa + i\beta)B_\varepsilon v\|_{L^2}, \quad v \in D(S), \ \varepsilon > 0 \tag{3.12}$$

because we employ (2.15) instead of (2.10) as noted in Remark 2.6; in this modification we are supposed to compute the integral $(Sv, B_\varepsilon v)_{L^2}$, $v \in D(S)$, using integration by parts (guaranteed by Lemma 4.5 (**a**) below). In the same way as in the case of $iS + B$ (3.12) yields the maximality of

$$A = A(\lambda + i\alpha, \kappa + i\beta) := (\lambda + i\alpha)S + (\kappa + i\beta)B. \tag{3.13}$$

Consequently, we can obtain the assertion on $(\mathbf{CGL})_\Omega$ corresponding to Theorem 2.1 for $(\mathbf{NCGL})_\Omega$. Proof of Theorem 3.1 (**I**) for $u_0 \in L^2$ is based on this assertion. Therefore, it is worth stating as a lemma. But, the difference from Theorem 2.1 is very little.

**Lemma 3.4.** *Let* $q \ge 2$. *Then for* $v_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ *there exists a unique strong solution* $v(\cdot) \in C([0,\infty); L^2(\Omega))$ *to* $(\mathbf{CGL})_\Omega$, *satisfying*

$$v(\cdot) \in C^{0,1}([0,T]; L^2(\Omega)) \cap C^{0,1/2}([0,T]; H_0^1(\Omega)) \cap C^{0,1/q}([0,T]; L^q(\Omega)),$$
$$(d/dt)v(\cdot), \ \Delta v(\cdot), \ |v|^{q-2}v(\cdot) \in L^\infty(0,T; L^2(\Omega)) \quad \forall \ T > 0.$$

*Besides, inequality* (2.2) *is replaced with*

$$\|U(t)v_0 - U(t)w_0\|_{L^2} \le e^{\gamma t}\|v_0 - w_0\|_{L^2}, \quad v_0, \ w_0 \in H^2 \cap H_0^1 \cap L^{2(q-1)}. \tag{3.14}$$

*Thus* $\{U(t); t \ge 0\}$ *can be extended to a quasi-contraction semigroup on* $L^2(\Omega)$.

The operator $iS + B$ in $(\mathbf{NCGL})_\Omega$ is replaced with quasi-$m$-accretive operator $A - \gamma = (\lambda + i\alpha)S + (\kappa + i\beta)B - \gamma$ in $(\mathbf{CGL})_\Omega$: $\operatorname{Re}((A-\gamma)u - (A-\gamma)v, u - v)_{L^2} \ge -\gamma\|u - v\|^2$. This explains the factor $e^{\gamma t}$ on the right-hand side of (3.14).

*Step* 2 (*Approximation of initial values in* $L^2$). The $m$-accretivity of $S + B = A(1 + i0, 1 + i0)$ is contained in those of $A$ (see (3.13)). This enables one to define $u_{0,n} := (1 + n^{-1}(S + B))^{-1}u_0$ for $u_0 \in L^2$. Then we see that $\{u_{0,n}\}$ is a sequence in $H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ satisfying

$$\|u_0 - u_{0,n}\|_{L^2} \to 0 \ (n \to \infty), \quad \|u_{0,n}\|_{L^2} \le \|u_0\|_{L^2} \ (n \in \mathbb{N}). \tag{3.15}$$

Hence (3.14) yields that there exists $u(\cdot) \in C([0,\infty); L^2(\Omega))$ such that

$$u(t) := U(t)u_0 = \lim_{n\to\infty} U(t)u_{0,n}, \quad t > 0. \tag{3.16}$$

It is clear that $u(0) = u_0$, however, we have to spend some time to prove that $\partial\varphi(u(\cdot))$, $\partial\psi(u(\cdot))$, $(d/dt)u(\cdot)$ make sense for $u(\cdot)$ given by (3.16) and hence the equation

$$\frac{du}{dt} + (\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0 \quad \text{a.e. on } (T^{-1}, T) \tag{3.17}$$

holds in $L^2(T^{-1}, T; L^2(\Omega))$ for every $T > 1$. Actually, because $T$ is arbitrary, the equation holds over the whole interval $(0,\infty)$. (3.17) is a new description of the CGL equation from the viewpoint of lower semicontinuous convex functions $\varphi$ and $\psi$. Here $\partial\varphi$ and $\partial\psi$ in the correspondence (between (3.17) and $(\mathbf{ACP})_0$)

$$\begin{cases} \partial\varphi(u) = Su = -\Delta u, & D(\partial\varphi) = H_0^1(\Omega) \cap H^2(\Omega), \\ \partial\psi(u) = Bu = |u|^{q-2}u, & D(\partial\psi) = L^{2(q-1)}(\Omega) \cap L^2(\Omega) \end{cases}$$

are respectively interpreted as the sub-differentials of $\varphi$ and $\psi$ on $L^2(\Omega)$:

$$\varphi(u) := \begin{cases} (1/2)\|\nabla u\|_{L^2}^2 & \text{for } u \in D(\varphi) := H_0^1(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{3.18}$$

$$\psi(u) := \begin{cases} (1/q)\|u\|_{L^q}^q & \text{for } u \in D(\psi) := L^2(\Omega) \cap L^q(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{3.19}$$

(for details see the beginning of Section 4). Now take a sequence $\{u_{0,n}\}$ as in (3.16). Then Lemma 3.4 yields that the approximate solution $u_n(t) := U(t)u_{0,n}$ in $C([0,\infty);\, L^2(\Omega))$ satisfies the following equation:

$$\frac{du_n}{dt} + (\lambda + i\alpha)\partial\varphi(u_n) + (\kappa + i\beta)\partial\psi(u_n) - \gamma u_n = 0 \quad \text{a.e. on } [0,\infty). \tag{3.20}$$

Put $\phi := \varphi$ or $\psi$, and $w := u_n$. Then as an advantage of the new viewpoint the chain rule for the composite function $\phi \circ w$ is available:

$$\frac{d}{dt}(\phi \circ w)(t) = \operatorname{Re}\Big(\partial\phi(w(t)), \frac{dw}{dt}\Big) \quad \text{a.a. } t \in [0,T] \quad \forall\, T > 0. \tag{3.21}$$

This leads us to several estimates of solutions by taking the inner product of (3.20) with $u_n(\cdot)$ and so on.

**Lemma 3.5.** *Let $\{\varphi,\, \psi\}$ be the pair of convex functions on $L^2$ with their sub-differentials $\{\partial\varphi,\, \partial\psi\}$ as defined above. Let $u_n(\cdot)$ be a strong solution to* $(\mathbf{CGL})_\Omega$ *in the sense of* Lemma 3.4, *satisfying equation* (3.20) *in terms of $\{\partial\varphi,\, \partial\psi\}$. For simplicity assume that $\lambda^{-1}|\alpha| \in [0, c_q^{-1}]$ in addition to $\kappa^{-1}|\beta| \in [0, c_q^{-1}]$. Then the inequality*

$$\begin{aligned} & t\,[\lambda\,\varphi(u_n(t)) + \kappa\,\psi(u_n(t))] \\ & + \int_0^t s\, e^{q\,\gamma_+(t-s)} \big(\lambda^2 \|\partial\varphi(u_n(t))\|_{L^2}^2 + \kappa^2 \|\partial\psi(u_n(t))\|_{L^2}^2\big)\, ds \\ & \le \frac{1}{4} e^{q\,\gamma_+ t} \|u_0\|_{L^2}^2 \end{aligned} \tag{3.22}$$

*holds for every $t > 0$. Here $\gamma_+ = \max\{0, \gamma\}$ as before.*

In fact, taking the inner product of (3.20) with $u_n(\cdot)$, it is easy to see that $\|u_n(t)\|_{L^2} \le e^{\gamma t}\|u_0\|_{L^2}$ $(t \ge 0)$ and

$$\int_0^t e^{q\,\gamma_+(t-s)}[\lambda\varphi(u_n(s)) + \kappa\psi(u_n(s))]\,ds \le \frac{1}{4}e^{q\,\gamma_+ t}\|u_0\|_{L^2}^2 \tag{3.23}$$

(in the latter case use condition $q \ge 2$). Next, we compute by taking the inner products of (3.20) with $\lambda\,\partial\varphi(u_n)$ and $\kappa\,\partial\psi(u_n)$, respectively. Adding these two equalities, we have, after some computations,

$$\begin{aligned}&\frac{d}{ds}\Big\{e^{q\,\gamma_+(t-s)}[\lambda\varphi(u_n(s)) + \kappa\psi(u_n(s))]\Big\}\\ &\quad + e^{q\,\gamma_+(t-s)}\big[\lambda^2\|\partial\varphi(u_n(s))\|_{L^2}^2 + \kappa^2\|\partial\psi(u_n(s))\|_{L^2}^2\big] \le 0.\end{aligned} \tag{3.24}$$

During the process we have used the assumption $\kappa^{-1}|\beta| \in [0, c_q^{-1}]$ and $\lambda^{-1}|\alpha| \in [0, c_q^{-1}]$ to delete the following two terms by their non-negativity:

$$\begin{aligned}&\mathrm{Re}\big\{(\lambda + i\alpha)(\partial\varphi(u_n), \partial\psi(u_n))_{L^2}\big\}\\ &= \lambda\mathrm{Re}\,(\partial\varphi(u_n), \partial\psi(u_n))_{L^2} - \alpha\mathrm{Im}\,(\partial\varphi(u_n), \partial\psi(u_n))_{L^2}\\ &\ge \big(c_q^{-1}\lambda - |\alpha|\big)|\mathrm{Im}\,(\partial\varphi(u_n), \partial\psi(u_n))_{L^2}| \ge 0,\end{aligned} \tag{3.25}$$

$$\begin{aligned}&\mathrm{Re}\big\{(\kappa + i\beta)(\partial\psi(u_n), \partial\varphi(u_n))_{L^2}\big\}\\ &= \kappa\mathrm{Re}\,(\partial\varphi(u_n), \partial\psi(u_n))_{L^2} + \beta\mathrm{Im}\,(\partial\varphi(u_n), \partial\psi(u_n))_{L^2}\\ &\ge \big(c_q^{-1}\kappa - |\beta|\big)|\mathrm{Im}\,(\partial\varphi(u_n), \partial\psi(u_n))_{L^2}| \ge 0.\end{aligned} \tag{3.26}$$

Actually, we do not need the restriction $\lambda^{-1}|\alpha| \in [0, c_q^{-1}]$ on the coefficient of $\partial\varphi = -\Delta$, however, we have shortened the reasoning. By the way, in the subsequent computation we want to make the left-hand side of (3.23). It is realized by the following integration by parts:

$$\begin{aligned}&\int_0^t s\,\frac{d}{ds}\Big\{e^{q\,\gamma_+(t-s)}[\lambda\varphi(u_n(s)) + \kappa\psi(u_n(s))]\Big\}\,ds\\ &= \Big[s\,e^{q\,\gamma_+(t-s)}\{\lambda\varphi(u_n(s)) + \kappa\psi(u_n(s))\}\Big]_{s=0}^{s=t}\\ &\quad - \int_0^t e^{q\,\gamma_+(t-s)}[\lambda\varphi(u_n(s)) + \kappa\psi(u_n(s))]\,ds.\end{aligned}$$

Taking this into account, we integrate (3.24) multiplied by $s$. Then (3.23) yields (3.22). This finishes the preparation for the final step in Proof of Theorem 3.1 (**I**).

*Step* 3 (*Initial values in* $L^2$). We shall show that $u(\cdot)$ defined by (3.16) is a strong solution to $(\mathbf{CGL})_\Omega$. To this end let $T > 1$. Then, replacing the interval $(0, t)$ of integration in (3.22) with a smaller one $(T^{-1}, T)$ (satisfying $0 < T^{-1} < T \le t$) and noting that $s \ge T^{-1}$, we can find a constant $M > 0$ such that

$$\int_{T^{-1}}^T \big(\|\partial\varphi(u_n(s))\|_{L^2}^2 + \|\partial\psi(u_n(s))\|_{L^2}^2\big)\,ds \le \frac{1}{4}MTe^{q\,\gamma_+ T}\|u_0\|_{L^2}^2.$$

This implies that $\{\partial\varphi(u_n(\cdot))\}$ and $\{\partial\psi(u_n(\cdot))\}$ are bounded in $L^2(T^{-1}, T;\, L^2(\Omega))$. The equation (3.20) yields that so is $\{(du_n/dt)(\cdot)\}$. Since $\partial\varphi$, $\partial\psi$ and $d/dt$ are demiclosed as operators in $L^2(T^{-1}, T;\, L^2(\Omega))$, it follows from (3.16) that for a.a. $t \in (T^{-1}, T)$,

$$u(t) = U(t)u_0 \in D(\partial\varphi) \cap D(\partial\psi) = H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega). \tag{3.27}$$

Besides, three sequences $\{\partial\varphi(u_n(\cdot))\}$, $\{\partial\psi(u_n(\cdot))\}$, $\{(du_n/dt)(\cdot)\}$ converges, respectively, to $\partial\varphi(u(\cdot))$, $\partial\psi(u(\cdot))$, $(du/dt)(\cdot)$ weakly in $L^2(T^{-1}, T; L^2(\Omega))$. Since $T > 1$ is arbitrary, (3.17) holds on the whole interval $(0, \infty)$.

To prove the property (3.2) of the solution take $t_0 \in (0,1)$ satisfying (3.27). Then it is possible to employ $u(t_0)$ as an initial value in Lemma 3.4. Therefore, we can get a new expression of the solution over the interval $[t_0, t_0^{-1}]$:

$$u(t) = U(t)u_0 = U(t-t_0)U(t_0)u_0 = U(t-t_0)u(t_0), \quad t \in [t_0, t_0^{-1}].$$

This implies that $u(\cdot)$ has the property stated in Lemma 3.4 over the interval $[t_0, t_0^{-1}]$. Since $t_0 > 0$ is almost arbitrary, it follows that every unique strong solution $u(\cdot)$ to $(\mathbf{CGL})_\Omega$ has property (3.2), and hence property (3.1), even when $u_0 \in L^2$.

## 4. Abstract theorem — Proof of Theorem 3.1 (II)

What was mentioned in previous Sections 2 and 3 makes it clear that abstract theory of evolution equations (with accretive non-linearity) is directly applicable to $(\mathbf{CGL})_\Omega$. In this section we introduce another abstract theorem (with non-accretive non-linearity) to prove Theorem 3.1 (**II**) and outline the verification of those conditions assumed in the theorem (for full details of this section see [49]).

Let $X$ be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $S$ be a non-negative selfadjoint operator in $X$, such as $-\Delta$ with Dirichlet boundary condition in Sections 2 and 3. Next, let $\psi : X \to (-\infty, \infty]$ be a lower semi-continuous convex function with effective domain $D(\psi) := \{u \in X;\ \psi(u) < \infty\} \neq \emptyset$. Then, generally speaking, the sub-gradient $\partial\psi(u)$ of $\psi$ at $u \in D(\psi)$ is defined by

$$\partial\psi(u) := \{f \in X;\ \psi(v) - \psi(u) \geq \operatorname{Re}(f, v-u)\ \ \forall\, v \in D(\psi)\}.$$

Here $\partial\psi(u)$ seems to be just like a set, that is, $\partial\psi$ is a (possibly multi-valued) mapping, called a **sub-differential operator**, from $D(\partial\psi) := \{u \in D(\psi);\ \partial\psi(u) \neq \emptyset\}$ to $X$. But we restrict ourselves here to the case of $\psi \geq 0$ and single-valued $\partial\psi$. It is quite important that we can prove the $m$-accretivity of $\partial\psi$ and that the non-negative self-adjoint operator $S$ is also expressed as the subdifferential $S = \partial\varphi$ of a convex function. Here $\varphi : X \to [0, \infty]$ is defined by

$$\varphi(u) := \begin{cases} (1/2)\|S^{1/2}u\|^2 & \text{if } \ u \in D(\varphi) := D(S^{1/2}), \\ \infty & \text{otherwise} \end{cases}$$

($S^{1/2}$ denotes the square root of $S$). Under this notation $(\mathbf{CGL})_\Omega$ is described as an abstract Cauchy problem in $X = L^2$:

$$(\mathbf{ACP}) \quad \begin{cases} (d/dt)u + (\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0 \quad \text{a.e. on } \ (0,\infty), \\ u(0) = u_0. \end{cases}$$

The validity of the viewpoint of sub-differential operators is well known, especially in the field of non-linear parabolic type evolution equations in *real* Hilbert spaces; a typical example is given by (3.17) with $\alpha = \beta = 0$. In fact, there are so many investigations the representatives of which are Brezis [4] (the case of $\kappa \geq 0$) and Otani [50], [51] (the case of $\kappa < 0$) ([63], [58] and [3] are expository books published relatively recently).

To write down the assumption imposed on (**ACP**) we need some deep properties of the family $\{(\partial\psi)_\varepsilon;\, \varepsilon > 0\}$ of Yosida approximations of $m$-sectorial operator $\partial\psi$:

$$(\partial\psi)_\varepsilon := \varepsilon^{-1}(1 - J_\varepsilon), \quad J_\varepsilon = J_\varepsilon(\partial\psi) := (1 + \varepsilon\partial\psi)^{-1} \quad \forall\, \varepsilon > 0.$$

Here we have the formal associative law $(\partial\psi)_\varepsilon = \partial(\psi_\varepsilon)$, where $\psi_\varepsilon$ is the **Moreau-Yosida regularization** of convex function $\psi$ defined by

$$\psi_\varepsilon(v) := \min_{w\in X}\Big\{\psi(w) + \frac{1}{2\varepsilon}\|w - v\|^2\Big\} \quad \forall\, v \in X \quad \forall\, \varepsilon > 0$$

(cf. [4, Proposition 2.11] or [3, Theorem 2.9]). Therefore, the Yosida approximation of $\partial\psi$ is simply denoted by $\partial\psi_\varepsilon := (\partial\psi)_\varepsilon = \partial(\psi_\varepsilon)$.

Now we introduce seven conditions imposed on $(\varphi,\psi)$ and $(\partial\varphi,\partial\psi)$:

(**A1**): $\exists\, q \in [2, \infty);\ \psi(\zeta u) = |\zeta|^q\psi(u)\ \ \forall\, u \in D(\psi)\ \forall\, \zeta \in \mathbb{C}\ \ (\mathrm{Re}\,\zeta > 0)$.

(**A2**): $\exists\, c_q \ge 0\ \forall\, u,\, v \in D(\partial\psi)$,

$$|\mathrm{Im}\,(\partial\psi(u) - \partial\psi(v), u - v)| \le c_q \mathrm{Re}\,(\partial\psi(u) - \partial\psi(v), u - v).$$

(**A3**): $|\mathrm{Im}\,(\partial\varphi(u), \partial\psi_\varepsilon(u))| \le c_q \mathrm{Re}\,(\partial\varphi(u), \partial\psi_\varepsilon(u))\ \forall\, u \in D(\partial\varphi)$; here $c_q$ is the same as in condition (**A2**) $\big(q \in [2, \infty)\big)$.

(**A4**): $(\partial\psi(u), \partial\psi_\varepsilon(u)) \ge 0\ \ \forall\, u \in D(\partial\psi)\ \forall\, \varepsilon > 0$.

(**A5**): $\exists\, \theta \in [0, 1]$ and for every $\eta > 0$ there exists a constant $C_1 = C_1(\eta) > 0$ such that for $\varepsilon > 0$ and $u,\, v \in D(\varphi) \cap D(\psi)$,

$$|\mathrm{Im}\,(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| \le \eta\, \varphi(u - v) + C_1\Big[\frac{\psi(u) + \psi(v)}{2}\Big]^\theta \|u - v\|^2.$$

(**A6**): For every $\eta > 0$ there exists a constant $C_2 = C_2(\eta) > 0$ such that for $\varepsilon > 0$ and $u \in D(\partial\varphi)$,

$$|\mathrm{Im}\,(\partial\varphi(u), \partial\psi_\varepsilon(u))| \le \eta\|\partial\varphi(u)\|^2 + C_2\psi(u)^\theta\varphi(u).$$

Here $\theta \in [0, 1]$ is the same as in condition (**A5**).

(**A7**): There exists a constant $C_3 > 0$ such that for $u,\, v \in D(\partial\psi)$ and $\nu$, $\mu > 0$,

$$|\mathrm{Im}\,(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)| \le C_3|\nu - \mu|\big(\sigma\|\partial\psi(u)\|^2 + \tau\|\partial\psi(v)\|^2\big).$$

Here $\sigma,\, \tau > 0$ are two constants satisfying $\sigma + \tau = 1$.

*Remark* 4.1. First, (**A1**) implies $\psi(0) = 0$ so that it follows from $\psi \ge 0$ that $0 \in D(\partial\psi)$ and $\partial\psi(0) = 0$. Thus, the $m$-accretivity of $\partial\varphi + \partial\psi$ (guaranteed by (**A2**), (**A3**)) yields the simple estimate $\|(1 + n^{-1}(\partial\varphi + \partial\psi)^{-1}\| \le 1$ $(n \in \mathbb{N})$. Second, we see from their contents that (**A5**), (**A6**) are, respectively, regarded as supplementary to (**A2**), (**A3**) which are the key in Section 2. Note further that the constant $\theta$ appearing in (**A5**), (**A6**) is computed as in (3.10). Finally, condition (**A7**) has been introduced in [49] for the first time to show that the family of approximate solutions satisfies Cauchy's convergence condition (without assuming the compactness of the sub-level set of $\varphi$) (see also [42]).

**Theorem 4.2.** *Let $c_q$ be the sectoriality constant of $\partial\psi$ as in condition* (**A2**) *$(q \ge 2)$ in addition to $\lambda$, $\kappa \in \mathbb{R}_+$ and $\alpha$, $\beta$, $\gamma \in \mathbb{R}$. Assume that conditions* (**A1**)$-$(**A7**) *are satisfied. Impose restriction $c_q^{-1} < \kappa^{-1}|\beta|$ on $\kappa + i\beta$ $(\kappa > 0)$. Then for every $u_0 \in \overline{D(\partial\varphi) \cap D(\partial\psi)}$ there exists a unique strong solution $u(\cdot) \in C([0, \infty); X)$ to* (**ACP**) *satisfying the following*:

(**a**) $u(\cdot) \in C^{0,1/2}_{\mathrm{loc}}(\mathbb{R}_+; X)$, *with* $\|u(t)\| \le e^{\gamma t}\|u_0\| \ \forall\, t \ge 0$,
(**b**) $\partial\varphi(u(\cdot))$, $\partial\psi(u(\cdot))$, $(du/dt)(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}_+; X)$,
(**c**) $\varphi(u(\cdot))$, $\psi(u(\cdot)) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$.

*Next let* $\theta$ *be as in conditions* (**A5**), (**A6**). *For two constants* $K_1 := \gamma + (1-\theta)C_0C_1$ *and* $K_2 := (\theta/2q\kappa)C_0C_1$ *put*

$$K(t, M) := K_1\, t + K_2\, e^{2\gamma_+ t} M^2, \tag{4.1}$$

*where* $C_0 := |\beta| - c_q^{-1}\kappa$ *and* $C_1$ *is as given in condition* (**A5**). *Then the solution operators* $U(t)$ (*induced by* $u_0 \mapsto u(\cdot)$ *and* $v_0 \mapsto v(\cdot)$) *satisfies* (*local*) *Lipschitz continuity*:

$$\|U(t)u_0 - U(t)v_0\| \le e^{K(t, \|u_0\| \vee \|v_0\|)}\|u_0 - v_0\| \quad \forall\, t \ge 0. \tag{4.2}$$

*In particular, if* $X = \overline{D(\partial\varphi) \cap D(\partial\psi)}$, *then the family* $\{U(t)\}$ *forms a semigroup on* $X$.

The proof of Theorem 4.2 is divided into two parts just like as in the proof of Theorem 3.1 (**I**) (though the abstract setting is emphasized in Theorem 4.2). In fact, the initial values in $D(\partial\varphi) \cap D(\partial\psi)$ and $\overline{D(\partial\varphi) \cap D(\partial\psi)}$ are respectively dealt with in the first and second parts. Therefore the key to the proof is to establish the next

**Proposition 4.3.** *Let* $c_q$ *be the sectoriality constant of* $\partial\psi$ ($q \ge 2$) *in addition to* $\lambda$, $\kappa \in \mathbb{R}_+$ *and* $\alpha$, $\beta$, $\gamma \in \mathbb{R}$. *Assume that conditions* (**A1**) $-$ (**A7**) *are satisfied. For simplicity assume that* $\lambda^{-1}|\alpha| \in [0, c_q^{-1}]$ *in addition to* $c_q^{-1} < \kappa^{-1}|\beta|$. *Then for every* $v_0 \in D(\partial\varphi) \cap D(\partial\psi)$ *there exists a unique strong solution* $v(\cdot) \in C([0,\infty); X)$ *to* (**ACP**) *satisfying the following*:
(**a**)′ $v(\cdot) \in C^{0,1/2}([0,T]; X) \ \forall\, T > 0$, *with* $\|v(t)\| \le e^{\gamma t}\|v_0\| \ \forall\, t \ge 0$,
(**b**)′ $\partial\varphi(v(\cdot))$, $\partial\psi(v(\cdot))$, $(dv/dt)(\cdot) \in L^2([0,T]; X) \ \forall\, T > 0$,
(**c**)′ $\varphi(v(\cdot))$, $\psi(v(\cdot)) \in W^{1,1}(0,T) \ \forall\, T > 0$, *with*

$$2\,\lambda \int_0^t \varphi(v(s))\, ds + q\,\kappa \int_0^t \psi(v(s))\, ds \le \frac{1}{2} e^{2\,\gamma_+ t}\|v_0\|^2 \quad \forall\, t \ge 0. \tag{4.3}$$

*Here for two solutions* $v_0 \mapsto v(\cdot)$ *and* $w_0 \mapsto w(\cdot)$ ($w_0 \in D(\partial\varphi) \cap D(\partial\psi)$) *one has*

$$\|v(t) - w(t)\| \le e^{K(t, \|v_0\| \vee \|w_0\|)}\|v_0 - w_0\| \quad \forall\, t \ge 0, \tag{4.2$'$}$$

*where* $K(t, M)$ *is the same function as in* Theorem 4.2.

To prove Proposition 4.3 we consider a family of problems approximate to (**ACP**):

$$(\mathbf{ACP}; A\text{-}B_\varepsilon) \qquad \begin{cases} (d/dt)v_\varepsilon(t) + Av_\varepsilon(t) + B_\varepsilon v_\varepsilon(t) = 0, & t > 0, \\ v(0) = v_0 \in D(\partial\varphi) \cap D(\partial\psi). \end{cases}$$

Here we define the accretive operator $A$ and its Lipschitz perturbation $B$ as follows:

$$A := (\lambda + i\alpha)\partial\varphi + (\kappa + i\,\delta c_q^{-1}\kappa)\partial\psi, \quad D(A) := D(\partial\varphi) \cap D(\partial\psi),$$
$$B_\varepsilon := i\,(\beta - \delta c_q^{-1}\kappa)\partial\psi_\varepsilon - \gamma, \quad D(B_\varepsilon) := D(\partial\psi_\varepsilon) = X;$$

note that $|\delta| = 1$ and $|\beta - \delta\, c_q^{-1}\kappa| = |\beta| - c_q^{-1}\kappa$ $(= C_0 > 0)$ because we take $\delta := \operatorname{sgn}\beta$ for $\beta \ne 0$. This means that the coefficient of $\partial\psi_\varepsilon$ (in the definition of $B_\varepsilon$) does not vanish. On the one hand, $A$ is nothing but the operator given by

(3.13) with $\beta = \beta_0 := \delta\, c_q^{-1}\kappa$. Since $\kappa^{-1}|\beta_0| = c_q^{-1}$ (this unifies two endpoint cases in Theorem 3.1 (**I**)), $A$ becomes $m$-accretive in $X$. Note further that

$$(4.4)\quad \lambda\|\partial\varphi(u)\| \le \|(\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\,\delta\, c_q^{-1}\kappa)\partial\psi(u)\| \quad \forall\, u \in D(\partial\varphi)\cap D(\partial\psi)$$

which is derived by letting $\varepsilon \downarrow 0$ in (3.12) with $\beta = \beta_0$. That is, in our strategy $A$ is selected at two endpoints of its $m$-accretivity (conditions (**A2**), (**A3**) are used here). On the other hand, $B_\varepsilon + \gamma$ is Lipschitz continuous on $X$ as a constant multiple of Yosida approximation of $\partial\psi$ (though it loses the accretivity because the coefficient of $\partial\psi$ is pure imaginary). Therefore, we can apply to (**ACP**; $A$-$B_\varepsilon$) the well-known fact that the solvability of (ACP) with $m$-accretive operator is stable under Lipschitz perturbation. Namely, for every $v_0 \in D(A)$ there exists a unique strong solution $v_\varepsilon(\cdot) \in C([0,\infty); X)$ to the approximate problem (**ACP**; $A$-$B_\varepsilon$) satisfying

$$(4.5)\qquad \begin{aligned} &v_\varepsilon(\cdot) \in C^{0,1}([0,T];\, X) \quad \forall\, T > 0,\\ &Av_\varepsilon(\cdot),\ (dv_\varepsilon/dt)(\cdot) \in L^\infty(0,T;X) \quad \forall\, T > 0\end{aligned}$$

(cf. [58, Corollary IV.4.1]). It follows from (4.5) and (4.4) that

$$\partial\varphi(v_\varepsilon(\cdot)) \in L^\infty(0,T;X).$$

This concludes that $\varphi\circ v_\varepsilon \in W^{1,1}(0,T)$ with its derivative given by (3.21) with $\phi := \varphi$ and $w := v_\varepsilon$:

$$\frac{d}{dt}\varphi(v_\varepsilon(t)) = \mathrm{Re}\Big(\partial\varphi(v_\varepsilon(t)), \frac{dv_\varepsilon}{dt}(t)\Big) \quad \text{a.a. } t \in [0,T]\ \ \forall\, T > 0.$$

Here condition (**A1**) yields that $\mathrm{Re}\,(\partial\psi(v_\varepsilon), v_\varepsilon) = q\,\psi(v_\varepsilon)$ and $\mathrm{Im}\,(\partial\psi(v_\varepsilon), v_\varepsilon) = 0 = \mathrm{Im}\,(\partial\psi_\varepsilon(v_\varepsilon), v_\varepsilon)$. Consequently, we can obtain $\|v_\varepsilon(t)\| \le e^{\gamma t}\|v_0\|$ $(t \ge 0)$ and (4.3) by taking the inner product of equation in (**ACP**; $A$-$B_\varepsilon$) with $v_\varepsilon(\cdot)$. This process is almost the same as that to (3.23) (though condition $q \ge 2$ is not required). Finally, we have

$$(4.6)\qquad \varphi(v_\varepsilon(t)) + \frac{\lambda}{2}\int_0^t \|\partial\varphi(v_\varepsilon(s))\|^2\,ds \le e^{k(t,\|v_0\|)}\varphi(v_0),$$

$$(4.7)\qquad \psi(v_\varepsilon(t)) + \frac{\kappa}{2}\int_0^t \|\partial\psi(v_\varepsilon(s))\|^2\,ds \le e^{q\gamma_+ t}\psi(v_0).$$

Here it is possible to give the proofs of (4.6) and (4.7) separately because of the extra assumption $\lambda^{-1}|\alpha| \in [0, c_q^{-1}]$. But, when $\lambda^{-1}|\alpha| > c_q^{-1}$, we need (4.6) to prove (4.7). Note further that $k(t,\|v_0\|)$ on the right-hand side of (4.6) is computed as

$$(4.8)\qquad k(t,\|v_0\|) := k_1 t + k_2 e^{2\,\gamma_+ t}\|v_0\|^2,$$

where $k_1 := 2\gamma_+ + (1-\theta)C_0$ and $k_2 := (\theta/(2\,q\,\kappa))C_0C_2$ with $C_0 = |\beta| - c_q^{-1}\kappa$. In fact, taking the inner product of the equation with $\partial\varphi(v_\varepsilon)$, we have

$$\frac{d}{ds}\varphi(v_\varepsilon) + \frac{\lambda}{2}\|\partial\varphi(v_\varepsilon)\|^2 \le 2\,\gamma\,\varphi(v_\varepsilon) + C_0\big\{|\mathrm{Im}\,(\partial\varphi(v_\varepsilon), \partial\psi_\varepsilon(v_\varepsilon))| - \eta\|\partial\varphi(v_\varepsilon)\|^2\big\},$$

where we have set $\eta := \lambda/(2C_0)$. To simplify the computation here we have again used two endpoint cases of inequality (3.26) (in which $\beta = \delta\, c_q^{-1}\kappa$):

$$\mathrm{Re}\,\big\{(\kappa + i\,\delta\, c_q^{-1}\kappa)(\partial\psi(v_\varepsilon), \partial\varphi(v_\varepsilon))_{L^2}\big\} \ge 0.$$

Next, applying condition (**A6**), we obtain

$$\frac{d}{ds}\varphi(v_\varepsilon) + \frac{\lambda}{2}\|\partial\varphi(v_\varepsilon)\|^2 \le \big[2\,\gamma + C_0C_2\psi(v_\varepsilon)^\theta\big]\varphi(v_\varepsilon) \le k_\varepsilon(s)\varphi(v_\varepsilon). \tag{4.9}$$

Here we have $k_\varepsilon(s) := k_1 + k_2\big\{2\,q\,\kappa\,\psi(v_\varepsilon(s))\big\} \ge 0$ after applying Young's inequality to $\psi(v_\varepsilon)^\theta$ $\big((1/\theta)^{-1} + [1/(1-\theta)]^{-1} = 1\big)$. That is, (**A6**) is assumed to evaluate the term containing $B_\varepsilon v_\varepsilon$. It then follows from (4.3) and (4.8) that

$$\int_0^t k_\varepsilon(s)\,ds \le k(t, \|v_0\|) = k_1 t + k_2 e^{2\,\gamma_+ t}\|v_0\|^2. \tag{4.10}$$

Now (4.9) and (4.10) yield (4.6). On the other hand, the inner product of the equation with $\partial\psi(v_\varepsilon)$ yields

$$(d/ds)\psi(v_\varepsilon) + \kappa\|\partial\psi(v_\varepsilon)\|^2 \le q\gamma_+\psi(v_\varepsilon).$$

The computation to lead us to (4.7) is also simplified by the non-negativity of (3.25) which is a consequence of the extra restriction $\lambda^{-1}|\alpha| \in [0, c_q^{-1}]$.

**Lemma 4.4** (Convergence estimate)**.** *Let* $\{v_\varepsilon(\cdot)\}_{\varepsilon>0}$ *be a family of solutions to approximate problem* (**ACP**; $A$-$B_\varepsilon$) *with constant* $C_0 = |\beta| - c_q^{-1}\kappa > 0$. *Assume that conditions* (**A1**)$-$(**A7**) *are satisfied. Then there exists a limit function* $v(\cdot) \in C([0,\infty); X)$ *of the family* $\{v_\varepsilon(\cdot)\}$ *satisfying the initial condition* $v(0) = v_0 \in D(\partial\varphi) \cap D(\partial\psi)$ *with the rate of uniform convergence*:

$$\max_{t\in[0,T]} \|v(t) - v_\varepsilon(t)\|^2 \le 2\,\kappa^{-1}C_0C_3\psi(v_0)e^{q\gamma_+ T + 2K(T,\|v_0\|)}\varepsilon \quad (\varepsilon > 0). \tag{4.11}$$

*Here* $C_3$ *is the constant in condition* (**A7**) *and* $K(t, M)$ *is the increasing function in* $t$ *introduced in* Theorem 4.2.

We use $\nu$, $\mu > 0$ instead of the original suffix $\varepsilon$ to outline the proof of convergence (4.11). Then it suffices to show that

$$\|v_\nu(t) - v_\mu(t)\|^2 \le 2\,\kappa^{-1}C_0C_3\psi(v_0)e^{q\gamma_+ t + 2K(t,\|v_0\|)}|\nu - \mu|. \tag{4.12}$$

First it follows from the equation in (**ACP**; $A$-$B_\varepsilon$) and condition (**A2**) that

$$\begin{aligned} &2^{-1}(d/dt)\|v_\nu - v_\mu\|^2 - \gamma\|u_\nu - u_\mu\|^2 \\ &\le -2\lambda\varphi(v_\nu - v_\mu) + C_0|\mathrm{Im}\,(\partial\psi_\nu(v_\nu) - \partial\psi_\mu(v_\mu), v_\nu - v_\mu)|. \end{aligned} \tag{4.13}$$

Noting again that $\mathrm{Im}(\partial\psi_\varepsilon(u_\mu), u_\mu) = 0$ ($\varepsilon = \mu$ or $\nu$), the inner product in the second term on the right-hand side of (4.13) is rewritten as

$$\begin{aligned} &\mathrm{Im}\,(\partial\psi_\nu(v_\nu) - \partial\psi_\mu(v_\mu), v_\nu - v_\mu) \\ &= \mathrm{Im}\,(\partial\psi_\nu(v_\nu) - \partial\psi_\nu(v_\mu), v_\nu - v_\mu) + \mathrm{Im}\,(\partial\psi_\nu(v_\mu) - \partial\psi_\mu(v_\mu), v_\nu). \end{aligned}$$

Setting $\eta := 2\lambda/C_0$, we see from (4.13) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_\nu - v_\mu\|^2 - \gamma\|v_\nu - v_\mu\|^2 \\ &\le C_0\big\{|\mathrm{Im}\,(\partial\psi_\nu(v_\nu) - \partial\psi_\nu(v_\mu), v_\nu - v_\mu)| - \eta\,\varphi(v_\nu - v_\mu)\big\} \\ &\quad + C_0|\mathrm{Im}\,(\partial\psi_\nu(v_\mu) - \partial\psi_\mu(v_\mu), v_\nu)|. \end{aligned}$$

Now conditions (**A5**) and (**A7**) apply to give a differential inequality

$$\frac{1}{2}\frac{d}{dt}\|v_\nu - v_\mu\|^2 - C_0C_3|\nu - \mu|\big(\sigma\|\partial\psi(v_\mu)\|^2 + \tau\|\partial\psi(v_\nu)\|^2\big) \le \Psi(v_\nu, v_\mu)\|v_\nu - v_\mu\|^2.$$

Here we employ the form of $\Psi(v,w)$ after applying Young's inequality to $[\psi(v_\nu)+\psi(v_\mu)]^\theta$:

$$\begin{aligned}\Psi(v,w) &:= \{\gamma + C_0C_1(1-\theta)\} + C_0C_1\theta\frac{\psi(v)+\psi(w)}{2}\\ &= K_1 + K_2\{q\kappa(\psi(v)+\psi(w))\},\end{aligned}$$

where $K_j$ $(j=1,\,2)$ is given as in (4.1). Hence (4.3) yields that

$$\int_0^t \Psi(v_\nu(s),v_\mu(s))\,ds \le K(t,\|v_0\|) = K_1\,t + K_2e^{2\gamma_+t}\|v_0\|^2.$$

Besides, we see from (4.7) that

$$\int_0^t \big(\sigma\|\partial\psi(v_\nu(s))\|^2 + \tau\|\partial\psi(v_\mu(s))\|^2\big)\,ds \le \frac{1}{\kappa}e^{q\gamma_+t}\psi(v_0).$$

Consequently, by integrating the differential inequality we can obtain (4.12). This finishes the essential part of Proof of Proposition 4.3. In fact, it will be easily understood that the computation started from (4.13) can be employed to derive the continuous dependence $(4.2)'$ of solutions on their initial values.

**Verification of conditions in Theorem 4.2.** First conditions (**A5**), (**A6**) have already appeared in [48] where a compactness condition is assumed instead of conditions (**A2**), (**A3**), and (**A7**). That is, it is observed that condition (**A7**) in addition to those conditions so far introduced is assumed in [49] to avoid the compactness condition. In the integration by parts when we compute $(-\Delta u,\partial\psi_\varepsilon(u))_{L^2}$ in conditions (**A3**) and (**A6**) it is required to differentiate $\partial\psi_\varepsilon(u) = \varepsilon^{-1}(u-(1+\varepsilon\partial\psi)^{-1}u)$ as a function of the space variable $x$. On the other hand, when we verify condition (**A7**), the question is the differentiation of $\partial\psi_\varepsilon(u)$ as a function of the parameter $\varepsilon$, exemplified as

$$\partial\psi_\nu(f) - \partial\psi_\mu(f) = \int_\mu^\nu \frac{\partial}{\partial\varepsilon}\big[\partial\psi_\varepsilon(f)\big]\,d\varepsilon.$$

To carry out these computations it suffices to prepare the following lemma on the differentiation of inverse functions. In books on calculus it is usually stated that such formulas of the derivatives can be constructed only locally. But thanks to the accretivity of $\partial\psi$ the assertion in Lemma 4.5 makes sense globally.

**Lemma 4.5.** *For $\varepsilon\in[0,\infty)$ and $x\in\Omega$ put*

$$u_\varepsilon(x) := \begin{cases}(1+\varepsilon\,\partial\psi)^{-1}f(x), & (\varepsilon>0),\\ f(x), & (\varepsilon=0),\end{cases}$$

*that is, $u_\varepsilon(x)+\varepsilon|u_\varepsilon(x)|^{q-2}u_\varepsilon(x) = f(x)$. Then one has the following:*

(**a**) *If $f\in H_0^1(\Omega)$, then $u_\varepsilon\in H_0^1(\Omega)$ (as a function of $x$) with*

$$\nabla_x u_\varepsilon = \begin{cases}\dfrac{1}{1+\varepsilon|u_\varepsilon|^{q-2}}\nabla_x f - \varepsilon\dfrac{q-2}{\mathrm{Jac}}|u_\varepsilon|^{q-4}u_\varepsilon\mathrm{Re}(\overline{u_\varepsilon}\,\nabla_x f), & (\varepsilon>0),\\ \nabla_x f, & (\varepsilon=0).\end{cases}$$

*Here* $\mathrm{Jac} := (1+\varepsilon|u_\varepsilon|^{q-2})(1+\varepsilon(q-1)|u_\varepsilon|^{q-2})$. *In particular, $H_0^1(\Omega)\cap C^1(\overline{\Omega})$ is invariant under $(1+\varepsilon\,\partial\psi)^{-1}$ for $\varepsilon\in[0,\infty)$.*

(**b**) *If* $f \in D(\partial\psi)$, *then for every* $E > 0$, $u_\varepsilon \in C^1([0,E];L^2(\Omega))$ (*as a function of* $\varepsilon$) *with*

$$\frac{\partial u_\varepsilon}{\partial \varepsilon} = \begin{cases} -\dfrac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}}\partial\psi_\varepsilon(f), & (0 < \varepsilon \le E), \\ -\partial\psi(f), & (\varepsilon = 0). \end{cases}$$

(**c**) *If* $f \in L^2(\Omega)$, *then for every* $\varepsilon_0 \in (0,E)$ $u_\varepsilon \in H^1(\varepsilon_0,E;L^2(\Omega))$ (*as a function of* $\varepsilon$) *with*

$$\frac{\partial u_\varepsilon}{\partial \varepsilon} = -\frac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}}\partial\psi_\varepsilon(f) \quad \text{a.e. on } (\varepsilon_0, E).$$

For a proof of Lemma 4.5 (**a**) see [47, Lemma 6.1]; Lemma 4.5 (**b**) and (**c**) are proved in [49, Proposition 4.4].

## 5. Complex Ginzburg-Landau equation (2) compactness methods

This section is devoted to the compactness methods which can be applied to the Cauchy problem $(\mathbf{CGL})_{\mathbb{R}^N}$. The key idea is to consider approximate problems where $-\Delta$ is replaced with such a family of Schrödinger operators $-\Delta + V_R$ with compact resolvent:

$$(\mathbf{CGL})_R \qquad \begin{cases} \dfrac{\partial u}{\partial t} + (\lambda + i\alpha)(-\Delta + V_R)u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 \\ \hfill \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), \hfill x \in \mathbb{R}^N. \end{cases}$$

Here $\{V_R;\, R \ge 0\}$ is a family of potentials diverging at infinity as defined by

$$V_R(x) := \begin{cases} (|x| - R)^2, & (|x| > R), \\ 0, & (|x| \le R). \end{cases}$$

That is, $V_0(x) = |x|^2$ is the usual **harmonic potential**. Let $Q$ be the multiplication operator by $V_R$. Then $S + Q = -\Delta + V_R$ becomes selfadjoint with separation property

$$\|Su\|_{L^2}^2 + \|Qu\|_{L^2}^2 \le \|(S+Q)u\|_{L^2}^2 + 2\|u\|_{L^2}^2, \quad u \in D(S) \cap D(Q).$$

The proof is completed by computing $\mathrm{Re}\,(Sv, Q_\varepsilon v)$, $v \in D(S)$, as was done in Okazawa [39, Theorem 5.4] ($Q_\varepsilon$ is the Yosida approximation of $Q$). Here $v \in D(S)$ can be replaced with $v \in C_0^\infty(\mathbb{R}^N)$. In fact, by virtue of Kato's inequality $C_0^\infty(\mathbb{R}^N)$ becomes a core for $S + Q$ (see Reed-Simon [55, Theorem X.28]). The compactness of the resolvent of $S + Q$ is also well known together with its quantum mechanical meaning of those eigenvalues of $S + Q$ (see [56, Theorem XIII.67]). This concludes the compactness of $(S + Q)^{-1/2}$.

We can construct an abstract existence theorem applicable to $(\mathbf{CGL})_R$ when we introduce the following compactness condition,
(**K**) For every $c > 0$ the sub-level set $\{u \in D(\varphi);\, \varphi(u) \le c\}$ of $\varphi$ is compact in $X$, in addition to conditions (**A1**)–(**A3**) in Section 4 (cf. [47, Theorem 4.1], [11]). A few remarks are in order. First we do not require the restriction $\kappa^{-1}|\beta| \le c_q^{-1}$, which is indispensable in Theorem 3.1 (**I**), because the reasoning here does not depend on the accretivity of non-linear operator $u \mapsto |u|^{q-2}u$. Second, the pair $(\alpha/\lambda, \beta/\kappa)$ has to belong to the planar domain $CGL(c_q^{-1})$ ($c_q$ is the constant appearing in

condition (**A2**)) since we need those estimates of solutions to the approximation problem

$$(\mathbf{ACP};\partial\varphi\text{-}\partial\psi_\varepsilon)\qquad \begin{cases} \dfrac{du_\varepsilon}{dt} + (\lambda+i\alpha)\partial\varphi(u_\varepsilon) + (\kappa+i\beta)\partial\psi_\varepsilon(u_\varepsilon) - \gamma u = 0, \\ \hfill t>0, \\ u(0) = u_0, \end{cases}$$

in which $\partial\psi$ in (**ACP**) is replaced with $\partial\psi_\varepsilon$. Here $CGL(c)$ denotes the domain

$$CGL(c) := \Big\{(x,y)\in\mathbb{R}^2;\ xy\ge 0 \ \text{ or } \ \frac{|xy|-1}{|x|+|y|} < c\Big\}$$

when $0 < c < \infty$, and $CGL(\infty) := \mathbb{R}^2$ in the limiting case as $c\to\infty$. That is, exceptional domains appear in the second and fourth quadrants (see [47, Figure 1] or [11, Figure 2] for the exact shape of $CGL(c)$).

We have to decide the forms of convex functions $\varphi$ and $\psi$ previously given by (3.18) and (3.19) when we apply the abstract existence theorem to $(\mathbf{CGL})_R$. The form of $\psi$ is the same as before, while as for $\varphi$ (3.18) should be modified as

$$\varphi(u) := \begin{cases} 2^{-1}\big(\|\nabla u\|_{L^2}^2 + \|V_R^{1/2}u\|_{L^2}^2\big) & \text{for } u\in D(\varphi) := H^1(\mathbb{R}^N)\cap D(V_R^{1/2}), \\ \infty & \text{otherwise.} \end{cases}$$

In fact, noting that $\varphi(u) = (1/2)\|(S+Q)^{1/2}u\|_{L^2}^2$ when $u\in D(\varphi)$, we can verify condition (**K**) because of the equivalence

$$\{u\in D(\varphi);\ \varphi(u)\le c\} = \{(S+Q)^{-1/2}v;\ v\in L^2,\ \|v\|_{L^2}^2\le 2\,c\}.$$

Then we can obtain the following existence theorem for $(\mathbf{CGL})_{\mathbb{R}^N}$ by letting $R\to\infty$. We have also the assertion on uniqueness under a condition weaker than (3.5) (cf. [11]).

**Theorem 5.1.** *Assume that* $(\alpha/\lambda,\beta/\kappa)\in CGL(c_q^{-1})$ *for* $\lambda,\ \kappa\in\mathbb{R}_+$ *and* $\alpha,\ \beta\in\mathbb{R}$ *in addition to* $\gamma\in\mathbb{R}$.
(**I**) (**Existence of solutions.**) *For every* $u_0\in H^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ *there exists at least one strong solution* $u(\cdot)\in C([0,\infty);L^2)$ *to* $(\mathbf{CGL})_{\mathbb{R}^N}$ *satisfying the following*:

$$u(\cdot)\in C([0,\infty);H^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)).$$

*Next put* $\gamma_+ = \max\{0,\gamma\}$. *Then* (3.3) *and the energy estimate hold*:

$$\begin{aligned} (5.1)\qquad & \frac{\delta^2}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{q}\|u(t)\|_{L^q}^q + \eta\int_0^t \big\{\delta^2\|\Delta u(s)\|_{L^2}^2 + \|u(s)\|_{L^{2(q-1)}}^{2(q-1)}\big\}\,ds \\ & \le e^{\gamma_+ q\,t}\Big(\frac{\delta^2}{2}\|\nabla u_0\|_{L^2}^2 + \frac{1}{q}\|u_0\|_{L^q}^q\Big) \quad \forall\, t>0. \end{aligned}$$

*Here* $\delta>0$ *and* $\eta>0$ *are constants depending only on* $\lambda,\ \kappa,\ \alpha,\ \beta$ *and* $q$.
(**II**) (**Uniqueness of solutions.**) *Impose the so-called* **sub-critical condition** *on the exponent of the power of non-linear term*:

$$2\le q< 2^* := \begin{cases} \dfrac{2N}{N-2} = 2+\dfrac{4}{N-2} & (N\ge 3), \\ \infty, & (N=1,2). \end{cases}$$

*Then for every* $u_0 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *there exists a unique strong solution* $u(\cdot) \in C([0,\infty); L^2)$ *to* $(\mathbf{CGL})_{\mathbb{R}^N}$*. For the mapping such that* $u_0 \mapsto u(\cdot)$, $v_0 \mapsto v(\cdot)$ *one has*

$$\|u(t) - v(t)\|_{L^2}^2 + \lambda \int_0^t \exp\biggl(\int_s^t K(r)\,dr\biggr) \|\nabla u(t) - \nabla v(t)\|_{L^2}^2\,ds$$
$$\leq \exp\biggl(\int_0^t K(r)\,dr\biggr) \|u_0 - v_0\|_{L^2}^2 \quad \forall\ t > 0,$$

*where* $K(\cdot) \in C[0,\infty)$ *depends only on* $\lambda$, $\kappa$, $\beta$, $\gamma$, $q$ *and the pair* $(E_\infty(u_0), E_\infty(v_0))$ *in which* $E_\infty(\cdot)$ *is defined by* $E_\infty(w) := (\delta^2/2)\|\nabla w\|_{L^2}^2 + (1/q)\|w\|_{L^q}^q$*; note that* $E_\infty(\cdot)$ *has already appeared in the energy estimate* (5.1).

## 6. Concluding remarks

**6.1** ($L^p$-theory). As mentioned in Section 1 an $L^p$-theory ($1 < p < \infty$) of $(\mathbf{CGL})_\Omega$ is dealt with in [32]. However, what they derived is only an $L^p$-version of Theorem 3.1 (**II**). Nevertheless, the restriction (3.5) on the power of non-linear term is replaced with

$$2 \leq q \leq 2 + 2\,p/N \tag{6.1}$$

and hence large values of $q$ is allowable in $L^p$-theory with large $p$. Professor H. Amann (Zurich) predicted that (3.5) is replaced with (6.1) where the local solvability is concerned (cf. [41, Proposition 1.1]). An $L^p$-version of the whole of this article might be obtained if one can construct a (reflexive) Banach space version of the Hilbert space case [49] in the direction of Akagi-Otani [1].

**6.2** (Distribution-valued initial values). Next, we want to mention the problem with initial values in the space of distributions (a typical example is the Dirac measure $\delta$). By Brezis-Friedman [7] it has been shown that solutions to $(\mathbf{NLH})_\Omega$ ($0 \in \Omega$) with $u_0 = \delta$ exist if and only if

$$1 < q < 2 + 2/N, \tag{6.2}$$

where $(\mathbf{NLH})_\Omega$ denotes $(\mathbf{CGL})_\Omega$ with $\lambda = \kappa = 1$ and $\alpha = \beta = \gamma = 0$, that is, NLH means the non-linear heat equation. Note that the right-hand side of (6.2) seems to be the limit of that in (6.1) as $p \downarrow 1$. The author has no information on the generalization to $(\mathbf{CGL})_\Omega$ ($0 \in \Omega$) (cf. Levermore-Oliver [27] and [41, Corollary 1.1] (when $N = 1$)).

**6.3** (Modification of non-linear terms). In this article we have considered only $f(u) := (\kappa + i\beta)|u|^{q-2}u$ as non-linear term of $(\mathbf{CGL})_\Omega$. But it is possible to generalize the non-linearity to that of non-power type such as $f(u) := (\kappa + i\beta)g(|u|^2)u$. An example of $g$ is given by $g(s) := |s|^{(q-2)/2}\log(e + c_0 s)$ ($c_0 \geq 0$) (see [30], [46]). There are more challenging cases. In fact, Yokota [65] dealt with the case of $f(u) := \kappa|u|^{q-2}u + i\beta|u|^{r-2}u$ ($q > r \geq 2$), while Ozawa-Yamazaki [53] is concerned with the case of $f(u) := (\kappa + i\beta)|u|^p$.

**6.4** (The case of negative $\kappa$). In this case blow-up of solutions will happen; however, what we know is rather unsatisfactory because it is left untouched as a question in complex spaces. There are few papers (cf. [10] and [53]) in the list of references on this subject.

**6.5** (The resolvent problem). The resolvent problem for $(\mathbf{CGL})_\Omega$ is dealt with only in [42] though the assertion is unsatisfactory.

**6.6** (Duhamel's principle). A recent author's experience is related to the Cauchy problem for the non-linear Schrödinger equation perturbed by inverse-square potential:

$$(\mathbf{NLS})_a \qquad \begin{cases} \partial u/\partial t + i(-\Delta + a|x|^{-2})u + i\,f(u) = 0 & \text{on } \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Here we assume that $f(u) = |u|^{p-1}u$ $(p \geq 1)$ and

$$a > -(N-2)^2/4 \quad (N \geq 3) \tag{6.3}$$

to ensure that $P_a := -\Delta + a|x|^{-2}$ is selfadjoint in $H^{-1}(\mathbb{R}^N)$ (use Hardy's inequality and the Lax-Milgram theorem). Then we can define the unitary group $\{S_a(t)\} = \{\exp(itP_a)\}$. Of concern is the difference of $(\mathbf{NLS})_a$ from the so-called potential free case $(\mathbf{NLS})_0$. Usually, $(\mathbf{NLS})_0$ is transformed by Duhamel's principle into the integral equation

$$(\mathbf{INT})_0 \qquad u(t) = S_0(t)u_0 - i\int_0^t S_0(t-s)f(u(s))\,ds, \quad t \in \mathbb{R}.$$

It is well known that by virtue of the Strichartz estimates for free Schrödinger group $\{S_0(t)\}$ the contraction principle applies to give the existence and uniqueness of weak solution for every $u_0 \in H^1(\mathbb{R}^N)$ (see Cazenave [9, Section 4.4], Tsutsumi [61, Chapter 4]). This suggests that the Strichartz estimates for $\{S_a(t)\}$ have the same possibility to solve $(\mathbf{NLS})_a$ $(a \neq 0)$. But, in the computation the coefficient $a$ of inverse-square potential is restricted as

$$a > (N/2)^2[(p-1)/(p+1)]^2 - (N-2)^2/4 \tag{6.4}$$

(cf. [43]), where $p \in [1, (N+2)/(N-2)]$ is the exponent of power of the non-linear term. But we can assert the uniqueness of solutions under the same condition as (6.3) so that (6.4) is not necessary. Hence the first term on the right-hand side of (6.4) is expected to vanish. This seems to be caused by the fact that the validity of $\nabla S_0(t) = S_0(t)\nabla$ is not stable under the replacement of $P_0 = -\Delta$ with $P_a$ $(a \neq 0)$, that is, $\nabla S_a(t) \neq S_a(t)\nabla$.

This inconvenience on the existence of solutions is eliminated in [44] in which we have given up the idea of transformation into integral equations. The separation of the existence of solutions from the Strichartz estimates was begun by Cazenave [9, Chapter 3] (originally in [8]) and his idea seems to be indispensable to the transition from Cauchy problems to initial-boundary value problems $(\mathbf{NLS})_\Omega$ (cf. also [59]). It is expected that we can deal with the singular perturbation problem (1.1) in a natural way after the necessary progress in the study of $(\mathbf{NLS})_\Omega$.

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