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CURVE COUNTING THEORIES ON CALABI-YAU 3-FOLDS: APPROACH VIA STABILITY CONDITIONS ON DERIVED CATEGORIES

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ABSTRACT. This is an English translation of the expository article, *Curve counting theories on Calabi-Yau 3-folds: Approach via stability conditions for derived categories* (Japanese), Sugaku **66** (2014), no. 4, 337–365.

1. INTRODUCTION

It is a problem in classical algebraic geometry to count algebraic curves on a given algebraic variety (when we can). In recent years, the curve counting theory is more important in connection with string theory beyond the classical concerns on it. In particular, the curve counting theory on a Calabi-Yau 3-fold is expected to be equivalent to period integrals on its mirror manifold, and, for instance, there is a famous calculation by physicists counting rational curves on quintic hypersurfaces in $\mathbb{P}^4$. In general, curves on varieties form families, so it is not a trivial problem to give definitions of their counting invariants. However now, the counting invariants are constructed using virtual fundamental cycles, and formulated as Gromov-Witten (GW) invariants.

On the other hand, in his paper [61] in 1998, Thomas introduced holomorphic Casson invariants as a holomorphic version of Casson invariants on real 3-manifolds. These invariants are now called Donaldson-Thomas (DT) invariants. The DT invariants count holomorphic vector bundles (more precisely, stable coherent sheaves) on complex three-dimensional Calabi-Yau manifolds, and provide higher dimensional generalization of Donaldson invariants on algebraic surfaces. The DT invariants are expected to correspond to BPS state counting in string theory (for instance, see [22]), and they are interesting not only for mathematicians but also physicists. Now if we consider DT invariants counting rank one stable sheaves, then they count algebraic curves on Calabi-Yau 3-folds. The DT invariants in this case are, in general, different from GW invariants, but Maulik-Nekrasov-Okounkov-Pandharipande [48] conjectured in 2006 that these invariants are equivalent after taking the generating functions and some variable change. This conjecture is called the MNOP conjecture, and has been a motivation in the study of GW theory and DT theory.

After some years since the proposal of the MNOP conjecture, the central techniques in the study of DT invariants have been torus localizations and degeneration formulas. These techniques are quite powerful in computing DT invariants on explicit varieties, say toric Calabi-Yau manifolds. However, it has been difficult to

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give intrinsic derivations of some expected properties of DT invariants from these techniques, which should hold for any Calabi-Yau 3-folds. In this article, we explain the author's approach to the study of DT invariants via derived categories of coherent sheaves and the stability conditions on them. A feature of this approach is that it is applied to any Calabi-Yau 3-fold, and it provides an intrinsic explanation of some properties of DT invariants via symmetries in derived categories. It enabled us to study DT invariants on Calabi-Yau 3-folds without passing through explicit computations, which have been impossible by the techniques so far.

We first explain basic definitions of GW invariants and DT invariants, and then explain the MNOP conjecture and the DT/PT conjecture by Pandharipande-Thomas [57]. Then we introduce main results in [77], [76], and the ideas for the proofs. These are the Euler number version of the rationality conjecture of the generating series of DT invariants (this is required in the formulation of the MNOP conjecture) and the DT/PT conjecture, and are proved using the wall-crossing theory in the derived category. These results were obtained in 2008–2009, and pioneered further developments on DT type invariants [63], [79], [78], [82], [62]. In particular, we proved a flop transformation formula of DT type invariants [63], and a product expansion formula of DT type invariants on local K3 surfaces [82], which we also briefly explain. Finally we introduce the Bogomolov-Gieseker type inequality conjecture [5] on certain two term complexes on smooth projective 3-folds, which Bayer, Macri, and the author proposed in 2011. Although this conjecture was motivated to construct stability conditions on projective 3-folds, we also see that this conjecture derives some relationship among DT invariants counting two dimensional torsion sheaves on Calabi-Yau 3-folds and DT or PT invariants counting curves [62]. This relationship is closely related to the Ooguri-Strominger-Vafa conjecture in string theory, and hence has an important meaning. We also refer to the author's expository articles [81], [64].

2. Curve counting theories on algebraic varieties

2.1. **Compactifications of moduli spaces.** Let X be a smooth projective variety defined over the complex number field. We shall define invariants counting algebraic curves contained in X. By fixing $g \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$, we consider the moduli space of algebraic curves $C \subset X$ with genus g and fundamental homology class β. If this moduli space is zero dimensional, then the space of global sections of the structure sheaf of the moduli space is an Artin ring, so we may define the counting invariants to be its length. In general, the moduli space of curves in X may have positive dimension (i.e., curves may form a family), so we are not able to define the counting invariants in the above naive way. However, even when the moduli space has a positive dimension, there exist some situations in which it has zero virtual dimension. In that case, we can define the counting invariants by constructing the zero-dimensional virtual fundamental cycle and integrating it.

We need the compactness of the moduli spaces in order to make sense of the integration of algebraic cycles. So we first discuss compactification of the moduli spaces. In general, the moduli spaces of smooth curves in X are not compact, and also their compactifications are not unique. At least we have the following two kinds of compactifications:

(i) Stable map compactification.

(ii) Hilbert scheme compactification. In (i), the curve $C \subset X$ is smooth as far as possible (but allows nodal singularities), but is not necessarily embedded into X. In (ii), the curve C is embedded into X but allows any singularity in C. As we mention later, they correspond to GW theory and DT theory, respectively. We first explain the stable map compactification in (i).

Definition 2.1. Let C be a projective curve, and let $f\colon C \to X$ be a morphism. The pair (C, f) is called a stable map if C has at worst nodal singularities and the group of automorphisms of C preserving f is a finite group.

Given an integer $g \ge 0$ and $\beta \in H_2(X, \mathbb{Z})$, we denote by $\overline{\mathcal{M}}_g(X, \beta)$ the moduli space of stable maps (C, f) such that the genus of C is g and $f_*[C] = \beta$. Although the automorphism groups of stable maps are finite, they are not necessarily trivial. Hence, the moduli space $\overline{\mathcal{M}}_g(X, \beta)$ is not necessarily represented by a scheme, but so by a proper Deligne-Mumford stack. Since it is a stack, there is information of stabilizer groups of stable maps in $\overline{\mathcal{M}}_g(X, \beta)$, which is important in studying GW theory.

In the Hilbert scheme compactification (ii), we don't put any restriction on singularities of curves. Therefore, it is more convenient to use holomorphic Euler characteristics of the structure sheaves of subschemes instead of genus. Given an integer n and $\beta \in H_2(X, \mathbb{Z})$, the moduli space of subscheme $C \subset X$ with $\dim C \le 1$, $\chi(\mathcal{O}_C) = n$, $[C] = \beta$ is called the Hilbert scheme, and denoted by $\mathrm{Hilb}_n(X, \beta)$. Contrary to stable maps, the space $\mathrm{Hilb}_n(X, \beta)$ is always represented by a projective scheme. Let us observe a difference between compactifications (i), (ii) in the following example.

Example 2.2. Let $X = \mathbb{P}^2$, and consider for $t \in \mathbb{C}^*$ a smooth conic

$$C_t = \{[X : Y : Z] \in \mathbb{P}^2 : Y^2 = t^2 XZ\}.$$

We consider the limit of the above family of curves at $t \to 0$. If we take the limit $t \to 0$ of the defining equation of C_t, we obtain the double curve $C_0 = \{Y^2 = 0\}$. C_0 is embedded into $\mathbb{P}^2$, but it is singular, not even reduced. C_0 is the limit of C_t in the Hilbert scheme. On the other hand, the curve C_t is the image of the following map $f_t\colon \mathbb{P}^1 \to \mathbb{P}^2$:

$$f_t([u : v]) = [u^2 : tuv : v^2].$$

If we take the limit $t \to 0$ of the map f_t, then we obtain the map f_0. The map f_0 is a double cover from $\mathbb{P}^1$ to the line $Y = 0$, in particular, it is not an embedding. The map f_0 is the limit as stable maps.

Remark 2.3. In the case that X is a Grassmannian manifold, there is another compactification via stable quotients (cf. [52], [47], [80]). The stable quotient compactification is also constructed when X is a GIT quotient of an affine algebraic variety (cf. [26]).

2.2. **Gromov-Witten theory.** The moduli spaces $\overline{\mathcal{M}}_g(X, \beta)$, $\mathrm{Hilb}_n(X, \beta)$ constructed in the previous subsection do not necessarily have the correct dimensions (virtual dimensions) which they should have. The virtual dimension is given by the difference between the dimension of the tangent space of the moduli space and the dimension of the obstruction space which appears in the deformation theory. Roughly speaking, if the virtual dimension is independent of a point in the moduli space, we can construct the virtual cycle on the moduli space having the virtual dimension [8].

We first consider $\overline{\mathcal{M}}_g(X,\beta)$. Let $f\colon C \to X$ be a stable map, and consider deformations of f fixing the complex structure of C. As is well known, the space of infinitesimal deformations of f is $H^0(C, f^*T_X)$, and the obstruction space is $H^1(C, f^*T_X)$. The dimension of the deformations of the complex structures of C is $3g-3$, and by taking into account that they are unobstructed, we see that the virtual dimension of $\overline{\mathcal{M}}_g(X,\beta)$ $f\colon C \to X$ is

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) + 3g - 3.$$

By the Riemann-Roch theorem, the above virtual dimension is calculated as

$$\int_\beta c_1(X) + (\dim X - 3)(1-g). \tag{2.1}$$

The virtual dimension (2.1) is independent of a point in $\overline{\mathcal{M}}_g(X,\beta)$, hence there is a virtual cycle on $\overline{\mathcal{M}}_g(X,\beta)$.

Although we don't give details on the construction of virtual cycles in this article, we just say a few words on them. In the framework of [8], the data of tangent spaces, the obstruction spaces of the above moduli spaces is unified by the notion of perfect obstruction theory. The definition is given as follows.

Definition 2.4. Let $\mathcal{M}$ be a Deligne-Mumford stack, and let $L_{\mathcal{M}}$ be the cotangent complex of $\mathcal{M}$. A perfect obstruction theory on $\mathcal{M}$ consists of a two term complex of vector bundles $E^\bullet = (E^{-1} \to E^0)$ on $\mathcal{M}$ together with the morphism in the derived category

$$h\colon E^\bullet \to L_{\mathcal{M}}$$

such that $\mathcal{H}^0(h)$ is an isomorphism and $\mathcal{H}^{-1}(h)$ is surjective.

Roughly speaking, the $\mathcal{H}^0$ of the dual of the two term complex $E^\bullet$ stands for the tangent space of $\mathcal{M}$, and the $\mathcal{H}^1$ stands for the obstruction space. If there is a perfect obstruction theory, we can construct the virtual cycle on $\mathcal{M}$ with dimension $\operatorname{rank} E^0 - \operatorname{rank} E^{-1}$. However, the virtual cycle does not only depend on the stack structure of $\mathcal{M}$ but also on the choice of a perfect obstruction theory. We refer to [8], [64] for details on the construction of virtual cycles.

There is a natural perfect obstruction theory on $\mathcal{M} = \overline{\mathcal{M}}_g(X,\beta)$ determined by the deformation theory of stable maps. Hence, we can construct the virtual cycle

$$[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}} \in A_*(\overline{\mathcal{M}}_g(X,\beta)).$$

Here the right hand side is the Chow group of $\overline{\mathcal{M}}_g(X,\beta)$. The dimension of the above virtual cycle may not be zero, but be zero if X is a Calabi-Yau 3-fold by (2.1). Here we give the definition of Calabi-Yau manifolds.

Definition 2.5. A smooth projective variety X is called a Calabi-Yau manifold if its canonical divisor is trivial and $H^1(X, \mathcal{O}_X) = 0$ holds.

The condition $H^1(X, \mathcal{O}_X) = 0$ is not essential here, but will be used later in developing DT theory (cf. Remark 2.13). A quintic hypersurface in $\mathbb{P}^4$ is a famous example of a Calabi-Yau 3-fold.

Definition 2.6. Let X be a Calabi-Yau 3-fold. For $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(X, \mathbb{Z})$, the GW invariant $\mathrm{GW}_{g,\beta}$ is defined as follows:

$$\mathrm{GW}_{g,\beta} = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

We remark that, since $\overline{\mathcal{M}}_g(X, \beta)$ is a Deligne-Mumford stack, the GW invariant is not necessarily an integer. Indeed, the GW invariants are not integers in the following example.

Example 2.7. Let X be a Calabi-Yau 3-fold, which admits a birational contraction $f \colon X \to Y$ contracting a rational curve $C \cong \mathbb{P}^1 \subset X$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ to an ordinary double point. In this case, the invariants $\mathrm{GW}_{g,d[C]}$ are computed in [25]. The result is as follows:

$$\mathrm{GW}_{0,d[C]} = \frac{1}{d^3}, \quad \mathrm{GW}_{1,d[C]} = \frac{1}{12d},$$
$$\mathrm{GW}_{g,d[C]} = \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!}, \ g \geq 2.$$

Here B_{2g} is the $2g$-th Bernoulli number.

We can also show the deformation invariance of GW invariants using a general theory of perfect obstruction theories and virtual cycles.

Theorem 2.8 ([8])**.** *The GW invariants* $\mathrm{GW}_{g,\beta}$ *are invariants under deformations of complex structures of* X*.*

2.3. **Donaldson-Thomas theory.** Next, we discuss virtual fundamental cycles on Hilbert schemes. Let $C \subset X$ be a subscheme which corresponds to a point in $\mathrm{Hilb}_n(X, \beta)$. For instance, if one looks at a textbook such as [39] on the Hilbert schemes, one sees that the tangent space of the Hilbert scheme at $C \subset X$ is given by $\mathrm{Hom}(I_C, \mathcal{O}_C)$, and the obstruction space is given by $\mathrm{Ext}^1(I_C, \mathcal{O}_C)$. Here I_C is the ideal sheaf which defines C. Hence, we expect that the virtual dimension of $\mathrm{Hilb}_n(X, \beta)$ at $C \subset X$ is given by

$$\hom(I_C, \mathcal{O}_C) - \mathrm{ext}^1(I_C, \mathcal{O}_C).$$

If $\dim X \leq 2$, then we have $\mathrm{Ext}^i(I_C, \mathcal{O}_C) = 0$ for $i \geq 2$, and the above virtual dimension depends only on β and n by the Riemann-Roch theorem. However, if $\dim X \geq 3$, then $\mathrm{Ext}^i(I_C, \mathcal{O}_C)$ does not necessarily vanish for $i \geq 2$. Hence, the virtual dimension may jump depending on a point of $\mathrm{Hilb}_n(X, \beta)$. In that case, we cannot construct the virtual cycle in the framework of [8].

The idea of how to avoid this issue is by regarding the Hilbert scheme as the moduli space of coherent sheaves. We regard β, n as elements of $H^4(X, \mathbb{Q})$, $H^6(X, \mathbb{Q})$ by Poincaré duality, and we define $I_n(X, \beta)$ to be the moduli space of torsion free coherent sheaves $I \in \mathrm{Coh}(X)$ satisfying the following numerical condition:

$$\mathrm{ch}(I) = (1, 0, -\beta, -n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X). \tag{2.2}$$

Then, we have the following map:

$$\mathrm{Hilb}_n(X, \beta) \ni C \mapsto I_C \in I_n(X, \beta). \tag{2.3}$$

For instance, if X is a Calabi-Yau manifold with dimension greater than or equal to three, the above map becomes an isomorphism. Here by the deformation theory of coherent sheaves (for instance, see [33]), the tangent space and the obstruction space of $I_n(X,\beta)$ at I_C are given by $\mathrm{Ext}^1(I_C,I_C)$, $\mathrm{Ext}^2(I_C,I_C)$, respectively. As the map (2.3) is an isomorphism, both of the tangent spaces are isomorphic, but the obstruction spaces can be different. Then if we apply the obstruction of $I_n(X,\beta)$ instead of that of $\mathrm{Hilb}_n(X,\beta)$, the virtual dimension is given by

$$\mathrm{ext}^1(I_C,I_C)-\mathrm{ext}^2(I_C,I_C). \tag{2.4}$$

If X is a three-dimensional Calabi-Yau manifold, the virtual dimension defined by (2.4) is always zero. In fact, in this case, we have $\mathrm{Ext}^2(I_C,I_C)\cong \mathrm{Ext}^1(I_C,I_C)^{\vee}$ by Serre duality, and the virtual dimension (2.4) becomes zero. However, we remark that, if $\dim X\ge 4$, then the virtual dimension (2.4) may depend on a point of $I_n(X,\beta)$ even if X is Calabi-Yau. By the above argument, we have the perfect obstruction theory and the zero dimensional virtual cycle on $I_n(X,\beta)$

$$[I_n(X,\beta)]^{\mathrm{vir}}\in A_0(I_n(X,\beta))$$

similarly to the GW theory. By integrating the above virtual cycle, we can define the DT invariant:

Definition 2.9. Let X be a Calabi-Yau 3-fold. For $n\in\mathbb{Z}$ and $\beta\in H_2(X,\mathbb{Z})$, the DT invariant $I_{n,\beta}$ is defined as follows:

$$I_{n,\beta}=\int_{[I_n(X,\beta)]^{\mathrm{vir}}}1\in\mathbb{Z}.$$

Remark 2.10. Similarly to Theorem 2.8, $I_{n,\beta}$ is also invariant under deformations of complex structures of X. On the other hand, contrary to the GW invariants, DT invariants are always integers. This is due to the fact that the moduli spaces of subschemes are projective schemes, while stable map moduli spaces are stacks.

We set the generating series of DT invariants in the following way:

$$I_\beta(X)=\sum_{n\in\mathbb{Z}}I_{n,\beta}q^n,\ I(X)=\sum_{\beta\in H_2(X,\mathbb{Z})}I_\beta(X)t^\beta.$$

Example 2.11. (i) In the case of $\beta=0$, the invariant $I_{n,0}$ counts zero dimensional subschemes with length n. Then the generating series $I_0(X)$ is computed in the following way [42], [9], [41]:

$$I_0(X)=M(-q)^{\chi(X)}.$$

Here $M(q)$ is the MacMahon function defined by

$$M(q)=\prod_{k\ge1}\frac{1}{(1-q^k)^k}.$$

(ii) Let $C\subset X$ be a rational curve as in Example 2.7. Then the generating series $I_{d[C]}(X)$ is computed by [7]

$$\sum_{d\ge0}I_{d[C]}(X)t^d=M(-q)^{\chi(X)}\prod_{k\ge1}(1-(-q)^kt)^k.$$

2.4. **Description via Behrend function.** Comparing GW theory and DT theory, there exist at least two different features. The one point is that, while the GW invariants take values in rational numbers, the DT invariants take values in integers. Th second point is that, while the GW invariants are not determined by the stack structure of $\overline{\mathcal{M}}_g(X,\beta)$, the DT invariants are determined by the scheme structure of $I_n(X,\beta)$. This is due to the fact that, since the tangent space $\mathrm{Ext}^1(I_C,I_C)$ of $I_n(X,\beta)$ is dual to the obstruction space $\mathrm{Ext}^2(I_C,I_C)$, the perfect obstruction theory for the DT invariant is uniquely reconstructed from the scheme structure of $I_n(X,\beta)$. Furthermore, Behrend [6] constructed a canonical constructible function χ_B on any complex scheme M, and proved that if $M = I_n(X,\beta)$, the χ_B-weighted topological Euler number of M coincides with the DT invariant. The constructible function χ_B (called the Behrend function) is rather easily calculated using the following result by Joyce-Song.

Theorem 2.12 ([35]). *Suppose that X is a Calabi-Yau 3-fold. For each point $p \in I_n(X,\beta)$, there exists an analytic neighborhood $p \in U \subset I_n(X,\beta)$, complex manifold V and a holomorphic function $f\colon V \to \mathbb{C}$ such that U is isomorphic to $\{df = 0\}$ as a complex analytic space.*

Remark 2.13. The proof of Theorem 2.12 requires the condition $H^1(X,\mathcal{O}_X) = 0$.

Using the above theorem, the Behrend function χ_B on $I_n(X,\beta)$ is described in the following way:

$$\chi_B(p) = (-1)^{\dim V}(1 - \chi(M_p(f))).$$

Here $M_p(f)$ is the Milnor fiber of f at $p \in U \subset V$. Moreover, the following result follows from [6].

Theorem 2.14 ([6]). *We have the following identity:*

$$I_{n,\beta} = \int_{I_n(X,\beta)} \chi_B \, d\chi. \tag{2.5}$$

The right-hand side of the identity (2.5) is the integration of χ_B over $I_n(X,\beta)$ with measure the topological Euler numbers. More explicitly, it is written in terms of the weighted Euler numbers:

$$\int_{I_n(X,\beta)} \chi_B \, d\chi = \sum_{k \in \mathbb{Z}} k \cdot \chi(\chi_B^{-1}(k)).$$

Theorem 2.14 provides a strong method in computing DT invariants. For example, if $I_n(X,\beta)$ is non-singular and connected, then χ_B is a constant function with value $(-1)^{\dim I_n(X,\beta)}$, so the following holds:

$$I_{n,\beta} = (-1)^{\dim I_n(X,\beta)} \chi(I_n(X,\beta)). \tag{2.6}$$

Also there are some cases where χ_B is a constant function while $I_n(X,\beta)$ is singular. For instance, χ_B is a constant function $(-1)^n$ on $I_n(X,0)$ (cf. [1]). Hence, we see that $I_{n,\beta}$ is closely related to the Euler number of $I_n(X,\beta)$.

Remark 2.15. The invariants $I_{n,\beta}$ count rank one sheaves, and in general it is also possible to define generalized DT invariants counting higher rank coherent sheaves, or rank zero (i.e., torsion) sheaves. (More precisely, they are invariants counting Gieseker semistable sheaves [33] on $\mathrm{Coh}(X)$ with respect to a fixed ample divisor.) The moduli stacks which define generalized DT invariants are not necessarily schemes, and they may have complicated stabilizer groups. In this case, we need to interpret the moduli stacks as elements of motivic Hall algebras, and integrate the Behrend functions over their logarithms. As rational numbers appear in the log, the generalized DT invariants take values in rational numbers. We will later discuss some special cases in Subsection 3.5. Also, the motivic version of DT invariants is expected to be defined. We refer to [35], [40] for details.

2.5. **MNOP conjecture.** In 2006, Maulik-Nekrasov-Okounkov-Pandharipande [48] proposed a conjecture relating GW invariants $\mathrm{GW}_{g,\beta}$ and DT invariants $I_{n,\beta}$ after taking their generating series. As we have seen so far, both invariants have different features, so this is an amazing conjecture. In particular, as DT invariants are integers, this predicts a hidden integrality of GW invariants. We first state the conjecture:

Conjecture 2.16 ([48])**.** (i) (*Rationality conjecture*) *The generating series $I_\beta(X)/I_0(X)$ is the Laurent expansion of a rational function of q at $q = 0$, which is invariant under $q \leftrightarrow 1/q$.*

(ii) (*GW/DT correspondence*) *Under the variable change $q = -e^{i\lambda}$, we have the following identity:*

$$\exp\left(\sum_{g \geq 0, \beta > 0} \mathrm{GW}_{g,\beta} \lambda^{2g-2} t^\beta\right) = I(X)/I_0(X).$$

Here $\beta > 0$ means that β is a homology class of an effective algebraic one cycle.

We explain the meaning of Conjecture 2.16 in more detail. First, as $I_{n,\beta} = 0$ for $n \ll 0$ by an easy argument, the series $I_\beta(X)$ is a Laurent series. However, the structure sheaves of subschemes $C \subset X$ which contribute to $I_{n,\beta}$ may contain zero dimensional subsheaves, and in that case $I_{n,\beta}$ may not count honest curves in X. In order to cancel out the contributions from such zero dimensional sheaves, we take the quotient of the generating series $I_\beta(X)$ by $I_0(X)$. For example, in the situation of Example 2.11 (ii), the generating series $I_{[C]}(X)/I_0(X)$ is computed as follows:

$$I_{[C]}(X)/I_0(X) = q - 2q^2 + 3q^3 - \cdots . \tag{2.7}$$

This is the Laurent expansion of the rational function $q/(1+q)^2$ at $q = 0$. Furthermore, this rational function is invariant under taking q to $1/q$. Here we note that the series of the right-hand side of (2.7) itself becomes a different series if we place q by $1/q$. The rationality conjecture (i) means that $I_\beta(X)/I_0(X)$ is analytically continued as a rational function of q to $\mathbb{P}^1$, and after analytic continuation, the behavior at $q = 0$ and $q = \infty$ are the same by the variable change $q \mapsto 1/q$. There is no symmetric property $q \mapsto 1/q$ on the level of the series. The GW/DT correspondence (ii) makes sense if we assume (i), i.e., we expand the rational function obtained by (i) at $q = -1$, and then apply the variable change.

Remark 2.17. Now, Conjecture 2.16 (i) is completely solved, and (ii) is solved for many Calabi-Yau 3-folds. We will discuss the rationality conjecture (i) in Corollary 3.3. In 2012, Pandharipande-Pixton [56] showed the GW/DT correspondence (ii) for complete intersection Calabi-Yau 3-folds in the product of projective spaces. In particular, Conjecture 2.16 is true for quintic Calabi-Yau 3-folds. What they proved is the equivalence between PT invariants and DT invariants which we discuss in the next subsection. Since the equivalence between PT invariants and DT invariants is proved (cf. Corollary 3.3), Conjecture 2.16 holds as a result.

2.6. **Pandharipande-Thomas conjecture.** In 2009, Pandharipande-Thomas [57] introduced the notion of stable pairs. It was an attempt to give a geometric understanding of the quotient $I(X)/I_0(X)$ of the generating series. More precisely, they conjectured that each coefficient of $I(X)/I_0(X)$ counts stable pairs. We first give the definition of stable pairs.

Definition 2.18. A pair (F, s) is called a stable pair if F is a pure one dimensional coherent sheaf and $s\colon \mathcal{O}_X \to F$ is a morphism of coherent sheaves whose cokernel has at most zero dimensional support.

Here we say that F is pure one dimensional if the support of F is one dimensional, and F does not have a zero dimensional subsheaf.

Example 2.19. Let $C \subset X$ be a smooth curve, and let $D \subset C$ be an effective divisor. By setting $F = \mathcal{O}_C(D)$ and $s\colon \mathcal{O}_X \to \mathcal{O}_C(D)$ the natural morphism, the pair (F, s) determines a stable pair.

We explain the difference between the notion of stable pairs and that of ideal sheaves I_C for subschemes $C \subset X$. The ideal sheaf I_C determines a pair $(\mathcal{O}_C, s)$ by setting s to be the natural surjection $s\colon \mathcal{O}_X \to \mathcal{O}_C$. This pair determines a stable pair if and only if $\mathcal{O}_C$ does not have a zero dimensional sheaf. On the other hand, the stable pair (F, s) is determined by an ideal sheaf if and only if s is surjective. In other words, although stable pairs guarantee that F does not have a zero dimensional subsheaf, they lose the surjectivity of s.

Similarly to the DT theory, for a given $\beta \in H_2(X, \mathbb{Z})$, $n \in \mathbb{Z}$, we consider the moduli space of stable pairs (F, s) satisfying the numerical condition

$$[F] = \beta, \quad \chi(F) = n. \tag{2.8}$$

This is proved to be realized as a projective scheme $P_n(X, \beta)$ in [57]. An important point here is that $P_n(X, \beta)$ is regarded as the moduli space of two term complexes

$$I^\bullet = (\mathcal{O}_X \stackrel{s}{\to} F) \in D^b \operatorname{Coh}(X) \tag{2.9}$$

determined by stable pairs. Here $D^b \operatorname{Coh}(X)$ is the bounded derived category of coherent sheaves on X.

Then if we regard the deformation theory of stable pairs as the deformation theory of objects $I^\bullet$ in the derived category, the tangent space of $P_n(X, \beta)$ at (F, s) is $\operatorname{Ext}^1(I^\bullet, I^\bullet)$ and the obstruction space if $\operatorname{Ext}^2(I^\bullet, I^\bullet)$. Similarly to the case of DT theory, these spaces are dual to each other by Serre duality. Hence there exists a perfect obstruction theory and the zero dimensional virtual cycle $[P_n(X, \beta)]^{\mathrm{vir}}$ on $P_n(X, \beta)$.

Definition 2.20. Let X be a Calabi-Yau 3-fold. For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, the PT invariant $P_{n,\beta}$ is defined as follows:

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Remark 2.21. Similarly to Theorem 2.8, the invariant $P_{n,\beta}$ is also invariant under deformations of complex structures of X. Also it is expected that $P_n(X, \beta)$ satisfies the property similar to Theorem 2.12, although its proof is not available at this moment. However, $P_{n,\beta}$ is given by the integration of the Behrend constructible function on $P_n(X,\beta)$ by [6]. Therefore, $P_{n,\beta}$ is closely related to $\chi(P_n(X,\beta))$.

The generating series of the PT invariants are defined by

$$P(X) = 1 + \sum_{n\in\mathbb{Z}, \beta>0} P_{n,\beta} q^n t^\beta.$$

The following is the main conjecture in [57], called DT/PT correspondence.

Conjecture 2.22. *The following equality holds:*

$$I(X)/I_0(X) = P(X).$$

As we discuss in the next subsection, the above conjecture is already solved. Let us observe the above conjecture in the following example.

Example 2.23. In the situation of Example 2.7, the stable pairs which contribute to $P_{n,[C]}$ are only $(\mathcal{O}_C(n-1), s \neq 0)$. Hence $P_n(X,[C])$ is isomorphic to $\mathbb{P}(H^0(C, \mathcal{O}_C(n-1))) \cong \mathbb{P}^{n-1}$, and $P_{n,[C]}$ is

$$(-1)^{\dim P_n(X,[C])}\chi(P_n(X,[C])) = (-1)^{n-1}n$$

similarly to (2.6). Hence,

$$\sum_{n\in\mathbb{Z}} P_{n,[C]} q^n = q - 2q^2 + 3q^3 - \cdots,$$

and Conjecture 2.22 is checked in this case.

3. The product expansion formula of the generating series

3.1. **The main result.** We explained the MNOP conjecture and the PT conjecture in the previous section. In this section, we introduce the approach of these conjectures using the notions of derived categories of coherent sheaves and stability conditions on them. We first state the main theorem in [77], [76].

Theorem 3.1 (Toda [77], [76] (Euler number version), Bridgeland [19])**.** *Let X be a Calabi-Yau 3-fold. For each $n \in \mathbb{Z}$ and $\beta \in H_2(X,\mathbb{Z})$, there exist invariants $N_{n,\beta} \in \mathbb{Q}$, $L_{n,\beta} \in \mathbb{Q}$ satisfying the following conditions:*

- *We have $N_{n,\beta} = N_{-n,\beta}$ and $L_{n,\beta} = L_{-n,\beta}$.*
- *We have $N_{n,\beta} = N_{n+\beta\cdot H,\beta}$ for any ample divisor H.*
- *$L_{n,\beta}$ is zero for $|n| \gg 0$, depending on β.*

Moreover, we have the following product expansion formula using the above invariants $N_{n,\beta}$, $L_{n,\beta}$:

$$I(X) = \prod_{n>0} \exp\left((-1)^{n-1} N_{n,0} q^n\right)^n P(X), \tag{3.1}$$

$$P(X) = \prod_{n>0,\beta>0} \exp\left((-1)^{n-1} N_{n,\beta} q^n t^\beta\right)^n \left(\sum_{n,\beta} L_{n,\beta} q^n t^\beta\right). \tag{3.2}$$

Remark 3.2. Here the 'Euler number version' of DT invariants and PT invariants mean the invariants replacing $I_{n,\beta}$, $P_{n,\beta}$ by the topological Euler numbers $\chi(I_n(X,\beta))$, $\chi(P_n(X,\beta))$ without the Behrend weight. As we discussed in Subsection 2.4 and Remark 2.21, the latter invariants are closely related to $I_{n,\beta}$, $P_{n,\beta}$, respectively. On the other hand, in order to obtain the results in [77], [76] for the honest DT invariants and PT invariants, we need to prove a result similar to Theorem 2.12 for the moduli spaces of objects in the derived category. This was announced by Behrend-Getzler [10], but later Bridgeland [19] showed Theorem 3.1 without relying on [10]. However the geometric meaning of $L_{n,\beta}$ is not obvious by the method of [19]. Although the paper [10] is not available, almost the same result is obtained by Ben-Bassat, Brav, Bussi, and Joyce [12].

The PT conjecture and the MNOP rationality conjecture easily follow from the properties of $N_{n,\beta}$, $L_{n,\beta}$ in the above theorem, and the product expansion formulas (3.1), (3.2).

Corollary 3.3. (Toda [77], [76] (Euler number version), Bridgeland [19])
Conjecture 2.16 (*i*) *and Conjecture* 2.22 *are true.*

The notion of 'stability conditions on the derived category' plays a key role in the proof of Theorem 3.1, which we explain in detail below.

3.2. **Stability conditions.** There exists the classical notion of 'stability' on vector bundles on algebraic curves. By definition, a vector bundle E on an algebraic curve is called (semi)stable if for any non-trivial subvector bundle $0 \neq F \subset E$, the following inequality holds:

$$\frac{\deg F}{\operatorname{rank} F} < (\leq) \frac{\deg E}{\operatorname{rank} E}.$$

The notion of Bridgeland stability conditions on triangulated categories generalizes the notion of (semi)stable vector bundles on algebraic curves to arbitrary triangulated categories. Its original motivation was to give a mathematical formulation of Π-stability [23] in string theory. The mathematical definition is given as follows.

Definition 3.4 ([17])**.** Let $\mathcal{D}$ be a triangulated category. A stability condition on $\mathcal{D}$ consists of data $(Z, \mathcal{A})$ of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ together with the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$, satisfying the following:

- For any non-zero object $E \in \mathcal{A}$, we have

$$Z(E) \in \mathbb{H} := \{r \exp(i\pi\phi) : r > 0, 0 < \phi \leq 1\}. \tag{3.3}$$

- For any $E \in \mathcal{A}$, there exists a filtration (called Harder-Narasimhan filtration) in $\mathcal{A}$

$$0 = E_0 \subset E_1 \subset \cdots \subset E_N = E$$

  such that each $F_i = E_i/E_{i-1} \in \mathcal{A}$ is Z-semistable satisfying $\arg Z(F_i) > \arg Z(F_{i+1})$.

Here $K(\mathcal{D})$ is the Grothendieck group of $\mathcal{D}$. Also we say $E \in \mathcal{A}$ is Z-(semi)stable if for any subobject $0 \neq F \subset E$, we have $\arg Z(F) < (\leq) \arg Z(E)$. Here by the condition (3.3), we define $\arg Z(*)$ in $(0, \pi]$. We give some examples.

Example 3.5. (i) Let X be a smooth projective 3-fold and let $\operatorname{Coh}_{\leq 1}(X)$ be the abelian category of coherent sheaves on X whose supports have dimension less than or equal to one. We set $\mathcal{D} = D^b \operatorname{Coh}_{\leq 1}(X)$, and define the group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ determined by the ample divisor H:

$$Z(E) = -\operatorname{ch}_3(E) + H \cdot \operatorname{ch}_2(E)\sqrt{-1}.$$

Then $(Z, \mathrm{Coh}_{\le 1}(X))$ is a stability condition on $\mathcal{D}$. If E is supported on a smooth curve $C \subset X$ scheme theoretically, then E is Z-(semi)stable if and only if E is a (semi)stable sheaf on C in the classical sense.

(ii) Let A be a finite dimensional algebra, let $\mathrm{mod}\, A$ be the abelian category of finitely generated right A-modules and set $\mathcal{D} = D^b \mathrm{mod}\, A$. Then there exist finite numbers of simple objects $S_1, \cdots, S_k \in \mathrm{mod}\, A$ such that $K(\mathcal{D})$ is a free abelian group with basis $[S_i]$. We choose $z_i \in \mathbb{H}$, and set $Z\colon K(\mathcal{D}) \to \mathbb{C}$ by $Z([S_i]) = z_i$. Then $(Z, \mathrm{mod}\, A)$ is a stability condition on $\mathcal{D}$.

We consider the following property of a stability condition $\sigma = (Z, \mathcal{A})$ on $\mathcal{D}$:

- (numerical property): The group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ factors through the fixed surjective group homomorphism $\mathrm{cl}\colon K(\mathcal{D}) \to \Gamma$ for a fixed free abelian group Γ.
- (support property): By fixing a norm on $\|*\|$ on $\Gamma_{\mathbb{R}}$, we have the following finiteness:

$$\sup \left\{ \frac{\|\mathrm{cl}(E)\|}{|Z(E)|} : E \in \mathcal{A} \text{ is } Z\text{-semistable} \right\} < \infty. \tag{3.4}$$

As for the first 'numerical property', for example in the case of $\mathcal{D} = D^b \mathrm{Coh}(X)$, we usually set (Γ, cl) by the following:

$$\Gamma = \mathrm{Im}(\mathrm{ch}\colon K(X) \to H^*(X, \mathbb{Q})), \quad \mathrm{cl} = \mathrm{ch}\,. \tag{3.5}$$

The second 'support property' is a little technical condition, and we omit a detailed explanation. We denote by $\mathrm{Stab}(\mathcal{D})$ the set of stability conditions on a triangulated category $\mathcal{D}$ satisfying the above properties. (Although the set $\mathrm{Stab}(\mathcal{D})$ depends on (Γ, cl) by the definition, we do not include these symbols in $\mathrm{Stab}(\mathcal{D})$ for simplicity.) By Bridgeland [17], it is shown that $\mathrm{Stab}(\mathcal{D})$ has a natural topology. Indeed, the following theorem holds:

Theorem 3.6 ([17]). *The forgetful map*

$$\mathrm{Stab}(\mathcal{D}) \to \Gamma_{\mathbb{C}}^{\vee}, \quad (Z, \mathcal{A}) \mapsto Z \tag{3.6}$$

is a local homeomorphism. In particular, $\mathrm{Stab}(\mathcal{D})$ *has a structure of a complex manifold.*

Remark 3.7. The support property we put above was not considered in Bridgeland's original paper [17]. Without this property, the forgetful map is only a local homeomorphism onto some vector subspace of $\Gamma_{\mathbb{C}}^{\vee}$. (In fact, the main theorem of [17] is stated like that.) The support property was introduced by Kontsevich-Soibelman [40], which ensured the forgetful map (3.6) to be a local homeomorphism.

Let us consider Example 3.5 (ii) to observe the above theorem. We set $\mathcal{D} = D^b \mathrm{mod}\, A$ as in Example 3.5 (ii), and set $(\Gamma, \mathrm{cl}) = (K(\mathcal{D}), \mathrm{id})$. Then, there is a one-to-one correspondence between stability conditions $(Z, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$ with $\mathcal{A} = \mathrm{mod}\, A$ and points in $\mathbb{H}^k$. In particular, the inner points of $\mathbb{H}^k$ give an open complex submanifold of $\mathrm{Stab}(\mathcal{D})$. However, $\mathbb{H}^k$ has boundaries. The heart of a t-structure of a stability condition after crossing the boundary is no longer $\mathrm{mod}\, A$, and some other heart $\mathcal{B}$ appears. If $\mathcal{B}$ is also written as $\mathcal{B} = \mathrm{mod}\, B$ for a finite dimensional algebra B, then again the stability conditions of the form $(Z, \mathrm{mod}\, B)$ are one-to-one correspondence with the points in $\mathbb{H}^k$, which determine another chamber in $\mathrm{Stab}(\mathcal{D})$. In this way, we may have the chamber structure on $\mathrm{Stab}(\mathcal{D})$

such that each chamber corresponds to the heart of a t-structure. This picture is not true in general, but provides an amenable model to have an understanding of the complex manifold $\mathrm{Stab}(\mathcal{D})$.

Suppose that $\mathcal{D} = D^b \operatorname{Coh}(X)$ and (Γ, cl) is given by (3.5). In this case, we denote $\mathrm{Stab}(X) := \mathrm{Stab}\, D^b \operatorname{Coh}(X)$. This complex manifold is in particular important when X is a Calabi-Yau manifold. In this case, the universal covering space of the moduli space of complex structures of the manifold $X^{\vee}$ mirror to X is expected to be embedded into $\mathrm{Stab}(X)$. Moreover, this embedding should be described by the solutions of the Picard-Fuchs equation which the period map of $X^{\vee}$ satisfies. However, it is quite difficult to understand the geometric structure of $\mathrm{Stab}(X)$, and at this moment the satisfactory answers are obtained only in the cases of $\dim X = 1$ [17], X is a K3 surface or an abelian surface [18], X is a local Calabi-Yau manifold [16], [73], [75], [4].

Remark 3.8. If X is a projective Calabi-Yau 3-fold, it is not even known whether $\mathrm{Stab}(X)$ is non-empty or not. In fact if $\dim X \geq 2$, one can show that there is no stability condition $(Z, \mathcal{A})$ on $D^b \operatorname{Coh}(X)$ with $\mathcal{A} = \operatorname{Coh}(X)$ (cf. [74, Lemma 2.7]). In the $\dim X = 2$ case, one can construct stability conditions by taking the heart to be the tilting of $\operatorname{Coh}(X)$ (cf. [18], [2]), which requires the Bogomolov inequality among Chern numbers of semistable sheaves. In the $\dim X = 3$ case, we conjectured in [5] that the stability conditions are constructed by taking the double tilting of $\operatorname{Coh}(X)$, which we explain in Section 5 in detail.

3.3. **The idea of the proof of Theorem 3.1.** Let X be a Calabi-Yau 3-fold. The DT invariant $I_{n,\beta}$ and the PT invariant $P_{n,\beta}$ count curves on X, and at the same time count ideal sheaves of subschemes, two term complexes (2.9), respectively. The idea of the proof of Theorem 3.1 is to regard these objects as stable objects in the derived category of coherent sheaves $D^b \operatorname{Coh}(X)$ with respect to different stability conditions. Below, we explain how the above idea is related to the DT/PT correspondence.

Remark 3.9. In this subsection, we just explain a rough idea, which is not mathematically rigorous. For example, as we stated in Remark 3.8, the existence of a Bridgeland stability condition on a Calabi-Yau 3-fold is not known, but we ignore this issue for a moment.

First, suppose that there is a stability condition σ_I on $D^b \operatorname{Coh}(X)$ such that any ideal sheaf I_C for a one dimensional subscheme $C \subset X$ is σ_I-stable. We have the surjection $s \colon \mathcal{O}_X \to \mathcal{O}_C$ associated to I_C, and $(\mathcal{O}_C, s)$ is a stable pair in the sense of Definition 2.18 if and only if $\mathcal{O}_C$ is pure one dimensional. Suppose that $\mathcal{O}_C$ is not pure one dimensional, and let $Q \subset \mathcal{O}_C$ be the maximum zero dimensional subsheaf. Then there is an exact triangle in $D^b \operatorname{Coh}(X)$

$$Q[-1] \to I_C \to I_{C'}.$$

Here C' is the one dimensional subscheme of X defined by $\mathcal{O}_{C'} = \mathcal{O}_C/Q$. The above exact triangle is expected to destabilize I_C with respect to another stability condition σ_P on $D^b \operatorname{Coh}(X)$. Thus we consider an exact triangle given by a 'flip' of the above exact triangle:

$$I_{C'} \to E \to Q[-1].$$

In a typical situation, the object $E \in D^b \operatorname{Coh}(X)$ is σ_P-stable, and isomorphic to the two term complex (2.9). In this way, σ_I may correspond to the DT invariants, σ_P may correspond to the PT invariants, and the comparison of these stability conditions may be relevant to prove DT/PT correspondence.

By developing the above discussion, we would like to approach Theorem 3.1 following the steps below:

- For each $\sigma = (Z, \mathcal{A}) \in \operatorname{Stab}(X)$ and $v \in H^*(X, \mathbb{Q})$, we construct the DT type invariant

$$\mathrm{DT}_\sigma(v) \in \mathbb{Q}. \tag{3.7}$$

It counts Z-semistable objects $E \in \mathcal{A}$ satisfying $\operatorname{ch}(E) = v$.

Remark 3.10. Here we need to construct the moduli spaces of Bridgeland semistable objects in order to construct the invariant (3.7). The existence of such moduli spaces is not known in general, but in some particular case, for example in [72], the moduli spaces of Bridgeland semistable objects are realized as algebraic stacks of finite type.

Remark 3.11. Here we remark that, although the invariants $I_{n,\beta}$ and $P_{n,\beta}$ are integers, the invariant $\mathrm{DT}_\sigma(v)$ obtained in (3.7) should be a rational number in general. This is due to the fact that, while there exist no non-trivial automorphisms of ideal sheaves and two term complexes (2.9), the semistable objects which contribute to (3.7) may not be necessarily stable, hence there may exist non-trivial automorphisms. The construction of the invariant $\mathrm{DT}_\sigma(v)$ in this case requires the motivic Hall algebra, as we discussed in Remark 2.15 for generalized DT invariants.

- Let $v \in H^*(X, \mathbb{Q})$ be a cohomology class given as the right-hand side of (2.2). Then, we try to find stability conditions σ_I, σ_P such that the following holds:

$$\mathrm{DT}_{\sigma_I}(v) = I_{n,\beta}, \ \mathrm{DT}_{\sigma_P}(v) = P_{n,\beta}.$$

  Moreover, we try to find another $\sigma_L \in \operatorname{Stab}(X)$ such that $L_{n,\beta} := \mathrm{DT}_{\sigma_L}(v)$ satisfies the properties required in Theorem 3.1.
- We investigate how the invariant $\mathrm{DT}_\sigma(v)$ behaves if we change σ. In general, there exist real codimension one submanifolds (walls) in $\operatorname{Stab}(X)$ such that the invariant $\mathrm{DT}_\sigma(v)$ is constant if σ is contained in a connected component (chamber) of the complement of walls but jumps if it crosses a wall. We then try to describe the change of $\mathrm{DT}_\sigma(v)$ under wall-crossing (wall-crossing formula), and apply it to obtain relations among $I_{n,\beta}$, $P_{n,\beta}$, and $L_{n,\beta}$. We then try to show that these relations imply the formulas (3.1), (3.2) in Theorem 3.1.

3.4. **The issues and their solutions.** There exist several technical obstructions to realize the steps in the previous subsection in a mathematically rigorous way. The largest obstruction is, as we stated in Remark 3.8, quite difficult to construct Bridgeland stability conditions. It is easy to construct stability conditions on $\mathcal{D} = D^b \operatorname{Coh}_{\leq 1}(X)$ as discussed in Example 3.5 (i), but there exist many more objects in $D^b \operatorname{Coh}(X)$, and the construction of stability conditions on it is much harder than Example 3.5. However, if we just focus on the proof of Theorem 3.1, we do not have to consider all the objects in $D^b \operatorname{Coh}(X)$. In fact, it is enough to consider the following triangulated subcategory of $D^b \operatorname{Coh}(X)$:

$$\mathcal{D}_X := \langle \mathcal{O}_X, \operatorname{Coh}_{\le 1}(X) \rangle_{\mathrm{tr}} \subset D^b \operatorname{Coh}(X).$$

Here for a set of objects $S \subset D^b \operatorname{Coh}(X)$, we denote by $\langle S \rangle_{\mathrm{tr}}$ the smallest triangulated subcategory of $D^b \operatorname{Coh}(X)$ which contains S. The triangulated category $\mathcal{D}_X$ is called the category of D0-D2-D6 bound states using the terminology in string theory. For example, it is easy to see that the ideal sheaves I_C for one dimensional subschemes $C \subset X$, two term complexes (2.9), are objects in $\mathcal{D}_X$. It is easier to construct Bridgeland stability conditions on $\mathcal{D}_X$, so we can discuss the previous steps replacing $D^b \operatorname{Coh}(X)$ by $\mathcal{D}_X$.

Now the situation is easier by considering $\mathcal{D}_X$, but still there exist some problems. There are two issues. One of them is that, while it is possible to construct a stability condition corresponding to σ_P, it is more difficult to construct a stability condition corresponding to σ_I. Another one concerns the last step on the derivation of the wall-crossing formula. If we apply the arguments of Joyce [34] and Joyce and Song [35], we see that the combinatorial data of numerical classes associated to wall-crossing contribute to the wall-crossing formula of DT type invariants. This combinatorial data is still too complicated for stability conditions on $\mathcal{D}_X$ to write down.

The notion of weak stability conditions on triangulated categories was introduced in [76] to solve these issues. Let $\mathcal{D}$ be a triangulated category and Γ a finitely generated free abelian group. We fix a group homomorphism $\mathrm{cl}\colon K(\mathcal{D}) \to \Gamma$ together with a saturated filtration

$$0 \subset \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_N = \Gamma.$$

A weak stability condition on $\mathcal{D}$ is defined to be a pair $(Z, \mathcal{A})$ satisfying the similar axioms of Bridgeland stability conditions. Here $\mathcal{A} \subset \mathcal{D}$ is the heart of a t-structure, and Z is an element

$$Z = \{Z_i\}_{i=0}^{N} \in \prod_{i=0}^{N} \operatorname{Hom}(\Gamma_i/\Gamma_{i-1}, \mathbb{C}).$$

If the filtration $\Gamma_\bullet$ is trivial, i.e., $N = 0$, the notion of weak stability conditions is essentially the same as that of Bridgeland stability conditions. If we set $\operatorname{Stab}_{\Gamma_\bullet}(\mathcal{D})$ to be the set of weak stability conditions on $\mathcal{D}$ satisfying the support property, one can show that it has a structure of a complex manifold similarly to the Bridgeland stability. (The property corresponding to the numerical property is automatically satisfied by the definition.) This space is expected to appear as limiting degeneration points of the space of Bridgeland stability conditions $\operatorname{Stab}(\mathcal{D})$, and the choice of a filtration $\Gamma_\bullet$ determines the direction of the degeneration. However, a mathematically rigorous result on it is not available at this moment.

The result of Theorem 3.1 was obtained by applying the steps in the previous subsection for the space of weak stability conditions on $\mathcal{D}_X$. More precisely, in the case of $\mathcal{D} = \mathcal{D}_X$ for a Calabi-Yau 3-fold X, we set

$$\Gamma = H_0(X, \mathbb{Z}) \oplus H_2(X, \mathbb{Z})_{\mathrm{fr}} \oplus H^0(X, \mathbb{Z})$$

and set cl as $\mathrm{cl}(E) = (\mathrm{ch}_3(E), \mathrm{ch}_2(E), \mathrm{ch}_0(E))$. Here $(*)_{\mathrm{fr}}$ means taking the free quotient. For the derivation of the formula (3.1), we set the filtration $\Gamma_\bullet$ as

$$\Gamma_0 = H_0(X,\mathbb{Z}),\ \Gamma_1 = H_0(X,\mathbb{Z}) \oplus H_2(X,\mathbb{Z})_{\mathrm{fr}},\ \Gamma_2 = \Gamma$$

and for the derivation of the formula (3.2), we set the filtration $\Gamma_\bullet$ as

$$\Gamma_0 = H_0(X,\mathbb{Z}) \oplus H_2(X,\mathbb{Z})_{\mathrm{fr}},\ \Gamma_1 = \Gamma.$$

The above technical issues are settled using the above setting.

3.5. **The invariant $N_{n,\beta}$.** Finally, in this section, we explain the invariant $N_{n,\beta} \in \mathbb{Q}$ which appears in Theorem 3.1. This invariant is a particular case of generalized DT invariants discussed in Remark 2.15, and appears as the difference of rank one DT type invariants under wall-crossing in the weak stability conditions on $\mathcal{D}_X$. The geometric meaning of the invariant $N_{n,\beta}$ is that it counts one dimensional semistable sheaves F satisfying the numerical condition (2.8). More precisely, let H be an ample divisor on X and let $\mathcal{M}_n^H(X,\beta)$ be the moduli stack of H-semistable sheaves $F \in \mathrm{Coh}_{\le 1}(X)$ satisfying the numerical condition (2.8). Here, the H-semistable sheaves are defined as in Example 3.5 (i). If we assume that $\mathcal{M}_n^H(X,\beta)$ is represented by a projective scheme (for example, this is satisfied for a suitable choice of H if n and β are coprime), then $N_{n,\beta}$ is defined in the following way using the Behrend constructible function χ_B on $\mathcal{M}_n^H(X,\beta)$:

$$N_{n,\beta} = \int_{\mathcal{M}_n^H(X,\beta)} \chi_B \, d\chi. \tag{3.8}$$

The moduli space $\mathcal{M}_n^H(X,\beta)$ depends on H, but the invariant $N_{n,\beta}$ can be shown to be independent of H. The invariant $N_{n,\beta}$ is an integer if it is defined by (3.8), but this is a special case of the assumption that $\mathcal{M}_n(X,\beta)$ is realized as a projective scheme. In general, $\mathcal{M}_n^H(X,\beta)$ is an algebraic stack, and one needs to take the logarithm of $\mathcal{M}_n^H(X,\beta)$ in the motivic Hall algebra and integrate the Behrend function on it to define the invariant $N_{n,\beta}$, as we discussed in Remark 2.15. It is in general difficult to compute $N_{n,\beta}$, but it is possible to do in some cases by combining the result of Example 2.11 and Theorem 3.1. We give some examples below (for details, we refer to [35], [81]):

Example 3.12. (i) In the case of $\beta = 0$, the invariant $N_{n,0}$ is given as follows:

$$N_{n,0} = -\chi(X) \sum_{k\ge 1, k|n} \frac{1}{k^2}.$$

(ii) Let $C \subset X$ be a rational curve as in Example 2.7. In this case, the invariant $N_{n,d[C]}$ is given as follows:

$$N_{n,d[C]} = \frac{1}{k^2}, \quad k = \mathrm{g.c.d.}(n,d).$$

The invariants $N_{n,\beta}$ in the above examples are rational numbers, and the squares of the integers which divide (n,β) appear in the denominators. This is the property expected in general, and formulated in the following conjecture (cf. [81, Conjecture 6.3]):

Conjecture 3.13. *We have the following formula:*

$$N_{n,\beta} = \sum_{k\geq 1, k|(n,\beta)} \frac{1}{k^2} N_{1,\beta/k}.$$

The above formula is called a multiple cover formula. Here we remark that the invariant $N_{1,\beta}$ is always an integer. By combining the result of Theorem 3.1 and Conjecture 3.13, the series $P(X)$ is expected to be described in the following way:

$$P(X) = \prod_{\beta>0} \prod_{j=1}^{\infty} (1-(-q)^j t^\beta)^{jN_{1,\beta}} \left(\sum_{n,\beta} L_{n,\beta} q^n t^\beta \right).$$

The above formula is shown in [81, Theorem 6.4] to be equivalent to Pandharipande-Thomas's strong rationality conjecture [57] of $P(X)$. Also, some special cases of Conjecture 3.13 are discussed in [65], [66].

4. Other results on DT invariants

The arguments of Theorem 3.1 were later used to show several results on DT type invariants and Bridgeland stability conditions. Below we give some of them.

4.1. **Flop formula of DT invariants.** The paper [63] is a continuation of [76], where we obtained a birational transformation formula of DT invariants using the wall-crossing phenomena in the category of D0-D2-D6 bound states. Let X be a Calabi-Yau 3-fold and let $f: X \to Y$ be a flopping contraction. By definition, f is isomorphic in codimension one with relative Picard number one and Y has at worst Gorenstein singularities. Given a flopping contraction f, its flop $f^\dagger: X^\dagger \to Y$ is uniquely constructed. The birational map

$$\phi := f^{-1} \circ f^\dagger : X^\dagger \dashrightarrow X$$

is not isomorphic, but $D^b \operatorname{Coh}(X^\dagger)$ and $D^b \operatorname{Coh}(X)$ are known to be equivalent by Bridgeland [15]. For a flopping contraction $f: X \to Y$, we define the generating series $I(X/Y)$ in the following way:

$$I(X/Y) := \sum_{f_*\beta=0} I_\beta(X) t^\beta.$$

The following is the main theorem of [63]:

Theorem 4.1 (Toda [63] (Euler number version), Calabrese [20])**.** *We have the following formula:*

$$I(X/Y) = i \circ \phi_* I(X^\dagger/Y),$$
$$\frac{I(X)}{I(X/Y)} = \phi_* \frac{I(X^\dagger)}{I(X^\dagger/Y)}.$$

Here ϕ_* and i are variable changes $\phi_*(\beta,n) = (\phi_*\beta, n)$, $i(\beta,n) = (-\beta,n)$.

Remark 4.2. Similarly to Theorem 3.1, the result of Theorem 4.1 was first proved for the Euler number version in [63], and the result for the honest DT invariants relied on [10]. Later Theorem 4.1 was proved without relying on [10] by Calabrese [20] following the argument of Bridgeland [19].

Remark 4.3. In the same situation as above, Van den Bergh [21] constructed the sheaf of non-commutative algebras $\mathcal{A}_Y$ on Y, and showed the derived equivalence

$$D^b \operatorname{Coh}(X) \xrightarrow{\sim} D^b \operatorname{Coh}(\mathcal{A}_Y) \xrightarrow{\sim} D^b \operatorname{Coh}(X^\dagger).$$

The relationship between DT invariants on X and 'non-commutative DT invariants' which count $\mathcal{A}_Y$-modules are also known at least for Euler number version [63]. The non-commutative DT invariants for the local $(-1,-1)$-curve are discussed in [60], [53].

4.2. **Stable pairs on local K3 surfaces.** In this subsection, we introduce the result of [82] on the Euler numbers of moduli spaces of stable pairs on local K3 surfaces. Let S be a K3 surface, i.e., two dimensional Calabi-Yau manifold, and set $X = S \times \mathbb{C}$. Although X is a Calabi-Yau manifold, it is not compact. Also, S admits deformations to non-algebraic K3 surfaces. Therefore, if we define DT invariants or PT invariants in a way similar to Subsection 2, (by the invariance under deformations of complex structures) the resulting invariants are always zero. In this case, in order to define reasonable invariants, one needs to construct reduced perfect obstruction theories on the moduli spaces by reducing the obstruction theory, and integrate the associated reduced virtual fundamental cycles. One can define reduced DT invariants or PT invariants by integrating the reduced virtual cycles. We refer to [49] for details. These reduced invariants are not invariant under deformations of complex structures, but invariant if the given curve classes are Hodge (1, 1)-type under deformations. It is not known whether these reduced DT (PT) invariants are described by Behrend functions or not, but they coincide with topological Euler numbers $\chi(I_n(X,\beta))$, $\chi(P_n(X,\beta))$ if the curve class $\beta \in H_2(X,\mathbb{Z})$ is irreducible. Here β is called irreducible if it is not written as $\beta = \beta_1 + \beta_2$ for effective algebraic one cycle classes β_1, β_2. By the above reason, instead of the above reduced invariants, we consider the Euler number version of PT invariants, and put the following generating series:

$$P^\chi(X) := 1 + \sum_{n\in\mathbb{Z}, \beta>0} \chi(P_n(X,\beta)) q^n t^\beta.$$

The main result of the paper [82] is to derive the product expansion formula of the generating series $P^\chi(X)$ in terms of invariants counting semistable sheaves with compact supports on X (the dimensions of the supports are not necessarily less than or equal to one, and can be two). The latter invariants are closely related to Euler numbers of Hilbert schemes of points on K3 surfaces. Using this result, we expect to obtain the Euler number version of the modular invariance conjecture of PT invariants by Katz-Klemm-Vafa [36].

Below, we introduce invariants counting semistable sheaves on X. Let $\operatorname{Coh}_c(X)$ be the category of coherent sheaves on X with compact supports, and let π be the projection from X to S. Let H be an ample divisor on S, and take a cohomology class

$$v = (r, \beta, n) \in H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}).$$

We denote by $\mathcal{M}^H(v)$ the moduli stack of π^*H-semistable sheaves $F \in \operatorname{Coh}_c(X)$ with Mukai vector v. Here for an object $F \in \operatorname{Coh}_c(X)$, its Mukai vector is defined in the following way:

$$v(F) := \operatorname{ch}(\pi_* F)\sqrt{\operatorname{td}_S} \in H^*(S,\mathbb{Z}).$$

The π^*H-semistability on $\mathrm{Coh}_c(X)$ is defined by reduced Hilbert polynomials with respect to π^*H similarly to the classical Gieseker semistability (cf. [33]). If the moduli stack $\mathcal{M}^H(v)$ is realized as a scheme, and any closed point of $\mathcal{M}^H(v)$ corresponds to a π^*H-stable sheaf, then $\mathcal{M}^H(v)$ is written as follows:

$$\mathcal{M}^H(v) = M^H(v) \times \mathbb{C}. \tag{4.1}$$

Here $M^H(v)$ is a $(v,v)/2+1$-dimensional holomorphic symplectic manifold (cf. [51]). The inner product on $H^*(S,\mathbb{Z})$ is defined in the following way, and called a Mukai product:

$$((r_1,\beta_1,n_1),(r_2,\beta_2,n_2)) = \beta_1\beta_2 - r_1n_2 - r_2n_1.$$

Under the above setting, we define the invariant $J(v) \in \mathbb{Q}$ as follows:

$$J(v) = \text{`}\chi\text{'}(\mathcal{M}^H(v)). \tag{4.2}$$

Remark 4.4. Here we denoted the Euler number of the right-hand side of (4.2) as 'χ' by the following reason. If the moduli stack $\mathcal{M}^H(v)$ is written as (4.1), then it is defined to be the usual Euler number $J(v) = \chi(M^H(v))$. However, in general, $\mathcal{M}^H(v)$ is an algebraic stack of finite type, and we are not able to define its Euler number if there exist non-trivial stabilizers groups. Hence, we denote the Euler number which defines $J(v)$ by 'χ' to emphasize that it is not a rigorous definition. In order to give a precise definition, similarly to generalized DT invariants in Remark 2.15, we need to take the logarithm of $\mathcal{M}^H(v)$ in the motivic Hall algebra, and its Euler number. Also, similarly to the invariant $N_{n,\beta}$, the invariant $J(v)$ is also independent of a choice of H. For the details, see [82].

In general it is difficult to compute the invariant $J(v)$, but we will later give a conjecture on the value of $J(v)$ in Conjecture 4.9 as an analogue of Conjecture 3.13. We first state the main result of [82]:

Theorem 4.5 (Toda [82])**.** *We have the following product expansion formula:*

$$\begin{aligned} P^{\chi}(X) = &\prod_{r\ge 0,\beta>0,n\ge 0} \exp\left(J(r,\beta,r+n)q^n t^\beta\right)^{n+2r} \\ &\cdot \prod_{r>0,\beta>0,n>0} \exp\left(J(r,\beta,r+n)q^{-n} t^\beta\right)^{n+2r}. \end{aligned} \tag{4.3}$$

Remark 4.6. If the curve class β is irreducible, then $\chi(P_n(X,\beta))$ was computed by Kawai-Yoshioka [37]. The result of Kawai-Yoshioka [37] is recovered by Theorem 4.5 together with the identities (4.4), (4.5) discussed later.

Remark 4.7. Similarly to Theorem 3.1, the result of Theorem 4.5 was proved using the wall-crossing argument in the space of weak stability conditions on the derived category. However, as the right-hand side of Theorem 4.5 contains contributions from sheaves with two dimensional supports, it is not enough to consider the category $\mathcal{D}_X$ of D0-D2-D6 bound states, and we need to consider the triangulated category which contains two dimensional sheaves.

By the identity (4.3), the computation of $\chi(P_n(X,\beta))$ is equivalent to that of $J(v)$. Let us discuss the invariant $J(v)$ in more detail. Suppose that $v \in H^*(S,\mathbb{Z})$ is primitive. Then for a suitable H, $\mathcal{M}^H(v)$ is written as (4.1). Furthermore, as the holomorphic symplectic manifold $M^H(v)$ is deformation equivalent to the Hilbert scheme of points on S (cf. [86]), the invariant $J(v)$ is written as

$$J(v) = \chi(\mathrm{Hilb}^{(v,v)/2+1}(S)). \tag{4.4}$$

The right-hand side of the above identity was computed by Göttsche [29]:

$$\sum_{n\geq 0} \chi(\mathrm{Hilb}^n(S))q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^{24}}. \tag{4.5}$$

In the case that $v \in H^*(S,\mathbb{Z})$ is not primitive, the invariant $J(v)$ has some symmetry induced by automorphisms of the Mukai lattice $H^*(S,\mathbb{Z})$.

Theorem 4.8 ([82])**.** *Let $g \in \mathrm{Aut}(H^*(S,\mathbb{Z}))$ be an isomorphism of the Mukai lattice preserving the weight two Hodge structure. Then we have the following formula:*

$$J(gv) = J(v).$$

Here the weight two Hodge structure on $H^*(S,\mathbb{Z})$ is defined by

$$H^{*2,0} = H^{2,0},\ H^{*0,2} = H^{0,2},\ H^{*1,1} = H^{0,0} \oplus H^{1,1} \oplus H^{2,2}.$$

Still Theorem 4.8 does not give the complete computation of $J(v)$, but we have the following conjecture on the value of $J(v)$ based on the above result and Conjecture 3.13 (cf. [82, Conjecture 3.1]):

Conjecture 4.9. *If $v \in H^*(S,\mathbb{Z})$ is an algebraic class, we have the following formula:*

$$J(v) = \sum_{k\geq 1, k|v} \frac{1}{k^2}\chi(\mathrm{Hilb}^{(v/k,v/k)/2+1}(S)).$$

If we assume Theorem 4.5 and Conjecture 4.9, the series $P^{\chi}(X)$ is written as

$$\begin{aligned} P^{\chi}(X) = &\prod_{r\geq 0, \beta>0, n\geq 0} (1-t^{\beta}q^n)^{-(n+2r)\chi(\mathrm{Hilb}^{\beta^2/2-r(n+r)+1}(S))} \\ &\cdot \prod_{r>0,\beta>0,n>0} (1-t^{\beta}q^{-n})^{-(n+2r)\chi(\mathrm{Hilb}^{\beta^2/2-r(n+r)+1}(S))}. \end{aligned} \tag{4.6}$$

As we discussed in [82, Section 6], the identity (4.6) resembles the automorphic form given by the Borcherds infinite product [14], and gives the Euler number version of the Katz-Klemm-Vafa [36] conjecture on the modularity of the generating series of PT invariants.

5. Bogomolov-Gieseker type inequality conjecture and its application

We have obtained several results on DT invariants by using weak stability conditions. However, in order to obtain further applications, we need to construct the honest Bridgeland stability conditions. In this section, we explain the candidate of Bridgeland stability conditions on smooth projective 3-folds proposed by Bayer, Macri, and the author [5].

5.1. **Neighborhood at the large volume limit.** Let X be a smooth projective variety of dimension d. We would like to construct stability conditions on $D^b \operatorname{Coh}(X)$ in the sense of Definition 3.4. As we mentioned in Remark 3.8, we cannot take the heart of a t-structure to be $\operatorname{Coh}(X)$ if $d \geq 2$. However, there is a candidate of the group homomorphism Z in the context of string theory. We take $B+\sqrt{-1}\omega \in H^2(X,\mathbb{C})$ such that ω is ample. Also, we set the group homomorphism $Z_{B,\omega}\colon K(X) \to \mathbb{C}$ as follows:

$$Z_{B,\omega}(E) = -\int_X e^{-i\omega} \operatorname{ch}^B(E).$$

Here $\operatorname{ch}^B(E) = e^{-B} \operatorname{ch}(E)$. We expect that the following conjecture holds:

Conjecture 5.1. *There exists the heart of a bounded t-structure* $\mathcal{A}_{B,\omega} \subset D^b \operatorname{Coh}(X)$ *such that the following holds:*

$$\sigma_{B,\omega} = (Z_{B,\omega}, \mathcal{A}_{B,\omega}) \in \operatorname{Stab}(X). \tag{5.1}$$

The set of stability conditions obtained as above is called the neighborhood at the large volume limit. The $\omega = \infty$ is the large volume limiting point. If $d = 1$, we have

$$Z_{B,\omega}(E) = -\deg(E) + (B + \sqrt{-1}\omega)\operatorname{rank}(E)$$

and it is easy to check that $\mathcal{A}_{B,\omega} = \operatorname{Coh}(X)$ satisfies the condition (5.1).

If $d = 2$, we have

$$Z_{B,\omega}(E) = -\operatorname{ch}_2^B(E) + \frac{\omega^2}{2}\operatorname{ch}_0^B(E) + \sqrt{-1}\operatorname{ch}_1^B(E)\omega.$$

In this case, $\mathcal{A}_{B,\omega}$ is obtained as a tilting of $\operatorname{Coh}(X)$. We first explain the tilting of $\operatorname{Coh}(X)$ for a general d. We define the function μ_ω on $\operatorname{Coh}(X) \setminus \{0\}$ as follows:

$$\mu_\omega(E) := \frac{c_1(E)\cdot \omega^{d-1}}{\operatorname{rank}(E)} \in \mathbb{R}\cup\{\infty\}.$$

Here we set $\mu_\omega(E) = \infty$ if $\operatorname{rank}(E) = 0$. The function μ_ω determines the μ_ω-stability on $\operatorname{Coh}(X)$ similarly to the classical stability.

Definition 5.2. An object $E \in \operatorname{Coh}(X)$ is called μ_ω-semistable if for any subobject $0 \neq F \subset E$, we have the inequality $\mu_\omega(F) \leq \mu_\omega(E)$.

Remark 5.3. In the case of $d \geq 2$, the μ_ω-semistability does not determine a stability condition on $\operatorname{Coh}(X)$ in the sense of Definition 3.4. For instance, the group homomorphism

$$Z\colon K(X) \ni E \mapsto -c_1(E)\cdot\omega^{d-1} + \sqrt{-1}\operatorname{rank}(E) \in \mathbb{C}$$

satisfies $Z(\mathcal{O}_x) = 0$, hence does not satisfy the axioms in Definition 3.4.

Using the μ_ω-stability, we determine the pair of subcategories $(\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})$ on $\operatorname{Coh}(X)$ as follows:

$$\mathcal{T}_{B,\omega} := \langle E \in \operatorname{Coh}(X) : E \text{ is } \mu_\omega\text{-semistable with } \mu_\omega(E) > B\omega^{d-1}\rangle_{\mathrm{ex}},$$
$$\mathcal{F}_{B,\omega} := \langle E \in \operatorname{Coh}(X) : E \text{ is } \mu_\omega\text{-semistable with } \mu_\omega(E) \leq B\omega^{d-1}\rangle_{\mathrm{ex}}.$$

Here $\langle * \rangle_{\mathrm{ex}}$ means the extension closure, i.e., the smallest extension closed subcategory of $\operatorname{Coh}(X)$ which contains $*$. By the existence of HN filtrations with respect to the μ_ω-stability, it is easy to check that

- for any $T \in \mathcal{T}_{B,\omega}$ and $F \in \mathcal{F}_{B,\omega}$, we have $\mathrm{Hom}(T, F) = 0$.
- For any $E \in \mathrm{Coh}(X)$, there exists an exact sequence $0 \to T \to E \to F \to 0$ such that $T \in \mathcal{T}_{B,\omega}$, $F \in \mathcal{F}_{B,\omega}$.

A pair of subcategories $(\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})$ satisfying the above conditions is called a torsion pair (cf. [32]). Given a torsion pair, we have the associated heart of a t-structure

$$\mathcal{B}_{B,\omega} = \langle \mathcal{F}_{B,\omega}[1], \mathcal{T}_{B,\omega} \rangle_{\mathrm{ex}} \subset D^b \,\mathrm{Coh}(X).$$

This is called the tilting of $\mathrm{Coh}(X)$ with respect to the torsion pair $(\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})$. In the case of $d = 2$, we can construct stability conditions by using the above heart of the t-structure:

Proposition 5.4. *Let X be a smooth projective surface. We take $B + \sqrt{-1}\omega \in H^2(X, \mathbb{C})$ such that ω is ample. Then we have the following:*

$$(Z_{B,\omega}, \mathcal{B}_{B,\omega}) \in \mathrm{Stab}(X).$$

We briefly explain the above proposition. First, we immediately have $\mathrm{Im}\, Z_{B,\omega}(E) \geq 0$ for any object $0 \neq E \in \mathcal{B}_{B,\omega}$ by the construction. In order to show the condition (3.3) in Definition 3.4, we need to show that $\mathrm{Re}\, Z_{B,\omega}(E) < 0$ if $\mathrm{Im}\, Z_{B,\omega}(E) = 0$. This is proved using the Hodge index theorem and the following Bogomolov-Gieseker inequality. We refer to [18], [2] for details.

Theorem 5.5 (Bogomolov [13], Gieseker [28])**.** *Let X be a smooth projective d-fold with $d \geq 2$. We take $B + \sqrt{-1}\omega \in H^2(X, \mathbb{C})$ such that ω is an ample class. Then for any torsion free μ_ω-semistable sheaf $F \in \mathrm{Coh}(X)$, we have*

$$\omega^{d-2} \left(\mathrm{ch}_1^B(F)^2 - 2\,\mathrm{ch}_0^B(F)\,\mathrm{ch}_2^B(F) \right) \geq 0. \tag{5.2}$$

It is not difficult to show the Harder-Narasimhan property once the condition (3.3) is proved. The proof of the support property in (3.4) requires more argument, and is proved in [83].

5.2. **The BG type inequality conjecture for smooth projective 3-folds.** In this subsection, we set $d = 3$ and assume that B, ω are defined over rational coefficients for some technical reason. In this case, $Z_{B,\omega}$ is as follows:

$$Z_{B,\omega}(E) = -\,\mathrm{ch}_3^B(E) + \frac{\omega^2}{2}\,\mathrm{ch}_1^B(E) + \sqrt{-1}\left(\omega\,\mathrm{ch}_2^B(E) - \frac{\omega^3}{6}\,\mathrm{ch}_0^B(E) \right). \tag{5.3}$$

Contrary to the $d = 2$ case, the pair $(Z_{B,\omega}, \mathcal{B}_{B,\omega})$ does not satisfy the condition (3.3). Therefore, we need to do one more tilting of $\mathcal{B}_{B,\omega}$. A hint for the further tilting is the following lemma:

Lemma 5.6. *For an object $0 \neq E \in \mathcal{B}_{B,\omega}$, one of the following conditions holds:*

- $\omega^2\,\mathrm{ch}_1^B(E) > 0$.
- $\omega^2\,\mathrm{ch}_1^B(E) = 0$, $\mathrm{Im}\, Z_{B,\omega}(E) > 0$.
- $\omega^2\,\mathrm{ch}_1^B(E) = \mathrm{Im}\, Z_{B,\omega}(E) = 0$, $-\mathrm{Re}\, Z_{B,\omega}(E) > 0$.

The proof is similar to Proposition 5.4, and uses the inequality in Theorem 5.5. We refer to [5, Lemma 3.2.1] for details. Lemma 5.6 means that the triple

$$(\omega^2\,\mathrm{ch}_1^B(E), \mathrm{Im}\, Z_{B,\omega}(E), -\mathrm{Re}\, Z_{B,\omega}(E))$$

behaves as if it were a triple

$$(\mathrm{rank}(F), \mathrm{ch}_1(F)\omega, \mathrm{ch}_2(F))$$

for a coherent sheaf $0 \neq F$ on an algebraic surface S. Therefore, by mimicking the μ_ω-stability on algebraic surfaces, we consider the following function on $\mathcal{B}_{B,\omega} \setminus \{0\}$:

$$\nu_{B,\omega}(E) := \frac{\mathrm{Im}\, Z_{B,\omega}(E)}{\omega^2 \mathrm{ch}_1^B(E)} \in \mathbb{Q} \cup \{\infty\}.$$

Here we set $\nu_{B,\omega}(E) = \infty$ if $\omega^2 \mathrm{ch}_1^B(E) = 0$.

Definition 5.7. An object $E \in \mathcal{B}_{B,\omega}$ is called $\nu_{B,\omega}$-semistable if for any subobject $0 \neq F \subset E$ in $\mathcal{B}_{B,\omega}$, we have the inequality $\nu_{B,\omega}(F) \leq \nu_{B,\omega}(E)$.

Similarly to the μ_ω-stability, we can show the existence of Harder-Narasimhan filtrations with respect to the $\nu_{B,\omega}$-stability using Lemma 5.6. Therefore, similarly to the torsion pair $(\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})$ on $\mathrm{Coh}(X)$, we can define the torsion pair on $\mathcal{B}_{B,\omega}$ as follows: :

$$\mathcal{T}'_{B,\omega} := \langle E \in \mathcal{B}_{B,\omega} : E \text{ is } \nu_{B,\omega}\text{-semistable } \nu_{B,\omega}(E) > 0\rangle_{\mathrm{ex}},$$
$$\mathcal{F}'_{B,\omega} := \langle E \in \mathcal{B}_{B,\omega} : E \text{ is } \nu_{B,\omega}\text{-semistable } \nu_{B,\omega}(E) \leq 0\rangle_{\mathrm{ex}}.$$

By taking the tilting with respect to the above torsion pair, we obtain the heart of a t-structure

$$\mathcal{A}_{B,\omega} := \langle \mathcal{F}'_{B,\omega}[1], \mathcal{T}'_{B,\omega}\rangle_{\mathrm{ex}} \subset D^b \mathrm{Coh}(X). \tag{5.4}$$

By the construction, we have $\mathrm{Im}\, Z_{B,\omega}(E) \geq 0$ for any $E \in \mathcal{A}_{B,\omega}$. Together with Bayer and Macri, the author conjectured that the above hearts of the t-structures give Bridgeland stability conditions on smooth projective 3-folds:

Conjecture 5.8 (Bayer-Macri-Toda [5])**.** *Let X be a smooth projective* 3*-fold, and take $B + \sqrt{-1}\omega \in H^2(X, \mathbb{C})$ such that ω is an ample class. Then the following holds:*

$$(Z_{B,\omega}, \mathcal{A}_{B,\omega}) \in \mathrm{Stab}(X).$$

Here $Z_{B,\omega}$ is given by (5.3)*, and $\mathcal{A}_{B,\omega}$ is given by* (5.4)*.*

In order to show the above conjecture, we need to show that any object $0 \neq E \in \mathcal{A}_{B,\omega}$ satisfying $\mathrm{Im}\, Z_{B,\omega}(E) = 0$ satisfies $\mathrm{Re}\, Z_{B,\omega}(E) < 0$, similarly to the surface case. In the surface case, we used Theorem 5.5. Unfortunately a generalization of Theorem 5.5 which evaluates ch_3 is not known. So far, even a conjecture has not been proposed. However, Conjecture 5.8 led to the following conjectural 3-fold version of Theorem 5.5:

Conjecture 5.9 (Bayer-Macri-Toda [5])**.** *In the same situation of Conjecture* 5.8*, let $F \in \mathcal{B}_{B,\omega}$ be a $\nu_{B,\omega}$-semistable object with $\nu_{B,\omega}(F) = 0$. Then we have the inequality*

$$\mathrm{ch}_3^B(E) \leq \frac{\omega^2}{18} \mathrm{ch}_1^B(E). \tag{5.5}$$

Remark 5.10. If Conjecture 5.9 holds, then it is shown in [5] that the pair $(Z_{B,\omega}, \mathcal{A}_{B,\omega})$ gives a stability condition on $D^b \mathrm{Coh}(X)$. On the other hand, in order to show the latter statement, it is enough to show the weaker inequality $\mathrm{ch}_3^B(E) < \omega^2 \mathrm{ch}_1^B(E)/2$. The coefficient $1/18$ in the inequality (5.5) is determined so that the equality holds for objects satisfying the equality in the classical BG inequality (5.2). Also, even if Conjecture 5.9 holds, the support property is not proved and one cannot conclude Conjecture 5.8.

5.3. **Evidence of Conjecture 5.9.** At this moment, the author does not have any idea of how to give a complete solution of Conjecture 5.9. For some explicit examples of smooth projective 3-folds, we have the following results. First, in the case of $X = \mathbb{P}^3$, Conjecture 5.9 was proved by Macri [46]. This was also partially proved in [5]. The idea is to set the non-commutative algebra A as

$$A = \mathrm{End}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(3))$$

and compare $\mathrm{mod}(A)$ with the heart $\mathcal{A}_{B,\omega}$ on $D^b \mathrm{Coh}(\mathbb{P}^3)$ through the Beilinson equivalence [11]

$$D^b \mathrm{Coh}(\mathbb{P}^3) \cong D^b \mathrm{mod}(A).$$

It is easy to construct stability conditions on $\mathrm{mod}(A)$, and their existence is used to show Conjecture 5.9. A similar idea was used by Schmidt [59] to show Conjecture 5.9 for the smooth quadric.

Conjecture 5.9 is in particular important for Calabi-Yau manifolds. In 2013, in the works [44], [45], Maciocia-Piyaratne proved Conjecture 5.9 for principally polarized abelian 3-folds with Picard number one. As abelian varieties have trivial canonical divisors, they are closer to Calabi-Yau manifolds than the above $\mathbb{P}^3$ or the quadric. Their work makes progress toward the construction problem of stability conditions on Calabi-Yau 3-folds. The idea of Maciocia-Piyaratne was to use Fourier-Mukai transforms between abelian varieties and their dual abelian varieties. In the case of principally polarized abelian varieties, their dual abelian varieties are isomorphic to the original abelian varieties, so it gives an autoequivalence of the derived category. Maciocia-Piyaratne proved that the hearts of the form $\mathcal{A}_{B,\omega}$ are preserved by the autoequivalences given by Fourier-Mukai transforms. The proof of this fact is quite technical, but this enables us to reduce the evaluation of ch_3 in Conjecture 5.9 to the classical BG inequality in Theorem 5.5, and prove Conjecture 5.9 in this case.

As another kind of evidence, one can (almost) prove the Fujita conjecture for smooth projective 3-folds assuming Conjecture 5.9. Here the Fujita conjecture is stated as follows:

Conjecture 5.11 (Fujita [27]). *Let X be a smooth projective d-fold, and let L be an ample divisor on X. Then*

- $K_X + (d+1)L$ *is free.*
- $K_X + (d+2)L$ *is very ample.*

In the two dimensional case, the above conjecture was proved by Reider [58] by applying Theorem 5.5. In the three dimensional case, the freeness part of the above conjecture is proved (cf. [24], [38]). However, the very ampleness part is still an open problem. As Conjecture 5.9 is regarded as a three dimensional version of Theorem 5.5, one naturally expects that Conjecture 5.9 would imply the Fujita conjecture for 3-folds, similarly to the result of Reider [58]. it would imply the Fujita conjecture for 3-folds, similarly to the result of Reider [58]. Indeed, we have the following result:

Theorem 5.12 (Bayer-Bertram-Macri-Toda [3]). *Let X be a smooth projective 3-fold satisfying Conjecture 5.9. Then for any ample divisor L on X, we have the following:*

- $K_X + 4L$ *is free.*
- $K_X + 6L$ *is very ample.*

Remark 5.13. As the Fujita conjecture states that $K_X + 5L$ is very ample, the result of Theorem 5.12 is slightly weaker than the Fujita conjecture. However, for instance, if we assume that the intersection number of K_X with any curve is an even number (e.g., X is a Calabi-Yau manifold), then one can also conclude that $K_X + 5L$ is very ample.

In the next subsection, we explain another example of this kind of evidence of Conjecture 5.9.

5.4. **Conjecture 5.9 and the Denef-Moore conjecture.** Conjecture 5.9 also implies an interesting relationship between invariants counting two dimensional semistable sheaves on Calabi-Yau 3-folds and curve counting invariants. This is a conjecture discussed by Denef-Moore [22] in the context of string theory, and has a close relation to the Ooguri-Strominger-Vafa (OSV) conjecture in [54]. Here the OSV conjecture is (here we state a mathematically non-rigorous way using the terminology of string theory) a conjectural approximation

$$\mathcal{Z}_{\mathrm{BH}} \sim |\mathcal{Z}_{\mathrm{top}}|^2. \tag{5.6}$$

The left-hand side is the partition function of black hole entropy, and the right-hand side is the partition function of topological string. A mathematical interpretation of the above conjecture is a certain approximation between the generating series of DT invariants counting two dimensional torsion sheaves on Calabi-Yau 3-folds and the square of the generating series of GW invariants counting curves. Denef-Moore [22] proposed a certain conjectural relationship between DT invariants counting two dimensional torsion sheaves and DT invariants counting curves, and used it to give an interpretation of the approximation (5.6). Below, we discuss Denef-Moore's conjecture.

Let X be a projective Calabi-Yau 3-fold, and let H be an ample divisor on X. For simplicity, we assume that $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(H)$. (For example, a quintic hypersurface in $\mathbb{P}^4$ is such an example.) For given $m \in \mathbb{Z}_{\geq 1}$, $\beta \in H_2(X, \mathbb{Q})$ and $n \in \mathbb{Q}$, let

$$\mathcal{M}(0, m[H], -\beta, -n) \tag{5.7}$$

be the moduli stack of H-semistable sheaves $E \in \mathrm{Coh}(X)$ satisfying the following numerical condition:

$$\mathrm{ch}(E) = (0, m[H], \beta, n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X).$$

Here the H-semistability is the classical H-Gieseker stability (cf. [33]), and we have regarded $\beta \in H^4(X, \mathbb{Q})$, $n \in H^6(X, \mathbb{Q})$ by the Poincaré duality. By the condition of the Chern character, the sheaf E is a torsion sheaf with two dimensional support, and the fundamental cycle $[E]$ determined by E is an element of the linear system $|mH|$. If the stack (5.7) is realized as a projective scheme, then the DT invariant counting two dimensional torsion sheaves is defined as follows:

$$\mathrm{DT}(0, m[H], \beta, n) = \int_{\mathcal{M}(0,m[H],\beta,n)} \chi_B \, d\chi.$$

Here χ_B is the Behrend constructible function on $\mathcal{M}(0, m[H], \beta, n)$. In general, the stack (5.7) is an algebraic stack of finite type, and the invariant $\mathrm{DT}(0, m[H], \beta, n) \in \mathbb{Q}$ is defined by integrating the Behrend constructible function of the log of (5.7) in the motivic Hall algebra as in Subsection 3.5. We define the generating series $\mathcal{Z}^m_{\mathrm{D4}}(X)$ as follows:

$$\mathcal{Z}^m_{\mathrm{D4}}(X) = \sum_{\beta,n} \mathrm{DT}(0, m[H], -\beta, -n) q^n t^\beta. \tag{5.8}$$

This is the generating series corresponding to the left-hand side of (5.6).

On the other hand, we define the following subseries of $I(X)$, $P(X)$:

$$I^m(X) = \sum_{(\beta,n)\in C(m)} I_{n,\beta} q^n t^\beta, \quad P^m(X) = \sum_{(\beta,n)\in C(m)} P_{n,\beta} q^n t^\beta.$$

Here $C(m)$ is defined to be

$$C(m) = \{(\beta, n) : \beta H < mH^3/2, |n| < m^2 H^3/2\}.$$

The generating series related to the right-hand side of (5.6) is given by

$$\begin{aligned}\mathcal{Z}^m_{\mathrm{D6}-\overline{\mathrm{D6}}}(X) = \sum_{m_2 - m_1 = m} & q^{H^3(m_1^3 - m_2^3)/6} t^{H^2(m_1^2 - m_2^2)/2} w^{H^3 m^3/6 + Hc_2(X)m/12} \\ & \cdot I^m(qw^{-1}, q^{m_2 H} t w^{-mH}) P^m(qw^{-1}, q^{-m_1 H} t^{-1} w^{-mH}).\end{aligned}$$

Denef-Moore's conjecture [22] is stated as follows:

Conjecture 5.14 (Denef-Moore [22]). *For sufficiently large $m > 0$, we have the identity*

$$\mathcal{Z}^m_{\mathrm{D4}}(X) = \frac{\partial}{\partial w} \mathcal{Z}^m_{\mathrm{D6}-\overline{\mathrm{D6}}}(X)|_{w=-1} \tag{5.9}$$

modulo the terms of $q^n t^\beta$ with

$$-\frac{H^3}{24} m^3 \left(1 - \frac{1}{m}\right) \le n + \frac{(\beta \cdot H)^2}{2mH^3}. \tag{5.10}$$

Remark 5.15. The left-hand side of (5.9) is the series corresponding to the left-hand side of (5.6) for $m \gg 0$. The identity (5.9) only holds modulo (5.10), but the left-hand side of (5.9) is expected to be (almost) a Fourier development of a Jacobi form, and assuming it implies that (5.9) modulo (5.10) almost recovers the left-hand side of (5.9). On the other hand, the right-hand side of (5.9) can be related to the square of the generating series of GW invariants if we assume the MNOP conjecture.

Mathematically, the identity (5.9) indicates a relationship between invariants counting two dimensional torsion sheaves on Calabi-Yau 3-folds and curve counting invariants. Conjecture 5.14 was derived from physics arguments in the paper of Denef-Moore [22], and it was not clear what was mathematically essential. In fact, Conjecture 5.14 essentially follows from Conjecture 5.9. This fact was proved in the author's paper [62].

Theorem 5.16 (Toda [62]). *If Conjecture* 5.9 *is true, then the Euler number version of Conjecture* 5.14 *also holds.*

Conjecture 5.9 was proposed independently from Conjecture 5.14, and Theorem 5.16 was not supposed at that time. The result of Theorem 5.16 is an indirect evidence of Conjecture 5.9.

6. Future subjects

It turns out that Conjecture 5.9 is an important conjecture which derives Fujita conjecture and Denef-Moore conjecture. It is an important subject to solve this conjecture, while there also exist other important research subjects. We will describe some of them.

6.1. **DT type invariants counting matrix factorizations.** The set of stability conditions which are expected to be constructed via Conjecture 5.8 corresponds to the 'neighborhood at the large volume limit' in terms of string theory. It is also an important subject to construct limiting stability conditions which are far from the large volume limit. For example, let $X = (W = 0) \subset \mathbb{P}^4$ be a quintic hypersurface. Then, the moduli space of complex structures of a manifold mirror to X becomes:

$$\mathcal{M}_K = \left[\{ \psi \in \mathbb{C} : \psi^5 \neq 1 \} / \mu_5 \right].$$

The point $\psi = \infty$ is the large volume limit. There exist other special points $\psi^5 = 1$ (conifold point), $\psi = 0$ (Gepner point). Here the Gepner point is the fix point with respect to the μ_5-action.

By the mirror symmetry, the universal covering space of the above $\mathcal{M}_K$ is expected to be embedded into $\mathrm{Stab}(X)$. The Gepner point corresponds to a certain special stability condition under this embedding. This is a stability condition which corresponds to a 'natural' stability condition

$$\sigma_G \in \mathrm{Stab}(\mathrm{HMF}(W)) \tag{6.1}$$

in a certain sense on the right-hand side of the Orlov equivalence [55]

$$D^b \,\mathrm{Coh}(X) \cong \mathrm{HMF}(W).$$

Here $\mathrm{HMF}(W)$ is the triangulated category of graded matrix factorizations of W, whose objects consist of data

$$P^0 \xrightarrow{f} P^1 \xrightarrow{g} P^0(5).$$

Here each P^i is a finitely generated graded free $A = \mathbb{C}[x_1, \cdots, x_5]$-module, f, g are homomorphisms as graded A-modules, satisfying $f \circ g = \cdot W$, $g \circ f = \cdot W$.

In the paper [67], the author defined some properties which should characterize σ_G using the symmetry of μ_5, and called stability conditions satisfying such properties as Gepner type stability conditions. Some evidence of the existence of Gepner type stability conditions on the category of graded matrix factorizations is given in [68], [69]. If a Gepner type stability condition (6.1) exists, then it should be possible to define DT type invariants counting semistable matrix factorizations by constructing moduli spaces of σ_G-semistable objects. If one can construct such invariants, it should be possible to relate them with the usual DT invariants counting stable sheaves via wall-crossing phenomena. On the other hand, σ_G has a symmetry with respect to μ_5 which should lead to certain symmetry among the usual DT invariants. It should be an interesting problem to understand the meaning of such a symmetry.

6.2. **Modularity problem of DT invariants counting two dimensional torsion sheaves.** As we mentioned in Remark 5.15, the generating series (5.8) is conjectured to have a close relationship with Fourier developments of certain Jacobi forms. It was conjectured by Vafa-Witten [84] that the similar generating series (the generating series of Euler numbers of moduli spaces of stable sheaves) on smooth algebraic surfaces have modularity, and are called S-duality conjecture. The modularity conjecture of the generating series (5.8) is a 3-fold version of Vafa-Witten's S-duality conjecture. However, the S-duality conjecture on algebraic surfaces is an open problem except the special cases such as rational surfaces, K3 surfaces (cf. [31]), and also the proof all rely on explicit calculations of the generating series. At this moment, the mathematical origin of the S-duality conjecture is mysterious. In the 3-fold version of the S-duality conjecture, the sheaves which contribute to the generating series (5.8) are two dimensional torsion sheaves, whose supports may have singularities. It is more difficult to count sheaves on singular surfaces than the case of smooth surfaces. On the other hand, it is known that the difference of the generating series under a blow-up of a smooth surface is described by a modular form [85], [43], [30], which gives an indirect evidence of the S-duality conjecture for algebraic surfaces. In the paper [70], we obtained a flop transformation formula of the generating series (5.8) under flops betweens Calabi-Yau 3-folds. Under a flop, the difference of the generating series is described by a Jacobi form, which gives an indirect evidence of a 3-fold version of the S-duality conjecture. This is a 3-fold version of the blow-up formula for algebraic surfaces, and a two dimensional torsion sheaf version of Theorem 4.1. Moreover, applying the result of [70], we proved in [71] that the generating series of Euler numbers of Hilbert schemes of points on algebraic surfaces with at worst A_n-type singularities is a modular form. In a view of 3-fold version of S-duality conjecture, we need to develop counting theory of sheaves on algebraic surfaces with singularities.

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